Universidade de Aveiro Departamento de Matemática
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Resultados de existência para equações elípticas com termos singulares

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Dissertação apresentada à Universidade de Aveiro e à Universidade do Minho para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática e Aplicações, realizada sob a orientação científica do Doutor Eugénio Alexandre Miguel Rocha, Professor Auxiliar do Departamento de Matemática da Universidade de Aveiro e sob coorientação científica do Doutor Jianqing Chen, investigador do Departamento de Matemática da Universidade de Aveiro.

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Dedico este trabalho aos meus dois maiores tesouros: a minha mãe e o meu marido.

o júri

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Doutor Eugénio Alexandre Miguel Rocha
Professor auxiliar da Universidade de Aveiro (Orientador)

Doutor Hugo Ricardo Nabais Tavares
Professor auxiliar convidado da Universidade de Lisboa

Doutor Jianqing Chen
Investigador da Universidade de Aveiro (Coorientador)

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palavras-chave

Métodos variacionais, equações diferenciais elípticas, equação Laplaciana não homogénea, termo singular, soluções positivas e nódais, termo concâvo, espaço de Lorentz, operador do tipo Leray-Lions.

resumo

Esta dissertação estuda em detalhe três problemas elípticos: (I) uma classe de equações que envolve o operador Laplaciano, um termo singular e não-linearidade com o exponente crítico de Sobolev, (II) uma classe de equações com singularidade dupla, o expoente crítico de Hardy-Sobolev e um termo côncavo e (III) uma classe de equações em forma divergente, que envolve um termo singular, um operador do tipo Leray-Lions, e uma função definida nos espaços de Lorentz.

As não-linearidades consideradas nos problemas (I) e (II), apresentam dificuldades adicionais, tais como uma singularidade forte no ponto zero (de modo que um "blow-up" pode ocorrer) e a falta de compacidade, devido à presença do exponente crítico de Sobolev (problema (I)) e Hardy-Sobolev (problema (II)). Pela singularidade existente no problema (III), a definição padrão de solução fraca pode não fazer sentido, por isso, é introduzida uma noção especial de solução fraca em subconjuntos abertos do domínio.

Métodos variacionais e técnicas da Teoria de Pontos Críticos são usados para provar a existência de soluções nos dois primeiros problemas. No problema (I), são usadas uma combinação adequada de técnicas de Nehari, o princípio variacional de Ekeland, métodos de minimax, um argumento de translação e estimativas integrais do nível de energia. Neste caso, demonstramos a existência de (pelo menos) quatro soluções não triviais onde pelo menos uma delas muda de sinal. No problema (II), usando o método de concentração de compacidade e o teorema de passagem de montanha, demostramos a existência de pelo menos duas soluções positivas e pelo menos um par de soluções com mudança de sinal. A abordagem do problema (III) combina um resultado de surjectividade para operadores monótonos, coercivos e radialmente contínuos com propriedades especiais do operador de tipo Leray-Lions. Demonstramos assim a existência de pelo menos, uma solução no espaço de Lorentz e obtemos uma estimativa para esta solução.

keywords

Variational methods, elliptic differential equations, inhomogeneous Laplacian equation, singular term, sign-changing solution, concave term, Lorentz spaces, Leray-Lions operator.

abstract

This dissertation study mainly three elliptical problems: (I) a class of equations, which involves the Laplacian operator, a singular term and a nonlinearity with the critical Sobolev exponent, (II) a class of equations with double singularity, the critical Hardy-Sobolev exponent and a concave term and (III) a class of equations in divergent form, which involves a singular term, a Leray-Lions operator, and a function defined on Lorentz spaces.

The nonlinearities considered in problems (I) and (II), bring additional difficulties which, as the strong singularity at zero (so blow-up may occur) and the lack of compactness due to the presence of a Sobolev critical exponent (problem (I)) and a Hardy-Sobolev critical exponent (problem (II)). In problem (III), the singularity implies that the standard definition of weak solution may not make sense. Therefore is necessary to introduce a special notion of weak solution on open subsets of the domain.

Variational methods and Critical Point Theory techniques are used to prove the existence of solutions in the two first problems. In problem (I), our method combines Nehari's techniques, Ekeland's variational principle, minimax methods, a translation argument and integral estimates of the energy level. In this case, we prove the existence of (at least) four nontrivial solutions where at least one of them is sign-changing. In problem (II), we prove the existence of at least two positive solutions and a pair of sign-changing solutions, using the concentration-compactness method and the mountain pass theorem. The approach in problem (III) combines a surjectivity result for monotone, coercive and radially continuous operators with special properties of Leray-Lions operators. We prove the existence of at least one solution in a Lorentz space and obtain an estimative for the solution.

Contents

List of symbols

$C_0^\infty(\Omega)$	space of smooth functions on Ω with compact support
$L^p(\Omega)$	space of Lebesgue-measurable functions. $u:\Omega\to\mathbb{R}$ such that $\|u\|_{L^p}=\left(\int_\Omega \|u\|^p\,dx\right)^{1/p}<\infty$, with $1\le p<\infty$
$L^\infty(\Omega)$	space of Lebesgue-measurable and essentially bounded functions $u:\Omega\to\mathbb{R}$ with norm $\|u\|_{L^\infty}=ess\sup\limits_{x\in\Omega}\|u(x)\|$
$W^{m,p}(\Omega)$	Sobolev space of functions $u\in L^p(\Omega)$ with $\left\|\nabla^k u\right\|\in L^p(\Omega)$ for all $k\in\mathbb{N}_0$, $\|k\|\le m$, with norm $\|u\|_{W^{m,p}}=\sum\limits_{0\le\|k\|\le m}\left\|\nabla^k u\right\|_{L^p}<\infty$
$W_0^{m,p}(\Omega)$	completion of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{W^{m,p}}$; if Ω is bounded an equivalent norm is given by $\|u\|_{W_0^{m,p}}=\sum\limits_{\|k\|=m}\left\|\nabla^k u\right\|_{L^p}$
$W^{-m,q}(\Omega)$	dual of $W_0^{m,p}(\Omega)$, where $\frac{1}{p}+\frac{1}{q}=1$
$H_0^1(\Omega)$	standard Sobolev space equivalent to $W_0^{1,2}(\Omega)$
$\|u\|$	norm of u in the Sobolev space $H_0^1(\Omega)$
$H^{-1}(\Omega)$	dual of $H_0^1(\Omega)$
$\langle\cdot,\cdot\rangle$	dual pairing between a Banach space and its dual
$\Delta=\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}$	Laplacian operator
a.e.	almost everywhere
$\hookrightarrow$	continuous injection
$\overset{c}{\hookrightarrow}$	compact injection
$\liminf$	infimum limit
$\limsup$	supremum limit
$B_r(x)$	ball centered at x with radius r
$O(\varepsilon^\beta)$	means that $\|O(\varepsilon^\beta)\varepsilon^{-\beta}\|\le K$ for some constant $K>0$, for ε small
$o(\varepsilon^\beta)$	means $\|o(\varepsilon^\beta)\varepsilon^{-\beta}\|\to 0$ as $\varepsilon\to 0$
$o(1)$	is just an infinitesimal value
$\to$	strong convergence
$\rightharpoonup$	weak convergence
$\int\cdot$	integral $\int_\Omega\cdot\,dx$
$\doteq$	to emphasize a new definition
$\mathcal{D}^{1,2}(\mathbb{R}^N)$	functions u in $L^{2^*}(\mathbb{R}^N)$ such that $\nabla u\in L^2(\mathbb{R}^N)$, with the norm $\|\cdot\|_{\mathcal{D}}^2=\int_{\mathbb{R}^N}\|\nabla\cdot\|^2dx$
$L^t(\Omega,\|x\|^s dx)$	weighted Sobolev spaces with the norm $\|u\|_{t,s}^t=\int_{\mathbb{R}^N}\|x\|^s\|u\|^t dx$
$L^p_{loc}(\Omega)$	space of p-integrable functions in compact subsets of Ω
$L^{p,q}(\Omega)$	Lorentz space of measurable functions defined on Ω
$L^{n,1}_{loc}(\Omega)$	space of measurable functions $g:\Omega\to\mathbb{R}$ such that $g_{\chi_A}\in L^{p,q}(\Omega)$ for each compact set $A\subset\Omega$, where χ_A is the characteristic function of A
$\Lambda^q(w)$	weighted Lorentz spaces of measurable functions such that $\|f\|_{\Lambda^q(w)}<\infty$

Introduction

Partial Differential Equations of elliptic type have been studied by many authors, due to their multiple applications in different contexts of sciences and engineering (see Brezis-Nirenberg [16] and Debnath [50]). Recently, the study of existence results for elliptic problems containing singularities have increased significantly (see for instance Abdellaoui-Colorado-Peral [1], Abdellaoui-Felli-Peral [2], Azorero-Peral [59], Ghoussoub-Yuan [62], Peral [97]). The methods used for solving such problems depend mainly on the type of singularities and parameters involved.

The main goal of this thesis is the study of existence and multiplicity results for nonlinear elliptic problems that contain singularities and/or terms with a critical exponent. We are interested in two classes of elliptic equations with critical exponent and nonlinearities defined on Sobolev spaces and one class of elliptic equations in divergence form with nonlinearities defined on Lorentz spaces. Specifically, we consider $\Omega \subset \mathbb{R}^N$ a bounded domain with smooth boundary and we study the following three nonlinear problems with Dirichlet boundary conditions.

(I) The problem $P_1(\lambda, \mu, \alpha, f, \gamma)$ involving the Sobolev critical exponent, a Hardy-type singular term and other two subcritical terms

$$-\Delta u(x) - \frac{\lambda}{|x|^2} u(x) = |u(x)|^{2^*-2} u(x) + \mu |x|^{\alpha-2} u(x) + f(x)|u(x)|^{\gamma}, \ \ in \ \ \Omega \backslash \{0\},$$

where $N \geq 3$, and $2^* \doteq 2N/(N-2)$ denotes the Sobolev critical exponent. The function $f \in L^{\infty}(\Omega)$ and the positive parameters λ, μ, α and γ satisfy additional conditions.

(II) The problem $P_2(\lambda, \zeta, q, s, f)$ involving the Hardy-Sobolev exponent, a concave term and a double singularity on the boundary

$$-\Delta u(x) - \frac{\lambda}{|x|^2} u(x) = \zeta f(x) |u(x)|^{q-2} u(x) + \frac{|u(x)|^{p*(s)-2} u(x)}{|x|^s}, \ \ in \ \ \Omega \backslash \{0\},$$

where $N \geq 3$ and $p^*(s) \doteq 2(N-s)/(N-2)$ denotes the Hardy-Sobolev critical exponent. Here f is a real function on Ω with an additional condition and the parameters λ, ζ, q and s are suitably defined.

(III) The problem $P_3(\psi, a, f)$ in divergence form, involving a Leray-Lions operator and a term that may have a singularity

$$-div(\psi(x,u(x),\nabla u(x)))+a(x)u(x)=f(x),\ \ in\ \ \Omega,$$

where $2\leq p<N$, $a\in L^{\infty}_{loc}(\Omega)$ satisfies an additional condition and f is a function defined in a Lorentz space $L^{q,q_1}(\Omega)$ with suitable exponents q and q_1.

In problems $P_1(\lambda,\mu,\alpha,f,\gamma)$ and $P_2(\lambda,\zeta,q,s,f)$, we study nontrivial solutions in the Sobolev space $H^1_0(\Omega)$. Due to the presence of the term $\frac{\lambda}{|x|^2}$ in problem $P_1(\lambda,\mu,\alpha,f,\gamma)$ and the terms $\frac{\lambda}{|x|^2}$ and $\frac{|u|^{p^s-2}}{|x|^s}$ in problem $P_2(\lambda,\zeta,q,s,f)$, we have strong singularity at zero, so blow-up may occur (see Smets [111]). To make sense, we consider that the equation hold on Ω with $\Omega\backslash\{0\}$ but still look for solutions on $0\in\Omega$. The singularity in both cases is overcomed using the Hardy inequality (see 1.1.8).

The problems $P_1(\lambda,\mu,\alpha,f,\gamma)$ and $P_2(\lambda,\zeta,q,s,f)$ are variational, due to the Hardy-Sobolev embedding (see Theorem A.1.6 and the Hardy inequality 1.1.8). Therefore we use critical point theory (see Ambrosetti-Malchiodi [7], Costa [46], Rabinowitz [101]) to study them. By Caffarely-Kohn-Nirenberg inequality (see Theorem 1.1.11), the associated functionals are well defined on $H^1_0(\Omega)$ and there exists a one-to-one correspondence between the critical points of the functionals and the solutions of the problems. Thus, we say that the solutions of $P_1(\lambda,\mu,\alpha,f,\gamma)$ and $P_2(\lambda,\zeta,q,s,f)$ are functions $u\in H^1_0(\Omega)$, which correspond to critical points of the associated Euler functionals.

Since neither $H^1_0(\Omega)\hookrightarrow L^2\left(\Omega,|x|^{-2}dx\right)$, $H^1_0(\Omega)\hookrightarrow L^{2^*}(\Omega)$ nor $H^1_0(\Omega)\hookrightarrow L^{p*(s)}\left(\Omega,|x|^{-s}dx\right)$ are compacts, the action functionals associated to problems as $P_1(\lambda,\mu,\alpha,f,\gamma)$ and $P_2(\lambda,\zeta,q,s,f)$, satisfy the Palais-Smale condition (Definition 1.1.12) only in a suitable range (see Brezis-Nirenberg [16], Chen [31, 35]). Furthermore, due to a lack of compactness, generated by the presence of the Sobolev critical exponent and Hardy-Sobolev critical exponent in problems $P_1(\lambda,\mu,\alpha,f,\gamma)$ and $P_2(\lambda,\zeta,q,s,f)$ respectively, standard variational arguments do not apply without some extra care.

We point out that, although the problems $P_1(\lambda,\mu,\alpha,f,\gamma)$ and $P_2(\lambda,\zeta,q,s,f)$ seem similar, there is no intersection between them. In fact, we observe that, if we consider the problem $P_2(\lambda,\zeta,q,s,f)$ plus the term $\mu|x|^{\alpha-2}u(x)$ with $\zeta=1$, $s=0$, $p*(s)=2^*$ and $q=\gamma+1$ for u positive and defining zero as a possible value for q; we have the same form as the problem $P_1(\lambda,\mu,\alpha,f,\gamma)$ when the function u is positive; but although we found some similarities between these problems and we can do comparisons, it is not possible to say that, one of them is a particular case of the other, due to the restrictions $0\leq\gamma<1$ in $P_1(\lambda,\mu,\alpha,f,\gamma)$ and $\gamma>1$ in $P_2(\lambda,\zeta,q,s,f)$. On the other hand the hypotheses considered for each problem are different.

In problem $P_3(\psi, a, f)$, we study nontrivial solutions in the Sobolev space $W_0^{1,p}(\Omega) \cap L^{r,s}(\Omega)$ where $2 \leq p < N$ and $L^{r,s}(\Omega)$ is a suitable Lorentz space. This problem has an additional difficulty since the function a is defined in $L^{\infty}_{loc}(\Omega)$, the problem may have a singularity on the boundary and therefore the standard definition of weak solution may not make sense (i.e. with test functions in $W_0^{1,p}(\Omega)$). The singularity is overcomed considering an increasing sequence of open subsets of the domain Ω (see Chapter 4, for more details). The main point here is to take advantage of the best fitted embedding of the Sobolev space $W_0^{1,p}(\Omega)$ into a Lorentz space, compared with the standard Sobolev embedding into a Lebesgue space.

In more detail, we prove that problem $P_1(\lambda, \mu, \alpha, f, \gamma)$ under suitable hypotheses (Subsection 2.2.4), has two nontrivial solutions and under less strong hypotheses (Subsection 2.2.5), has at least four nontrivial solutions in the Sobolev space $H_0^1(\Omega)$ where at least one of them is sign-changing. The problem $P_2(\lambda, \zeta, q, s, f)$ has at least two positive solutions and at least one pair of sign-changing solutions in $H_0^1(\Omega)$. We prove the existence of at least a solution $u \in W_0^{1,p}(\Omega) \cap L^{r,s}(\Omega)$ of problem $P_3(\psi, a, f)$, the uniqueness under suitable conditions and also obtain an apriori estimate for the solution with respect to the Lorentz space norm of $f \in L^{q,q_1}(\Omega)$ for suitable values p, q, q_1, r and s.

The techniques described later in the Chapters 2, 3 and 4, are mainly based or improvements of the results obtained in the works of Chen-Rocha [42] and Tarantello [118] for the problem $P_1(\lambda, \mu, \alpha, f, \gamma)$. In addition to these works, we consider the results of Bouchekif-Matallah [14] as a starting point for the problem $P_2(\lambda, \zeta, q, s, f)$. For problem $P_3(\psi, a, f)$, the existence result generalizes some previous results, e.g. in Napoli-Mariani [91], besides others.

At this point, we call the especial attention of the reader for the notation that we will use for the different problems. We will consider the notation defined for the problems $P_1(\lambda, \mu, \alpha, f, \gamma)$, $P_2(\lambda, \zeta, q, s, f)$ and $P_3(\psi, a, f)$ as standard and we give specifications on the parameters when referring to subclasses. In this sense, for example, the problem $P_1(0, \mu, 2, f, 0)$ represents the problem $P_1(\lambda, \mu, \alpha, f, \gamma)$ when $\lambda = 0$, $\alpha = 2$ $\gamma = 0$, and μ and f are general but satisfy additional hypotheses, which may be different from ours.

The literature on elliptic problems is rather extensive. It would be impossible to cover all different aspects of this type of problems even restricting it to some classes. Let us describes the situation of a simple model for this type of equations. The solvability of the problem

$$\begin{cases} -\Delta u(x) & = |u(x)|^{p-2}u(x) \quad \text{in} \quad \Omega, \\ u(x) & > 0 \qquad\qquad\qquad \text{in} \quad \Omega, \\ u(x) & = 0 \qquad\qquad\qquad \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \in R^N$ is a bounded domain with smooth boundary $\partial\Omega$, depends on the value of

p and sometimes is related with the shape of Ω. Some results are well known:
(i) In the subcritical case, i.e. $p < \frac{2N}{N-2}$, the problem admits solution. The existence of positive and sign-changing solutions of problem (1) does not depend on the shape of Ω;
(ii) In the supercritical case, i.e. $p > \frac{2N}{N-2}$, Passaseo [96] proved that there exist contractible domains (intuitively are spaces that can be continuously shrunk to a point), where the number of positive solutions of problem (1) is arbitraly large. For some examples of domains where problem (1) has no solutions see Passaseo [94, 95];
(iii) In the critical case, i.e. $p = \frac{2N}{N-2}$, if Ω is not contractible, then problem (1) has a solution for $N = 3$ (see Bahri-Coron [12]). If Ω is an annulus, problem (1) has a solution (see Kazdan-Warner [80]). If Ω has a "small hole", problem (1) has also a solution (see Bahri-Coron [45]). If Ω is a star-shaped with $p \geq \frac{2N}{N-2}$ then problem (1) has no solution. This follows by the application of the Pohozaev identity (Pohozaev [100]).

These example clearly shows that the use of a critical exponent changes the problem characteristics and its difficulty in proving the existence of solutions. Another model example and one of the starting points for the study of elliptic problems is the well known Yamabe's problem (see Yamabe [123]), which is one of the celebrated problems in Differential Geometry and concerns the existence of a Riemannian metric with constant scalar curvature for a given (compact) manifold. Such problem can be modeled as a Dirichlet elliptic problem, for example written as $P_1(0, \mu, 2, 0, 0)$:

$$\begin{cases} -\Delta u(x) & = |u(x)|^{2^*-2}u(x) + \mu u(x) \quad \text{in} \quad \Omega, \\ u(x) & > 0 \quad \text{in} \quad \Omega, \\ u(x) & = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Concerning this problem, we mention two relevant results about the existence of solutions which show the importance of the geometry of the domain and the behavior of the coefficients.

Theorem 0.1.1. *(Brezis-Nirenberg [16]) Suppose $\Omega \subset \mathbb{R}^N$, $(N \geq 3)$ and let $\mu_1 > 0$ denote the first eigenvalue of the operator $(-\Delta, H_0^1(\Omega))$ with homogeneous Dirichlet boundary conditions:*
(i) If $N \geq 4$, then for any $\mu \in (0, \mu_1)$ there exists a (positive) solution of Yamabe's problem;
(ii) If $N = 3$, there exists $\mu_* \in (0, \mu_1)$ such that for any $\mu \in (\mu_*, \mu_1)$, Yamabe´s problem admits a solution;
(iii) If $N = 3$ and $\Omega = B_1(0) \subset \mathbb{R}^3$, then $\mu_* = \frac{\mu_1}{4}$ and for $\mu \leq \frac{\mu_1}{4}$ there is no solution to Yamabe's problem.

Theorem 0.1.2. *(Ceramini-Solimini-Struwe [27]) Suppose $\Omega = B_R(0)$ is a ball in $\mathbb{R}^N$, $N \geq 7$. Then for any $\mu > 0$, Yamabe´s problem admits infinitely many radially symmetric solutions.*

In [75], Janelli considered the problem $P_1(\lambda, \mu, 2, 0, 0)$ and prove that there exist $\bar{\lambda}$ such that:

(i) If $0 \leq \lambda < \bar{\lambda} - 1$ and $0 < \mu < \mu_1(\lambda)$, then the problem has at least a positive solution, where $\mu_1(\lambda)$ is the first eigenvalue of $(-\Delta - \frac{\lambda}{|x|^2}, H_0^1(\Omega))$ with Dirichlet boundary condition;

(ii) If $\lambda > 0$ and $\bar{\lambda} - 1 < \lambda < \bar{\lambda}$, then there exists $\mu_*(\lambda) > 0$ such that the problem has at least a positive solution provided $\lambda \in (\mu_*(\lambda), \mu_1(\lambda))$.

Now, concerning singular terms, one the best studied elliptic problems with a singular term is a problem involving a term with a negative power of the solution, i.e. a problem of form, under Dirichlet boundary condition,

$$-\Delta_p u(x) = \beta(x)u(x)^{-\eta} + f(x, u(x)), \text{ with } \eta \geq 0,$$

which was first studied in the context of semilinear equations ($p = 2$). Among the first works in this direction are the papers of Crandall-Rabinowitz-Tártar [48] and Stuart [114]. Since then, there have been several other papers on the subject. We mention the relevant works of Coclite-Palmieri [44], Diaz-Morel-Oswald [51], Lair-Shaker [82], Shaker [106], Shi-Yao [107], Sun-Wu-Long [115], and Zhang [125]. In particular, Lair-Shaker [82] assumed that $f \equiv 0$ and $\beta \in L^2(\Omega)$ and established the existence of a unique positive weak solution. Their result was extended by Shi-Yao [107] to the case of a "sublinear" reaction, namely when

$$f(x, u) = \lambda u^{r-1} \text{ with } \lambda > 0 \text{ and } 1 < r \leq 2.$$

The case of a "superlinear-subcritical" nonlinearity, i.e. when $2 < r < 2^*$, was investigated by Coclite-Palmieri [44] under the assumption that $\beta \equiv 1$. In both works (i.e. [44] and [107]), it is shown that there exists a critical value $\lambda^* > 0$ of the parameter λ, such that for every $\lambda \in (0, \lambda^*)$ the problem admits a nontrivial positive solution. Subsequently, Sun-Wu-Long [115] using the Ekeland variational principle (Proposition 1.1.7), obtained two nontrivial positive weak solutions for more general functions β. The work of Zhang [125] extended their results to more general nonnegative superlinear perturbations, using critical point theory on closed convex sets. For the same problem but driven by the p-Laplacian, we mention the works of Agarwal-Lü-O'Regan [3], Agarwal-O'Regan [4], where $N = 1$ (ordinary differential equations), and Perera-Silva [98], Perera-Zhang [99], where $N \geq 2$ (partial differential equations) and the reaction term has the parametric form

$$\beta(x)u(x)^{-\eta} + \lambda f(x, u(x)) \text{ with } \lambda > 0.$$

For such a parametric nonlinearity, the authors prove existence and multiplicity results (two positive weak solutions), valid for all $\lambda \in (0, \lambda^*)$. Moreover, the perturbation term f exhibits a strict $(p-1)$-superlinear growth near $+\infty$ and, more precisely, it satisfies on $[0, +\infty)$, the well-known Ambrosetti-Rabinowitz condition. Chen-Papageorgiou-Rocha

[40] considered the reaction term nonparametric and the perturbation as $(p-1)$-linear near $+\infty$ and proved the existence of an ordered pair of smooth positive strong solutions.

Existence results, for other type of singularities and particular results for the problems $P_1(\lambda, \mu, \alpha, f, \gamma)$, $P_2(\lambda, \zeta, q, s, f)$ and $P_3(\psi, a, f)$, are presented in the stated of art (Previous Results) of the Chapters 2, 3 and 4, respectively.

This document is organized as follows. In Chapter 1, we briefly introduce some of the mathematical background needed for this work, namely basic notions of Critical Point Theory and Theory of Monotone Operators.

In Chapter 2, by variational methods, careful integral estimates combined with Nehari set techniques, we study multiplicity results for the problem $P_1(\lambda, \mu, \alpha, f, \gamma)$ in two parts: (a) the existence of two nontrivial solutions (see Proposition 2.2.26 and Proposition 2.2.28) and (b) the existence of four nontrivial solutions with less restrictive hypotheses (see Theorem 2.2.25). In this part, minimax methods are used to prove the existence of a sign-changing solution (see Proposition 2.2.32). A fourth solution (see Proposition 2.2.36) is obtained applying a translation argument and verifying that the mountain pass theorem is satisfied in the range where the Palais-Smale condition holds.

In Chapter 3, we study multiplicity results for the problem $P_2(\lambda, \zeta, q, s, f)$. Here we use the concentration compactness principle (see Proposition 1.1.15) to prove the existence of the first solution and a mountain pass theorem (see Theorem 1.1.16) for the second solution. The existence of sign-changing solutions are obtained combining Nehari techniques (see Subsection 1.1.1) with energy estimates, in which it is essential to know the exact local behavior of the solution. We use the fact that the problem is odd to obtain other solutions.

In Chapter 4, we replace the Laplacian operator by a more general nonlinear elliptic second order partial differential operator with a divergence structure and we study the existence and uniqueness of solutions of problem $P_3(\psi, a, f)$, when the function f is defined on a Lorentz space. The existence of a solution of this problem is obtained combining a surjectivity result for monotone, coercive and radially continuous operators with special properties of Leray-Lions operators, namely to be of type M and pseudomonotone. Moreover, we obtain an *apriori* estimate for the solution in terms of the norm of the nonlinearity (see Theorem 4.3.13). Here we use some ideas of An et al [8] and Drivaliaris-Yannakakis [52]. The proof of the estimate is inspired in Napoli-Mariani [91].

In Chapter 5, we present some final considerations on the three problems studied and give some direction on a possible future research.

In Appendix A, we make a breve introduction to the space of functions, considered in this work: the Sobolev spaces (see Section A.1) and Lorentz spaces (see Section A.2), both play an important role in the theory of interpolation of operators and in partial differential equations. In Appendix B, we present some integral estimates relevant to our results.

Chapter 1

Preliminary results for the solvability of nonlinear elliptic equations

In this chapter we present some mathematical preliminaries that are relevant for the understanding of our work. The literature on this subjects is quite extensive for instance for Critical Point Theory see Ambrosetti-Malchiodi [7], Costa [46], Rabinowitz [101] and Struwe [113]. For Theory of Monotone Operators see Showalter [108], Zeidler [124] and Zuchi-Xiaodong [126].

1.1 Variational approach for elliptic equations

In this section, we give some concepts directly related to Critical Point Theory.

In the study of second order semilinear elliptic boundary value problems, the following result due to Rabinowitz [101], is frequently used to establish when an class of functionals is $C^1(H_0^1(\Omega);\mathbb{R})$.

Proposition 1.1.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ whose boundary is a smooth manifold. Let p be a function which satisfy:*

(P_1) $p \in C(\bar{\Omega} \times \mathbb{R};\mathbb{R})$*;*

(P_2) *There are constants $a_1, a_2 > 0$ such that $|p(x,\xi)| \leq a_1 + a_2|\xi|^s$, where $0 \leq s < (N+2)(N-2)^{-1}$ and $N \geq 3$.*

If

$$I(u) \doteq \int_\Omega \frac{1}{2}|\nabla u|^2 - P(x,u)dx,$$

where $P(x,\xi) \doteq \int_0^{\xi} p(x,t)dt$, *then* $I \in C^1(H_0^1(\Omega);\mathbb{R})$ *and*

$$I'(u)\varphi = \int_{\Omega} \nabla u \cdot \nabla \varphi - p(x,u)\varphi dx$$

for all $\varphi \in H_0^1(\Omega)$. *Moreover* $J(u) \doteq \int_{\Omega} P(x,u(x))dx$ *is weakly continuous and* $J'(u)$ *is compact.*

1.1.1 Nehari's set method

The Nehari method, introduced by Z. Nehari [92, 93], is very useful in Critical Point Theory and plays an important role in obtaining ours results.

Definition 1.1.2. *Let* E *be a Hilbert space and* $I : E \to \mathbb{R}$ *be of class* $C^1(E;\mathbb{R})$. *We define*

$$M = \{u \in E\backslash\{0\} : \langle I'(u), u\rangle = 0\}.$$

M *is called the Nehari set associated with the functional* I.

We set $S_E = \{u \in E : \|u\|_E = 1\}$. Under some assumptions, we can see that M is a differentiable manifold homeomorphic to the unit sphere of E and bounded away from 0 (see Szulkin-Weth [116]). Consider the assumptions:

(i) There exists a normalization function φ (i.e $\varphi(0) = 0$, φ is strictly increasing and $\varphi(t) \to \infty$ as $t \to \infty$) such that

$$u \mapsto \psi(u) \doteq \int_0^{\|u\|_E} \varphi(t)dt$$

$\varphi \in C^1(E\backslash\{0\};\mathbb{R})$, $J \doteq \psi'$ is bounded on bounded sets, and $\langle J(u), u\rangle = 1$ for all $u \in S_E$;

(ii) For each $u \in E\backslash\{0\}$ there exists $\bar{t} \equiv \bar{t}(u)$ such that if $\alpha_u(t) \doteq I(tu)$ then

$$\begin{cases} \alpha_u'(t) > 0, & \text{for } 0 < t < \bar{t}, \\ \alpha_u'(t) < 0, & \text{for } t > \bar{t}; \end{cases}$$

(iii) There exists $\delta > 0$ such that $\bar{t} > \delta$ for all $u \in S_E$;

(iv) For each compact subset $K \subset S_E$, there exists a constant c_k such that $\bar{t} \leq c_k$ for all $u \in K$.

The following result guarantees that $M \neq \emptyset$.

Lemma 1.1.3. *(Szulkin-Weth [116]) Suppose* I *satisfies* (ii), *then for any* $u \in W_0^{1,p}(\Omega)\backslash\{0\}$ *there exists a unique* $\bar{t} \equiv \bar{t}(u) \in \mathbb{R}$ *such that* $\bar{t}u \in M$.

By (iii), M is closed in E and bounded away from 0. We also have the following result.

Proposition 1.1.4. *(Szulkin-Weth [116]) Suppose I satisfies $(iii)-(iv)$, then*

(a) *The mapping $\alpha : E\backslash\{0\} \to M$ defined by $\alpha_u(t) \doteq \bar{t}u$ is continuous.*

(b) *The mapping $\beta : S_E \to M$ defined by $\beta \doteq \alpha\,|_{S_E}$ is a homeomorphism and the inverse of β is given by $\beta^{-1}(u) = \frac{u}{\|u\|_E}$.*

Remark 1.1.5. *Under assumptions which imply that the functional I satisfies:*

(v) *$I \in C^2(E;\mathbb{R})$;*

(vi) *$\langle I''(u)u, u\rangle \neq 0$,*

is possible to guarantee that the set M is a manifold. In fact, set $G(u) \doteq \langle I'(u), u\rangle$, so $M = G^{-1}(0)\backslash\{0\}$ and $G \in C^1(E;\mathbb{R})$. Now, considering $u \in M$, since (v) and (vi) hold, one has

$$\langle G'(u), u\rangle = \langle I''(u)u, u\rangle + \langle I'(u), u\rangle = \langle I''(u)u, u\rangle \neq 0. \tag{1.1}$$

Thus, $G'(u) \neq 0$ for all $u \neq 0$ and this implies using the Implicit Function theorem that M is a C^1-manifold of codimension one (see Ambrosetti-Malchiodi [7], Guillemin-Pollack [64], Szulkin-Weth [116]).

Now, we emphasize the application of the Nehari method. The main idea of this technique is the following: Consider the existence of functions $u \in W_0^{1,p}(\Omega)$ satisfying the following variational problem (P):

$$\mathbf{L}u(x) = f(x, u(x)) \;\; in \;\; \Omega,$$

where $\mathbf{L}$ is a nonlinear second order differential operator.

Let $I : W_0^{1,p}(\Omega) \to \mathbb{R}$, with $I \in C^1(W_0^{1,p}(\Omega);\mathbb{R})$ be the Euler functional associated to problem (P).

There exists a one to one correspondence between the critical points of Euler functional I and the solutions of problem (P). Then, we say that $u \in W_0^{1,p}(\Omega)$ is a solution of problem (P), if and only if, u is a critical point of the Euler functional I. Therefore we are interested in the following set of solutions

$$S = \{u \in W_0^{1,p}(\Omega) : \langle I'(u), v\rangle = 0 \text{ for any } v \in W_0^{1,p}(\Omega)\}.$$

Here $\langle\cdot,\cdot\rangle$ represents the duality between the spaces $W_0^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$.

We define, the Nehari set

$$M = \{u \in W_0^{1,p}(\Omega)\backslash\{0\} : \langle I'(u), u\rangle = 0\}.$$

Remark 1.1.6. *Note that u is a nontrivial critical point of I if and only if $u \in M$ and u is a critical point of the restriction of I to M. In fact, suppose that $\bar{u}$ is a nontrivial* *critical point of I, i.e., $I'(\bar{u}) = 0$ and $\bar{u} \not\equiv 0$. Then $\langle I'(\bar{u}), \bar{u}\rangle = 0$. Hence $\bar{u} \in M$. Conversely, if $\bar{u}$ is a critical point of I on M, by method of Lagrange multipliers (see Costa [46]), there holds that $I'(\bar{u}) = \lambda G'(\bar{u})$ and*

$$\langle I'(\bar{u}), \bar{u}\rangle = \lambda\langle G'(\bar{u}), \bar{u}\rangle.$$

Since $\langle I'(\bar{u}), \bar{u}\rangle = 0$ and by (1.1), $\langle G'(\bar{u}), \bar{u}\rangle \neq 0$. Then it follows that $\lambda = 0$ and hence $I'(\bar{u}) = 0$.

In view of the previous remark, one may apply Critical Point Theory on M, in order to find critical points of I.

Now, we choose wisely some sets $M_i \subseteq M$, and study the corresponding minimization problems on them

$$c_i \doteq \inf_{u \in M_i} I(u).$$

The main idea is then to prove the existence of critical points u_i such that $c_i = I(u_i)$.

1.1.2 Ekeland's variational principle

The following principle was proven by I. Ekeland in [54]. This principle has been a very useful tool in studying of optimization problems in Control Theory, Differential Geometry and Differential Equations.

Proposition 1.1.7. *Let (M, d) be a complete metric space and $\phi : M \to \mathbb{R} \cup \{+\infty\}$ be a lower-semicontinuous function which is bounded from below. Suppose $\varepsilon > 0$ and $u \in M$ are such that*

$$\phi(u) \leq \inf_M \phi + \varepsilon.$$

Then, given any $\lambda > 0$, there exists $v \in M$ such that:

(i) $\phi(v) \leq \phi(u)$;

(ii) $d(u, v) \leq \lambda$;

(ii) $\phi(v) < \phi(w) + \frac{\varepsilon}{\lambda} d(v, w)$ *for any* $v \neq w$.

1.1.3 Some inequalities

The following is a classical result essentially due to Hardy (see Hardy-Littewood-Polya [66]).

Lemma 1.1.8. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. If $u \in H_0^1(\Omega)$, then*

(i) $\frac{u}{|x|^2} \in L^2(\Omega)$;

(ii) (Hardy inequality) $\int \frac{u^2}{|x|^2} \leq \frac{1}{[(N-2)^2/4]} \int |\nabla u|^2$.

Remark 1.1.9. *The Hardy inequality, can be extended to functions in the space* $\mathcal{D}^{1,2}(\mathbb{R}^N)$ *which is the completion of* $C_0^\infty(\Omega)$ *with respect to the norm* $\|u\|_{\mathcal{D}}^2 = \int_{\mathbb{R}^N} |\nabla u|^2 dx$.

Remark 1.1.10. *By the Hardy inequality, for* $0 \leq \lambda < \Lambda = (N-2)^2/4$ *the norm*

$$\|u\|_\lambda = \left(\int_{\mathbb{R}^N} |\nabla u|^2 - \lambda \frac{u^2}{|x|^2} dx \right)^{1/2}$$

is equivalent to the usual norm

$$\|u\|_{H_0^1(\Omega)} = \left[\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right]^{1/2}.$$

The following inequality is an extension of the Hardy and Sobolev inequalities due to Caffarelli-Kohn-Nirenberg [18].

Lemma 1.1.11. *For* $1 < p < N$ *and any* $u \in C_0^\infty(R^N)$, *there exists a constant* k *such that*

$$\left(\int_{\mathbb{R}^N} |x|^{-bq} |u|^q \right)^{p/q} \leq k \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p$$

where $0 \leq a \leq \frac{(N-p)}{p}$, $a \leq b < a+1$ *and* $q = \frac{Np}{N-(p(a+1-b))} > p$.

1.1.4 Compactness analysis

The following definition is a compactness condition, which is a tool used in the proof of existence of critical points of functionals defined in Banach spaces.

Definition 1.1.12. *Let* $c \in \mathbb{R}$, E *be a Banach space and* $I \in C^1(E, \mathbb{R})$. *We say that* I *satisfies the Palais-Smale condition at* c, *which we denote by* $(PS)_c$*-condition, if any sequence* $(u_n)_{n \in \mathbb{N}}$ *in* E *satisfying* $I(u_n) \to c$ *and* $\|I'(u_n)\|_{E^{-1}} \to 0$ *has a convergent subsequence. We say that* I *satisfies the* (PS)*-condition if* I *satisfies the* $(PS)_c$*-condition for every* $c \in \mathbb{R}$.

To establish a local version of the Palais-Smale condition, we introduce an important principle due to Lions [85, 86, 87, 88], which is similar to that of [103, 109, 110]. But before we recall the following notion of convergence.

Definition 1.1.13. *Let* (X, μ) *be a measure space. A sequence* $(f_n)_{n \in \mathbb{N}}$ *of measurable real value functions is said to converge in measure to a measurable real-function* f *if*

$$\lim_{n \to \infty} \mu \left(\{ x \in X : |f_n(x) - f(x)| \geq \alpha \} \right) = 0$$

for each $\alpha > 0$, *where* μ *is a measure.*

Lemma 1.1.14. *Let* Ω *be a bounded domain and* $\{u_n\} \subset H_0^1(\Omega)$ *a bounded sequence. There then exist two nonnegative and bounded measures on* $\bar{\Omega}$, τ, ν, *and there exists a* *subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that*

$$|\nabla u_n|^2 - \lambda \frac{u_n^2}{|x|^2} \rightharpoonup \tau \text{ and } \frac{|u_n|^{p^*(s)}}{|x|^s} \rightharpoonup \nu$$

weakly in the sense of measures.

Now, let us introduce the so-called concentration compactness principle.

Proposition 1.1.15. *Let $\{u_n\} \subset H_0^1(\Omega)$ be such that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$,*

$$|\nabla u_n|^2 - \lambda \frac{u_n^2}{|x|^2} \rightharpoonup \tau \text{ and } \frac{|u_n|^{p^*(s)}}{|x|^s} \rightharpoonup \nu$$

weakly in the sense of measures, where τ and ν non-negative and bounded measures on $\bar{\Omega}$. Then there exist some at most countable index set J and a family $\{x_j : j \in J\}$ of points in $\bar{\Omega}$ such that:

(i) $\nu = \frac{|u|^{p^*(s)}}{|x|^s} + \sum_{j \in J} \nu_j \delta_{x_j}$;

(ii) $\tau \geq |\nabla u|^2 - \lambda \frac{u^2}{|x|^2} + \sum_{j \in J} \tau_j \delta_{x_j}$;

(iii) $\tau_j \geq S \nu_j^{p/p^*}$,

where δ_{x_j} is the Dirac measure at x_j, $\{\tau_j : j \in J\}$ is a family of positive numbers and $S \doteq \inf\{\|u\|_\lambda : u \in H_0^1(\Omega), \int \frac{|u|^{p^*(s)}}{|x|^s} = 1\}$, for $0 \leq s < 2$ and $p^*(s) \doteq 2(N-s)/(N-2)$. In particular $\sum_{j \in J} \nu_j^{p/p^*} < \infty$, $s \in [0, 2)$.

One common result used to find critical points is the mountain pass theorem of A. Ambrosetti and P. Rabinowitz.

Theorem 1.1.16. *(Rabinowitz [101]) Let E be a Hilbert space and $I \in C^1(E; \mathbb{R})$ be a functional that satisfies the Palais-Smale condition. Suppose $I(0) = 0$ and*

(i) There exist positive constants ρ and α; such that $I(u) \geq \alpha$ when $\|u\|_E = \rho$;

(ii) There is an element $w \in E$ such that $\|w\|_E > \rho$ and $I(w) \leq 0$.

then there is a critical value $c \geq \alpha$ of I.

Moreover

$$c \doteq \inf_{g \in \Gamma} \sup_{0 \leq t \leq 1} I(g(t))$$

where $\Gamma \doteq \{g \in C([0,1], E) : g(0) = 0, g(1) = w\}$.

1.1.5 Pseudo-gradient flow

Definition 1.1.17. *A functional $J : E \to \mathbb{R}$ is said to be locally Lipschitz provided that, for every $u \in E$, there exists a neighborhood V of u and a positive constant $k \equiv k(V)$, depending on V, such that*

$$|J(v) - J(w)| \leq k\|v - w\|$$

for each $v, w \in V$.

Definition 1.1.18. *Let E be a Hilbert space and $J : E \to \mathbb{R}$ be of class $C^1(E, \mathbb{R})$. Consider the set*

$$E_0 = \{u \in E : J'(u) \neq 0\}.$$

A pseudo-gradient vector field for J on E_0 is a locally Lipschitz continuous map X such that the following conditions hold

(i) $\|X(u)\| < 2\|J'(u)\|$;

(ii) $\langle J'(u), X(u)\rangle > \|J'(u)\|^2$;

for all $u \in E_0$.

Lemma 1.1.19. *(Rabinowitz [101]) Any functional $J \in C^1(E; \mathbb{R})$ admits a pseudo-gradient vector field for J on E_0.*

1.1.6 Strong maximum principle

Consider the semilinear equation

$$-\Delta u(x) + B(u(x)) = f(x) \text{ in } x \in \Omega,$$

where Ω is a domain in $\mathbb{R}^N (N \geq 1)$, B is a nondecreasing real function with $B(0) = 0$ and $f \geq 0$ a.e. in Ω.

Proposition 1.1.20. *(Vazquez [122]) Let $u \in L^1_{loc}(\Omega)$ be such that*

(i) $\Delta u \in L^1_{loc}(\Omega)$ *in the sense of distributions in* Ω;

(ii) $u \geq 0$ *a.e. in* Ω;

(iii) $\Delta u < B(u)$ *a.e. in* $\{x \in \Omega : 0 < u(x) < a\}$*, where a is a positive constant and $B : [0, a] \to R$ is a continuous nondecreasing function with $B(0) = 0$.*

Under the assumption that $B(S) = 0$ for some $S > 0$ or

$$\int_0^{\frac{a}{2}} (B(S)\,S)^{-\frac{1}{2}}\, dS = \infty$$

if $B(S) > 0$ for $S > 0$, then either $u \equiv 0$ a.e. in Ω or u is strictly positive in Ω in the sense that for every compact subset $K \subset \Omega$ there is a constant $c \equiv c(K) > 0$ such that $u \geq c$ a.e. in K.
In particular if u vanishes a.e. in a set of positive measure, it must vanish a.e. in Ω.

1.1.7 Spectrum of the negative Dirichlet p-Laplacian

Let us briefly recall some basic facts about the spectrum of the negative Dirichlet p-Laplacian.
We consider the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u(x) = \bar{\lambda}\, |u(x)|^{p-2}\, u(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

A number $\bar{\lambda} \in \mathbb{R}$ for which the above problem has a nontrivial solution is said to be an eigenvalue of the negative Dirichlet p-Laplacian. The set of eigenvalues is called their spectrum.

The smallest eigenvalue $\bar{\lambda}_1$ is positive, isolated, simple and admits the following variational characterization

$$\bar{\lambda}_1 = \inf \left\{ \frac{\|\nabla u\|_p^p}{\|u\|_p^p} \,:\; u \in W_0^{1,p}(\Omega)\,,\; u \neq 0 \right\}. \tag{1.2}$$

The infimum in (1.2) is attained on the corresponding one-dimensional eigenspace.

We say that a dimension N is critical for a second order linear elliptic positive operator L, if there exists a smooth bounded domain $\Omega \subset \mathbb{R}^N$ in which the equation

$$\begin{cases} Lu = f(x,u) + \beta u, & \text{in } \Omega; \\ u > 0, & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

has no solution for some $\beta \in (0, \beta_1)$, where β_1 is the first eigenvalue of L and $f(x,u)$ is a nonlinear term critical with respect to L.

Now, we study the operator $-\Delta - \frac{\lambda}{|x|^2}$ with Dirichlet boundary condition. When $\lambda < \Lambda$, where Λ is the best constant in the Hardy inequality, the spectrum is contained in the positive semi-axis, each eigenvalue $\bar{\lambda_k}$ $(k \geq 1)$ is isolated and has finite multiplicity. The smallest eigenvalue $\bar{\lambda_1}$ is simple and $\bar{\lambda_k} \to \infty$, as $k \to \infty$, moreover all eigenfunctions (for any such $\bar{\lambda_k}$) belong to the space $H_0^1(\Omega)$ (see Egnell [53], Ferrero-Gazzola [57]). Thus as a consequence of the Hardy inequality, the linear elliptic operator $-\Delta u - \frac{\lambda}{|x|^2}u$ is positive and has discrete spectrum if $\lambda < \Lambda = \left(\frac{N-2}{2}\right)^2$. On the other hand, the conditions under which critical dimension occur for operator $-\Delta - \frac{\lambda}{|x|^2}$ is when $\lambda > \Lambda - 1$.

1.1.8 Sign-changing solution

Let Ω be a smooth bounded domain in $\mathbb{R}^N$. For $u \in L^2(\Omega)$, we define $u^+(x) = \max\{u(x), 0\} \in L^2(\Omega)$ and $u^-(x) = \min\{u(x), 0\} \in L^2(\Omega)$. If $u \in H_0^1(\Omega)$, then $u^+, u^- \in H_0^1(\Omega)$ (see Kinderlehrer-Stampacchia [81]).

Definition 1.1.21. *(Castro-Cossio-Neuberger [24]) We say that $u \in L^2(\Omega)$ is sign-changing if $u^+ \neq 0$ and $u^- \neq 0$. For $u \neq 0$ we say that u is positive (and write $u > 0$) if $u^- = 0$, and similarly, u is negative $(u < 0)$, if $u^+ = 0$.*

1.2 Elliptic equations in divergence form

In this section, we present some results for more general elliptic operators of second-order having a divergence structure i.e. operator of the form

$$Lu \doteq \sum_{i,j=1}^{N} \partial/\partial x_i \left(a_{ij}(x) \partial u/\partial x_j \right) + \text{lower order terms}$$

1.2.1 Operators of monotone type

The theory of monotone operators applied to boundary value problems, has its origin in the works of Minty [90], Browder [17], Leray-Lions [83] and Hartman-Stampacchia [67].

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, X be a separable reflexive Banach space and X^* its dual space. We write $\langle u^*, u \rangle$ for $u^* \in X^*$ and $u \in X$, denoting the dual product in $X^* \times X$.

Definition 1.2.1. *Let $B : X \to X^*$ be an operator, then B is said to be*

- *Coercive when $\lim_{\|u\| \to \infty} \frac{\langle Bu, u \rangle}{\|u\|} = \pm\infty$;*
- *Monotone when $\langle Bu - Bv, u - v \rangle \geq 0$, for all $u, v \in X$;*
- *Strictly monotone when $\langle Bu - Bv, u - v \rangle > 0$, for all $u, v \in X$ with $u \neq v$;*
- *Hemicontinuous when $\lambda \in \mathbb{R} \mapsto \langle B(u + \lambda v), w \rangle$ is continuous, for all $u, v, w \in X$;*
- *Radially continuous: if $\lambda \in \mathbb{R} \mapsto \langle B(u + \lambda v), v \rangle$ is continuous, for all $u, v \in X$.*

A prototype of a nonlinear monotone coercive operator is the p-Laplacian Δp, $1 < p < \infty$, defined by

$$\Delta_p u = div \left(|\nabla u|^{p-2} \nabla u \right)$$

where $\nabla u = (\partial u/\partial x_1, ..., \partial u/\partial x_N)$ is the gradient of u.

Definition 1.2.2. *Let $B : X \to X^*$ be an operator, then B is said to be* pseudomonotone, *if $u_n \rightharpoonup u$ and $\limsup_{n\to\infty} \langle B(u_n), u_n - u\rangle \leq 0$ imply*

$$\liminf_{n\to\infty} \langle Bu_n, u_n - v\rangle \geq \langle Bu, u - v\rangle$$

for all $v \in X$.

Lemma 1.2.3. *(Zeidler [124]) Let $A, B : X \to X^*$ be some given operators on the real reflexive Banach space X, then it holds:*

(i) If A is monotone and hemicontinuous, then A is pseudomonotone;

(ii) If A is completely continuous, the A is pseudomonotone;

(iii) If A and B are pseudomonotone, then $A + B$ is pseudomonotone.

Now, we introduce another important class of operators, which is very stable under perturbations.

Definition 1.2.4. *Let $B : X \to X^*$ be an operator, then B is said to be a (S_+)–type* operator*, if $u_n \rightharpoonup u$ and*

$$\limsup_{n\to\infty} \langle Bu_n, u_n - u\rangle \leq 0$$

imply $u_n \to u$.

In this work, we will use the following notion M–type operator, restricted to a subspace.

Definition 1.2.5. *Let V be a linear subspace of X and $A : X \times V \to \mathbb{R}$, then A is said to be of type M with respect to V if for any sequence $(v_\lambda)_{\lambda\in\Lambda} \subset V$, $w \in X$ and $v^* \in V^*$, we have*

(a) $v_\lambda \rightharpoonup w$;

(b) $A(v_\lambda, v) \to \langle v^*, v\rangle$ for all $v \in V$;

(c) $A(v_\lambda, v_\lambda) \to \langle \bar{v}^*, w\rangle$, where $\bar{v}^*$ is the extension of v^* on the closure of V;

imply that $A(w, v) = \langle v^*, v\rangle$ for all $v \in V$.

Lemma 1.2.6. *(Zeidler [124]) Any monotone and hemicontinuous operator is a M–type operator.*

Now we introduce a class of operators of monotone type, the Leray-Lions operator, which appear in the functional analytical treatment of nonlinear elliptic and parabolic problems. In what follows, we introduce these operators and give some examples.

The operator $\Psi : X \to X^*$ defined by $\Psi(u) = -div(\psi(x, u(x), \nabla u(x)))$ is called a *Leray-Lions operator* if satisfies the following conditions:

(i) The map $\psi : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a *Carathéodory function*, i.e.

- the map $x \mapsto \psi(x, s, \xi)$ is measurable for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$;
- the map $(s, \xi) \mapsto \psi(x, s, \xi)$ is continuous for almost all $x \in \Omega$;

(ii) *Elliptic condition*: there exists $\alpha > 0$ such that

$$\psi(x, s, \xi)\xi \geq \alpha|\xi|^p$$

for all $(x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$;

(iii) *Growth condition*: there exist $\beta > 0$ and $a \in L^{p'}(\Omega)$ such that

$$|\psi(x, s, \xi)| \leq a(x) + \beta(|s|^{p-1} + |\xi|^{p-1})$$

for all $(x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$;

(iv) *Monotonicity condition*: for $\xi, \eta \in \mathbb{R}^N$, $\xi \neq \eta$ and almost all $x \in \Omega$, we have

$$[\psi(x, s, \xi) - \psi(x, s, \eta)](\xi - \eta) > 0.$$

Common examples of Leray-Lions operators are the generalized mean curvature operator

$$\psi(x, s, \xi) = (1 + |\xi|^2)^{(p-2)/2}\xi$$

and the p-Laplacian

$$\psi(x, s, \xi) = |\xi|^{p-2}\xi,$$

but weighted versions of this operators can also be considered, beside others.

Lemma 1.2.7. *(Zeidler [124]) Any Leray-Lions operator is pseudomonotone and a (S_+)–type operator.*

1.2.2 Existence theorems

The following is a version of the so-called Browder-Minty theorem.

Lemma 1.2.8. *(Gajewski-Greger-Zacharias [58], Roubick [102]) Let V bet a reflexive Banach space and $A : V \to V^*$ be a radially continuous, coercive and monotone operator, then A is surjective.*

Since the monotonicity assumption made in the above theorem is general not easy to test, we introduce a weaker condition. The following main Lemma on pseudomonotone operators is due to Brézis [15].

Lemma 1.2.9. *Let X be a real reflexive Banach space and let $A : X \to X^*$ be a pseudomonotone, bounded and coercive operator, and $b \in X^*$, then there exists a solution of the equation $Au = b$.*

As a very special case of Browder-Minty theorem, one gets another important result known as the Lax-Milgram theorem.

Theorem 1.2.10. *Let H be a Hilbert space with the inner product $(\cdot,\cdot) : H \times H \to \mathbb{R}$, and $b : H \times H \to \mathbb{R}$ be a bilinear form on H. Further, assume that there exist constants $C_{1,}C_2 > 0$, such that*

(i) $b(u,u) \geq C_1 \|u\|_H^2$, *for all* $u \in H$;

(ii) $|b(u,v)| \leq C_2 \|u\|_H \|v\|_H$, *for all* $u, v \in H$,

then for every bounded linear functional $f : H \to \mathbb{R}$ there exists a unique element $u \in H$, such that

$$\langle f, v\rangle = b\,(u,v) \text{ for all } v \in H.$$

The following Lemma is a result of An et al. [8], that will be applied in our work.

Lemma 1.2.11. *Let X bet a reflexive Banach space over $\mathbb{R}$, $(X_n)_{n\in\mathbb{N}}$ be a increasing sequence of closed subspaces of X, and $V = \underset{n\in\mathbb{N}}{\cup} X_n$. Suppose that*

$$A : X \times V \to \mathbb{R}$$

is a real-valued function on $X \times V$ for which the following hold:

(a) $A_n = A|_{X_n \times X_n}$ *is a bounded bilinear form, for all* $n \in \mathbb{N}$;

(b) $A(\cdot, v)$ *is a bounded linear functional on* X, *for all* $v \in V$;

(c) There exists $c > 0$ such that for all $v \in V$,

$$A(v,v) \geq c\|v\|^2,$$

then, for each bounded linear functional v^* on V, there exists $u \in X$ such that $A(u,v) = \langle v^*, v\rangle$ for all $v \in V$.

The following result is a nonlinear extension of Lemma 1.2.11 due to Drivaliaris-Yannakakis [52].

Lemma 1.2.12. *Let X bet a reflexive Banach space, let Λ be a directed set, let $\{X_\lambda\}_{\lambda\in\Lambda}$ be an upwards directed family of closed subspaces of X, and let $V = \underset{\lambda\in\Lambda}{\cup} X_\lambda$. Suppose that*

$$A : X \times V \to \mathbb{R}$$

is a function for which the following hold:

(a) *A is of type M with respect to V;*

(b) *$lim_{\|x\|\to\infty} A(x,x)/\|x\| = \infty$;*

(c) *$A_\lambda(x,\cdot) \in X_\lambda^*$; for all $\lambda \in \Lambda$ and all $x \in X_\lambda$, where A_λ is the restriction of A on $X_\lambda \times X_\lambda$;*

(d) *the operator $T_\lambda : X_\lambda \to X_\lambda^*$ defined by $\langle T_\lambda x, y\rangle = A_\lambda(x,y)$ for all $x, y \in X_\lambda$, is monotone and hemicontinuous for all $\lambda \in \Lambda$,*

then, for each $v^* \in V^*$, there exists $x \in X$ such that

$$A(x,v) = \langle v^*, v\rangle$$

for all $v \in V$.

Chapter 2

Multiplicity results for a class of singular elliptic equations with the critical Sobolev exponent

Here, we consider $N \geq 3$ and study the existence of solutions $u \in H_0^1(\Omega)$ of a second order elliptic problem, on a bounded smooth domain $\Omega \subset R^N$, that involves a singular term, i.e. the problem $P_1(\lambda, \mu, \alpha, f, \gamma)$:

$$\begin{cases} -\Delta u(x) - \frac{\lambda}{|x|^2} u(x) & = |u(x)|^{2^*-2} u(x) + \mu |x|^{\alpha-2} u(x) + f(x)|u(x)|^{\gamma} \quad \text{in} \quad \Omega \backslash \{0\}, \\ \qquad\qquad u(x) & = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

where $2^* \doteq 2N/(N-2)$ denotes the critical Sobolev exponent in the sense that the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is continuous but is not compact.

Here we consider $f \in L^\infty(\Omega)$, which may be sign-changing, and the parameters $0 \leq \gamma < 1$, $0 \leq \lambda < \Lambda$, where Λ is the best constant in the Hardy inequality (see Lemma 1.1.8) and suitable values for α and μ.

The problem $P_1(\lambda, \mu, \alpha, f, \gamma)$ has loss of compactness and so the corresponding functional does not satisfy globally the classical Palais-Smale condition in $H_0^1(\Omega)$. In fact, as we have mention before the non-linearity has critical growth at the limiting exponent $2^* - 1$ for the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ (see Cerami-Fortunato-Struwe [26]). On other hand, due to term $\frac{\lambda}{|x|^2} u(x)$ the problem has strong singularity at zero and the non-compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^2\left(\Omega, |x|^{-2} dx\right)$ even locally in any neighborhood of zero, brings us to the question of the possibility of blow-up. To make sense, we define the equation on $\Omega \backslash \{0\}$, but still we assume that $0 \in \Omega$. Moreover the presence of term $\mu |x|^{\alpha-2} u(x)$ plays an important role, because it allows to control the singular term.

Considering suitable hypotheses and using special techniques, for overcoming the difficulties in dealing with problems like $P_1(\lambda,\mu,\alpha,f,\gamma)$, we prove the existence of two nontrivial solutions and under less strong hypothesis we prove the existence of (at least) four nontrivial solutions $u \in H_0^1(\Omega)$ and we prove that at least one of them is sign-changing.

The results obtained in this chapter are related with the publication Chen-Murillo-Rocha [39] and Chen-Murillo-Rocha [36].

2.1 Previous results

Equations that involve the critical Sobolev exponent, have been extensively investigated, since that, when $p = \frac{N+2}{N-2}$ the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ is not compact. Hence the Euler functional, does not satisfy the (PS)-condition globally, leading to difficulties in finding critical points by standard variational methods. Thus, if we consider the Yamabe's problem

$$\begin{cases} -\Delta u(x) & = |u(x)|^{2^*-2}u(x) + \xi u(x) \quad \text{in} \quad \Omega, \\ u(x) & > 0 \quad \text{in} \quad \Omega, \\ u(x) & = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{2.2}$$

the functional associated to problem

$$\psi(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{1}{2^*+1}\int |u|^{2^*+1} - \frac{1}{2}\int \xi\, |u|^2,$$

may lose compactness. However in a range, which is determined by the best constant for the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$, Brezis-Nirenberg [16], proved that, some compactness will hold. These type of equations have been studied by many other authors (e.g., see Kang-Deng [77] and Chaudhuri-Ramaswamy [29]). For the problem $P_1(0,0,\alpha,f,\gamma)$ and odd nonlinearity, Li-Zou [84] obtained infinitely many solutions. For more related results, we refer the interested readers to Costa-Silva [47], Ruiz-Willem [103] and Sang [104].

Elliptic equations containing simultaneously the critical exponent and a singular term ($\lambda \neq 0$), which are particular cases of the problems $P_1(\lambda,\mu,\alpha,f,\gamma)$, were considered in the literature as Ferrero-Gazzola [57]. They established the existence of solutions for the problem $P_1(\lambda,\mu,2,0,0)$ which depends the spatial dimension N and suitable restrictions on the coefficient of the singularity λ (for $N \geq 4$ with $\lambda \leq \Lambda - 1$ and $\Lambda - 1 < \lambda < \Lambda$).

Other relevant studies, are the works of He-Zou [70] for the problem $P_1(\lambda,0,\alpha,f,\gamma)$ and the works of Tarantello [118] and Chen [31], for the problem $P_1(\lambda,\mu,2,0,\gamma)$ under some conditions on $f(x,u)$. For problem $P_1(0,\mu,2,f,0)$ with Neumann condition, Tarantello [119] proved the existence of three solutions, one of which necessarily changes sign. When $N \geq 7$, Kang-Deng [77] proved the existence of two nontrivial solutions of the problem $P_1(\lambda, \mu, 2, f, 0)$ provided f satisfies some additional conditions.

Since we are facing with the singular term $\frac{\lambda}{|x|^2}$ and critical nonlinearity, we need to use the exact local behavior for the solutions of the problems obtained in Chen [31] and Chen [34] to estimate the energy, which is essential in the process of getting sign-changing solution. We also point out that similar techniques have been used in Chen-Rocha [42] to study

$$-\Delta u(x) - \frac{\lambda}{|x|^2}u(x) = |u(x)|^{\frac{4}{N-2}}u(x) + \overline{\lambda}\,|x|^{\alpha-2}u(x) + f(x),$$

where the existence of four nontrivial solutions was proved and at least one of them is sign-changing solution under some further conditions on λ, α and f.

In the present chapter, we emphasize in the results of Tarantello in [119] for the problem

$$\begin{cases} -\Delta u(x) & = |u(x)|^{2^*-2}u(x) + f(x) \quad \text{in} \quad \Omega, \\ \quad u(x) & = 0 \qquad\qquad\qquad\qquad\quad \text{on} \quad \partial\Omega, \end{cases} \tag{2.3}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded set and $f \in H^{-1}(\Omega)$ with $f \neq 0$ satisfying the following suitable condition

$$\int fu \leq C(\|\nabla u\|_2)^{\frac{N+2}{2}} \tag{2.4}$$

for all $u \in H_0^1(\Omega)$ such that $\|u\|_{2^*} = 1$ and an adequate positive constant C. She defined the infimum:

$$\mu_0 \doteq \inf_{\|u\|_{2^*}=1} \left\{ C(\|\nabla u\|_2)^{\frac{N+2}{2}} - \int fu \right\}.$$

and proved that for $f \neq 0$, μ_0 is achieved. Moreover, in particular if f satisfies the more restrictive assumption

$$\int fu < C(\|\nabla u\|_2)^{\frac{N+2}{2}} \tag{2.5}$$

for all $u \in H_0^1(\Omega)$ such that $\|u\|_{2^*} = 1$, one gets that $\mu_0 > 0$.

The functional

$$I(u) \doteq \frac{1}{2}\int |\nabla u|^2 - \frac{1}{2^*}\int |u|^{2^*} - \int fu,$$

associated to problem 2.3 is bounded below in the manifold

$$\bar{\Lambda} \doteq \left\{ u \in H_0^1(\Omega) : \langle I'(u), u \rangle = 0 \right\}.$$

The main result in Tarantello [119] is the following:

Theorem 2.1.1. *The problem (2.3), admits at least two weak solutions $u_0, u_1 \in H_0^1(\Omega)$ for $f \neq 0$ satisfying (2.4); and at least one weak solution for f satisfying (2.5). Moreover $u_0 \geq 0$, $u_1 \geq 0$ for $f \geq 0$.*

The following Lemma is very important for solving the problem (2.3) and permits to characterize these solutions in the subsets of Nehari:

$$\bar{\Lambda}^{+} \doteq \left\{ u \in \bar{\Lambda} : \|\nabla u\|_2^2 - (2^* - 1)\, \|u\|_{2^*}^{2^*} > 0 \right\}$$

and

$$\bar{\Lambda}^{-} \doteq \left\{ u \in \bar{\Lambda} : \|\nabla u\|_2^2 - (2^* - 1)\, \|u\|_{2^*}^{2^*} < 0 \right\}$$

Lemma 2.1.2. *Let $f \neq 0$ satisfy (2.5). For every $u \in H_0^1(\Omega)$, $u \neq 0$ there exists a unique $t^{+} = t^{+}(u) > 0$ such that $t^{+}u \in \bar{\Lambda}^{-}$. In particular:*

$$t^{+} > \left[\frac{\|\nabla u\|_2^2}{(2^* - 1)\, \|\nabla u\|_{2^*}^{2^*}} \right]^{1/(2^*-2)} \doteq t_{\max}$$

and $I(t^{+}u) = \max\limits_{t \geq t_{\max}} I(tu)$.

Moreover, if $\int\limits_{\Omega} fu > 0$, then there exists a unique $t^{-} = t^{-}(u) > 0$ such that $t^{-}u \in \bar{\Lambda}^{+}$. In particular

$$t^{-} > \left[\frac{\|\nabla u\|_2^2}{(2^* - 1)\, \|\nabla u\|_{2^*}^{2^*}} \right]^{1/(2^*-2)}$$

and $I(t^{-}u) \leq I(tu)$ for all $[0, t_{\max}]$.

Remark 2.1.3. *To prove the Lemma 2.1.2, Tarantello defined the function*

$$\varphi_u t \doteq t\, \|\nabla u\|_2^2 - t^{(2^*-1)}\, \|u\|_{2^*}^{2^*},$$

which achieves its maximum at

$$t_{max} \doteq \left[\frac{\|\nabla u\|_2^2}{(2^* - 1)\, \|u\|_{2^*}^{2^*}} \right]^{\frac{1}{2^*-2}}.$$

For better understand this Lemma, we can see graphically, the behavior of function $\varphi_u t$ (see Figure 1). Note that for all $t > 0$, if (2.5) holds, there exists a unique t^{+}. Moreover, if we consider $\bar{u}$, such that $\int f\bar{u} > 0$ there exists one additional point $\bar{t}_{-}$.

Remark 2.1.4. *The solutions u_0 and u_1 in the Theorem 2.1.1 of Tarantello are such that $u_0 \in \bar{\Lambda}^{+}$ and $u_1 \in \bar{\Lambda}^{-}$. Indeed, to prove the existence of u_0, Tarantello suppose that $f \neq 0$ satisfies (2.5) and using Ekeland variational principle prove that $\int fu > 0$. Then from Lemma 2.1.2, she conclude that there exists a unique t^{-} such that $t^{-}\bar{u_0} \equiv u_0 \in \bar{\Lambda}^{+}$, for $\bar{u_0} \in H_0^1(\Omega)$. For the existence of u_1. Tarantello suppose $f \neq 0$ satisfies (2.4) and using the Lemma 2.1.2, conclude that there exists a unique t^{+} such that $t^{+}\bar{u_0} \equiv u_0 \in \bar{\Lambda}^{-}$. In other words, Tarantello proved that there exists a unique function $\bar{u_0} \in H_0^1(\Omega)$, such that $\bar{u_0} \equiv \frac{u_1}{t_+} = \frac{u_0}{t_-}$.*

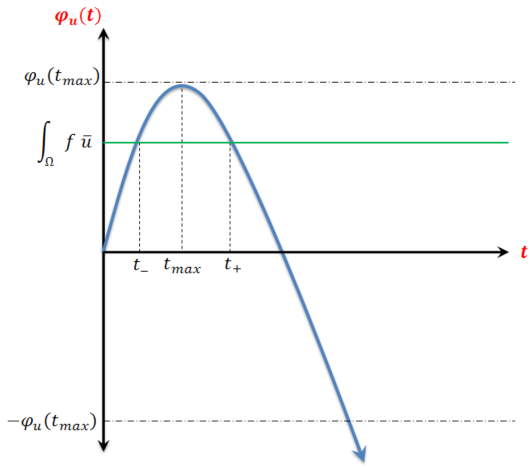

Figure 2.1: Behavior of the function $\varphi_u t$

2.2 Multiplicity results

The aim goal of this section is to study the existence of nontrivial solutions of problem $P_1(\lambda, \mu, \alpha, f, \gamma)$. We will start introducing some notation and remarks.

Define the functionals

$$\begin{aligned} T(u) &\doteq \int |\nabla u|^2 - \left(\tfrac{\lambda}{|x|^2} + \mu|x|^{\alpha-2}\right)|u|^2 \\ U(u) &\doteq \|u\|_{2^*}^{2^*}, \qquad F(u) \doteq \int f|u|^\gamma u, \\ Q(u) &\doteq T(u) - U(u) - F(u), \\ G(u) &\doteq 2\,T(u) - 2^*\,U(u) - (\gamma+1)F(u). \end{aligned}$$

Let μ_1 be the infimum defined in Chaudhuri-Ramaswamy [29]:

$$\mu_1 \doteq \inf\left\{\int\left(|\nabla u|^2 - \frac{\lambda}{|x|^2}|u|^2\right) : \int |x|^{\alpha-2}|u|^2 = 1\right\} > 0$$

and define the value

$$S_{\lambda,\mu} \doteq \inf\left\{\left(T(u)\right)^{\frac{1}{2}} : \int |u|^{2^*} = 1\right\} \tag{2.6}$$

Lemma 2.2.1. *If $0 \le \lambda < \Lambda$ and $0 < \mu < \mu_1$, then $S_{\lambda,\mu} > 0$, $T(u) > 0$ for all $u \in H_0^1(\Omega)\backslash\{0\}$ and $T(0) = 0$.*

Proof. For any $u \neq 0$, we have from the assumption $0 < \mu < \mu_1$ and the Hardy inequality that

$$T(u) \ge \left(1 - \frac{\mu}{\mu_1}\right)\int\left(|\nabla u|^2 - \frac{\lambda}{|x|^2}|u|^2\right) \ge \left(1 - \frac{\mu}{\mu_1}\right)\left(1 - \frac{\lambda}{\Lambda}\right)\int |\nabla u|^2.$$

Thus

$$\left(1 - \frac{\mu}{\mu_1}\right)\left(1 - \frac{\lambda}{\Lambda}\right)\int |\nabla u|^2 \le T(u) \le \int |\nabla u|^2. \tag{2.7}$$

Note that the best Sobolev constant (see Definition A.1.9)

$$S(\Omega) \doteq \inf\left\{ \int |\nabla u|^2 : \int |u|^{2^*} = 1 \right\} > 0.$$

Thus, from (2.7), we have

$$0 < S(\Omega) \leq \int |\nabla u|^2 \leq \left(1 - \frac{\mu}{\mu_1}\right)^{-1} \left(1 - \frac{\lambda}{\Lambda}\right)^{-1} T(u)$$

for all $u \in H_0^1(\Omega)$ such that $\int |u|^{2^*} = 1$. Therefore $0 < T(u)$ for all $u \in H_0^1(\Omega)$ such that $\int |u|^{2^*} = 1$ and therefore $S_{\lambda,\mu} > 0$. □

Remark 2.2.2. *(i) By the Gagliardo-Nirenberg-Sobolev inequality (see Lemma A.1.6), exists $K^U > 0$ such that $U(u) \leq K^U \|u\|^{2^*}$.*
(ii) For all $u \in H_0^1(\Omega)$,

$$F(u) \leq \left| \int f|u|^\gamma u \right| \leq \|f\|_\infty \|u\|_{\gamma+1}^{\gamma+1} \leq (\|f\|_\infty K_{\gamma+1}) \|u\|^{\gamma+1} \doteq K^T \|u\|^{\gamma+1}, \tag{2.8}$$

since $f \in L^\infty$, using the Hölder inequality and the Sobolev embedding of $H_0^1(\Omega)$ in $L^{\gamma+1}(\Omega)$ with constant $K_{\gamma+1} > 0$.

Define the following Euler–Lagrange energy functional

$$I(u) \doteq \frac{1}{2} T(u) - \frac{1}{2^*} U(u) - \frac{1}{\gamma+1} F(u).$$

Definition 2.2.3 (weak solution)**.** *We say that $u \in H_0^1(\Omega)$ is a (weak) solution of the problem $P_1(\lambda, \mu, \alpha, f, \gamma)$ if u is a critical point of the Euler functional I, i.e. for any $v \in H_0^1(\Omega)$ there holds*

$$\int (\nabla u \nabla v - \frac{\lambda}{|x|^2} uv - \mu |x|^{\alpha-2} uv - |u|^{2^*-2} uv - f|u|^\gamma v) = 0. \tag{2.9}$$

Remark 2.2.4. (*i*) *We can rewritte the problem $P_1(\lambda, \mu, \alpha, f, \gamma)$ as*

$$-\Delta u(x) - \frac{\lambda}{|x|^2} u(x) - \mu |x|^{\alpha-2} u(x) = |u(x)|^{2^*-2} u(x) + f(x)|u(x)|^\gamma, \;\; in \;\; \Omega \backslash \{0\}.$$

Then

$$-\Delta u^2 - \frac{\lambda}{|x|^2} u^2 - \mu |x|^{\alpha-2} u^2 = |u|^{2^*-2} u^2 + f(x)|u|^\gamma u, \;\; in \;\; \Omega \backslash \{0\}.$$

Since $f \in L^\infty(\Omega)$, we have

$$\frac{f(x)|u|^\gamma u}{|u|^{2^*-2} u^2} \leq |f||u|^{\gamma+1-2^*} \leq c|u|^{\gamma+1-2^*},$$

for some $c > 0$. Then $f(x)|u|^\gamma u$ is a lower-order perturbation of $|u|^{2^*-2} u^2$, in the sense

that $\frac{f(x)|u|^\gamma u}{|u|^{2^*-2}u(x)^2} \longrightarrow 0$ *as* $|u| \longrightarrow \infty$*, and we get for standard arguments due to Rabinowitz [101](see Proposition 1.1.1), that* $I \in C^1(H_0^1(\Omega), \mathbb{R})$*;*
(ii) If u *is solution of problem* $P_1(\lambda, \mu, \alpha, f, 0)$*, we have* $I'(u) = 0$*, then* $\langle I'(u), u\rangle = 0$ *and therefore* $u \in M$*.*

For any $u \in H_0^1(\Omega)\backslash\{0\}$ and $t \in \mathbb{R}$. Define $\phi_u(t) \doteq |t|^{-\gamma} \langle I'(tu), u\rangle + F(u)$, i.e.

$$\phi_u(t) = t\,|t|^{-\gamma}\, T(u) - t\,|t|^{2^*-\gamma-2}\, U(u). \tag{2.10}$$

This function attains its maximum at the positive value

$$t_{\max} \equiv t_{\max}(u) \doteq \left(\frac{1-\gamma}{2^*-\gamma-1}\, T(u)\, U(u)^{-1} \right)^{\frac{N-2}{4}}$$

We define $\phi_u(t_{\max}) = \Phi_*(u)$, where $\Phi_*(u)$ is the functional $\Phi_* : H_0^1(\Omega)\backslash\{0\} \to R$ given by

$$\Phi_*(u) \doteq t_{\max}(u)^{1-\gamma}\, T(u) - t_{\max}(u)^{2^*-\gamma-1}\, U(u) = C_{\gamma,N} T(u)^{\frac{2^*-\gamma-1}{2^*-2}} U(u)^{-\frac{1-\gamma}{2^*-2}},$$

with $C_{\gamma,N} \doteq \left(\frac{1-\gamma}{2^*-\gamma-1}\right)^{\frac{1-\gamma}{2^*-2}} \left(\frac{2^*-2}{2^*-\gamma-1}\right)$.

Let the set $B_\epsilon \doteq \left\{ w \in H_0^1(\Omega) : \|w\| < \epsilon \right\}$, the infimum

$$\widetilde{\mu}_f \doteq \inf_{u \in H_0^1(\Omega)} \{\Phi_*(u) - |F(u)|\}$$

and the infimum introduced by Tarantello:

$$\mu_f \doteq \inf_{U(u)=1} \left\{ C_{\gamma,N} T(u)^{\frac{2^*-\gamma-1}{2^*-2}} - F(u) \right\}.$$

Remark 2.2.5. **(i)** *If* $\widetilde{\mu}_f > 0$ *then* $\mu_f > 0$*. Indeed, since*

$$F(u) \leq |F(u)| < \Phi_*(u) = C_{\gamma,N} T(u)^{\frac{2^*-\gamma-1}{2^*-2}} U(u)^{-\frac{1-\gamma}{2^*-2}},$$

we have

$$C_{\gamma,N} T(u)^{\frac{2^*-\gamma-1}{2^*-2}} U(u)^{-\frac{1-\gamma}{2^*-2}} - F(u) > 0.$$

Therefore

$$\begin{aligned} \mu_f &:= \inf_{U(u)=1} \left\{ C_{\gamma,N} T(u)^{\frac{2^*-\gamma-1}{2^*-2}} - F(u) \right\} \\ &= \inf_{U(u)=1} \left\{ C_{\gamma,N} T(u)^{\frac{2^*-\gamma-1}{2^*-2}} U(u)^{-\frac{1-\gamma}{2^*-2}} - F(u) \right\} > 0. \end{aligned}$$

In the following three subsections, in order for obtains the results, we introduce some auxiliary results which are relevant to prove the results of this chapter, namely briefly describe the solution of an auxiliary problem, the local behavior of the solutions of the problem $P_1(\lambda,\mu,\alpha,f,\gamma)$ and some integral estimates.

2.2.1 Auxiliary problem

From Catrina-Wang [25], we have the following proposition:

Proposition 2.2.6. *For $0<\lambda<\Lambda\doteq(\frac{N-2}{2})^2$, the problem*

$$-\Delta u-\frac{\lambda}{|x|^2}u=|u|^{2^*-2}u \qquad x\in\mathbb{R}^N\backslash\{0\},\quad u(x)\to 0 \text{ as } |x|\to+\infty, \tag{2.11}$$

has a family of solutions

$$U_\varepsilon(x)=\frac{[4\varepsilon(\Lambda-\lambda)N/(N-2)]^{\frac{N-2}{4}}}{[\varepsilon|x|^{\gamma_1/\sqrt{\Lambda}}+|x|^{\gamma_2/\sqrt{\Lambda}}]^{\frac{N-2}{2}}} \quad \text{for } \varepsilon>0,$$

where $\gamma_1=\sqrt{\Lambda}-\sqrt{\Lambda-\lambda}$, $\gamma_2=\sqrt{\Lambda}+\sqrt{\Lambda-\lambda}$. Moreover, U_ε is the extremal function of the minimization problem

$$S_\lambda=\inf\left\{\int_{\mathbb{R}^N}\left(|\nabla u|^2-\frac{\lambda}{|x|^2}u^2\right)dx \,:\, u\in\mathcal{D}^{1,2}(\mathbb{R}^N),\ \int_{\mathbb{R}^N}|u|^{2^*}dx=1\right\}. \tag{2.12}$$

Clearly,

$$\int_{\mathbb{R}^N}|U_\varepsilon(x)|^{2^*}dx=\int_{\mathbb{R}^N}\left(|\nabla U_\varepsilon|^2-\frac{\lambda}{|x|^2}U_\varepsilon^2\right)dx=S_\lambda^{\frac{N}{2}}.$$

2.2.2 Local behavior of the solution

The local behavior of the solution of problem $P_1(\lambda,\mu,\alpha,f,\gamma)$, permits to calculate important estimates, that guaranties that the solutions obtained for the problem are different. The following proposition has been proved in Chen [33] and Chen [34], using the method of Moser iteration (see Chou-Chu[43] and Han-Lin [65]).

Proposition 2.2.7. *Let $0\le\lambda<\Lambda$. We have that*

- *if $u\in H_0^1(\Omega)$ is a solution of the problem $P_1(\lambda,\mu,\alpha,f,\gamma)$, then there holds*

$$|u(x)|\le K_1|x|^{-(\sqrt{\Lambda}-\sqrt{\Lambda-\lambda})}, \qquad x\in B_r(0)\backslash\{0\} \tag{2.13}$$

  for some positive constant K_1 and sufficiently small $r>0$;

- *if $u\in H_0^1(\Omega)$ is a positive solution of the problem $P_1(\lambda,\mu,\alpha,f,\gamma)$, then there holds*

$$K_2|x|^{-(\sqrt{\Lambda}-\sqrt{\Lambda-\lambda})}\le|u(x)|\le K_1|x|^{-(\sqrt{\Lambda}-\sqrt{\Lambda-\lambda})}, \qquad x\in B_r(0)\backslash\{0\} \tag{2.14}$$

  for $r>0$ sufficiently small and some positive constants $K_1,\ K_2$.

Remark 2.2.8. *Let u be a positive solution of the problem $P_1(\lambda,\mu,\alpha,f,\gamma)$*

(i) When $\lambda = 0$, $u(0)$ is positive and we come back to the usual case.

(ii) When $0 < \lambda < \Lambda$, the singular order at $x = 0$ of u stated in Proposition 2.2.7 coincide with the singularity of the explicit form $U_\varepsilon(x)$.

(iii) When $\lambda \to \Lambda$, the singularity of the positive solutions become more and more stronger.

2.2.3 Integral estimates

The following estimates are very relevant for obtaining of the results and to overcome the difficulties created by the singular term.

Define a cut-off function $\phi(x) = 1$ if $|x| \leq \delta$, $\phi(x) = 0$ if $|x| \geq 2\delta$, $\phi(x) \in C_0^1(\Omega)$ and $|\phi(x)| \leq 1$, $|\nabla\phi(x)| \leq C$. Let $v_\varepsilon(x) = \phi(x)U_\varepsilon(x)$, where $U_\varepsilon(x)$ is the family of solutions defined above.

From the work of Chen-Rocha [42], we have:

Proposition 2.2.9. *Let $0 \leq \lambda < \Lambda$ and $w \in H_0^1(\Omega)$ be a solution of the problem $P_1(\lambda,\mu,\alpha,f,\gamma)$, then for $\varepsilon > 0$ small enough we have that:*

$$\int w^{2^*-1}v_\varepsilon = O(\varepsilon^{\frac{N-2}{4}}) \quad \textit{and} \quad \int w v_\varepsilon^{2^*-1}dx = O(\varepsilon^{\frac{N-2}{4}}); \tag{2.15}$$

$$\int \left(|\nabla v_\varepsilon|^2 - \frac{\lambda}{|x|^2}v_\varepsilon^2\right) = S_\lambda^{\frac{N}{2}} + O(\varepsilon^{\frac{N}{2}}) + O(\varepsilon^{\frac{N-2}{2}}); \tag{2.16}$$

$$\int v_\varepsilon^{2^*} = S_\lambda^{\frac{N}{2}} - O(\varepsilon^{\frac{N}{2}}); \tag{2.17}$$

$$\int |x|^{\alpha-2}v_\varepsilon^2 = O(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}}), \ \textit{when } 0 < \alpha < 2\sqrt{\Lambda-\lambda}; \tag{2.18}$$

$$\int v_\varepsilon = O(\varepsilon^{\frac{N-2}{4}}); \tag{2.19}$$

$$\int w v_\varepsilon = O(\varepsilon^{\frac{N-2}{4}}); \tag{2.20}$$

Remark 2.2.10. *We emphasize that in the estimate (2.18), the local behavior of the solution of problem $P_1(\lambda,\mu,\alpha,f,\gamma)$ played on essential role.*

2.2.4 Existence of two nontrivial solutions

We consider the following hypotheses (H_2):

(i) $0 \leq \lambda < \Lambda$, $0 < \mu < \mu_1$, $0 < \alpha < \sqrt{\Lambda-\lambda}$, $0 \leq \gamma < 1$, $f \in L^\infty(\Omega)$ and $\widetilde{\mu}_f > 0$;

(ii) $\frac{N-\sqrt{\Lambda}}{\sqrt{\Lambda}+\sqrt{\Lambda-\lambda}} < \gamma < 1$, f is continuous at $0 \in \Omega$ and $f(0) > 0$;

(*iii*) $f > 0$.

We say that hypotheses (H_2) hold if $(H_2)(i)$ holds and one of the hypotheses $(H_2)(ii)$ or $(H_2)(iii)$ holds.

We start stabilizing when the condition $\widetilde{\mu}_f > 0$ is satisfied.

Lemma 2.2.11. *Let* $\bar{\alpha} = \frac{2^*-\gamma-1}{2^*-2}$ *and* $\beta = \frac{1-\gamma}{2^*-2}$. *If*

$$\|f\|_\infty < C_{(\gamma,N)}(K_1{}^T)^{\bar{\alpha}}(K^U)^{-\beta}K_{\gamma+1}{}^{-1}, \tag{2.21}$$

where $C_{(\gamma,N)} = \left(\frac{1-\gamma}{2^*-\gamma-1}\right)^{\frac{1-\gamma}{2^*-2}}\left(\frac{2^*-2}{2^*-\gamma-1}\right)$, $K_1{}^T = \left(1-\frac{\mu}{\mu_1}\right)\left(1-\frac{\lambda}{\Lambda}\right)$; K^U *and* $K_{\gamma+1}$ *are the best Sobolev constant for the embedding of* $H_0^1(\Omega)$ *into* $L^{2^*}(\Omega)$; *and* $H_0^1(\Omega)$ *into* $L^{\gamma+1}(\Omega)$ *respectively. Then* $\widetilde{\mu}_f > 0$.

Proof. From (2.7) and Remark 2.2.2, there exist positive constants $K_{\gamma+1}$, $K_1{}^T$ and K^U such that $F(u) = \|f\|_\infty K_{\gamma+1}\|u\|^{\gamma+1}$, $T(u) \geq K_1{}^T\|u\|^2$ and $U(u) \leq K^U\|u\|^{2^*}$. Then

$$\Phi_*(u) \geq C_{(\gamma,N)}(K_1{}^T)^{\bar{\alpha}}\|u\|^{2\bar{\alpha}}(K^U)^{-\beta}\|u\|^{-2^*\beta} = C_{(\gamma,N)}(K_1{}^T)^{\bar{\alpha}}(K^U)^{-\beta}\|u\|^{\gamma+1}.$$

Now, if the inequality (2.21) holds, from (2.8) we have

$$F(u) \leq C_{(\gamma,N)}(K_1{}^T)^{\alpha}(K^U)^{-\beta}\|u\|^{\gamma+1}$$

and therefore $F(u) < \Phi_*(u)$. Now we consider two cases

- If $F(u) > 0$, we have $-\Phi_*(u) < -F(u) < F(u) < \Phi_*(u)$ and $|F(u)| < \Phi_*(u)$. Therefore $\widetilde{\mu}_f > 0$.

- If $F(u) < 0$, we have $F(u) = -|F(u)|$, then

$$\widetilde{\mu}_f = \inf_{u\in H_0^1(\Omega)}\{\Phi_*(u) - |F(u)|\} = \inf_{u\in H_0^1(\Omega)}\{\Phi_*(u) + F(u)\}.$$

  Since $F(u) < \Phi_*(u)$ and $\Phi_*(u) > 0$, we have $\Phi_*(u) + F(u)$. Thus $\widetilde{\mu}_f > 0$.

Therefore, we have $\widetilde{\mu}_f > 0$. □

As the energy functional I is not bounded below on $H_0^1(\Omega)$, we consider the functional on the Nehari set

$$M \doteq \{u \in H_0^1(\Omega)\backslash\{0\} : Q(u) = 0\}.$$

and the subsets of M defined by the sign of G (second derivative of I)

$$M^+ \doteq \{u \in M : G(u) > 0\}, \quad M^0 \doteq \{u \in M : G(u) = 0\}, \quad M^- \doteq \{u \in M : G(u) < 0\}.$$

For $u \in M$, the functionals I and G, can be rewritten as

$$I_M(u) = -\frac{1-\gamma}{2(\gamma+1)}T(u) + \frac{2^*-\gamma-1}{2^*(\gamma+1)}U(u),$$

$$G_M(u) = (1-\gamma)\,T(u) - (2^*-\gamma-1)\,U(u),$$

where we have denoted the restrictions of I and G, to the set M, by I_M and G_M, respectively.

Remark 2.2.12. *(a) $I(u)$ is bounded from below in M. In fact for any $u \in M$, we have*

$$\begin{aligned} I_{\mathbf{M}}(u) &= \left(\frac{1}{2} - \frac{1}{2^*}\right) T\,(u) + \left(\frac{1}{2^*} - \frac{1}{\gamma+1}\right) F(u) \\ &\geq \frac{2^*-2}{2\,2^*} T\,(u) - \left(\frac{2^*-\gamma-1}{2^*(\gamma+1)} K^T\right) \|u\|^{\gamma+1}, \end{aligned}$$

using (2.8). From (2.7), we have

$$\begin{aligned} I_{\mathbf{M}}(u) &\geq \left(1-\frac{\mu}{\mu_1}\right)\left(1-\frac{\lambda}{\Lambda}\right)\frac{2^*-2}{2\,2^*}\|u\|^2 - \left[\frac{2^*-\gamma-1}{2^*(\gamma+1)}K^T\right]\|u\|^{\gamma+1} \\ &\geq -\frac{1}{4}\left[\left(1-\frac{\mu}{\mu_1}\right)\left(1-\frac{\lambda}{\Lambda}\right)\frac{2^*-2}{2\,2^*}\right]^{-1}\left[\frac{2^*-\gamma-1}{2^*(\gamma+1)}K^T\right]^2. \end{aligned}$$

(b) For any $u \in H_0^1(\Omega)\backslash\{0\}$, we have $I(tu) \to -\infty$ as $|t| \to \infty$.

The following Lemma is a generalization of Lemma 2.1 of Tarantello [119]:

Lemma 2.2.13. *Suppose the hypothesis $(H_2)(i)$ holds. For any $u \in H_0^1(\Omega)\backslash\{0\}$, define $s_f \doteq sign\,F(u) \in \{-1,+1\}$. Then there exist three values $t_0 \equiv t_0(u) \in \mathbb{R}$, $t_- \equiv t_-(u) \in \mathbb{R}$, $t_+ \equiv t_+(u) \in \mathbb{R}$ such that:*
(i) $t_+ > 0$, $t_+u \in M^-$, $t_+ > t_{\max}$ and $I(t_+u) = \max_{t \geq t_{\max}} I(tu)$;
(ii) $s_f\,t_- > 0$, $t_-u \in M^+$, $0 < s_f\,t_- < t_{\max}$ and $I(t_-u) = \min_{-t_{\max} \leq t \leq t_{\max}} I(tu)$;
(iii) $t_0 < 0$, $t_0u \in M^-$, $t_0 < -t_{\max}$ and $I(t_0u) = \max_{t \leq -t_{\max}} I(tu)$.

Proof. Let $t \in \mathbb{R}$. Define the function $\phi_u(t) \doteq |t|^{-\gamma}\,\langle I'(tu), u\rangle + F(u)$, i.e.

$$\phi_u(t) = t\,|t|^{-\gamma}\,T(u) - t\,|t|^{2^*-\gamma-2}\,U(u). \tag{2.22}$$

From the definition of ϕ_u, we have $\phi_u(0) = \lim_{t\to 0^\pm} \phi_u(t) = 0$,
$\lim_{t\to+\infty} \phi_u(t) = -\infty$, $\phi_u(-t) = -\phi_u(t)$ for all $t > 0$, and $\phi_u''(t) < 0$ for all $t > 0$, so ϕ_u (restricted to $t > 0$) is a concave function which attains its maximum at $t_{\max}$ and $\phi_u(t_{\max}) = \Phi_*(u) > 0$.
For simplicity of presentation, we first assume $s_f = +1$.
(i) Since ϕ_u (for $t > 0$) is a concave and continuous function and $0 < F(u) < \phi_u(t_{\max})$, there exists a unique $t_+ > t_{\max}$ such that $\phi_u(t_+) = F(u) > 0$. This implies, from the definition of ϕ_u, that $|t_+|^{-\gamma}\,\langle I'(t_+\,u), u\rangle = 0$ so $Q(t_+\,u) = 0$ and $t_+\,u \in M$. Moreover,

from $\phi'_u(t_+) < 0$ i.e. $T(u) < (2^*-\gamma-1)(1-\gamma)^{-1}|t_+|^{2^*-2}\,U(u)$, we have $G_M(t_+u) < 0$; thus $t_+u \in M^-$ and $I(t_+u) \geq I(tu)$ for all $t \geq t_{\max}$. The last statement is true because, if we set $r(t) = I(t\,u)$, then $r'(t) = t^{-1}Q(t\,u)$ so $r'(t_+) = 0$, and from $r'(t) = t^\gamma(\phi_u(t) - \phi_u(t_+))$ we have $r'(t) > 0$, when $t_{\max} \leq t < t_+$, and $r'(t) < 0$, when $t > t_+$.
(ii) By similar arguments to the ones used in (i), there exists a unique $t_- > 0$ such that $-t_{\max} < 0 < t_- < t_{\max}$ and $\phi_u(t_-) = F(u) > 0$ so $t_-u \in M$ and, from $\phi'_u(t_-) > 0$, $t_-u \in M^+$. From $r'(t) = t^\gamma(\phi_u(t) - \phi_u(t_-))$, we have $r'(t) > 0$, when $t_- < t \leq t_{\max}$, and $r'(t) < 0$, when $-t_{\max} \leq t < t_-$. Therefore, at least, $I(t_-u) \leq I(tu)$ for all $-t_{\max} \leq t \leq t_{\max}$.
(iii) Note that $\lim_{t\to-\infty}\phi_u(t) = +\infty$, $\phi_u(-t_{\max}) = -\Phi_*(u) < 0$, $\phi'_u(t) < 0$ for all $t < -t_{\max}$, and $\phi''_u(t) > 0$ for all $t < 0$, hence there exists a unique $t_0 < -t_{\max} < 0$ such that $\phi_u(t_0) = F(u) > 0$ so $t_0u \in M$ and, from $\phi'_u(t_0) < 0$, $t_0u \in M^-$. From $r'(t) = t^\gamma(\phi_u(t) - \phi_u(t_0))$, we have $r'(t) > 0$, when $t < t_0$, and $r'(t) < 0$, when $t_0 < t < -t_{\max}$. Therefore, $I(t_0u) \geq I(tu)$ for all $t < -t_{\max}$.
For the general situation $s_f \in \{-1,+1\}$, it is enough to observe that $(s_f)^{-1} = s_f$, $(s_f)^2 = 1$, $\phi_u(s_f\,t) = s_f\,\phi_u(t)$ for $t \in \mathbb{R}$, $F(s_f\,u) = s_f\,F(u)$, $G_M(s_f\,u) = G_M(u)$, and $r'(s_f\,t) = s_f\,r'(t)$ for $t \in \mathbb{R}$. □

Remark 2.2.14. *The above Lemma can be further improved. In fact, $\phi'_u(\pm t_{\max}) = 0$, $\phi'_u(t) > 0$ when $-t_{\max} < t < t_{\max}$ and $\phi'_u(t) < 0$ otherwise. So in fact, under the same hypotheses, we can say: (i) $I(t_+u) = \max_{t\geq t_-} I(tu)$; (ii) $I(t_-u) = \min_{t_0\leq t\leq t_+} I(tu)$; and (iii) $I(t_0u) = \max_{t\leq t_-} I(tu)$.*

Remark 2.2.15. *For $0 \leq \gamma < 1$, beside the situation in Lemma 2.2.13, i.e. when $|F(u)| < \Phi_*(u)$ where we have* ***three*** *values t_0, t_- and t_+, other situations are: (a) for $F(u) = \Phi_*(u)$ we have* ***two*** *values $t_0 < 0$ and $t_- = t_+ = t_{\max} > 0$; (b) $F(u) = -\phi_*(u)$ we have* ***two*** *values $t_0 = t_- = -t_{\max} < 0$ and $t_+ > 0$; (c) for $F(u) > \Phi_*(u)$ we have* ***one*** *value $t_0 < 0$; and (d) for $F(u) < \Phi_*(u)$ we have* ***one*** *value $t_+ > 0$. Therefore, we can rewrite Lemma 2.2.13 in the following (more general) way.*

Lemma 2.2.16. *Let $0 \leq \gamma < 1$. For any $u \in H_0^1(\Omega)\backslash\{0\}$, we have:*
(i) if $F(u) < \Phi_*(u)$, exists $t_+ > 0$ such that $t_+u \in M^-$, $t_+ > t_{\max}$ and $I(t_+u) = \max_{t\geq t_{\max}} I(tu)$;
(ii) if $-\Phi_*(u) < F(u) < \Phi_*(u)$, exists $t_- \in \mathbb{R}$ such that $s_f\,t_- > 0$, $t_-u \in M^+$, $0 < s_f\,t_- < t_{\max}$ and $I(t_-u) = \min_{-t_{\max}\leq t\leq t_{\max}} I(tu)$;
(iii) if $F(u) > -\Phi_*(u)$, exists $t_0 < 0$ such that $t_0u \in M^-$, $t_0 < -t_{\max}$ and $I(t_0u) = \max_{t\leq -t_{\max}} I(tu)$.

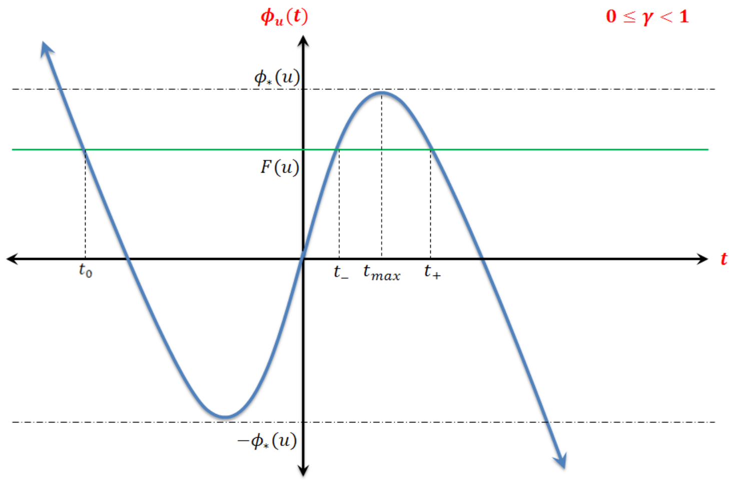

Figure 2.2: Behavior of the function ϕ_u

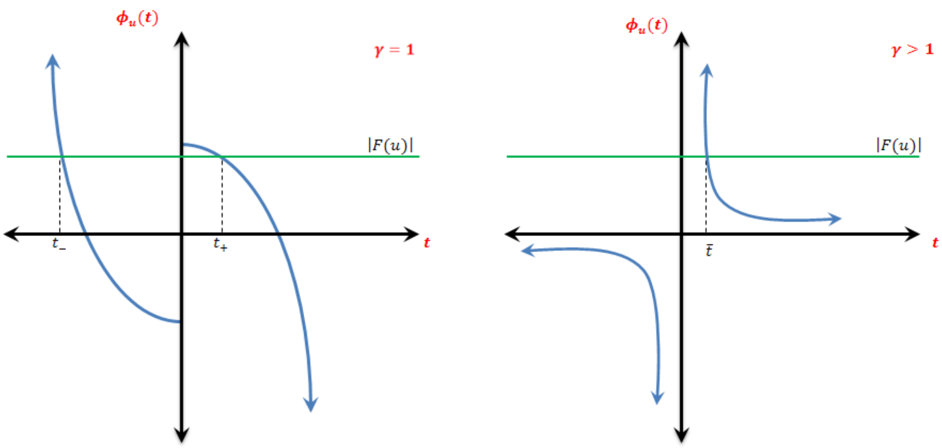

Figure 2.3: Behavior of the function ϕ_u for different values

Remark 2.2.17. *For better understand this Lemma, we can see graphically the behavior of function ϕ_u (see Figure 2.2). Note that for $0 \leq \gamma < 1$, $\phi_u(\text{tmax}) = \phi_*(u)$ and if $0 < |F(u)| < \phi_*(u)$, we have **three** values t_0, t_- and t_+.If we consider Other values for γ, the behavior of the function $\phi_u(t)$ is quite different (see Figure 2.3). When $\gamma = 1$, we have $\lim_{t\to 0^\pm} \phi_u(t) = \pm T(u)$ and $\lim_{t\to\pm\infty} \phi_u(t) = \mp\infty$, thus: (a) for $|F(u)| < T(u)$, we have **two** values $t_- < 0$ and $t_+ > 0$; (b) for $F(u) \geq T(u)$, we have **one** value $t_- < 0$; and (c) for $F(u) \leq -T(u)$, we have **one** value $t_+ > 0$. When $\gamma > 1$, we have $\lim_{t\to 0^\pm} \phi_u(t) = \pm\infty$ and $\lim_{t\to\pm\infty} \phi_u(t) = 0$, thus: (a) for $F(u) > 0$, we have **one** value $t_+ > 0$; and (b) for $F(u) < 0$, we have **one** value $t_- < 0$.*

We prove the existence of two nontrivial solution, using Ekeland variational principle and Nehari techniques.

Set
$$c_+ \doteq \inf_{u\in M^+} I(u) \quad \text{and} \quad c_- \doteq \inf_{u\in M^-} I(u).$$
Let $u \in H_0^1(\Omega)\backslash\{0\}$. From Lemma 2.2.13, there is a real value $t \equiv t\,(u)$ such that $tu \in M^-$ so $M^- \neq \emptyset$ (following the same idea $M^+ \neq \emptyset$) and $M \neq \emptyset$. Recall M is a manifold, and I is continuous and bounded from below on M.

Ekeland's variational principle 1.1.7 applied to the optimization problem

$$c_0 \doteq \inf_{u\in M} I(u) \tag{2.23}$$

gives a bounded minimizing sequence $(u_n)_{n\in\mathbb{N}} \subset M$ satisfying:

(E_a) $c_0 \leq I(u_n) < c_0 + \frac{1}{n}$;

(E_b) $I(u) \geq I(u_n) - \frac{1}{n}\|u - u_n\|$ for all $u \in M$.

The following result will be used below, in a contradiction argument, to show that the minimizing sequence converges strongly in $H_0^1(\Omega)$.

Proposition 2.2.18. *Assume hypothesis $(H_2)(i)$ holds. Let $u \in H_0^1(\Omega)$, $(u_n)_{n\in\mathbb{N}} \subset M^-$ be such that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $I(u_n) \to c \in \mathbb{R}$ but u_n does not converge strongly to u in $H_0^1(\Omega)$. Recall the definitions of $s_f \equiv s_f(u)$, $t_+ \equiv t_+(u)$ and $t_- \equiv t_-(u)$ in Lemma 2.2.13. Then the following holds:*

(i) If $u \not\equiv 0$ and $t_+ \leq 1$, then $c > I(t_+u)$;

(ii) If $u \not\equiv 0$ and $t_+ > 1$, then $c \geq I(t_-u) + \frac{1}{N}S_\lambda^{\frac{N}{2}}$;

(iii) If $u \equiv 0$, then $c \geq \frac{1}{N}S_\lambda^{\frac{N}{2}}$.

Proof. Firstly, following the same idea of Chen-Li-Li [37], (Lemma 2.6), we prove that $u_n \rightharpoonup u$ and $\int |x|^{\alpha-2}|u_n - u|^2 \to 0$ as

$$n \to \infty.$$

Let $\{u_n\} \subset H_0^1(\Omega)$ be bounded. We may assume that, up to a subsequence,

$$u_n \rightharpoonup u \text{ in } H_0^1(\Omega)\,,\ u_n \to u \text{ a.e in } \Omega.$$

Then $u_n \to u$ in $L^p(\Omega)$ for $1 < p < 2^*$. So that, letting $\frac{2N}{N+\alpha-2} < s < 2^*$, we have from Hölder inequality that

$$\int |x|^{\alpha-2} |u_n - u|^2 \leq \left(\int |u_n - u|^s\right)^{\frac{2}{s}} \left(\int |x|^{\frac{(\alpha-2)s}{s-2}}\right)^{\frac{s-2}{s}}$$

and from the choice of s, $\int |x|^{\frac{(\alpha-2)s}{s-2}} < \infty$ holds. It follows from $\int |u_n - u|^s \to 0$ as $n \to \infty$ that

$$\int |x|^{\alpha-2} |u_n - u|^2 \to 0.$$

We may assume that there exist $a, b \geq 0$ such that

$$T(u_n - u) = \int (|\nabla u_n - \nabla u|^2 - \frac{\lambda}{|x|^2}|u_n - u|^2) + o(1) \to a^2,$$

and $\int |u_n - u|^{2^*} \to b^{2^*}$. Note that, since u_n does not converge strongly to u, we have $a \neq 0$. On the other hand, from $f \in L^\infty$ and the compactness of the Sobolev embedding, we have $\int f|u_n - u|^\gamma (u_n - u) \to 0$.

For $t \in \mathbb{R}$, we set

$$r(t) \doteq I(tu), \quad \beta(t) \doteq \frac{a^2}{2}t^2 - \frac{b^{2^*}}{2^*}t^{2^*}$$

and $\theta(t) \doteq r(t) + \beta(t)$. So, for $t > t_+$,

$$r'(t) = \langle I'(tu), u\rangle = t^\gamma \left(\phi_u(t) - \int f(x)|u|^\gamma u\right) = t^\gamma \left(\phi_u(t) - \phi_u(t_+)\right) < 0, \tag{2.24}$$

since ϕ_u is a decreasing function for $t > t_+$. From

$$\begin{aligned}|I(tu_n) - \theta(t)| &= \left|\frac{1}{2}t^2 T(u_n) - \frac{t^{2^*}}{2^*}\|u_n\|_{2^*}^{2^*} - \frac{t^{\gamma+1}}{\gamma+1}\int f|u_n|^\gamma u_n - I(tu) - \beta(t)\right| \\ &\leq \left|\frac{1}{2}t^2 T(u_n - u) - \frac{t^{2^*}}{2^*}\|u_n - u\|_{2^*}^{2^*} - \beta(t)\right|\end{aligned}$$

we see that $I(tu_n) \to \theta(t)$ as $n \to +\infty$. We now prove the three statements:

(i) Suppose $u \neq 0$ and $t_+ \leq 1$. From (2.24), $r'(1) \leq 0$. Since $Q(u_n) = \langle I'(u_n), u_n\rangle \to \theta'(1)$ and $u_n \in M$, we have $Q(u_n) = 0$ and $\theta'(1) = 0$. Thus $\beta'(1) \geq 0$ and hence $a^2 - b^{2^*} \geq 0$. So, we have

$$\beta(t_+) = b^{2^*}\left(\frac{{t_+}^2}{2} - \frac{t_+^{2^*}}{2^*}\right) > 0.$$

Since $I(tu_n) \to \theta(t)$, then $I(u_n) \to \theta(1)$ and hence

$$c = \theta(1) \geq \theta(t_+) = I(t_+u) + \beta(t_+u) > I(t_+u).$$

(ii) Suppose $u \neq 0$ and $t_+ > 1$. First, from $t_+ > 1$, $b \neq 0$. Indeed, since $0 = Q(u_n) = \langle I'(u_n), u_n \rangle \to \theta'(1)$ and $u_n \in M^-$, then $\theta'(1) = 0$ and $\theta''(1) \leq 0$. If $b = 0$, we have $r'(1) = -a^2 < 0$ and

$$r''(1) = \theta''(1) - a^2 - (2^* - 1)b^{2^*} \leq -a^2 < 0,$$

which contradicts to $t_+ > 1$. So we have $b \neq 0$. We know that β attains its maximum at $t_* = (a^2/b^{2^*})^{\frac{1}{2^*-2}}$ and $\beta'(t) > 0$ for $0 < t < t_*$ and $\beta'(t) < 0$ for $t > t_*$. Therefore $\beta(t_*) = \frac{1}{N}(a/b)^N$. Now, since

$$S_\lambda = \inf \frac{\int_{\mathbb{R}^N} \left(|\nabla u|^2 - \frac{\lambda}{|x|^2} u^2\right) dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{\frac{2}{2^*}}} \leq \frac{a^2}{\left(b^{2^*}\right)^{\frac{2}{2^*}}},$$

we have $a^2 \geq S_\lambda \left(b^{2^*}\right)^{\frac{2}{2^*}}$ and

$$\beta(t_*) = \frac{1}{N} \left(\frac{a^2}{b^2}\right)^{\frac{N}{2}} \geq \frac{1}{N} \left(\frac{S_\lambda \left(b^{2^*}\right)^{\frac{2}{2^*}}}{b^2}\right)^{\frac{N}{2}} \geq \frac{1}{N} S_\lambda^{\frac{N}{2}}.$$

Next, we show that $t_* \leq t_+$. Suppose this is not the case, i.e., $1 < t_+ < t_*$. As $0 > \theta'(t) = r'(t) + \beta'(t)$ for all $t > 1$, we have $r'(t) \leq -\beta'(t) < 0$ for $t \in (1, t_*)$, which contradicts to $1 < t_+ < t_*$ and $r'(t_+) = 0$. So, in fact, $t_* \leq t_+$.

Note that $\theta(1) = \lim_{n\to\infty} I(u_n)$ and $I(u_n) = \max_{t>0} I(tu_n)$. Hence, we obtain

$$\theta(1) = \lim_{n\to\infty} \left(\max_{t>0} I(tu_n)\right) \geq \lim_{n\to\infty} I(t_*u_n) = \theta(t_*)$$

and

$$c = \theta(1) \geq \theta(t_*) = I(t_*u) + \beta(t_*) \geq I(t_*u) + \frac{1}{N} S_\lambda^{\frac{N}{2}}.$$

Moreover, from $t_* \leq t_+$ and $I(t_-u) = \min_{0 \leq t \leq t_+} I(tu)$ (see Remark 2.2.14), we have $c \geq I(t_*u) + \frac{1}{N} S_\lambda^{\frac{N}{2}} \geq I(t_-u) + \frac{1}{N} S_\lambda^{\frac{N}{2}}$.

(iii) Suppose $u \equiv 0$. Since $u_n \in M^- \subset M$, we have

$$\int (|\nabla u_n|^2 - \frac{\lambda}{|x|^2} |u_n|^2) = \int |u_n|^{2^*} + o(1).$$

and

$$c \geq \frac{1}{2} \int \left(|\nabla u_n|^2 - \frac{\lambda}{|x|^2} |u_n|^2\right) - \frac{1}{2^*} \int |u_n|^{2^*} + o(1)$$

Using the fact that $S_\lambda|v|_{2^*}^2 \le \int(|\nabla v|^2 - \frac{\lambda}{|x|^2}|v|^2)$ for all $v \in H_0^1(\Omega)$ and $v \neq 0$, we obtain that

$$c \ \ge \left(\frac{1}{2}-\frac{1}{2^*}\right)\int\left(|\nabla u_n|^2 - \frac{\lambda}{|x|^2}|u_n|^2\right) + o(1) \ge \frac{1}{N}S_\lambda^{\frac{N}{2}}.$$

The proof is complete. □

We recall the definition of set

$$B_\epsilon \doteq \left\{w \in H_0^1(\Omega) : \|w\| < \epsilon\right\}.$$

Lemma 2.2.19. *Suppose hypotheses* $(H_2)(i)$ *holds, then:*
(i) For every $u \in M$, $G_M(u) \doteq (1-\gamma)T(u) - (2^*-\gamma-1)U(u) \neq 0$, *i.e.* $M^0 = \emptyset$;
(ii) For any sequence $(u_n)_{n\in\mathbb{N}} \subset M$, *we have*

$$\lim_{n\to+\infty} G_M(u_n) = 0 \quad \Rightarrow \quad \liminf_{n\to+\infty}\|u_n\| = 0;$$

(iii) Given $u \in M$, *there exists* $\varepsilon > 0$ *and a differentiable function* $t : H_0^1(\Omega) \to \mathbb{R}$, *satisfying* $t(w) > 0$ *for all* $w \in B_\varepsilon$, $t(0) = 1$, $t(w)(u-w) \in M$ *for all* $w \in B_\varepsilon$ *and*

$$\langle t'(0), w\rangle = \frac{\int\left(2\nabla u\nabla w - 2\frac{\lambda}{|x|^2}uw - 2\mu|x|^{\alpha-2}uw - 2^*|u|^{2^*-2}uw - (1+\gamma)f|u|^\gamma w\right)}{G_M(u)}. \tag{2.25}$$

Proof. (i) Assume, by contradiction, that $(1-\gamma)T(\bar u) - (2^*-\gamma-1)U(\bar u) = 0$ for some $\bar u \in M$, then we have

$$s_{\bar u} \doteq U(\bar u)^{\frac{1}{2^*}} \ge \left(\frac{1-\gamma}{2^*-\gamma-1}C\right)^{\frac{1}{2^*-2}} > 0$$

for some constant $C > 0$, by using the Gagliardo-Nirenberg-Sobolev inequality. On the other hand, since $\bar u \in M$, we have

$$F(\bar u) = \frac{2^*-2}{1-\gamma}U(\bar u).$$

Recall the definition of Φ_* in Lemma 2.2.13, and define $\Psi_*(u) \doteq \Phi_*(u) - F(u)$ for all $u \in M$. Hence, $\Psi_*(su) = s^{1+\gamma}\Psi_*(u)$, for any $s > 0$ and $u \in M$, and

$$\Psi_*(\bar u) \ge \inf_{U(u)^{1/2^*}=s_{\bar u}}\Psi_*(u) = s_{\bar u}^{1+\gamma}\left(\inf_{U(v)^{1/2^*}=1}\Psi_*(v)\right) \ge s_{\bar u}^{1+\gamma}\mu_f.$$

Let $K \doteq \frac{2^*-\gamma-1}{1-\gamma}$. Thus, from $\mu_f > 0$, we have

$$0 < s_{\bar u}^{1+\gamma}\mu_f \le \Psi_*(\bar u) \le \left[K^{-\frac{1-\gamma}{2^*-2}}(1-K)K^{\frac{2^*-\gamma-1}{2^*-2}} - (K-1)\right]U(\bar u) < 0.$$

This is a contradiction. Therefore $(1-\gamma)T(u) - (2^*-\gamma-1)U(u) \neq 0$ for all $u \in M$.

(ii) Arguing by contradiction again, assume there exists a subsequence $(u_n)_{n\in\mathbb{N}} \subset M$ such that

$$(1-\gamma)T(u_n) - (2^* - \gamma - 1)U(u_n) = o(1)$$

and $\|u_n\| > s$ for all $n \in N$ and some $s > 0$. Hence, $s_{u_n} \doteq U(u_n)^{\frac{1}{2^*}} > 0$ for all $n \in N$. Since $u_n \in M$, we get

$$F(u_n) = T(u_n) - U(u_n) = \left[(2^* - 2)/(1-\gamma)\right] U(u_n) + o(1).$$

These together with $\mu_f > 0$ and $\Psi_*(u_n) \geq \inf\limits_{U(u)^{1/2^*} = s_{u_n}} \Psi_*(u) \geq s_{u_n}^{1+\gamma}\mu_f$ implies

$$0 < s_{u_n}^{1+\gamma}\mu_f \leq \Psi_*(u_n) \leq \left(1 - K^2\right) U(u_n) + o(1) < 0,$$

which is a contradiction, so $(1-\gamma)T(u_n) - (2^* - \gamma - 1)U(u_n) = o(1)$ and $\|u_n\| = o(1)$.

(iii) Let $u \in M$ and $\phi : \mathbb{R} \times H_0^1(\Omega) \to \mathbb{R}$ be defined by

$$\phi(t, w) \doteq t|t|^{-\gamma} T(u - w) - t|t|^{2^* - \gamma - 2} U(u - w) - F(u - w).$$

Note that $\frac{\partial}{\partial t}\phi(1, 0) = G_M(u) \neq 0$ (by (i)) and $\phi(1, 0) = Q(u) = 0$. Hence applying the implicit function theorem at the point $(1, 0)$, we have that there exists a function $t \equiv t(w)$ with $t(0) = 1$ and

$$\left\langle t'(0), w \right\rangle = -\frac{\partial}{\partial w}\phi(1, 0)\left(\frac{\partial}{\partial t}\phi(1, 0)\right)^{-1}.$$

□

The following result prove the existence of a first solution for the problem $P_1(\lambda, \mu, \alpha, f, \gamma)$

Proposition 2.2.20. *Suppose hypotheses $(H_2)(i)$ hold. We have $c_0 < 0$, there is a critical point $w_0 \in M^+$ of I such that $I(w_0) = c_0$, and w_0 is a local minimizer for I. Moreover, $w_0 > 0$ whenever that $f > 0$.*

Proof. Let $u \in M^+ \neq \emptyset$ (see Lemma 2.2.13). From $G(u) > 0$, we have

$$U(u) < \frac{1-\gamma}{2^* - \gamma - 1} T(u), \tag{2.26}$$

so,

$$\begin{aligned} I_{\mathbf{M}}(u) &= \left(\frac{1}{2} - \frac{1}{\gamma+1}\right) T(u) - \left(\frac{1}{2^*} - \frac{1}{\gamma+1}\right) U(u) \\ &< \left(\frac{\gamma-1}{\gamma+1}\right) T(u) + \left(\frac{2^* - \gamma - 1}{2^*(\gamma+1)}\right)\left(\frac{1-\gamma}{2^* - \gamma - 1}\right) T(u) \\ &= \left(\frac{1-\gamma}{\gamma+1}\right)\left(\frac{1}{2^*} - 1\right) T(u) < 0. \end{aligned}$$

Hence $c_+ < 0$, since $c_+ \doteq \inf_{u\in\mathbf{M}^+} I(u) \leq I(t_- u) < 0$.

Moreover,

$$c_0 \doteq \inf_{u\in \mathbf{M}} I(u) \le \inf_{u\in \mathbf{M}^+} I(u) < 0.$$

From Ekeland's variational principle there exists a bounded minimization sequence $(u_n)_{n\in\mathbb{N}} \subset M$. We need to show that $\|I'(u_n)\|_{H^{-1}(\Omega)} \to 0$ as $n\to\infty$.

Choosing n where $I'(u_n)\neq 0$, applying the item (iii) of Lemma 2.2.19 for $\delta>0$ sufficiently small and setting $u\equiv u_n$, $w\equiv \delta\frac{I'(u_n)}{\|I'(u_n)\|}$, we have that exists $t_n(\delta)\doteq t\left(\delta\frac{I'(u_n)}{\|I'(u_n)\|}\right)$ such that

$$w_\delta \doteq t_n(\delta)\left(u_n - \delta\frac{I'(u_n)}{\|I'(u_n)\|}\right)\in M.$$

On the other hand, by (E_b) and the Taylor expansion of I, we have

$$\begin{aligned}\frac{1}{n}\|w_\delta - u_n\| &\ge \langle I'(w_\delta), u_n - w_\delta\rangle + o(\|u_n - w_\delta\|)\\ &= \langle I'(w_\delta), u_n(1-t_n(\delta))\rangle + \left\langle I'(w_\delta), t_n(\delta)\,\delta\frac{I'(u_n)}{\|I'(u_n)\|}\right\rangle\\ &+ o\left(\left\|u_n - t_n(\delta)\,u_n + \delta\frac{I'(u_n)}{\|I'(u_n)\|}u_n\right\|\right).\end{aligned}$$

Hence

$$\frac{1}{n}\|w_\delta - u_n\| \ge (1-t_n(\delta))\langle I'(w_\delta), u_n\rangle + \delta t_n(\delta)\left\langle I'(w_\delta), \frac{I'(u_n)}{\|I'(u_n)\|}\right\rangle + o(\delta). \tag{2.27}$$

Dividing (2.27) by $\delta>0$ and passing to the limit as $\delta\to 0$, we have

$$\frac{1}{n}(1+\|u_n\|\,\|t_n'(0)\|) \ge \left\langle I'(u_n), \frac{I'(u_n)}{\|I'(u_n)\|}\right\rangle = \|I'(u_n)\|.$$

Since $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence,

$$\|I'(u_n)\| \le \frac{1}{n}(1+\|u_n\|\,\|t_n'(0)\|) \le \frac{C}{n}(1+\|t_n'(0)\|)$$

for a suitable positive constant $C>0$. Note that $t_n'(0) = \left\langle t'(0), \frac{I'(u_n)}{\|I'(u_n)\|}\right\rangle$. Then by (2.25), since $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence and $\|w\|=\delta$, we have

$$|t_n'(0)| \le \frac{C_1}{|(1-\gamma)T(u_n) - (2^*-\gamma-1)U(u_n)|}$$

for a suitable positive constant C_1. From Lemma 2.2.19, we have

$$\liminf_{n\to+\infty}\,[(1-\gamma)T(u_n) - (2^*-\gamma-1)U(u_n)] > 0.$$

Thus $|t_n'(0)|\le K_1$, for a suitable constant $K_1>0$ and therefore $\|I'(u_n)\|_{H^{-1}(\Omega)}\to 0$ as $n\to\infty$.

Let w_0 be the weak limit in $H_0^1(\Omega)$ of (a subsequence of) the minimizing sequence u_n. Then $w_0\in M^+$. Indeed, suppose that $w_0\in M^-$ (since $M^0=\emptyset$), from Lemma 2.2.13 there exists $t_+ \equiv t_+(w_0)$ such that $t_+ > 0$ and $t_+ w_0 \in M^-$. But $w_0 \in M^-$ implies $t_+ = 1$. In this case, there exists also $t_- \equiv t_-(w_0) \in (-t_{\max}, t_{\max})$ such that $t_- < t_+ = 1$. Thus, we have

$$\frac{d}{dt} I(tw_0)\bigg|_{t=t_-} = \langle I'(t_- w_0), w_0\rangle = (t_-)^{-1} Q(t_- w_0) = 0$$

and

$$\begin{aligned}\frac{d^2}{dt^2} I(tw_0)\bigg|_{t=t_-} &= \frac{d}{dt}\left(|t|^{\gamma}\left[\phi_u(t) - F(u)\right]\right)\bigg|_{t=t_-} = \frac{d}{dt}\left(|t|^{\gamma}\left[\phi_u(t) - \phi_u(t_-)\right]\right)\bigg|_{t=t_-} \\ &= \gamma\,|t_-|^{\gamma-2} t_-\left[\phi_u(t_-) - \phi_u(t_-)\right] + |t_-|^{\gamma}\phi_u'(t_-) = |t_-|^{\gamma}\phi_u'(t_-) > 0.\end{aligned}$$

Hence, there exists $\bar{t} \in \mathbb{R}$ such that $t_- < \bar{t} < t_+$ and $I(\bar{t} w_0) > I(t_- w_0)$. But from Remark 2.2.14,

$$I(t_- w_0) < I(\bar{t} w_0) < I(t_+ w_0) = I(w_0) = c_0.$$

This is a contradiction. Therefore $w_0 \in M^+$. This implies that $F(w_0) > \frac{2^*-2}{1-\gamma} U(w_0) > 0$.

We have that w_0 is a weak solution of the problem, since $I'(u_n) \to 0$ as $n \to \infty$, we have $\langle I'(w_0), w\rangle = 0$, for all $w \in H_0^1(\Omega)$. Therefore

$$c_0 \le I(w_0) \le \lim_{n\to\infty} I(u_n) = c_0.$$

Then $u_n \to w_0$ (converges strongly) in $H_0^1(\Omega)$ and $I(w_0) = c_0 = \inf_{u\in M} I(u)$.

We now show that w_0 is a local minimum for I. From Lemma 2.2.13, for all $u \in M$, there exists $t_-(u) \in \mathbb{R}$ such that $t_-(u) < t_{\max}(u)$, $t_-(u)u \in M^+$ and

$$I(t_-(u)u) \le I(\xi u), \qquad \text{for all } 0 < \xi < t_{\max}(u). \tag{2.28}$$

So, from $w_0 \in M^+$, we have

$$t_{\max}(w_0) > t_-(w_0) = 1. \tag{2.29}$$

Let $\varepsilon > 0$ be sufficiently small. From item (iii) of Lemma 2.2.19 exists a differentiable function $t : H_0^1(\Omega) \to \mathbb{R}$ such that $t(w) > 0$, $t(0) = 1$ and $t(w)(w_0 - w) \in M$ for all $\|w\| < \varepsilon$. From (2.29), the continuity of $t_{\max}(u)$ and $t(w) \to 1$ as $\|w\| \to 0$, we can always find a sufficiently small $\varepsilon > 0$ for which $t(w) < t_{\max}(w_0 - w)$ for all $w \in H_0^1(\Omega)$ with $\|w\| < \varepsilon$.

Note that $t(w)(w_0 - w) \in M^+$ so $t(w) = t_-(w_0 - w)$ and $t(w) < t_{\max}(w_0 - w)$. From (2.28), with $u = w_0 - w$, and the fact that w_0 is a local minimum, we have

$$I(\xi(w_0 - w)) \ge I(t(w)(w_0 - w)) \ge I(w_0).$$

Now taking $\xi = 1$, we conclude that $I(w_0 - w) \ge I(w_0)$, for all $w \in H_0^1(\Omega)$ with $\|w\| < \varepsilon$. Therefore, w_0 is a local minimum for I.

From Lemma 2.2.13, for $|w_0| \in H_0^1(\Omega)$, there exists a unique value $t_-(|w_0|) \in \mathbb{R}$ such that $t_-(|w_0|)\,|w_0| \in M^+$, $t_-(|w_0|) < t_{\max}(|w_0|) = t_{\max}(w_0)$ and

$$I(t_-(|w_0|)\,|w_0|) = \min_{-t_{\max}\le t\le t_{\max}} I(t(|w_0|)\,|w_0|).$$

Since $w_0 \in M^+$, then $t_-(w_0) = 1$. Thus

$$c_0 \le I(t_-(w_0)\,w_0) = \min_{-t_{\max}\le t\le t_{\max}} I(tw_0) \le I(t_-(|w_0|)\,w_0).$$

Note that, from $f > 0$, we have $I(t_-(|w_0|)\,|w_0|) \le I(t_-(|w_0|)\,w_0) \le c_0$. Therefore $I(t_-(w_0)\,w_0) = c_0$ and we can always take $w_0 > 0$. □

Remark 2.2.21. *From Proposition 2.2.20, we have that there is a critical point w_0 of I such that $I(w_0) = c_0$. Hence, since $c_0 \doteq \inf_{u\in M} I(u)$, $c_+ \doteq \inf_{u\in M^+} I(u)$ and $w_0 \in M^+$, we have that $c_0 = c_+$.*

If $|F(u)| < \phi_*(u)$, from Remark 2.2.21, there exists $t_-u \in M^+$ such that

$$I(t_-u) = inf_{u\in M^+} I(u) = inf_{u\in M} I(u).$$

Lemma 2.2.22. *Suppose hypotheses $(H_2)(i)$ hold, then there is $s_0 > 0$ and $\varepsilon > 0$ sufficiently small such that $w_0 + s_0 v_\varepsilon \in M^-$, where $w_0 \in M^+$ is a critical point of I and v_ε is a truncated function.*

Proof. We use the same argument as in the Proposition 2.2 of Tarantello [119].
Set

$$\Sigma \doteq \left\{u \in H_1^0(\Omega) : \|u\|_T = T(u)^{\frac{1}{2}} = 1\right\}$$

and $\Psi : \Sigma \to M^-$ a map, such that $\Psi(u) = t_+(u)u$, where $t_+(u)$ is defined as Lemma (2.2.13).

First, note that M^- is closed. In fact, if $u \in M^-$, we have

$$U(u) \ge \frac{1-\gamma}{2^*-\gamma-1} T(u).$$

Since exists $K_1^T > 0$ such that $K_1^T \|u\|^2_{H_0^1(\Omega)} \le T(u)$ and $K^U > 0$ such that $U(u) \le K^U \|u\|^{2^*}$, we have

$$\left[\frac{1-\gamma}{2^*-\gamma-1}\frac{K_1^T}{K^U}\right]^{\frac{1}{2^*-2}} \le \|u\|.$$

Then $\|u\|_{H_0^1(\Omega)} \ge \widetilde{K}$, where $\widetilde{K} = \left[\frac{1-\gamma}{2^*-\gamma-1}\frac{K_1^T}{K^U}\right]^{\frac{1}{2^*-2}} > 0$. Therefore in view of Lemma (2.2.19), every sequence (u_n) in M^- satisfies

$$\lim_{n\to+\infty} G_M(u_n) < 0.$$

This is M^- is closed. By the uniqueness and extremal property (see Lemma (2.2.13)), $t_+(u)$ is a continuous function. Thus Ψ is continuous with continuous inverse

$$\Psi^{-1}(u) = \frac{u}{\|u\|_T}.$$

Hence, the map Ψ defined a homeomorphism.
Now, suppose u such that $t_+\left(\frac{u}{\|u\|_T}\right) = \|u\|_T$. Then

$$t_+\left(\frac{u}{\|u\|_T}\right)\frac{u}{\|u\|_T} = \|u\|_T \frac{u}{\|u\|_T} = u$$

and from the Lemma (2.2.13), we have

$$t_+\left(\frac{u}{\|u\|_T}\right)\frac{u}{\|u\|_T} \in M^-.$$

Hence $u \in M^-$. Thus M^- disconnects $H_1^0(\Omega)$ in exactly two components:

$$\begin{aligned}
U^- &\doteq \left\{u = 0 \text{ ou } u \neq 0 : \|u\|_T < t_+\left(\frac{u}{\|u\|_T}\right)\right\},\\
U^+ &\doteq \left\{u : \|u\|_T > t_+\left(\frac{u}{\|u\|_T}\right)\right\}.
\end{aligned}$$

Note that $M^+ \subset U^-$. In fact, if $u \in M^+$,

$$\frac{1-\gamma}{2^*-\gamma-1}\, T(u)\, U(u)^{-1} > 1,$$

and since that

$$t_+(u) > t_{\max}(u) \doteq \left(\frac{1-\gamma}{2^*-\gamma-1}\, T(u)\, U(u)^{-1}\right)^{\frac{N-2}{4}};$$

we have

$$\begin{aligned}
t_+\left(\frac{u}{\|u\|_T}\right) &> \left(\frac{1-\gamma}{2^*-\gamma-1}\, T\!\left(\frac{u}{\|u\|_T}\right) U\!\left(\frac{u}{\|u\|_T}\right)^{-1}\right)^{\frac{N-2}{4}}\\
&= \left(\frac{1-\gamma}{2^*-\gamma-1}\, \frac{\frac{1}{\|u\|_T^2}T(u)}{\frac{1}{\|u\|_T^{2^*}}U(u)}\right)^{\frac{1}{2^*-2}}\\
&= \|u\|_T \left(\frac{1-\gamma}{2^*-\gamma-1}\, \frac{T(u)}{U(u)}\right)^{\frac{1}{2^*-2}}\\
&> \|u\|_T\,.
\end{aligned}$$

This is $u \in U^-$. From Lemma (2.2.13), if $F(u) > 0$, and $t_2 \neq 0$ is a critical point of

$$\phi_u(t) = t\,|t|^{-\gamma}\, T(u) - t\,|t|^{2^*-\gamma-2}\, U(u);$$

we have

$$t_2 = \left(\frac{T(u)}{U(u)}\right)^{\frac{1}{2^*-2}} = \left(\frac{\|u\|_T^2}{\|u\|_{2^*}^{2^*}}\right)^{\frac{1}{2^*-2}}$$

and unique values $t_-(u)$ and $t_+(u)$ such that $0 < t_-(u) < t_{\max}(u) < t_+(u)$ and

$$t_+(u) \leq \left(\frac{\|u\|_T}{\|u\|_{2^*}}\right)^{\frac{2^*}{2^*-2}}.$$

Now, we consider the function w_0+sv_ε, where $s > 0$. We can assume that $F(w_0 + sv_\varepsilon) > 0$. In fact,

$$F(w_0 + sv_\varepsilon) = \int f(x) |w_0 + sv_\varepsilon|^\gamma (w_0 + sv_\varepsilon),$$

for ε small enough. If $f(x)$ is positive near 0, $F(w_0 + sv_\varepsilon) > 0$. If $f(x)$ is negative, we replace $w_0 + sv_\varepsilon$ by $w_0 - sv_\varepsilon$. Therefore for $F(w_0 + sv_\varepsilon) > 0$, we have

$$t_+\left(\frac{w_0 + sv_\varepsilon}{\|w_0 + sv_\varepsilon\|_T}\right) \leq \left(\frac{\|w_0 + sv_\varepsilon\|_T}{\|w_0 + sv_\varepsilon\|_{2^*}}\right)^{\frac{2^*}{2^*-2}} \to 1, \quad \text{as } s \to \infty \text{ and } \varepsilon \to 0^+.$$

Hence for $R_0 > 0$ sufficiently large and $\varepsilon_0 > 0$ sufficiently small, we have

$$\varsigma = \sup\left\{\left[t_+\left(\frac{w_0 + sv_\varepsilon}{\|w_0 + sv_\varepsilon\|_T}\right)\right]^2 : \ s \geq R_0,\, 0 < \varepsilon < \varepsilon_0\right\} < \infty.$$

Thus for $s \geq \varsigma S_\lambda^{\frac{N}{2}} + R_0$ and ε small, we have

$$\begin{aligned}
T(w_0 + sv_\varepsilon) &= T(w_0) + s^2 T(v_\varepsilon) + 2s \int (\nabla w_0 \nabla v_\varepsilon - \frac{\lambda}{|x|^2} w_0 v_\varepsilon - \mu |x|^{\alpha-2} w_0 v_\varepsilon) \\
&= T(w_0) + s^2 T(v_\varepsilon) + 2s \int (|w_0|^{2^*-2} w_0 v_\varepsilon + f(x)|w_0|^\gamma v_\varepsilon) \\
&= T(w_0) + s^2 T(v_\varepsilon) + 2s \left[O(\varepsilon^{\frac{N-2}{4}}) + O(\varepsilon^{\gamma_*})\right],
\end{aligned}$$

where $\gamma_* \equiv \gamma_*(\gamma)$. Note that, from Proposition 2.2.9, $T(v_\varepsilon) = S_\lambda^{\frac{N}{2}} + O(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}})$ for ε small.

We get that

$$\|w_0 + sv_\varepsilon\|_T = T(w_0 + sv_\varepsilon)^{\frac{1}{2}} > t_+\left(\frac{w_0 + sv_\varepsilon}{\|w_0 + sv_\varepsilon\|_T}\right),$$

which implies that $w_0 + sv_\varepsilon \in U^+$. Thus we have $\gamma_0 \in (0,1)$ such that $w_0 + \gamma_0 sv_\varepsilon \in M^-$. The conclusion follows by choosing $s_0 = \gamma_0 s$. □

Lemma 2.2.23. *If hypotheses (H_2) hold, then $c_- < c_0 + \frac{1}{N} S_\lambda^{\frac{N}{2}}$.*

Proof. From Lemma 2.2.22, we know that there is $s_0 > 0$ and $\varepsilon > 0$ sufficiently small such that $w_0 + s_0 v_\varepsilon \in M^-$, by using the arguments in Proposition 2.2 of Tarantello [119]. To prove $c_- < c_0 + \frac{1}{N}S_\lambda^{\frac{N}{2}}$, we only need to prove that $\sup_{s>0} I(w_0 + sv_\varepsilon) < c_0 + \frac{1}{N}S_\lambda^{\frac{N}{2}}$, since

$$c_- = \inf_{u\in M^-} I(u) \le I(w_0 + s_0 v_\varepsilon) \le \sup_{s>0} I(w_0 + sv_\varepsilon).$$

Moreover, we only need to consider bounded values for s, since, $I(w_0 + sv_\varepsilon) \to -\infty$ as $s \to +\infty$ implies that there is $s_0 > 0$ such that

$$\sup_{s>0} I(w_0 + sv_\varepsilon) \le \sup_{0<s<s_0} I(w_0 + sv_\varepsilon).$$

First, since w_0 is a solution of $P_1(\lambda, \mu, \alpha, f, \gamma)$., we get from direct computations that

$$\begin{aligned} I(w_0 + sv_\varepsilon) &= \frac{1}{2}T(w_0 + sv_\varepsilon) - \frac{1}{2^*}U(w_0 + sv_\varepsilon) - \frac{1}{\gamma+1}F(w_0 + sv_\varepsilon) \\ &= I(w_0) + I(sv_\varepsilon) + \int |w_0|^{2^*-2} w_0 (sv_\varepsilon) + \int f(x)|w_0|^\gamma (sv_\varepsilon) \\ &\quad - \frac{1}{2^*}\left[U(w_0 + sv_\varepsilon) - U(w_0) - U(sv_\varepsilon)\right] \\ &\quad - \frac{1}{\gamma+1}\left[F(w_0 + sv_\varepsilon) - F(w_0) - F(sv_\varepsilon)\right]. \end{aligned}$$

Suppose hypotheses $(H_2)(ii)$ hold. Using the elementary inequality

$$\left||a+b|^q - |a|^q - |b|^q\right| \le d_1 \left[|a|^{q-1}|b| + |a|\,|b|^{q-1}\right]$$

for $a, b \in \mathbb{R}$ and $q > 1$, we obtain that

$$\begin{aligned} I(w_0 + sv_\varepsilon) &\le I(w_0) + I(sv_\varepsilon) + \int |w_0|^{2^*-1}(sv_\varepsilon) + |f|_{L^\infty(\Omega)} \int |w_0|^\gamma (sv_\varepsilon) \\ &\quad + d_2 \int |w_0|^{2^*-1}|sv_\varepsilon| + d_3 \int |w_0|\,|sv_\varepsilon|^{2^*-1} \\ &\quad + d_4 \int |w_0|^\gamma |sv_\varepsilon| + d_5 \int |w_0|\,|sv_\varepsilon|^\gamma, \end{aligned}$$

where, here and below, d_j for $j \in \mathbb{N}$ denote positive constants.

Secondly, since f is continuous at 0 and $f(0) > 0$, there exist $d_6 > 0$ and $\delta_0 > 0$ such that $f(x) \ge d_6$ for any $x \in B_{\delta_0}(0)$, the ball with center at 0 and radius δ_0. Hence, we have

$$\begin{aligned} \sup_{s>0} I(w_0 + sv_\varepsilon) &\le I(w_0) + \sup_{s>0}\left[\frac{1}{2}T(sv_\varepsilon) - \frac{1}{2^*}U(sv_\varepsilon)\right] + d_9 \int |w_0|^{2^*-1} v_\varepsilon \\ &\quad + d_{10} \int |w_0|\,|v_\varepsilon|^{2^*-1} + d_{11} \int |w_0|^\gamma v_\varepsilon + d_{12} \int |w_0| v_\varepsilon^\gamma \\ &\quad - d_7 \int_{B_{\delta_0}(0)} v_\varepsilon^{\gamma+1} + d_8 \int_{\Omega\setminus B_{\delta_0}(0)} v_\varepsilon^{\gamma+1}. \end{aligned}$$

Note that for ε small enough,

$$\int |w_0|^\gamma v_\varepsilon = O(\varepsilon^{\frac{N-2}{4}}),$$

$$\int_{\Omega\setminus B_{\delta_0}(0)} v_\varepsilon^{\gamma+1} = O(\varepsilon^{\frac{N-2}{4}(\gamma+1)}),$$

$$\int |w_0| v_\varepsilon^\gamma = O(\varepsilon^{\frac{N-2}{4}\gamma})$$

and

$$\int_{B_{\delta_0}(0)} v_\varepsilon^{\gamma+1} = O\left(\varepsilon^{\frac{[N-(\gamma+1)\sqrt{\Lambda}]\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}}\right).$$

We obtain from the assumption $\frac{N-\sqrt{\Lambda}}{\sqrt{\Lambda}+\sqrt{\Lambda-\lambda}} < \gamma < 1$ and Proposition 2.2.9 that

$$\sup_{s>0} I(w_0 + sv_\varepsilon) \quad < \quad I(w_0) + \frac{1}{N} S_\lambda^{\frac{N}{2}} = c_0 + \frac{1}{N} S_\lambda^{\frac{N}{2}}.$$

When hypotheses $(H_2)(iii)$ hold, instead of $(H_2)(ii)$, the proof is similar so we omit the details. □

The following result prove the existence of a second solution for the problem $P_1(\lambda, \mu, \alpha, f, \gamma)$.

Proposition 2.2.24. *If hypotheses (H_2) hold, then there is a critical point $w_1 \in M^-$ of I such that $I(w_1) = c_-$. Moreover, if $f > 0$, then $w_1 > 0$.*

Proof. First we will prove that there is $w_1 \in M^-$ of I such that $I(w_1) = c_-$. Here we will use the same idea of the proof of the Proposition 3.7 in Chen-Rocha [42]. Let $(u_n)_{n\in\mathbb{N}} \subset M^-$ and $I(u_n) \to c_-$. Then by direct calculations we know that

$$0 < \inf T(u_n) \le \sup T(u_n) < \infty.$$

The definition of μ_1 and $0 < \mu < \mu_1$ implies that $(u_n)_{n\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$. We may assume that $(u_n)_{n\in\mathbb{N}}$ converges weakly to some w_1. By Proposition 2.2.18 we have that $w_1 \neq 0$. Now suppose that $(u_n)_{n\in\mathbb{N}}$ does not converge to w_1. Then by (1) and (2) of Proposition 2.2.18, we get that $c_- > I(t_+(w_1)w_1)$ or

$$c_- \ge I(t_-(w_1)w_1) + \frac{1}{N} S_\lambda^{\frac{N}{2}} \ge c_0 + \frac{1}{N} S_\lambda^{\frac{N}{2}}.$$

In any case we get a contradiction since $c_- < c_0 + \frac{1}{N} S_\lambda^{\frac{N}{2}}$. Therefore $(u_n)_{n\in\mathbb{N}}$ converges strongly to w_1. This means $w_1 \in M^-$ and $I(w_1) = c_-$.

Next we will show that such w_1 is a weak solution of equation in problem $P_1(\lambda, \mu, \alpha, f, \gamma)$. Choose any $v \in H_0^1(\Omega)$. For any $\rho \in (0, 1)$ we set $t_\rho = t_+(w_1 + \rho v)$ (where $t_+(w_1 + \rho v)$ is defined according to Lemma 2.2.13). Since w_1, $t_\rho(w_1 + \rho v) \in M^-$ and $I(w_1) = \inf_{u\in M^-} I(u)$, we have

$$I(t_\rho(w_1+\rho v)) \geq I(w_1).$$

On the other hand from $w_1 \in M^-$, we have that for any $t>0$, $I(w_1) \geq I(tw_1)$. In particular, $I(w_1) \geq I(t_\rho w_1)$. Thus we have for any $\rho \in (0,1)$,

$$I(t_\rho(w_1+\rho v)) \geq I(t_\rho w_1).$$

Hence, we get that

$$0 \quad \leq \frac{1}{\rho}\Big(I(t_\rho(w_1+\rho v)) - I(t_\rho w_1)\Big)$$

From Lemma (2.2.13), for all $u \in M$, there exists $t_+ = t_+(u) > 0$ such that $t_+(u)u \in M^-$. If $w_1 \in M^-$, then $t_+(w_1) = 1$. Thus for $\rho \to 0^+$, $t_\rho = t_+(w_1+\rho v) \to 1$. Letting $\rho \to 0^+$, we obtain

$$\begin{aligned}
0 &\leq \lim_{\rho\to 0^+} \frac{1}{\rho}\Big(I(t_\rho(w_1+\rho v)) - I(t_\rho w_1)\Big) \\
&= \lim_{\rho\to 0^+} \frac{\langle I'(w_1), v\rangle}{1} = \int \Big(\nabla w_1 \nabla v - \frac{\lambda}{|x|^2} w_1 v - \mu |x|^{\alpha-2} w_1 v - |w_1|^{2^*-2} w_1 v - f(x)|w_1|^\gamma v\Big).
\end{aligned}$$

As v is arbitrarily, we get that

$$\int \Big(\nabla w_1 \nabla v - \frac{\lambda}{|x|^2} w_1 v - \mu |x|^{\alpha-2} w_1 v - |w_1|^{2^*-2} w_1 v - f(x)|w_1|^\gamma v\Big) = 0.$$

Which means that w_1 is a solution of the problem $P_1(\lambda,\mu,\alpha,f,\gamma)$.

Now, we will show that $w_1 > 0$, if $f>0$. From Lemma 2.2.13, there exists $t_+(w_1) \in R$, such that $s_f\, t_+(|w_1|) > 0$, $t_+(|w_1|)\,|w_1| \in M^-$, $s_f\, t_+(|w_1|) > t_{\max}(|w_1|) = t_{\max}(w_1)$ and $I(t_+(|w_1|)\,|w_1|) = \max_{s_f t \geq 0} I(t(|w_1|)\,|w_1|)$. Since $w_1 \in M^-$, then $t_+(w_1) = 1$. Thus

$$I(t_+(w_1)\,w_1) = I(w_1) = \max_{s_f t\geq 0} I(tw_1) \geq I(t_+(|w_1|)\,w_1).$$

Note that, since $f>0$, we have

$$I(t_+(|w_1|)\,w_1) \geq I(t_+(|w_1|)\,|w_1|) \geq c_-.$$

Therefore $I(t_+(w_1)\,w_1) = c_-$ and we can always take $w_1 > 0$. □

Now, we are ready for the multiplicity theorem for problem $P_1(\lambda,\mu,\alpha,f,\gamma)$.

Theorem 2.2.25. *Suppose hypotheses (H_2) holds, then $P_1(\lambda,\mu,\alpha,f,\gamma)$ has two nontrivial solutions in $H_0^1(\Omega)$. Moreover, if $(H_2)(iii)$ hold, then both solutions are positive.*

Proof. This result is direct consequence from Proposition 2.2.20 and Proposition 2.2.24 □

2.2.5 Multiplicity theorem with less restrictive hypotheses

In this section, we will prove the existence of other solutions for the problem $P_1(\lambda, \mu, \alpha, f, \gamma)$ under less restricts hypotheses. As described in the introduction, the proof is divided into three steps. We start proving the existence of two nontrivial solutions; we prove the existence of a third solution which is a sign-changing solution and we prove the existence of a fourth solution using translated argument.

We consider the following hypothesis (H_3):

(i) $0 \leq \lambda < \Lambda$, $0 < \mu < \mu_1$, $0 < \alpha < \sqrt{\Lambda - \lambda}$, $0 \leq \gamma < 1$, $f \in L^\infty(\Omega)$ and $\widetilde{\mu}_f > 0$;

(ii) $0 < \alpha < \gamma\sqrt{\Lambda - \lambda}$ and $0 < \gamma \leq \frac{N}{\sqrt{\Lambda}+\sqrt{\Lambda-\lambda}} - 1$.

We say that hypotheses (H_3) hold if $(H_3)(i)$ holds and the hypotheses $(H_3)(ii)$ holds.

Under the above hypotheses we prove that the problem $P_1(\lambda, \mu, \alpha, f, \gamma)$ has at least four nontrivial solutions in $H_0^1(\Omega)$ and at least one of them is sign-changing.

Existence of two nontrivial solution

We consider as before

$$c_0 \doteq \inf_{u \in M} I(u) \quad \text{and} \quad c_- \doteq \inf_{u \in M^-} I(u).$$

Proposition 2.2.26. *Suppose hypotheses $(H_3)(i)$ hold. We have $c_0 < 0$, there is a critical point $w_0 \in M^+$ of I such that $I(w_0) = c_0$, and w_0 is a local minimizer for I. Moreover, $w_0 > 0$ whenever $f > 0$.*

Proof. Since the hypothesis $(H_2)(i)$ is equal to the hypothesis $(H_3)(i)$, the proof is the same as Proposition 2.2.20. □

In the rest of this section, we fix w_0 which is obtained in the proposition 2.2.26
On this point, we emphasize the importance of the estimate calculated in Appendix B, which guarantees that the solutions obtained are different.

Lemma 2.2.27. *If hypotheses (H_3) hold, then $c_- < c_0 + \frac{1}{N} S_\lambda^{\frac{N}{2}}$.*

Proof. First, from Lemma (2.2.22), we know that there are $s_0 > 0$ and $\varepsilon > 0$ sufficiently small such that $w_0 + s_0 v_\varepsilon \in M^-$. To prove $c_- < c_0 + \frac{1}{N} S_\lambda^{\frac{N}{2}}$, we only need to prove that $\sup_{s>0} I(w_0 + s v_\varepsilon) < c_0 + \frac{1}{N} S_\lambda^{\frac{N}{2}}$, since $c_- = \inf_{u \in M^-} I(u) \leq I(w_0 + s_0 v_\varepsilon) \leq \sup_{s>0} I(w_0 + s v_\varepsilon)$. Moreover, we only need to consider bounded values for s, since, $I(w_0 + s v_\varepsilon) \to -\infty$ as $s \to +\infty$ implies that there is $s_0 > 0$ such that

$$\sup_{s>0} I(w_0 + s v_\varepsilon) \leq \sup_{0<s<s_0} I(w_0 + s v_\varepsilon).$$

Firstly, since w_0 is a solution of the problem $P(\lambda,\mu,\alpha,f,\gamma)$, we get

$$\begin{aligned}
I(w_0+sv_\varepsilon) &= \frac{1}{2}T(w_0+sv_\varepsilon) - \frac{1}{2^*}U\left(w_0+sv_\varepsilon\right) - \frac{1}{\gamma+1}F\left(w_0+sv_\varepsilon\right)\\
&= \frac{1}{2}\left[T\left(w_0\right) + s^2T\left(v_\varepsilon\right) + 2s\int (|w_0|^{2^*-2}w_0v_\varepsilon + f(x)|w_0|^{\gamma}v_\varepsilon)\right]\\
&- \frac{1}{2^*}U\left(w_0+sv_\varepsilon\right) - \frac{1}{\gamma+1}F\left(w_0+sv_\varepsilon\right)\\
&= I(w_0) + I(sv_\varepsilon) + \int |w_0|^{2^*-2}w_0(sv_\varepsilon) + \int f(x)|w_0|^{\gamma}(sv_\varepsilon)\\
&- \frac{1}{2^*}\left[U\left(w_0+sv_\varepsilon\right) - U\left(w_0\right) - U\left(sv_\varepsilon\right)\right]\\
&- \frac{1}{\gamma+1}\left[F\left(w_0+sv_\varepsilon\right) - F\left(w_0\right) - F\left(sv_\varepsilon\right)\right].
\end{aligned}$$

Using the elementary inequality

$$||a+b|^q - |a|^q - |b|^q| \le d_1\left[|a|^{q-1}|b| + |a|\,|b|^{q-1}\right] \tag{2.30}$$

for $a,b\in\mathbb{R}$ and $q>1$, we obtain that

$$\begin{aligned}
I(w_0+sv_\varepsilon) &\le I(w_0) + I(sv_\varepsilon) + \int |w_0|^{2^*-1}(sv_\varepsilon) + |f|_{L^\infty(\Omega)}\int |w_0|^{\gamma}(sv_\varepsilon)\\
&+ d_2\int |w_0|^{2^*-1}|sv_\varepsilon| + d_3\int |w_0|\,|sv_\varepsilon|^{2^*-1}\\
&+ d_4\int |w_0|^{\gamma}|sv_\varepsilon| + d_5\int |w_0|\,|sv_\varepsilon|^{\gamma},
\end{aligned}$$

where, here and below, d_j for $j\in N$ denote positive constants.
Note that

$$I(sv_\varepsilon) = \frac{1}{2}T(sv_\varepsilon) - \frac{1}{2^*}U\left(sv_\varepsilon\right) - \frac{1}{\gamma+1}F(sv_\varepsilon) < \frac{1}{2}T(sv_\varepsilon) - \frac{1}{2^*}U\left(sv_\varepsilon\right) + \widetilde{K}\int sv_\varepsilon^{\gamma+1},$$

where $\widetilde{K}$ is a positive constant. Thus,

$$\begin{aligned}
I(w_0+sv_\varepsilon) &\le I(w_0) + \frac{1}{2}T(sv_\varepsilon) - \frac{1}{2^*}U\left(sv_\varepsilon\right) + \widetilde{K}s\int v_\varepsilon^{\gamma+1} + |f|_{L^\infty(\Omega)}\int |w_0|^{\gamma}(sv_\varepsilon)\\
&+ (s+d_2s)\int |w_0|^{2^*-1}v_\varepsilon + (d_3s^{2^*-1})\int |w_0|\,|v_\varepsilon|^{2^*-1}\\
&+ (d_4s)\int |w_0|^{\gamma}v_\varepsilon + (d_5s^{\gamma})\int |w_0|v_\varepsilon^{\gamma}
\end{aligned}$$

and

$$\begin{aligned}\sup_{s>0} I(w_0+sv_\varepsilon) \le I(w_0) + \sup_{s>0}\left[\frac{1}{2}T(sv_\varepsilon) - \frac{1}{2^*}U\left(sv_\varepsilon\right)\right] + d_6\int v_\varepsilon^{\gamma+1}\\ + d_7\int |w_0|^\gamma v_\varepsilon + d_8\int |w_0|^{2^*-1}v_\varepsilon\\ + d_9\int |w_0|\,|v_\varepsilon|^{2^*-1} + d_{10}\int |w_0|v_\varepsilon^\gamma.\end{aligned}$$

Let

$$g(s) = \frac{1}{2}T(sv_\varepsilon) - \frac{1}{2^*}U\left(sv_\varepsilon\right) = \frac{s^2}{2}T(v_\varepsilon) - \frac{s^{2^*}}{2^*}U(v_\varepsilon).$$

Then

$$g'(s) = sT(v_\varepsilon) - s^{2^*-1}U(v_\varepsilon)$$

Let $\widetilde{s} = [T(v_\varepsilon)U(v_\varepsilon)^{-1}]^{\frac{1}{2^*-1}}$, where $\widetilde{s}$ is such that: $g'(s) = 0$, if $s = \widetilde{s}$; $g'(s) > 0$, if $0 < s < \widetilde{s}$ and $g'(s) < 0$, if $s > \widetilde{s}$. Thus $\widetilde{s}$ is the maxima of $g(s)$ on $(0,\infty)$ and

$$\begin{aligned}\sup_{s>0} g(s) &= g(\widetilde{s})\\ &= \frac{1}{2}[T(v_\varepsilon)U(v_\varepsilon)^{-1}]^{\frac{1}{2^*-1}}T(v_\varepsilon) - \frac{1}{2^*}[T(v_\varepsilon)U(v_\varepsilon)^{-1}]^{\frac{2^*}{2^*-1}}U(v_\varepsilon)\\ &= (\frac{1}{2} - \frac{1}{2^*})T(v_\varepsilon)^{\frac{N}{2}}U(v_\varepsilon)^{-\frac{N-2}{2}}\\ &= \frac{1}{N}T(v_\varepsilon)^{\frac{N}{2}}U(v_\varepsilon)^{-\frac{N-2}{2}},\end{aligned}$$

so

$$\sup_{s>0} I(sv_\varepsilon) \le \frac{1}{N}T(v_\varepsilon)^{\frac{N}{2}}U(v_\varepsilon)^{1-\frac{N}{2}}.$$

Therefore

$$\sup_{s>0}\left[\frac{1}{2}T(sv_\varepsilon) - \frac{1}{2^*}U\left(sv_\varepsilon\right)\right] \le \frac{1}{N}T(v_\varepsilon)^{\frac{N}{2}}\bigg(U\left(v_\varepsilon\right)\bigg)^{1-\frac{N}{2}} < \frac{1}{N}S_\lambda^{\frac{N}{2}} - O(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}}).$$

Note that for ε small enough, $\int |w_0|^\gamma v_\varepsilon = O(\varepsilon^{\frac{N-2}{4}})$, $\int |w_0|v_\varepsilon^\gamma = O(\varepsilon^{\frac{N-2}{4}\gamma})$. Hence, from Proposition 2.2.9, we have

$$\begin{aligned}\sup_{s>0} I(w_0+sv_\varepsilon) \le I(w_0) + \frac{1}{N}S_\lambda^{\frac{N}{2}} - O(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}}) + d_6\int v_\varepsilon^{\gamma+1}\\ + d_7O(\varepsilon^{\frac{N-2}{4}}) + d_8O(\varepsilon^{\frac{N-2}{4}})\\ + d_9O(\varepsilon^{\frac{N-2}{4}}) + d_{10}O(\varepsilon^{\frac{N-2}{4}\gamma}).\end{aligned}$$

Thus,

$$\sup_{s>0} I(w_0+sv_\varepsilon) \le I(w_0) + \frac{1}{N}S_\lambda^{\frac{N}{2}} - O(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}}) + d_6\int v_\varepsilon^{\gamma+1}.$$

For ε small enough,

$$\int v_{\varepsilon}^{\gamma+1} = \begin{cases} O(\varepsilon^{\frac{N-2}{4}(\gamma+1)}), & 1 < 1+\gamma < \frac{N}{\sqrt{\Lambda}+\sqrt{\Lambda-\lambda}}; \\ O(\varepsilon^{\frac{N-2}{4}(\gamma+1)}|\ln\varepsilon|), & 1+\gamma = \frac{N}{\sqrt{\Lambda}+\sqrt{\Lambda-\lambda}}; \\ O\left(\varepsilon^{\frac{[N-(\gamma+1)\sqrt{\Lambda}]\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}}\right), & \frac{N}{\sqrt{\Lambda}+\sqrt{\Lambda-\lambda}} < 1+\gamma < 2. \end{cases} \tag{2.31}$$

Then:
i) If $1+\gamma < \frac{N}{\sqrt{\Lambda}+\sqrt{\Lambda-\lambda}}$, we have

$$\sup_{s>0} I(w_0 + sv_\varepsilon) \leq I(w_0) + \frac{1}{N}S_\lambda^{\frac{N}{2}} - O(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}}) + O(\varepsilon^{\frac{N-2}{4}\gamma}).$$

Note that for $a, b \geq 0$, we have $O(\varepsilon^a) - O(\varepsilon^b) < 0$, if and only if $a > b$, so $-O(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}}) + O(\varepsilon^{\frac{N-2}{4}\gamma}) < 0$. if α is such that $\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}} < \frac{N-2}{4}\gamma$, that is

$$\alpha < \gamma\sqrt{\Lambda-\lambda}.$$

Hence, we obtain from the assumption $(H_3)(ii)$ that

$$\sup_{s>0} I(w_0 + sv_\varepsilon) < I(w_0) + \frac{1}{N}S_\lambda^{\frac{N}{2}} = c_0 + \frac{1}{N}S_\lambda^{\frac{N}{2}}.$$

ii) If $1+\gamma = \frac{N}{\sqrt{\Lambda}+\sqrt{\Lambda-\lambda}}$, then 2.31 implies that

$$\sup_{s>0} I(w_0 + sv_\varepsilon) \leq I(w_0 + \frac{1}{N}S_\lambda^{\frac{N}{2}} - O(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}}) + O(\varepsilon^{\frac{N-2}{4}\gamma}|\ln\varepsilon|).$$

Therefore, we obtain from the assumption $(H_3)(ii)$ again that

$$\sup_{s>0} I(w_0 + sv_\varepsilon) < I(w_0) + \frac{1}{N}S_\lambda^{\frac{N}{2}} = c_0 + \frac{1}{N}S_\lambda^{\frac{N}{2}}. \tag{2.32}$$

The proof is complete. □

Proposition 2.2.28. *If (H_3) hold, then there is a critical point $w_1 \in M^-$ of I such that $I(w_1) = c_-$. Moreover, if $f > 0$, then $w_1 > 0$.*

Proof. First, we show that there is $w_1 \in M^-$ such that $I(w_1) = c_-$ and w_1 is a solution of $P_1(\lambda, \mu, \alpha, f, \gamma)$ for that, we use the item (i) and (ii) of Proposition 2.2.18, and the same idea of the proof of the Proposition 2.2.24. We omit the details here.

Next we will show that $w_1 > 0$, if $f > 0$. From Lemma 2.2.13, there exists $t_+(w_1) \in \mathbb{R}$, such that

$$t_+(|w_1|) > 0,$$

$$t_+(|w_1|)\,|w_1| \in M^-,$$

$$t_+(|w_1|) > t_{\max}(|w_1|) = t_{\max}(w_1)$$

and

$$I(t_+(|w_1|)\,|w_1|) = \max_{t\geq 0} I(t\,(|w_1|)\,|w_1|).$$

Since $w_1 \in M^-$, then $t_+(w_1) = 1$. Thus

$$I(t_+(w_1)\,w_1) = I(w_1) = \max_{t\geq 0} I(tw_1) \geq I(t_+(|w_1|)\,w_1).$$

Note that, since $f > 0$, we have

$$I(t_+(|w_1|)\,w_1) \geq I(t_+(|w_1|)\,|w_1|) \geq c_-.$$

Therefore $I(t_+(w_1)\,w_1) = c_-$ and we can always take $w_1 > 0$. $\square$

Existence of sign-changing solution

In this subsection we will study the existence of sign-changing solution of the problem $P_1(\lambda, \mu, \alpha, f, \gamma)$. We denote $u^+ \doteq \max\{0, u\}$ and $u^- \doteq \max\{0, -u\}$, for $u \in H_0^1(\Omega)$. Then $u^+,\ u^- \in H_0^1(\Omega)$ and $u = u^+ - u^-$.

Following Tarantello [119], we define

$$M_1^- \doteq \{u \in M; \quad u^+ \in M^-\} \quad \text{and} \quad M_2^- \doteq \{u \in M; \quad -u^- \in M^-\}.$$

Set also $M_*^- \doteq M_1^- \cap M_2^-$ and we have:

Lemma 2.2.29. *If $(H_3)(i)$ hold, then $M_*^- \neq \emptyset$.*

Proof. We will to prove that there exist $s_0 > 0$ and $t_0 \in \mathbb{R}$ such that

$$s_0(w_1 - t_0U_\varepsilon)^+ \in M^- \quad \text{and} \quad -s_0(w_1 - t_0U_\varepsilon)^- \in M^-,$$

where U_ε is defined as in Subsection 2.2.1. For this, we define

$$t_1 \doteq \min_{\bar{\Omega}\setminus\{0\}} \frac{w_1}{U_\varepsilon} \quad \text{and} \quad t_2 \doteq \max_{\bar{\Omega}\setminus\{0\}} \frac{w_1}{U_\varepsilon}.$$

For $t \in (t_1, t_2)$, we denote by $s^+(t)$ and $s^-(t)$ the positive values given by Lemma 2.2.13(i) and associated with the $t_+(u)$ value of $u = (w_1 - tU_\varepsilon)^+$ and $u = -(w_1 - tU_\varepsilon)^-$, respectively. Hence, we have

$$s^+(t)(w_1 - tU_\varepsilon)^+ \in M^- \quad \text{and} \quad -s^-(t)(w_1 - tU_\varepsilon)^- \in M^-.$$

Note that $s^+(t)$ and $s^-(t)$ are continuous with respect to t and satisfy:

$$\lim_{t\to t_1+0} s^+(t) = \lim_{t\to t_1+0} t_+(w_1 - tU_\varepsilon) = t_+(w_1 - t_1U_\varepsilon) < +\infty,$$

$$\lim_{t\to t_2-0} s^-(t) = \lim_{t\to t_2-0} t_+(-(tU_\varepsilon - w_1)) = t_+(t_2U_\varepsilon - w_1) < +\infty,$$

$$\lim_{t\to t_1+0} s^-(t) = +\infty \quad \text{and} \quad \lim_{t\to t_2-0} s^+(t) = +\infty.$$

The continuity of $s^+(t)$ and $s^-(t)$ implies that there is a point $t_0 \in (t_1, t_2)$ such that $s^+(t_0) = s^-(t_0) = s_0 > 0$. This proves the Lemma. $\square$

Lemma 2.2.30. *If $(H_3)(i)$ hold, then $M_1^-,\ M_2^- \subset M^-$.*

Proof. Let $u \in M_1^-$, i.e. $u \in M$ and $u^+ \in M^-$. Then

$$G_M(u) \quad = -(2^* - 2)T(u) + (2^* - \gamma - 1)\, F(u).$$

Since $\widetilde{\mu}_f > 0$, we have

$$|F(u)| < \Phi_*(u) \doteq \left(\frac{1-\gamma}{2^*-\gamma-1}\right)^{\frac{1-\gamma}{2^*-2}} \left(\frac{2^*-2}{2^*-\gamma-1}\right) T(u)^{\frac{2^*-\gamma-1}{2^*-2}} U(u)^{-\frac{1-\gamma}{2^*-2}},$$

Thus,

$$G_M(u) \quad \le (2^*-2)T(u)^{\frac{2^*-\gamma-1}{2^*-2}} \left[-T(u)^{\frac{\gamma-1}{2^*-2}} + \left(\frac{1-\gamma}{2^*-\gamma-1}\right)^{\frac{1-\gamma}{2^*-2}} U(u)^{-\frac{1-\gamma}{2^*-2}} \right].$$

From $u^+ \in M^-$, we have

$$(1-\gamma)\, T(u^+) - (2^*-\gamma-1)\, U(u^+) < 0$$

and from the definition of $S_{\lambda,\mu}$, we have $U(u) \le {S_{\lambda,\mu}}^{-1} T(u)^{\frac{1}{2}}$. Therefore, we obtain

$$T(u^+) \le \frac{2^*-\gamma-1}{1-\gamma} U(u^+) \le \frac{2^*-\gamma-1}{1-\gamma} {S_{\lambda,\mu}}^{-1} T(u^+)^{\frac{1}{2}}. \tag{2.33}$$

Thus

$$T(u^+)^{\frac{1}{2}} \le \frac{2^*-\gamma-1}{1-\gamma} {S_{\lambda,\mu}}^{-1} \tag{2.34}$$

and using $U(u)^{-1} = S_{\lambda,\mu} T(u)^{\frac{-1}{2}}$, we have

$$-T(u^+)^{\frac{\gamma-1}{2^*-2}} + \left(\frac{1-\gamma}{2^*-\gamma-1}\right)^{\frac{1-\gamma}{2^*-2}} U(u^+)^{-\frac{1-\gamma}{2^*-2}} < 0$$

Therefore

$$G_M(u) \quad \le (2^*-2)T(u)^{\frac{2^*-\gamma-1}{2^*-2}} \left[-T(u^+)^{\frac{\gamma-1}{2^*-2}} + \left(\frac{1-\gamma}{2^*-\gamma-1}\right)^{\frac{1-\gamma}{2^*-2}} U(u^+)^{-\frac{1-\gamma}{2^*-2}} \right] < 0.$$

i.e. $u \in M^-$. This proves that $M_1^- \subset M^-$. By a similar argument we can prove that

$M_2^- \subset M^-$. □

Define

$$c_*^- \doteq \inf_{u \in M_*^-} I(u).$$

Lemma 2.2.31. *If* (H_3) *hold, then* $c_*^- < c_- + \frac{1}{N} S_\lambda^{\frac{N}{2}}$.

Proof. We will estimate $I(sw_1 - tU_\varepsilon)$ for $s \geq 0$ and $t \in R$, since that $M_*^- \neq \emptyset$ (Lemma 2.2.29). Since at this time, ε can be sufficiently small, we replace U_ε by $v_\varepsilon = \phi(x)U_\varepsilon$ defined as before. The structure of I, guarantees that there is $R > 0$ possibly large such that $I(sw_1 - tv_\varepsilon) \leq c_-$ for all $s^2 + t^2 \geq R^2$. Thus it suffices to estimate $I(sw_1 - tv_\varepsilon)$ for all $s^2 + t^2 \leq R^2$. Since w_1 is a solution of the problem $P(\lambda, \mu, \alpha, f, \gamma)$, from inequality (2.30), we obtain for positive constants e_j with $j \in N$ that

$$\begin{aligned} I(sw_1 - tv_\varepsilon) \quad &\leq I(sw_1) + I(tv_\varepsilon) - \int |sw_1|^{2^*-1}(tv_\varepsilon) \\ &- |f|_{L^\infty(\Omega)} \int |sw_1|^\gamma (tv_\varepsilon) + e_2 \int |sw_1|^{2^*-1} |tv_\varepsilon| + e_3 \int |sw_1| \, |tv_\varepsilon|^{2^*-1} \\ &+ e_4 \int |sw_1|^\gamma |tv_\varepsilon| + e_5 \int |sw_1| \, |tv_\varepsilon|^\gamma \end{aligned}$$

and for positive constants g_j with $j \in N$ we have

$$\begin{aligned} I(sw_1 - tv_\varepsilon) \quad &\leq I(sw_1) + I(tv_\varepsilon) + g_1 \int |w_1|^{2^*-1} v_\varepsilon + g_2 \, |f|_{L^\infty(\Omega)} \int |w_1|^\gamma v_\varepsilon \\ &+ g_3 \int |w_1| |v_\varepsilon|^{2^*-1} + g_4 \int |w_1|^\gamma \, |v_\varepsilon| \\ &+ g_5 \int |w_1| \, |v_\varepsilon|^\gamma . \end{aligned}$$

Note that since $w_1 \in M$, we have $I(sw_1) \leq I(w_1)$ for all $s \geq 0$, we have

$$\begin{aligned} I(sw_1 - tv_\varepsilon) \quad &\leq I(w_1) + I(tv_\varepsilon) + g_1 \int |w_1|^{2^*-1}(v_\varepsilon) + g_2 \, |f|_{L^\infty(\Omega)} \int |w_1|^\gamma (v_\varepsilon) \\ &+ g_3 \int |w_1| |v_\varepsilon|^{2^*-1} + g_4 \int |w_1|^\gamma \, |v_\varepsilon| \\ &+ g_5 \int |w_1| \, |v_\varepsilon|^\gamma . \end{aligned}$$

Thus, from Proposition 2.2.9, the Proposition B.0.9 and following the similar argument that Lemma 2.2.27, we obtain that

$$\max_{s>0,\ t \in \mathbb{R}} I(sw_1 - tv_\varepsilon) < I(w_1) + \frac{1}{N} S_\lambda^{\frac{N}{2}} < c_- + \frac{1}{N} S_\lambda^{\frac{N}{2}}.$$

□

The following result prove the existence of a third solution for the problem $P_1(\lambda,\mu,\alpha,f,\gamma)$.

Proposition 2.2.32. *If (H_3) hold, then there is $w_2 \in M_*^-$ such that $I(w_2) = c_*^-$ and w_2 is a sign-changing solution of the problem $P_1(\lambda,\mu,\alpha,f,\gamma)$.*

Proof. We will prove that there is $w_2 \in M_*^-$ such that $I(w_2) = c_*^-$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence with $u_n \in M_*^-$ such that $I(u_n) \to c_*^-$. Note that $(u_n^+)_{n\in\mathbb{N}}$ is bounded, using the fact that $u_n^+ \in M^-$,

$$0 < \inf \|u_n^+\| \leq \sup \|u_n^+\| < +\infty,$$

and Sobolev inequality. Similar idea applies to $(u_n^-)_{n\in\mathbb{N}}$.

We consider $(u_n^+)_{n\in\mathbb{N}}$ and $(u_n^-)_{n\in\mathbb{N}}$ such that $u_n^+ \rightharpoonup u^+$ and $u_n^- \rightharpoonup u^-$ in $H_0^1(\Omega)$. Let $I(u_n^+) \to d_1$, $I(u_n^-) \to d_2$ and $c_*^- = d_1 + d_2$.

Note that $u^+ \neq 0$ and $u^- \neq 0$. By Proposition 2.2.18, we have that:
If $u^+ = 0$ and $u^- = 0$, then $d_1 \geq \frac{1}{N}S_\lambda^{\frac{N}{2}}$, $d_2 \geq \frac{1}{N}S_\lambda^{\frac{N}{2}}$ and hence $c_*^- \geq \frac{2}{N}S_\lambda^{\frac{N}{2}}$.
If $u^+ = 0$ and $u^- \neq 0$, then $d_1 \geq \frac{1}{N}S_\lambda^{\frac{N}{2}}$, $d_2 \geq c_-$ or $d_2 \geq c_0 + \frac{1}{N}S_\lambda^{\frac{N}{2}}$, which implies that $c_*^- \geq c_- + \frac{1}{N}S_\lambda^{\frac{N}{2}}$ or $c_*^- \geq c_0 + \frac{2}{N}S_\lambda^{\frac{N}{2}}$.
If $u^+ \neq 0$ and $u^- = 0$, the $d_2 \geq \frac{1}{N}S_\lambda^{\frac{N}{2}}$, $d_1 \geq c_-$ or $d_1 \geq c_0 + \frac{1}{N}S_\lambda^{\frac{N}{2}}$, which implies that $c_*^- \geq c_- + \frac{1}{N}S_\lambda^{\frac{N}{2}}$ or $c_*^- \geq c_0 + \frac{2}{N}S_\lambda^{\frac{N}{2}}$.

All the above three cases contradict Lemma 2.2.27 and Lemma 2.2.31. Therefore $u^+ \neq 0$ and $u^- \neq 0$. Thus according to (1) and (2) of Proposition 2.2.18, we have one of the following:

(i) $(u_n^+)_{n\in\mathbb{N}}$ converges strongly to u^+;

(ii) $d_1 > I(t_+(u^+)u^+)$;

(iii) $d_1 > I(t_-(u^+)u^+) + \frac{1}{N}S_\lambda^{\frac{N}{2}}$;

and, similarly, we have one of the following:

(iv) $(u_n^-)_{n\in\mathbb{N}}$ converges strongly to u^-;

(v) $d_2 > I(-t_+(-u^-)u^-)$;

(vi) $d_2 > I(-t_-(-u^-)u^-) + \frac{1}{N}S_\lambda^{\frac{N}{2}}$.

The key point is that only cases (i) and (iv) hold. In fact, all the following situations are contradictions.

If (ii) and (v) hold, then $t_+(u^+)u^+ - t_+(-u^-)u^- \in M_*^-$ and

$$\begin{aligned} c_*^- &\leq I(t_+(u^+)u^+ - t_+(-u^-)u^-) = I(t_+(u^+)u^+) + I(-t_+(-u^-)u^-) \\ &< d_1 + d_2 = c_*^-. \end{aligned}$$

If (iii) and (vi) hold, then $t_-(u^+)u^+ - t_-(-u^-)u^- \in M^+$ and hence

$$\begin{aligned} c_- + \frac{1}{N}S_\lambda^{\frac{N}{2}} &< c_0 + \frac{2}{N}S_\lambda^{\frac{N}{2}} \leq I(t_-(u^+)u^+ - t_-(-u^-)u^-) + \frac{2}{N}S_\lambda^{\frac{N}{2}} \\ &= I(t_-(u^+)u^+) + I(-t_-(u^-)u^-) + \frac{2}{N}S_\lambda^{\frac{N}{2}} \\ &\leq d_1 + d_2 = c_*^-. \end{aligned}$$

If (ii) and (vi) hold, then $t_+(u^+)u^+ - t_-(-u^-)u^- \in M^-$ and

$$c_- + \frac{1}{N}S_\lambda^{\frac{N}{2}} \leq I(t_+(u^+)u^+ + t_-(u^-)u^-) + \frac{1}{N}S_\lambda^{\frac{N}{2}} < d_1 + d_2 = c_*^-.$$

If (i) and (v) hold, then $u^+ - t_+(-u^-)u^- \in M_*^-$ and

$$c_*^- \leq I(u^+ - t_+(-u^-)u^-) < d_1 + d_2 = c_*^-.$$

All the above cases leave to a contradiction, therefore both $(u_n^+)_{n\in\mathbb{N}}$ and $(u_n^-)_{n\in\mathbb{N}}$ converge strongly to u^+ and u^-, respectively and we get that $u^+,\ u^- \in M^-$.

Let $w_2 = u^+ - u^-$. We have $I(w_2) = c_*^-$, since $I(w_2) = I(u^+ - u^-) = I(u)$ and $I(u_n)$ converge strongly to c_*^-.

Next we show that w_2 is a critical point of I. For that we suppose that w_2 is not a critical point of I and we define

$$W_\delta(u) = \psi(u))V(u),$$

where: (i) $V(u)$ is the pseudo-gradient vector field for $I(u)$:

$$V(u) = \nabla I(u) - \left\langle \nabla I(u), \frac{\nabla Q(u)}{\|\nabla Q(u)\|} \right\rangle \frac{\nabla Q(u)}{\|\nabla Q(u)\|}, \qquad u \in M^-,$$

since that $I \in C^1(H_0^1(\Omega), \mathbb{R})$ and $\langle \nabla Q(u), u\rangle < 0$.
The pseudo-gradient $V(u)$ satisfies

$$\|V(u)\| \leq 2\|I'(u)\|, \tag{2.35}$$

$$I'(u)V(u) \geq \|I'(u)\|^2. \tag{2.36}$$

(ii) $\psi:\ M^- \to [0,1]$ is a Lipschitz mapping such that

$$\psi(v) = \begin{cases} 1 & \text{for } v \in M^- \quad \text{with} \quad \|v - w_2\| \leq \delta, \\ 0 & \text{for } v \in M^- \quad \text{with} \quad \|v - w_2\| \geq 2\delta, \end{cases}$$

where $\delta \in (0, \min\{\|u^+\|, \|u^-\|\}/3)$ is such that $\|V(v) - V(w_2)\| \leq \frac{1}{2}\|V(w_2)\|$ for each

$v \in M^-$ with $\|v - w_2\| \le 2\delta$.

Let $\eta : [0, s_0] \times M^- \to M^-$ denote the pseudo-gradient flow associated to I on $H_0^1(\Omega)$, that is the solution of the differential equation

$$\eta(0, v) = v, \quad \frac{d}{ds}\eta(s, v) = -W_\delta(\eta(s, v)), \tag{2.37}$$

for some positive number s_0 and $(s, v) \in [0, s_0] \times M^-$.
Since $W_\delta(u)$ is locally Lipschitz continuous and $\|W_\delta(u)\| \le 1$, then (2.37) has a unique solution depending continuously on v.

For $0 \le t \le 1$, we set

$$\chi(t) = t_+((1-t)u^+ - tu^-) \cdot ((1-t)u^+ - tu^-) \quad \text{and} \quad \xi(t) = \eta(s_0, \chi(t))$$

By the definition of $W_\delta(u)$ and (2.36), we have,

$$\begin{aligned}
\frac{d}{ds}I(\eta(s, v)) &= I'(\eta(s, v))\eta'(s, v) \\
&= -I'(\eta(s, v))W_\delta(\eta(s, v)) \\
&= -I'(\eta(s, v))\psi(\eta(s, v))V(\eta(s, v)) \\
&= -\psi(\eta(s, v))I'(\eta(s, v))V(\eta(s, v)) \\
&\le -\psi(\eta(s, v))\|I'(\eta(s, v))\|^2 \\
&\le 0.
\end{aligned}$$

The last inequality, means that $I(\eta(s, v)) \le I(\eta(0, v))$ for any $s \ge 0$.
Thus, if $t \in (0, 1/2) \cup (1/2, 1)$ then

$$\begin{aligned}
I(\xi(t)) &= I(\eta(s_0, \chi(t))) \\
&\le I(\eta(0, \chi(t))) \\
&= I(\chi(t)) = I(\chi(t)^+) + I(\chi(t)^-) < I(u^+) + I(u^-) = I(w_2).
\end{aligned}$$

and $I(\xi(1/2)) < I(\chi(1/2)) = I(w_2)$. Therefore $I(\xi(t)) < I(w_2)$ for $t \in (0, 1)$.
Note that, as $t \to 0+$,

$$t_+(\xi(t)^+) - t_+(-\xi(t)^-) = \eta(s_0, t_+(\chi(t)^+) - t_+(-\chi(t)^-)) \to -\infty$$

and as $t \to 1^-$,

$$t_+(\xi(t)^+) - t_+(-\xi(t)^-) = \eta(s_0, t_+(\chi(t)^+) - t_+(-\chi(t)^-)) \to \infty$$

Hence, the continuity of $\eta(s_0, t_+(\chi(t)^+) - t_+(-\chi(t)^-))$ implies that there is $t_1 \in (0, 1)$ such that $t_+(\xi(t_1)^+) = t_+(-\xi(t_1)^-)$.
Thus, $\xi(t_1) = \xi(t_1)^+ - \xi(t_1)^- \in M_*^-$ and $I(\xi(t_1)) < I(w_2)$, which is a contradiction with $I(w_2) = c_*^- = \inf_{u \in M_*^-} I(u)$. Therefore w_2 is a critical point of I. $\square$

Existence of a fourth solution

In this subsection, we prove the existence of another solution for the problem $P_1(\lambda, \mu, \alpha, f, \gamma)$ by a translated argument. For this, we need to prove a local Palais-Smale condition, due to non-compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$.

Let w_0 as before and we define a C^1 functional $\bar{I}: H_0^1(\Omega) \to R$ by

$$\bar{I}(v) \quad \doteq I(w_0 + v^+) - I(w_0)$$

for $v \in H_0^1(\Omega)$. Thus, we have $\langle \bar{I}'(v), \phi \rangle = \langle I'(w_0 + v^+), \phi \rangle$. Therefore, if v is a critical point of $\bar{I}$, then $w_0 + v^+$ is a critical point of I.

Consider the following minimax value

$$\bar{c} \doteq \inf_{\gamma \in \Gamma} \sup_{0 \leq t \leq 1} \bar{I}(\gamma(t)),$$

where $\Gamma \doteq \{\gamma \in C([0,1],\ H_0^1(\Omega)) : \gamma(0) = 0,\ \gamma(1) = kv_\varepsilon\}$ with suitable ε and k.

Lemma 2.2.33. *If $(H_3)(i)$ hold, we have $\bar{c} < \frac{1}{N} S_\lambda^{\frac{N}{2}}$.*

Proof.

$$\begin{aligned}\bar{I}(v) \quad &= \frac{1}{2} T(w_0 + v^+) - \frac{1}{2^*} U(w_0 + v^+) - \frac{1}{\gamma+1} F(w_0 + v^+) \\ &- \frac{1}{2} T(w_0) + \frac{1}{2^*} U(w_0) + \frac{1}{\gamma+1} F(w_0).\end{aligned}$$

Note that for $v_\varepsilon(x) = \phi(x) U_\varepsilon(x)$, defined as before,

$$\begin{aligned}\sup_{s>0} \bar{I}(sv_\varepsilon^+) \quad &= \sup_{s>0} [I(w_0 + sv_\varepsilon^+) - I(w_0)] \\ &= \sup_{s>0} [I(w_0 + sv_\varepsilon^+)] - c_0.\end{aligned}$$

From (2.32), we have

$$\sup_{s>0} I(w_0 + sv_\varepsilon^+) < I(w_0) + \frac{1}{N} S_\lambda^{\frac{N}{2}} = c_0 + \frac{1}{N} S_\lambda^{\frac{N}{2}}.$$

Hence, by the definition of $\bar{c}$, we have

$$\bar{c} < \sup_{s>0} \bar{I}(sv_\varepsilon^+) = \sup_{s>0} [I(w_0 + sv_\varepsilon^+)] - c_0 < c_0 + \frac{1}{N} S_\lambda^{\frac{N}{2}} - c_0 = \frac{1}{N} S_\lambda^{\frac{N}{2}}.$$

□

Lemma 2.2.34. *The origin is a local minimum of $\bar{I}$.*

Proof. Let $v \in H_0^1(\Omega)$ and $v = v^+ - v^-$. We have

$$\begin{aligned}
\bar{I}(v) &= \frac{1}{2}T(w_0+v^+) - \frac{1}{2^*}U(w_0+v^+) - \frac{1}{\gamma+1}F(w_0+v^+) - I(w_0)\\
&= \frac{1}{2}T(w_0+v+v^-) - \frac{1}{2^*}U(w_0+v^+) - \frac{1}{\gamma+1}F(w_0+v^+) - I(w_0)\\
&= \frac{1}{2}T(w_0+v) + \frac{1}{2}T(v^-) + 2\int_\Omega (\nabla(w_0+v^+)\nabla(v^-)\\
&-\frac{\lambda}{|x|^2}(w_0+v^+)(v^-) - \mu|x|^{\alpha-2}(w_0+v^+)(v^-) - \frac{1}{2^*}U(w_0+v^+)\\
&-\frac{1}{\gamma+1}F(w_0+v^+) - I(w_0)\\
&= \frac{1}{2}T(w_0+v) + \frac{1}{2}T(-v^-) - \frac{1}{2^*}U(w_0+v^+) - \frac{1}{\gamma+1}F(w_0+v^+) - I(w_0)\\
&= \frac{1}{2}T(w_0+v^+) + \frac{1}{2}T(-v^-) - \frac{1}{2^*}U(w_0+v^+) - \frac{1}{\gamma+1}F(w_0+v^+) - I(w_0)\\
&= \frac{1}{2}T(-v^-) + I(w_0+v^+) - I(w_0).
\end{aligned}$$

Since w_0 is a local minimum of I, then exists $\varepsilon > 0$, such that $I(w) \geq I(w_0)$, for all $\|w - w_0\| \leq \varepsilon$, $w \in H_0^1(\Omega)$. Thus, in particular for $w_0 + v^+ \in H_0^1(\Omega)$, we have that $I(w_0+v^+) - I(w_0) \geq 0$ and

$$\bar{I}(v) \geq \frac{1}{2}T(v^-) \geq 0 = \bar{I}(0)$$

as $\|v\| \leq \varepsilon$. □

We will prove the existence of a four solution of the problem $P_1(\lambda, \mu, \alpha, f, \gamma)$ by contradiction. Assume that $v = 0$ is the only critical point of $\bar{I}$ in $H_0^1(\Omega)$.

Lemma 2.2.35. *If* 0 *is the only critical point of $\bar{I}$. Then $\bar{I}$ satisfies the $(PS)_c$-condition for any $c < \frac{1}{N}S_\lambda^{\frac{N}{2}}$.*

Proof. Let $v_n \subset H_0^1(\Omega)$, such that

$$\begin{cases} \bar{I}(v_n) & \to c,\\ \bar{I}'(v_n) & \to 0. \end{cases} \tag{2.38}$$

Then $\bar{I}(v_n) = c + o(1)$ and $\langle \bar{I}'(v_n), \phi\rangle = o(1)\|\phi\|$.

First we prove that v_n is bounded in $H_0^1(\Omega)$. Note that

$$
\begin{aligned}
2^*c+o(1)+o(1)\|w_0+v_n^+\| & \\
&= 2^*\bar{I}(v_n)-\langle \bar{I}'(v_n), w_0+v_n^+\rangle \\
&= 2^*I(w_0+v_n^+)-2^*I(w_0)-\langle I'(w_0+v_n^+), w_0+v_n^+\rangle \\
&= \frac{2^*}{2}T(w_0+v_n^+)-U(w_0+v_n^+)-\frac{2^*}{\gamma+1}F(w_0+v_n^+)-2^*I(w_0) \\
&\quad -T(w_0+v_n^+)+U(w_0+v_n^+)+F(w_0+v_n^+) \\
&\geq (\frac{2^*}{2}-1)T(w_0+v_n^+)+(1-\frac{2^*}{\gamma+1})F(w_0+v_n^+)-2^*I(w_0) \\
&\geq (\frac{2^*}{2}-1)T(w_0+v_n^+)+(1-\frac{2^*}{\gamma+1})\|w_0+v_n^+\|^{\gamma+1}-2^*I(w_0) \\
&= (\frac{2^*}{2}-1)T(w_0+v_n)+(1-\frac{2^*}{\gamma+1})\|w_0+v_n^+\|^{\gamma+1}-2^*I(w_0).
\end{aligned}
$$

Note that for all $u\neq 0$, from assumption $0<\mu<\mu_1$, $0\leq\lambda<\Lambda$ and the Hardy inequality, we have

$$T(u)\geq\left(1-\frac{\mu}{\mu_1}\right)\int\left(|\nabla u|^2-\frac{\lambda}{|x|^2}u^2\right)\geq\left(1-\frac{\mu}{\mu_1}\right)\left(1-\frac{\lambda}{\Lambda}\right)\int|\nabla u|^2.$$

Hence

$$
\begin{aligned}
&2^*c+o(1)+o(1)\|w_0+v_n\| \\
&\geq (\tfrac{2^*}{2}-1)(1-\tfrac{\mu}{\mu_1})(1-\tfrac{\lambda}{\Lambda})\textstyle\int|\nabla(w_0+v_n)|^2+(1-\tfrac{2^*}{\gamma+1})\|w_0+v_n^+\|^{\gamma+1}-2^*I(w_0) \\
&\geq (\tfrac{2^*}{2}-1)(1-\tfrac{\mu}{\mu_1})(1-\tfrac{\lambda}{\Lambda})\|w_0+v_n\|^2+(1-\tfrac{2^*}{\gamma+1})\|w_0+v_n^+\|^{\gamma+1}-2^*I(w_0) \\
&\geq (\tfrac{2^*}{2}-1)(1-\tfrac{\mu}{\mu_1})(1-\tfrac{\lambda}{\Lambda})\|v_n\|^2+(1-\tfrac{2^*}{\gamma+1})\|w_0+v_n^+\|^{\gamma+1}-2^*I(w_0).
\end{aligned}
$$

Therefore v_n is bounded in $H_0^1(\Omega)$.
Now, we prove that $(v_n)_{n\in\mathbb{N}}\to 0$ in $H_0^1(\Omega)$. Since v_n is bounded in $H_0^1(\Omega)$, we can assume if necessary to a subsequence that

$$
\begin{cases}
v_n & \rightharpoonup \sigma \quad \text{in } H_0^1(\Omega), \\
v_n & \to \sigma \quad \text{a.e in } \Omega, \\
v_n & \to \sigma \quad \text{in } L^t(\Omega), 1<t<2^*.
\end{cases}
\tag{2.39}
$$

Denote $u_n=v_n-\sigma$, then Brezis-Lieb Lemma (see Costa [46]), implies that

$$\int|\nabla v_n|^2=\int|\nabla u_n|^2+\int|\nabla\sigma|^2+o(1),$$

$$\int\frac{\lambda}{|x|^2}v_n^2=\int\frac{\lambda}{|x|^2}u_n^2+\int\frac{\lambda}{|x|^2}\sigma^2+o(1),$$

$$\int\mu|x|^{\alpha-2}v_n^2=\int\mu|x|^{\alpha-2}u_n^2+\int\mu|x|^{\alpha-2}\sigma^2+o(1),$$

$$\int|v_n|^{2^*}=\int|u_n|^{2^*}+\int|\sigma|^{2^*}+o(1)$$

and $\langle \bar{I}'(\sigma), \phi\rangle = 0$ for any $\phi \in H_0^1(\Omega)$. That is σ is a weak solution of $P_1(\lambda, \mu, \alpha, f, \gamma)$. Therefore $\bar{I}(\sigma) = 0$ and $I(w_0 + \sigma^+) = 0$, that is σ is a critical point of $\bar{I}$ in $H_0^1(\Omega)$ and $w_0+\sigma^+$ is a critical point of I in $H_0^1(\Omega)$. Since σ is a critical point of $\bar{I}$, by the assumption, we have $\sigma = 0$. Then $v_n \to 0$ in $L^t(\Omega), 1 < t < 2^*$. By the Brezis-Lieb Lemma

$$\int |w_0 + v_n^+|^{2^*} - \int |w_0|^{2^*} = \int |v_n^+|^{2^*} + o(1). \tag{2.40}$$

Then

$$\begin{aligned}
\bar{I}(v_n) &= I(w_0 + v_n^+) - I(w_0) \\
&= \tfrac{1}{2}T(w_0 + v_n^+) - \tfrac{1}{2^*}U(w_0 + v_n^+) - \tfrac{1}{\gamma+1}F(w_0 + v_n^+) - I(w_0) \\
&= \tfrac{1}{2}T(w_0 + v_n^+) - \tfrac{1}{2^*}\textstyle\int |v_n^+|^{2^*} - \tfrac{1}{2^*}\int |w_0|^{2^*} - \tfrac{1}{\gamma+1}F(w_0 + v_n^+) \\
&-I(w_0) + o(1).
\end{aligned}$$

Note that, since $1 < \gamma + 1 < 2^*$, we have $v_n \to 0$ in $L^{\gamma+1}(\Omega)$. Thus

$$\begin{aligned}
F(w_0 + v_n^+) &= \int f(x)|(w_0 + v_n^+)|^\gamma (w_0 + v_n^+) \\
&= \int f(x)|(w_0 + v_s^+)|^\gamma (w_0 + v_s^+) + o(1) \\
&= \int f(x)|w_0|^\gamma (w_0) + o(1) = F(w_0) + o(1).
\end{aligned}$$

Therefore, since w_0 is a solution of $P_1(\lambda, \mu, \alpha, f, \gamma)$, by the previous result and

$$T(w_0 + v_n^+) = T(w_0) + T(v_n^+) + 2\int |w_0|^{2^*-2} w_0 v_n^+ + 2\int f|w_0|^\gamma {v_n}^+,$$

we have

$$\begin{aligned}
\bar{I}(v_n) &= \frac{1}{2}T(w_0 + v_n^+) - \frac{1}{2^*}\int |v_n^+|^{2^*} - \frac{1}{2^*}\int |w_0|^{2^*} - \frac{1}{\gamma+1}F(w_0 + v_n^+) \\
&-I(w_0) + o(1) \\
&= \frac{1}{2}T(w_0) + \frac{1}{2}T(v_n^+) + \int |w_0|^{2^*-2} w_0 v_n^+ + \int f|w_0|^\gamma {v_n}^+ - \frac{1}{2^*}\int |v_n^+|^{2^*} \\
&-\frac{1}{2^*}\int |w_0|^{2^*} - \frac{1}{\gamma+1}F(w_0) - I(w_0) + o(1) \\
&= \frac{1}{2}T(v_n^+) + \int |w_0|^{2^*-2} w_0 v_n^+ + \int f|w_0|^\gamma {v_n}^+ - \frac{1}{2^*}\int |v_n^+|^{2^*} + o(1).
\end{aligned}$$

Then since, $v_n \to 0$ in $L^t(\Omega)$ for $1 < t < 2^*$ we have

$$\bar{I}(v_n) = \frac{1}{2}T(v_n^+) - \frac{1}{2^*}\int |v_n^+|^{2^*} + o(1).$$

Now,

$$\begin{aligned}
\langle \bar{I}'(v_n), w_0 + v_n^+ \rangle &= \langle I'(w_0 + v_n^+), w_0 + v_n^+ \rangle \\
&= T(w_0 + v_n^+) - U(w_0 + v_n^+) - F(w_0 + v_n^+). \\
&= T(w_0) + T(v_n^+) + 2\int |w_0|^{2^*-2} w_0 v_n^+ + 2\int f|w_0|^{\gamma} {v_n}^+ \\
&- \int |w_0 + v_n^+|^{2^*} - F(w_0 + v_n^+) \\
&= T(w_0) + T(v_n^+) + 2\int |w_0|^{2^*-2} w_0 v_n^+ + 2\int f|w_0|^{\gamma} {v_n}^+ \\
&- \int |v_n^+|^{2^*} - \int |w_0|^{2^*} - F(w_0) + o(1).
\end{aligned}$$

Since w_0 is a solution of $P_1(\lambda, \mu, \alpha, f, \gamma)$ and $v_n \to 0$ in $L^t(\Omega), 1 < t < 2^*$, we have

$$\langle \bar{I}'(v_n), w_0 + v_n^+ \rangle = T(v_n) - U(v_n^+) + o(1) \to 0$$

We assume that $T(v_n) \to d$ and $U(v_n^+) = \int |v_n^+|^{2^*} \to d$. We will prove that $d = 0$. Note that, since $v_n \in M$ and $v_n \rightharpoonup \sigma = 0$ in $H_0^1(\Omega)$, we have

$$\int (|\nabla v_n|^2 - \frac{\lambda}{|x|^2}|v_n|^2) = \int |v_n|^{2^*} + o(1).$$

We assume that $d \neq 0$. Using the fact, that $S_\lambda |v|_{2^*}^2 \leq \int (|\nabla v|^2 - \frac{\lambda}{|x|^2}|v|^2)$ for all $v \in H_0^1(\Omega)$ and $v \neq 0$, where

$$S_\lambda = \inf\left\{ \int_{\mathbb{R}^N} \left(|\nabla u|^2 - \frac{\lambda}{|x|^2} u^2 \right) dx \, : \, u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \, \int_{\mathbb{R}^N} |u|^{2^*} dx = 1 \right\},$$

we obtain that

$$S_\lambda (\int |v_n^+|^{2^*})^{\frac{2}{2^*}} \leq \int (|\nabla v_n^+|^2 - \frac{\lambda}{|x|^2}|v_n^+|^2) = \int |v_n^+|^{2^*} + o(1).$$

Then $S_\lambda d^{\frac{2}{2^*}} \leq d$ and $d \geq S_\lambda^{\frac{2^*}{2^*-2}} = S_\lambda^{\frac{N}{2}}$. Thus

$$\begin{aligned}
c &= o(1) + \bar{I}(v_n) \\
&= o(1) + \frac{1}{2}T(v_n) - \frac{1}{2^*}\int |v_n^+|^{2^*} - \frac{1}{\gamma+1}\int f(x)|(v_n^+)|^{\gamma}(v_n^+) + o(1) \\
&> o(1) + \frac{1}{2}d - \frac{1}{2^*}d \\
&= o(1) + \frac{1}{N}d \\
&\geq \frac{1}{N} S_\lambda^{\frac{N}{2}}.
\end{aligned}$$

Which contradicts $c < \frac{1}{N} S_\lambda^{\frac{N}{2}}$. Then $d = 0$.

Now, by the Hardy inequality again, and since $0 < \mu < \mu_1$ and $\lambda < \Lambda$,

$$T(v_n) \geq \left(1 - \frac{\mu}{\mu_1}\right)\left(1 - \frac{\lambda}{\Lambda}\right) \int \left(|\nabla v_n|^2\right) \geq \int \left(|\nabla v_n|^2\right) = \|v_n\|^2.$$

Therefore, since $T(v_n) \to d = 0$, we have $\|v_n\|^2 \to 0$. Hence $v_n \to 0 \in H_0^1(\Omega)$. The proof is complete. □

Proposition 2.2.36. *If $(H_3)(i)$ hold, there exists a critical point $w_{1,1} \in H_0^1(\Omega)$ of I such that $w_{1,1} > w_0$ in Ω. Moreover, $w_2 \neq w_{1,1}$.*

Proof. By Lemma 2.2.34 and since $I(tv) \to -\infty$, $t \to \infty$ we have the conditions (i) and (ii) of mountain pass theorem (Theorem 1.1.16) respectively. Thus by Lemmas 2.2.33 and 2.2.35, we obtain that there is a critical point $v \neq 0$ of $\bar{I}$. By the Strong Maximum Principle (Theorem 1.1.20), we have that $v > 0$ in Ω.
Set $w_{1,1} = w_0 + v^+$. Then $w_{1,1}$ is a critical point of I and $w_{1,1} > w_0$ in Ω.

We will prove that $w_2 \neq w_{1,1}$. Suppose that $w_2 = w_{1,1}$. Note that:
i) Since $-w_2^- \in M^-$ and $w_0 \in M^+$, we have

$$T(-w_2^-) < \left(\frac{2^* - \gamma - 1}{1 - \gamma}\right) U(-w_2^-)$$

and

$$T(w_0) > \left(\frac{2^* - \gamma - 1}{1 - \gamma}\right) U(w_0)$$

respectively.
Since $0 \geq -w_2^- = w_{1,1} \geq -w_0^- \geq w_0$, then $U(-w_2^-) \geq U(-w_0^-) \geq U(w_0)$ Therefore, we get that

$$\begin{aligned} T(-w_2^-) &< \left(\frac{2^* - \gamma - 1}{1 - \gamma}\right) U(-w_2^-) \\ &\leq \left(\frac{2^* - \gamma - 1}{1 - \gamma}\right) U(-w_0-) \\ &\leq \left(\frac{2^* - \gamma - 1}{1 - \gamma}\right) U(w_0) \\ &< T(w_0). \end{aligned}$$

ii) Since $w_0 \in M^+$ and from the definition of S_λ (see (2.12)), we have
$U(u) = S_\lambda^{-1} T(u)^{\frac{1}{2}}$ and

$$T(w_0) > \left(\frac{2^* - \gamma - 1}{1 - \gamma}\right) U(w_0) = \left(\frac{2^* - \gamma - 1}{1 - \gamma}\right) S_\lambda^{-1} T(w_0)^{\frac{1}{2}}.$$

Then

$$T(w_0)^{\frac{1}{2}} < \left(\frac{1 - \gamma}{2^* - \gamma - 1}\right) S_\lambda.$$

For other hand, since $-w_2^- \in M^-$, we have

$$T(-w_2^-)^{\frac{1}{2}} > \left(\frac{1-\gamma}{2^*-\gamma-1}\right) S_\lambda.$$

Hence

$$T(w_0) < \left(\frac{1-\gamma}{2^*-\gamma-1}\right)^2 S_\lambda^2 < T(-w_2^-).$$

Thus by (i) and (ii) we have a contradiction. Therefore, we have proved $w_2 \neq w_{1,1}$. $\square$

Proposition 2.2.37. *If $(H_3)(i)$ hold, there exists a critical point $w_{1,2} \in H_0^1(\Omega)$ of I such that $w_{1,2} < w_0$ in Ω. Moreover, $w_2 \neq w_{1,2}$.*

Proof. For $v \in H_0^1(\Omega)$, we define the following functional

$$\widehat{I}(v) \doteq I(w_0 + v^-) - I(w_0).$$

Now using the same procedure as in getting the solution $w_{1,1}$, we can easily get the existence of a critical point $w_{1,2} \in H_0^1(\Omega)$ of I and $w_{1,2}$ satisfies all the requirement of Proposition 2.2.37. $\square$

We are now ready for the multiplicity theorem for problem $P_1(\lambda, \mu, \alpha, f, \gamma)$ under strong less hypothesis.

Theorem 2.2.38. *Suppose hypotheses (H_3) hold, then $P_1(\lambda, \mu, \alpha, f, \gamma)$. has at least four nontrivial solutions in $H_0^1(\Omega)$ and at least one of them is sign-changing.*

Proof. From the previous subsections, we got five weak solutions of the problem $P_1(\lambda, \mu, \alpha, f, \gamma)$, i.e. w_0, w_1, w_2, $w_{1,1}$ and $w_{1,2}$. However, since we are not able to prove that w_1 is different from $w_{1,1}$ or $w_{1,2}$, we can only state the existence of (at least) four different solutions w_0, w_2, $w_{1,1}$ and $w_{1,2}$ of $P_1(\lambda, \mu, \alpha, f, \gamma)$. Moreover, we know w_2 is sign-changing. $\square$

Chapter 3

Multiplicity results for a class of singular elliptic equations with critical Hardy-Sobolev exponent and involving a concave term

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $N \geq 3$ and $0 \in \Omega$. Here, we study the existence of multiple positive and sign-changing solutions $u \in H_0^1(\Omega)$ of the problem $P_2(\lambda, \zeta, q, s, f)$:

$$\begin{cases} -\Delta u - \frac{\lambda}{|x|^2} u &= \zeta f(x) |u|^{q-2} u + \frac{|u|^{p^*(s)-2} u}{|x|^s} \quad \text{in } \Omega \backslash \{0\}, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $p^*(s) \doteq 2(N-s)/N-2$, is the critical Hardy-Sobolev exponent, $1 < q < 2$, $0 \leq s < 2$, f is a real function on Ω, and the parameters λ and ζ are positive. Note that, when $s = 0$, we have the critical Sobolev exponent $p^* = \frac{2N}{N-2}$.

The problem $P_2(\lambda, \zeta, q, s, f)$ is constituted by a semilinear elliptic equation with critical nonlinearity, due to the term $\frac{|u|^{p^*(s)-2}u}{|x|^s}$, which in addition with the term $\frac{\lambda}{|x|^2}u$ leads to the problem showing a double singularity at zero. This singularity and the non-compactness of the embeddings $H_0^1(\Omega) \hookrightarrow L^2\left(\Omega; |x|^{-2}dx\right)$ and $H_0^1(\Omega) \hookrightarrow L^{p^*(s)}(\Omega; |x|^{-s}dx)$, even locally in any neighborhood of zero, brings us to the possibility of blow-up (Smets [111]). However, we will see that the presence of the term $\zeta f(x)|u|^{q-2}u$ controls this question.

Without the critical term $\frac{|u|^{p^*(s)-2}u}{|x|^s}$, it should be easy to deduce that the associated Euler functional of problem $P_2(\lambda, \zeta, q, s, f)$ satisfies the Palais-Smale condition and existence results are obtained under some proper assumptions. To overcome the compactness issue, we use the concentration compactness principle in order to obtain the existence solutions, under some certain hypotheses.

The starting point of this study is the work of Bouchekif et al. [14], which studied the subclass $P_2(\lambda, \zeta, q, s, 1)$ and established the following result.

Theorem 3.0.39. *If $0 \in \Omega, 0 \leq \lambda < \Lambda - 1$, $1 < q < 2$, and $0 \leq s < 2$, then there is $\bar{\Lambda} > 0$ such that $P_2(\lambda, \zeta, q, s, 1)$ has at least two positive solutions in $H_0^1(\Omega)$ for $\zeta \in (0, \bar{\Lambda})$.*

The purpose here is to prove, under suitable assumptions, that $P_2(\lambda, \zeta, q, s, f)$ not only has two positive solutions, but exists Λ^* such that also possesses an additional pair of sign-changing solutions for $\zeta \in (0, \Lambda^*)$. Note that our result extends Bouchekif et al. [14] even in the case of $f \equiv 1$.

The results obtained in this chapter are related with the work of Chen-Murillo-Rocha in [38].

3.1 Previous results

In the literature, there are some very known results related with problems involving concave and convex nonlinearity. The problem

$$\begin{cases} -\Delta u &= \lambda \left|u\right|^{q-1} u + \left|u\right|^{p-1} u, \quad \text{in} \quad \Omega, \\ \quad u &= 0, \qquad\qquad\qquad\qquad\qquad \text{on} \quad \partial\Omega, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^N$, was studied by Ambrosetti-Brezis-Cerami [6] with the following result:

Theorem 3.1.1. *There exists $\lambda^* > 0$, such that for all $\lambda \in (0, \lambda^*)$,*

(i) if $0 < q < 1 < p \leq \frac{N+2}{N-2}$, the problem (3.1) has infinitely many solutions with negative energy.

(ii) if $0 < q < 1 < p < \frac{N+2}{N-2}$, the problem (3.1) has infinitely many solutions with positive energy.

One particular case of problem (3.1), is just considering positive solutions, i.e. when $u > 0$. In this case, we have the following result by Ambrosetti-Malchiodi [7].

Theorem 3.1.2. *Let $0 < q < 1 < p$, then there exists $\bar{\Lambda} > 0$ such that one has*

(i) for all $\lambda \in (0, \bar{\Lambda})$, the problem (3.1) has a positive solution;

(ii) for all $\lambda = \bar{\Lambda}$, the problem (3.1) has at least a weak positive solution;

(iii) for all $\lambda > \bar{\Lambda}$, the problem (3.1) has no solutions.

Theorem 3.1.3. *Let $0 < q < 1 < p \leq \frac{N+2}{N-2}$, then for all $\lambda \in (0, \bar{\Lambda})$, the problem (3.1) has at least two positive solutions.*

Problems of the same type as $P_2(\lambda, \zeta, q, s, f)$ have been a central theme in the past several years. We refer the interested readers to Ambrosetti-Brezis-Cerami [6], Bouchekif-Matallah [14], Cao-Kang [21], Ekeland-Ghoussoub [55] and Ferrero-Gazzola [57] for similar equations with Dirichlet boundary condition and Chabrowski [28] for a similar equation with Neumann boundary condition.

Problems involving a Hardy–type singular term $-\frac{\lambda}{|x|^2}u$, where $0 \in \Omega$, a term with the critical exponent (compactness loss) and singularity $-\mu u^{-q}$ with $0 < q < 1$, were studied by Chen-Rocha [41], showing the existence of two positive solutions under adequate hypotheses. In that work, Nehari optimization techniques and precise estimates of the energy of critical points are important tools.

When the problem does not involve the term $\frac{|u|^{p^*(s)-2}u}{|x|^s}$, Garcia–Azorero-Peral-Primo [60] obtained a pair of positive solutions, under the condition $0 \leq \lambda < \Lambda$; see also Abdellaoui-Colorado-Peral [1], where a similar problem with a class of more general operators was considered. The problem $P_2(\lambda, \zeta, q, 0, 0)$ without the term $\frac{|u|^{p^*(s)-2}u}{|x|^s}$, when $\frac{\lambda}{|x|^2}u$ has the form $\frac{\lambda}{|x|^2}u^r$, where $1 < r < \frac{N+2}{N-2}$, was studied recently by Davila-Peral [49]. They proved that the existence of positive solutions depends on the geometry of the domain, specifically, using Pohozaev's identity, proved that there are no energy solutions, when the domain is star-shaped, but via a perturbation argument, they proved that the problem has solutions in dumbbell domains.

There are also in the literature some results about problems with double singularity, which generally involve the critical Hardy-Sobolev exponent with $0 \leq s < 2$. When $0 \leq \lambda < \Lambda - 4$, Chen [31] proved that for any $\zeta > 0$, the problem $P_2(\lambda, \zeta, 2, s, 1)$ possesses a nontrivial solution with critical level in the range of $\left(0, \frac{2-s}{2(N-s)} S_{\lambda,s}^{(N-s)/(2-s)}\right)$, where $S_{\lambda,s}$ is the best constant defined in (3.2). For $f \equiv 1$ and $\max\left\{2, \frac{N}{\sqrt{\Lambda}+\sqrt{\Lambda-\lambda}}, \frac{N-2\sqrt{\Lambda-\lambda}}{\sqrt{\Lambda}}\right\} < q < 2^*$, Kang-Peng [78] proved that problem $P_2(\lambda, \zeta, q, s, f)$ has a positive solution in $H_0^1(\Omega)$ when $0 \leq \lambda < \Lambda$. He-Zou [70] proved using the same condition on λ, the existence of infinitely many solutions for a suitable positive number ζ, when the term $\zeta f(x)|u|^{q-2}u$ has the form $\zeta f(x, u)$, where $f(x, 0) \equiv 0$ and $f(x, u)$ is a lower order perturbation of $u^{p^*(s)-1}$, in the sense that $\frac{f(x,u)}{|u|^{p^*(s)-2}u} \to 0$ as $|u| \to \infty$ uniformly.

3.2 Multiplicity results

This section is concerned with the existence of solutions of problem $P_2(\lambda, \zeta, q, s, f)$.

We define the minimization problem

$$S_{\lambda,s} = \inf\left\{\int_{\mathbb{R}^N}\left(|\nabla u|^2 - \frac{\lambda}{|x|^2}u^2\right)dx \,:\, u \in \mathcal{D}^{1,2}(\mathbb{R}^N),\ \int_{\mathbb{R}^N}\frac{|u|^{p^*(s)}}{|x|^s}dx = 1\right\} \tag{3.2}$$

and denote

$$\Lambda^* \doteq \left(\frac{2-q}{p^*(s)-q}\right)^{\frac{2-q}{p^*(s)-q}}\left(\frac{p^*(s)-2}{(p^*(s)-q)|f|_\infty}\right)|\Omega|^{\frac{q-p^*(s)}{p^*(s)}} S_{\lambda,s}^{\frac{p^*(s)-q}{p^*(s)-2}}$$

where $|\Omega|$ is the measure of Ω.

Since $P_2(\lambda, \zeta, q, s, f)$ is variational in nature, we use variational methods to solve it and our main result is obtained by studying several minimization problems. For this, define the functional $J : H_0^1(\Omega) \to \mathbb{R}$, associated to problem $P_2(\lambda, \zeta, q, s, f)$, by

$$J(u) \doteq \frac{1}{2}\int \left(|\nabla u|^2 - \frac{\lambda}{|x|^2}u^2\right) - \frac{\zeta}{q}\int f(x)|u|^q - \frac{1}{p^*(s)}\int \frac{|u|^{p^*(s)}}{|x|^s}. \tag{3.3}$$

Definition 3.2.1 (weak solution)**.** *We say that $u \in H_0^1(\Omega)$ is a solution of problem $P_2(\lambda, \zeta, q, s, f)$ if for any $\phi \in H_0^1(\Omega)$ there holds*

$$\langle J'(u), \phi\rangle \equiv \int \left(\nabla u \nabla \phi - \frac{\lambda}{|x|^2}u\phi dx - \zeta f(x)|u|^{q-2}u\phi - \frac{|u|^{p^*(s)-2}u\phi}{|x|^s}\right) = 0.$$

Remark 3.2.2. *The problem $P_2(\lambda, \zeta, q, s, f)$ can be rewritten as*

$$-\Delta u(x) - \frac{\lambda}{|x|^2}u(x) = \frac{|u|^{p^*(s)-2}u}{|x|^s} + g(x,u), \ \ in \ \ \Omega\backslash\{0\},$$

where $g(x,u) = \zeta f(x)|u(x)|^{q-2}u(x)$. Note that g is a lower perturbation of $|u|^{p^*(s)-2}u$. In fact, since $f \in L^\infty(\Omega)$, we have

$$g(x,u)\left(|u|^{p^*(s)-2}u\right)^{-1} \leq \zeta|f||u|^{q-p^*} \leq c|u|^{q-p^*},$$

thus g is a lower-order perturbation of $|u|^{p^*(s)-2}u$, in the sense that

$$g(x,u)\left(|u|^{p^*(s)-2}u\right)^{-1} \longrightarrow 0 \ \ as \ |u| \longrightarrow \infty.$$

Therefore we get from Proposition 1.1.1 due to Rabinowitz [101]), that $J \in C^1(H_0^1(\Omega), \mathbb{R})$.

We consider the following hypotheses (H_4):

(i) $0 \leq \lambda < \Lambda - 4$, $\frac{N+\sqrt{\Lambda-\lambda}}{\sqrt{\Lambda}+\sqrt{\Lambda-\lambda}} < q < 2$, $0 \leq s < 2$;

(ii) There is $\delta_1 > 0$ such that $f(x) > \delta_1$ for all $x \in \Omega$ and $f \in C(\bar{\Omega})$.

We define the functional

$$\bar{G}(u) \doteq (2-q)\int \left(|\nabla u|^2 - \frac{\lambda}{|x|^2}u^2\right) - (p^*(s) - q)\int \frac{|u|^{p^*(s)}}{|x|^s}.$$

Motivated by Tarantello [119], we define

$$M \doteq \{u \in H_0^1(\Omega)\backslash\{0\} : \langle J'(u), u\rangle = 0\}$$

and consider the following subsets of M, defined by the sign of $\bar{G}$ (second derivative of J)

$$M^+ \doteq \{u \in M : \bar{G}(u) > 0\}, \quad M^0 \doteq \{u \in M : \bar{G}(u) = 0\}, \quad M^- \doteq \{u \in M : \bar{G}(u) < 0\}.$$

In what follows, for $u \in H_0^1(\Omega)$ we use the norm

$$\|u\|_\lambda^2 = \int \left(|\nabla u|^2 - \frac{\lambda}{|x|^2} u^2 \right).$$

We present now, a equivalent result to Lemma 2.2.13, for problem $P_2(\lambda, \zeta, q, s, f)$, which correspondents to a generalization of Lemma 2.1 of Tarantello [119].

Lemma 3.2.3. *Let $\zeta \in (0, \Lambda^*)$ and suppose (H_4) hold. For any $u \in H_0^1(\Omega)$ and $u \neq 0$, there exist values $t_-(u), t_+(u)$ and $t_{\max} \doteq \left(\frac{(2-q)\|u\|_\lambda^2}{(p^*(s)-q)\int \frac{|u|^{p^*(s)}}{|x|^s}} \right)^{\frac{1}{p^*(s)-2}}$ such that $0 < t_-(u) < t_{\max} < t_+(u)$. Moreover,*

$$t_-(u)\, u \in M^+ \text{ and } J(t_-(u)\, u) = \min_{0 \le t \le t_{\max}} J(tu),$$

$$t_+(u)\, u \in M^- \text{ and } J(t_+(u)\, u) = \max_{t \ge t_{\max}} J(tu).$$

Proof. The proof is similar to the one in Chapter 2, following Chen [32] and Tarantello [118]. Since

$$J(tu) = \frac{1}{2} \int \left(|\nabla tu|^2 - \frac{\lambda}{|x|^2} |tu|^2 \right) - \frac{\zeta}{q} \int f\, |tu|^q - \frac{1}{p^*(s)} \int \frac{|tu|^{p^*(s)}}{|x|^s}$$

we have

$$\frac{\partial J}{\partial t}(tu) = \int \left(t\, |\nabla u|^2 - \frac{\lambda}{|x|^2} t u^2 \right) - \zeta \int f(x) t^{q-1} |u|^{q-1} u - \int t^{p^*(s)-1} \frac{|u|^{p^*(s)-1}}{|x|} u.$$

Thus

$$\frac{\partial J}{\partial t}(tu) = t^{q-1} \left(t^{2-q} \int \left(|\nabla u|^2 - \frac{\lambda}{|x|^2} u^2 \right) - t^{p^*(s)-q} \int \frac{|u|^{p^*(s)}}{|x|^s} - \zeta \int f(x)\, |u|^q \right).$$

The function $\phi(t) \doteq t^{2-q} \int \left(|\nabla u|^2 - \frac{\lambda}{|x|^2} u^2 \right) - t^{p^*(s)-q} \int \frac{|u|^{p^*(s)}}{|x|^s}$, achieves its maximum at the point

$$t_{\max} \doteq \left(\frac{(2-q)\, \|u\|_\lambda^2}{(p^*(s)-q) \int \frac{|u|^{p^*(s)}}{|x|^s}} \right)^{\frac{1}{p^*(s)-2}},$$

and $\phi'(t) > 0$ if $t < t_{\max}$ and $\phi'(t) < 0$ if $t > t_{\max}$. Moreover,

$$\phi(t_{\max}) = \left(\frac{2-q}{p^*(s)-q} \right)^{(2-q)/(p^*(s)-2)} \left(\frac{p^*(s)-2}{p^*(s)-q} \right) \|u\|_\lambda^{\frac{2(p^*(s)-q)}{p^*(s)-2}} \left(\int \frac{|u|^{p^*(s)}}{|x|^s} \right)^{\frac{q-2}{p^*(s)-2}}.$$

Using the definition of $S_{\lambda,s}$, we have

$$\phi\left(t_{\max}\right) \geq \left(\frac{2-q}{p^*(s)-q}\right)^{(2-q)/(p^*(s)-2)} \left(\frac{p^*(s)-2}{p^*(s)-q}\right) S_{\lambda,s}^{\frac{p^*(s)(2-q)}{2(p^*(s)-2)}} \|u\|_{\lambda}^{q}.$$

Now for $\zeta \in (0, \Lambda^*)$, we obtain by Hölder inequality and the definition of $S_{\lambda,s}$ that

$$\zeta \int f(x)|u|^q \leq \zeta |f|_{\infty} |\Omega|^{1-\frac{q}{p^*(s)}} S_{\lambda,s}{}^{-\frac{q}{2}} \|u\|_{\lambda}^{q} < \phi\left(t_{\max}\right). \tag{3.4}$$

It follows that there are $t_+ \doteq t_+(u) > t_{\max} > t_- \doteq t_-(u)$ such that

$$\phi\left(t_+\right) = \zeta \int f(x)|u|^q = \phi\left(t_-\right)$$

and

$$\phi'\left(t_+\right) < 0 < \phi'\left(t_-\right).$$

Equivalently, we have $t_+u \in M^-$ and $t_-u \in M^+$. Also $J(t_+u) \geq J(tu)$, for any $t \geq t_-$ and $J(t_-u) \leq J(tu)$ for any $t \in [0, t_+]$. □

Remark 3.2.4. *Using a similar idea to Chapter 2, we can see graphically the behavior of function ϕ defined in the Lemma 3.2.3. Consider $t > 0$ and define $\bar{F}(u) \doteq \zeta \int f(x)|u|^q$. From (3.4), if $\zeta \in (0, \Lambda^*)$, then $\bar{F}(u) < \phi\left(t_{\max}\right)$. For $\frac{N+\sqrt{\Lambda-\lambda}}{\sqrt{\Lambda}+\sqrt{\Lambda-\lambda}} < q < 2$, we have* ***two*** *values t_- and t_+ (see Figure 3.1). If we consider other values for q, the behavior of the function ϕ is quite different (see Figure 3.2). When $q = 2$, we have $\phi(t) = \|u\|_{\lambda}^2 - t^{p-2}\int \frac{|u|^p}{|x|^s}$, $\lim_{t\to 0^\pm}\phi(t) = \pm\|u\|_{\lambda}^2$ and $\lim_{t\to\pm\infty}\phi_u(t) = \mp\infty$, thus, since $\bar{F}(u) < \phi\left(t_{\max}\right)$, we have* ***one*** *value $t_+ > 0$. When $q > 2$, we have $\phi(t) = t^{-(q-2)}\|u\|_{\lambda}^2 - t^{p-q}\int \frac{|u|^p}{|x|^s}$, $\lim_{t\to 0^\pm}\phi(t) = \mp\infty$ and $\lim_{t\to\pm\infty}\phi(t) = 0$, thus for $\bar{F}(u) > 0$, we have* ***one*** *value $t_+ > 0$.*

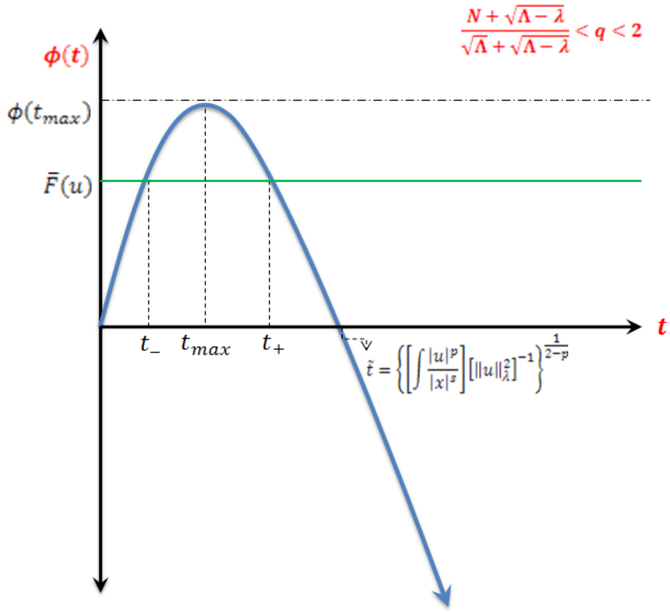

Figure 3.1: Behavior of the function ϕ

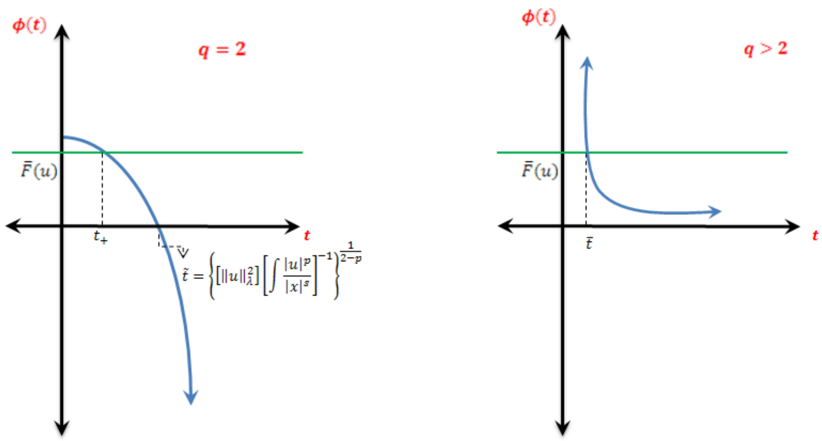

Figure 3.2: Behavior of the function ϕ for different values of q.

Proposition 3.2.5. *Assume $0 < \zeta < \Lambda^*, 0 < \lambda < \Lambda - 4$. Let $(u_n)_{n\in\mathbb{N}} \subset M^-$ be such that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $J(u_n) \longrightarrow c$ but u_n does not converge strongly to u in $H_0^1(\Omega)$. Then the following holds:*

(i) $c > J(t_+(u)\,u)$ in the case $u \neq 0$ and $t_+(u) \leq 1$;

(ii) $c \geq J(t_-(u)\,u) + \frac{2-s}{2(N-s)} S_{\lambda,s}^{\frac{N-s}{2-s}}$ in the case $u \neq 0$ and $t_+(u) > 1$;

(iii) $c \geq \frac{2-s}{2(N-s)} S_{\lambda,s}^{\frac{N-s}{2-s}}$ in the case $u = 0$.

Proof. Keep the expression of J in mind. Note that from $u_n \rightharpoonup u$, we have

$$\int f(x)\,|u_n - u|^q \to 0 \text{ as } n \to \infty.$$

We may assume that

$$\|u_n - u\|_\lambda^2 \to a^2 \text{ and } \int \frac{|u_n - u|^{p^*(s)}}{|x|^s} \to b^{p^*(s)},$$

for some $a, b \in \mathbb{R}$. Since u_n does not converge strongly to u in $H_0^1(\Omega)$, we have $a \neq 0$. Set

$$r(t) \doteq J(tu)\,, \quad \beta(t) \doteq \frac{a^2}{2}t^2 - \frac{b^{p^*(s)}}{p^*(s)}t^{p^*(s)}$$

and $\theta(t) \doteq r(t) + \beta(t)$, then $J(tu_n) \to \theta(t)$ as $n \to +\infty$. We consider three situations:

(i) Suppose $u \neq 0$ and $t_+(u) \leq 1$. We use the notation as in the proof of Lemma 3.2.3. For this u and

$$\phi(t) \doteq t^{2-q} \int \left(|\nabla u|^2 - \frac{\lambda}{|x|^2}u^2 \right) - t^{p^*(s)-q} \int \frac{|u|^{p^*(s)}}{|x|^s},$$

we have $\phi'(t) < 0$ for $t > t_{\max}$. Therefore $\phi(1) \leq \phi(t_+(u))$. From $\phi(t_+(u)) = \zeta \int f(x)\,|u|^q$ and

$$r'(t) = \frac{\partial}{\partial t} J(tu) = t^{q-1} \left(\phi(t) - \zeta \int f(x)\,|u|^q \right),$$

we obtain $r'(1) \leq 0$. Since $u_n \in M^-$ for any $n \in \mathbb{N}$, we have $\theta'(1) = 0$. Thus $\beta'(1) \geq 0$ and hence $a^2 - b^{p^*(s)} \geq 0$. Hence $\beta(t_+(u)) > 0$ and

$$c \geq \theta(1) \geq \theta(t_+(u)) = J(t_+(u)\,u) + \beta(t_+(u)) > J(t_+(u)\,u)\,.$$

(ii) Suppose $u \neq 0$ and $t_+(u) > 1$. Firstly, from $t_+(u) > 1$ we claim that $b \neq 0$. Indeed if $b = 0$ then, on one hand, from the proof of Lemma 3.2.3, we know that $r'(t) < 0$ for $t > t_+(u)$ or $t \in (0, t_-(u))$. On the other hand from $\theta'(1) = 0$ and $\theta''(1) \leq 0$, we have that $r'(1) = -a^2 < 0$ and $r''(1) \leq -a^2 < 0$, which contradicts $t_+(u) > 1$. Thus we prove that $b \neq 0$.

Denote

$$t_* \doteq \left(a^2/b^2\right)^{\frac{1}{p^*(s)-2}}.$$

We know that β attains its maximum at t_* and $\beta'(t) > 0$ for $0 < t < t_*$ and $\beta'(t) < 0$ for $t > t_*$. Therefore we obtain from $S_{\lambda,s}b^2 \le a^2$ that

$$\beta(t_*) = \left(\frac{1}{2} - \frac{1}{p^*(s)}\right)\left(a^2/b^2\right)^{\frac{p^*(s)}{p^*(s)-2}} \ge \frac{2-s}{2(N-s)} S_{\lambda,s}{}^{\frac{N-s}{2-s}}.$$

Next, we show that $t_* \le t_+(u)$. Suppose this is not the case, i.e., $1 < t_+(u) < t_*$. As $0 > \theta'(t) = r'(t) + \beta'(t)$ for all $t > 1$, we have $r'(t) \le -\beta'(t) < 0$ for $t \in (1, t_*)$, which contradicts $1 < t_+(u) < t_*$ and $r'(t_+(u)) = 0$. We have shown that $t_* \le t_+(u)$. Hence we obtain

$$c = \theta(1) \ge \theta(t_*) = J(t_* u) + \beta(t_*) \ge J(t_-(u)\, u) + \frac{2-s}{2(N-s)} S_{\lambda,s}{}^{\frac{N-s}{2-s}}.$$

This implies that (ii) holds.

(iii) Suppose $u \equiv 0$. Since $u_n \in M^- \subset M$, we have

$$\int \left(|\nabla u|^2 - \frac{\lambda}{|x|^2}|u_n|^2\right) = \int \frac{|u_n|^{p^*(s)}}{|x|^s} + o(1).$$

Using the fact that

$$S_{\lambda,s}\left(\int \frac{|v|^{p^*(s)}}{|x|^s}\right)^{\frac{2}{p^*(s)}} \le \int \left(|\nabla v|^2 - \frac{\lambda}{|x|^2}|v|^2\right)$$

for all $v \in H_0^1(\Omega)$ and $v \neq 0$, we obtain

$$\begin{aligned} c &\ge \frac{1}{2}\int \left(|\nabla u|^2 - \frac{\lambda}{|x|^2}|u_n|^2\right) - \frac{1}{p^*(s)}\int \frac{|u_n|^{p^*(s)}}{|x|^s} + o(1) \\ &\ge \left(\frac{1}{2} - \frac{1}{p^*(s)}\right)\int \left(|\nabla u|^2 - \frac{\lambda}{|x|^2}|u_n|^2\right) + o(1) \ge \frac{2-s}{2(N-s)} S^{\frac{N-s}{2-s}}. \end{aligned}$$

The proof is complete. □

For $\zeta \in (0, \Lambda^*)$, denote

$$c_{0,f} \doteq \inf_{u\in M^+} J(u) \text{ and } c_{1,f} \doteq \inf_{u\in M^-} J(u).$$

Remark 3.2.6. *In the case of $f \equiv 1$, Bouchekif-Matallah [14] have proved that*

$$c_{0,1} < 0 \text{ and } c_{1,1} < c_{0,1} + \frac{2-s}{2(N-s)} S_{\lambda,s}{}^{\frac{N-s}{2-s}}, \tag{3.5}$$

and $c_{0,1}$ and $c_{1,1}$ achieve their minimum at v_0 and v_1, respectively, i.e. $c_{0,1} = J(v_0)$ and $c_{1,1} = J(v_1)$. Moreover, v_0 and v_1 are positive solutions of $P_2(\lambda, \zeta, q, s, f)$ in the case of $f \equiv 1$.

3.2.1 Local behavior of the solution

Since we are facing with the singular term $\frac{\lambda}{|x|^2}u$ and a critical nonlinearity, to proceed with, we need to use the exact local behavior for the solutions of the problem $P_2(\lambda,\zeta,q,s,f)$ to estimate the energy. We point out that Smets [111] has essentially proved that for any positive solution u of $P_2(\lambda,\zeta,q,s,f)$, there holds $u \in L^r(\Omega)$ for any $r < 2^*\frac{\sqrt{\Lambda}}{\sqrt{\Lambda}-\sqrt{\Lambda-\lambda}}$. However from Chen [31] and Chen [34], we have the following refined result.

Proposition 3.2.7. *Let $0 \le \lambda < \Lambda - 4$ and $0 \le s < 2$. If $u \in H_0^1(\Omega)$ is a positive solution of the problem $P_2(\lambda,\zeta,q,s,f)$, then there holds*

$$K_1|x|^{-(\sqrt{\Lambda}-\sqrt{\Lambda-\lambda})} \le u(x) \le K_2|x|^{-(\sqrt{\Lambda}-\sqrt{\Lambda-\lambda})}, \qquad x \in B_\rho(0)\backslash\{0\} \tag{3.6}$$

for $\rho > 0$ sufficiently small and some positive constants $K_1$and K_2.

3.2.2 Integral estimates

From Catrina et al [25] and Chou et al [43], we have that $S_{\lambda,s}$ is achieved by a family of functions with parameters $\varepsilon > 0$,

$$U_\varepsilon(x) = \frac{\left(\frac{2\varepsilon B(N-s)}{A}\right)^{\frac{A}{(2-s)}}}{|x|^{A-B}(\varepsilon + |x|^{(2-s)\frac{B}{A}})^{\frac{N-2}{2-s}}} \tag{3.7}$$

where $A = \sqrt{\Lambda}$ and $B = \sqrt{\Lambda-\lambda}$. Moreover, there holds

$$\int_{\mathbb{R}^N}\left(|\nabla U_\varepsilon|^2 - \frac{\lambda}{|x|^2}U_\varepsilon^2\right)dx = \int_{\mathbb{R}^N}\frac{|U_\varepsilon|^{p^*(s)}}{|x|^s}dx = S_{\lambda,s}^{\frac{N-s}{2-s}}. \tag{3.8}$$

For further details, see also Chen [31].

Next, choose $\delta_2 > 0$ such that $B(0,2\delta_2) \subset \Omega$ and $2\delta_2 < \rho$ (ρ is as in Proposition 3.2.7). Define a cut-off function $\psi \in C_0^2(\Omega)$ satisfying

$$\psi(x) = \begin{cases} 1, & |x| \le \delta_2, \\ 0, & |x| \ge 2\delta_2, \end{cases}$$

$|\psi(x)| \le 1$, and $|\nabla\psi(x)| \le C$ for some positive constant C. Denote $u_\varepsilon(x) = \psi(x)U_\varepsilon(x)$.

Using Proposition 3.2.7, we have the following integral estimates which will play an essential role in what follows.

Proposition 3.2.8. *If $0 \le \lambda < \Lambda - 1$, $1 < q < 2$, $0 \le s < 2$ and $w \in H_0^1(\Omega)$ is a positive solution of $P_2(\lambda,\zeta,q,s,f)$, then for ε small enough, there holds*

$$\|u_\varepsilon\|_\lambda^2 = S^{\frac{N-s}{2-s}} + O\left(\varepsilon^{\frac{N-s}{2-s}}\right), \qquad \int\frac{u_\varepsilon}{|x|^s} \ge S^{\frac{N-s}{2-s}} - O\left(\varepsilon^{\frac{N-s}{2-s}}\right) \tag{3.9}$$

$$\int \frac{|w|^{p^*(s)-1}|u_\varepsilon|}{|x|^s} = O\left(\varepsilon^{\frac{N-s}{2(2-s)}}\right), \qquad \int \frac{|u_\varepsilon|^{p^*(s)-1}|w|}{|x|^s} = O\left(\varepsilon^{\frac{N-s}{2(2-s)}}\right) \tag{3.10}$$

$$\int |w|^{q-1} u_\varepsilon = O\left(\varepsilon^{\frac{\sqrt{\Lambda}}{2-s}}\right), \qquad \int |u_\varepsilon|^{q-1}|w| = O\left(\varepsilon^{\frac{(q-1)\sqrt{\Lambda}}{2-s}}\right) \tag{3.11}$$

and

$$\int |u_\varepsilon|^q = \begin{cases} O\left(\varepsilon^{\frac{q\sqrt{\Lambda}}{2-s}}\right), & \text{if } 1 < q < \frac{N}{\sqrt{\Lambda}+\sqrt{\Lambda-\lambda}}, \\ O\left(\varepsilon^{\frac{q\sqrt{\Lambda}}{2-s}}|\ln\varepsilon|\right), & \text{if } q = \frac{N}{\sqrt{\Lambda}+\sqrt{\Lambda-\lambda}}, \\ O\left(\varepsilon^{\frac{(N-q\sqrt{\Lambda})\sqrt{\Lambda}}{(2-s)\sqrt{\Lambda-\lambda}}}\right), & \text{if } \frac{N}{\sqrt{\Lambda}+\sqrt{\Lambda-\lambda}} < q < 2. \end{cases} \tag{3.12}$$

Proof. For the proofs of (3.9) and (3.10) see Chen [31]. We use Proposition 3.2.7 to estimate $\int |w|^{q-1} u_\varepsilon$. Hence, we get

$$\begin{aligned} \int w^{q-1}u_\varepsilon &= K_3\varepsilon^{\frac{\sqrt{\Lambda}}{2-s}} + K\int_{B(0,\delta_2)} \left[|x|^{q\left(\sqrt{\Lambda}-\sqrt{\Lambda-\lambda}\right)}\left(\varepsilon + |x|^{\frac{(2-s)\sqrt{\Lambda-\lambda}}{\sqrt{\Lambda}}}\right)^{\frac{N-2}{2-s}}\right]^{-1}\varepsilon^{\frac{\sqrt{\Lambda}}{2-s}}dx \\ &= K_3\varepsilon^{\frac{\sqrt{\Lambda}}{2-s}} + K\int_0^{\delta_2} \left[\rho^{q\left(\sqrt{\Lambda}-\sqrt{\Lambda-\lambda}\right)}\left(\varepsilon + \rho^{\frac{(2-s)\sqrt{\Lambda-\lambda}}{\sqrt{\Lambda}}}\right)^{\frac{N-2}{2-s}}\right]^{-1}\varepsilon^{\frac{\sqrt{\Lambda}}{2-s}}\rho^{N-1}d\rho. \end{aligned}$$

Since $-1 + N - q - q(\sqrt{\Lambda} - \sqrt{\Lambda-\lambda}) - 2\sqrt{\Lambda-\lambda} > -1$, we get that

$$\int_{B(0,\delta_2)} \left[|x|^{q\left(\sqrt{\Lambda}-\sqrt{\Lambda-\lambda}\right)}\left(\varepsilon + |x|^{\frac{(2-s)\sqrt{\Lambda-\lambda}}{\sqrt{\Lambda}}}\right)^{\frac{N-2}{2-s}}\right]^{-1}\varepsilon^{\frac{\sqrt{\Lambda}}{2-s}}dx = O\left(\varepsilon^{\frac{\sqrt{\Lambda}}{2-s}}\right).$$

Therefore

$$\int |w|^{q-1} u_\varepsilon = O\left(\varepsilon^{\frac{\sqrt{\Lambda}}{2-s}}\right).$$

The proofs of $\int |u_\varepsilon|^{q-1}|w|\,dx$ and (3.12) are similar. We omit the details. □

3.2.3 Existence of two nontrivial solutions

We will prove the existence of two nontrivial solutions for problem $P_2(\lambda,\zeta,q,s,f)$, following the same ideas in Bouchekif-Matallah [14]. The first solution is obtained using the concentration-compactness method, introduced by Lions (see 1.1.15) and the second solution by contradiction, applying the mountain pass theorem.

Lemma 3.2.9. *Suppose $(H_4)(ii)$ holds. If there exists a constant $C \equiv C(N, \Omega, q, s) > 0$, such that, for all sequences $(u_n)_{n\in\mathbb{N}}$ in $H_0^1(\Omega)$ satisfying*

$$J(u_n) \to c < \frac{2-s}{2(N-s)} S_{\lambda,s}^{\frac{N-s}{2-s}} - C \tag{3.13}$$

and

$$J'(u_n) \to 0 \quad \text{in } H^{-1}(\Omega), \tag{3.14}$$

then there exists a subsequence strongly convergent in $H_0^1(\Omega)$.

Proof. First we prove that the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$ and therefore $u_n \rightharpoonup u$ in $H_0^1(\Omega)$. In fact, for $(u_n)_{n\in\mathbb{N}}$ in $H_0^1(\Omega)$, we have

$$\begin{aligned} J(u_n) &= \frac{1}{2}\int (|\nabla u_n|^2 - \frac{\lambda}{|x|^2}|u_n|^2) - \frac{\zeta}{q}\int f|u_n|^q - \frac{1}{p^*(s)}\int \frac{|u_n|^q}{|x|^s} \\ &= c + o(1) \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} \langle J'(u_n), u_n\rangle &= \frac{1}{2}\int (|\nabla u_n|^2 - \frac{\lambda}{|x|^2}|u_n|^2) - \zeta\int f|u_n|^q - \int \frac{|u_n|^q}{|x|^s} \\ &= o(1)\,\|u_n\|. \end{aligned} \tag{3.16}$$

It follows from (3.15) and (3.16) that

$$\begin{aligned} J(u_n) - \frac{1}{2}\langle J'(u_n), u_n\rangle &= \left(\frac{1}{p^*(s)} - \frac{1}{2}\right)\int \frac{|u_n|^q}{|x|^s} - \zeta\left(\frac{1}{q} - \frac{1}{2}\right)\int f|u_n|^q \\ &= c + o(1)\,\|u_n\| \end{aligned}$$

i.e. we have

$$\begin{aligned} \left(\frac{1}{p^*(s)} - \frac{1}{2}\right)\int \frac{|u_n|^q}{|x|^s} &= \zeta\left(\frac{1}{q} - \frac{1}{2}\right)\int f|u_n|^q + c + o(1)\,\|u_n\| \\ &\leq \zeta\left(\frac{1}{q} - \frac{1}{2}\right)|f|_\infty |\Omega|^{1-\frac{q}{p^*(s)}} S_{\lambda,s}^{\frac{-q}{2}} \|u_n\|_\lambda^q + c + o(1)\,\|u_n\|. \end{aligned}$$

Then

$$\int \frac{|u_n|^q}{|x|^s} \leq \zeta\left(\frac{1}{q} - \frac{1}{2}\right)|f|_\infty |\Omega|^{1-\frac{q}{p^*(s)}} S_{\lambda,s}^{\frac{-q}{2}} \|u_n\|_\lambda^q + c + o(1)\,\|u_n\|. \tag{3.17}$$

On the other hand

$$\begin{aligned} c + 0(1)\,\|u_n\| &= J(u_n) \\ &= \frac{1}{2}\int (|\nabla u_n|^2 - \frac{\lambda}{|x|^2}|u_n|^2) - \frac{\zeta}{q}\int f|u_n|^q - \frac{1}{p^*(s)}\int \frac{|u_n|^q}{|x|^s} \\ &\geq \frac{1}{2}\|u_n\|_\lambda^q - \frac{\zeta}{q}|f|_\infty |\Omega|^{1-\frac{q}{p^*(s)}} S_{\lambda,s}^{\frac{-q}{2}} \|u_n\|_\lambda^q - \frac{1}{p^*(s)}\int \frac{|u_n|^q}{|x|^s}. \end{aligned} \tag{3.18}$$

Then by (3.17) and (3.18) imply that $(u_n)_{n\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$. Therefore going if necessary to a subsequence we may assume that

$$\begin{cases} u_n & \rightharpoonup \sigma \quad \text{in } H_0^1(\Omega), \\ u_n & \to \sigma \quad \text{a.e. in } \Omega, \\ u_n & \to \sigma \quad \text{in } L^t(\Omega), 1 \le t < 2^*, \\ u_n & \rightharpoonup \sigma \quad \text{in } L^2(\Omega, |x|^{-2}dx), \\ u_n & \rightharpoonup \sigma \quad \text{in } L^{p^*(s)}(\Omega, |x|^{-s}dx). \end{cases} \tag{3.19}$$

Denote $v_n = u_n - \sigma$, then Brezis-Lieb Lemma (see Costa [46]) implies that

$$\int |\nabla u_n|^2 = \int |\nabla v_n|^2 + \int |\nabla \sigma|^2 + o(1),$$

$$\int \frac{\lambda}{|x|^2} u_n^2 = \int \frac{\lambda}{|x|^2} v_n^2 + \int \frac{\lambda}{|x|^2}\sigma^2 + o(1),$$

$$\int \frac{u_n^2}{|x|^s} = \int \frac{v_n^2}{|x|^s} + \int \frac{\sigma^2}{|x|^s} + o(1),$$

and $\langle J'(\sigma), \phi\rangle = 0$ for any $\phi \in H_0^1(\Omega)$. That is $\sigma \in H_0^1(\Omega)$ is a weak solution of the problem $P_2(\lambda, \zeta, q, s, f)$. From concentration compactness principle 1.1.15 and the Hardy-Sobolev inequality, we get a subsequence still denoted by $(u_n)_{n\in\mathbb{N}}$, an at most countable set D, a set of distinct points $(x_j)_{j\in D} \subset \Omega$ and sets of nonnegative numbers $(\widehat{u}_j)_{j\in D}$ and $(\widehat{v}_j)_{j\in D}$ such that:

(a) $|\nabla u_n|^2 - \lambda\frac{|u_n|^2}{|x|^2} \rightharpoonup \widehat{u} \ge |\nabla\sigma|^2 - \lambda\frac{|\sigma|^2}{|x|^2} + \sum\limits_{j\in D} \widehat{u}_j\delta_{x_j}$;

(b) $\frac{|u_n|^{p^*(s)}}{|x|^s} \rightharpoonup \widehat{v} = \frac{|\sigma|^{p^*(s)}}{|x|^s} + \sum\limits_{j\in D} \widehat{v}_j\delta_{x_j}$;

(c) $\widehat{v}_j^{\frac{2}{p^*(s)}} \le S_{\lambda,s}^{-1}\widehat{u}_j$

for all $j \in D$. Here δ_{x_j} is the Dirac mass at x. We assume that there exists some $j \in D$ such that $\widehat{u}_j \ne 0$. Let $\varepsilon > 0$ and Ψ be a cut-off function centered at x_j with

$$\Psi(x) \doteq \begin{cases} 1, & \text{if } |x - x_j| \le \frac{1}{2}\varepsilon, \\ 0, & \text{if } |x - x_j| \ge \varepsilon, \end{cases}$$

and $|\nabla\Psi| \le \frac{4}{\varepsilon}$. Then $\langle J'(u_n), \Psi u_n\rangle \to 0$, i.e.

$$\begin{aligned} 0 &= \lim_{\varepsilon\to 0}\lim_{n\to\infty} \left\langle J'(u_n), \Psi u_n \right\rangle \\ &= \lim_{\varepsilon\to 0}\lim_{n\to\infty} \int \left(|\nabla u_n|^2\Psi + u_n\nabla u_n\nabla\Psi - \frac{\lambda}{|x|^2}u_n^2\Psi - \zeta f(x)\,|u_n|^q\,\Psi - \frac{|u_n|^{p^*(s)}\,\Psi}{|x|^s} \right) dx \\ &\ge \widehat{u}_j - \widehat{v}_j \\ &\ge \widehat{u}_j - S_{\lambda,s}^{-p^*(s)/2}\widehat{u}_j^{p^*(s)/2}. \end{aligned}$$

Thus $\widehat{u}_j^{\frac{2-p^*(s)}{2}} \leq S_{\lambda,s}^{-p^*(s)/2}$ and since $\frac{2-p^*(s)}{2} < 0$, we have

$$S_{\lambda,s}^{p^*(s)/p^*(s)-2} = S_{\lambda,s}^{N-s/2-s} \leq \widehat{u}_j.$$

Therefore $\widehat{u}_j = 0$ or $\widehat{u}_j \geq S_{\lambda,s}^{N-s/2-s}$.
From (3.13) and (3.14) we have

$$\begin{aligned}
& J\left(u_n\right) - \frac{1}{p^*(s)}\left\langle J'\left(u_n\right), u_n\right\rangle \\
= \; & \frac{1}{2}\int\left(\left|\nabla u_n\right|^2 - \frac{\lambda}{|x|^2}\left|u_n\right|^2\right) - \frac{\zeta}{q}\int f\left|u_n\right|^q - \frac{1}{p^*(s)}\int\frac{\left|u_n\right|^{p^*(s)}}{|x|^s} \\
& -\frac{1}{p^*(s)}\int\left(\left|\nabla u_n\right|^2 - \frac{\lambda}{|x|^2}\left|u_n\right|^2\right) - \zeta\frac{1}{p^*(s)}\int f\left|u_n\right|^q - \frac{1}{p^*(s)}\int\frac{\left|u_n\right|^{p^*(s)}}{|x|^s} \\
= \; & \left(\frac{1}{2} - \frac{1}{p^*(s)}\right)\int\left(\left|\nabla u_n\right|^2 - \frac{\lambda}{|x|^2}\left|u_n\right|^2\right) - \zeta\left(\frac{1}{q} - \frac{1}{p^*(s)}\right)\int f\left|u_n\right|^q \\
= \; & \frac{2-s}{2\left(N-s\right)}\left\|u_n\right\|_\lambda^2 - \zeta\left(\frac{1}{q} - \frac{1}{p^*(s)}\right)\int f\left|u_n\right|^q.
\end{aligned}$$

Using (3.4), we obtain

$$\begin{aligned}
J\left(u_n\right) - \frac{\left\langle J'\left(u_n\right), u_n\right\rangle}{p^*(s)} & \geq \frac{2-s}{2\left(N-s\right)}\left\|u_n\right\|_\lambda^2 - \zeta\left(\frac{1}{q} - \frac{1}{p^*(s)}\right)|f|_\infty |\Omega|^{1-\frac{q}{p^*(s)}} S_{\lambda,s}{}^{-\frac{q}{2}}\left\|u_n\right\|_\lambda^q \\
& \geq \frac{2-s}{2\left(N-s\right)}\left\|u_n\right\|_\lambda^2 - \zeta\left(\frac{1}{q} - \frac{1}{p^*(s)}\right)\bar{C}\left\|u_n\right\|_\lambda^q
\end{aligned}$$

Thus there exists $C \equiv C(N, \Omega, q, s)$ such that

$$\frac{2-s}{2\left(N-s\right)}t^2 - \zeta\left(\frac{1}{q} - \frac{1}{p^*(s)}\right)\bar{C}t^q \geq -C\zeta^{\frac{2}{2-q}}$$

for all $t \geq 0$. If we assume that $\widehat{u}_j \neq 0$ for some $j \in D$, then

$$\begin{aligned}
c & \geq \frac{2-s}{2\left(N-s\right)}S_{\lambda,s}^{N-s/2-s} + \frac{2-s}{2\left(N-s\right)}\|\sigma\|_\lambda^2 - \zeta\left(\frac{1}{q} - \frac{1}{p^*(s)}\right)\int f\sigma^q \\
& \geq \frac{2-s}{2\left(N-s\right)}S_{\lambda,s}^{N-s/2-s} - C\zeta^{\frac{2}{2-q}},
\end{aligned}$$

which contradicts our assumption (3.13). Consequently

$$\widehat{u}_j = 0 \text{ for all } j \in D \text{ and } u_n \longrightarrow \sigma \text{ strongly in } H_0^1(\Omega) \text{ as } n \text{ goes to } +\infty.$$

□

Remark 3.2.10. *Using (3.4), the Sobolev and Hardy inequalities we have*

$$J(u) \geq \frac{1}{2}\|u\|_\lambda^2 - \frac{1}{q}\zeta c_1\|u\|_\lambda^q - \frac{1}{p^*(s)}c_2\|u\|_\lambda^{p^*(s)}$$

Let $\rho = \|u\|_\lambda$. By the above inequality, we can choose two positive constants ρ_0 and Λ, such that, for $\zeta \in (0, \Lambda^*)$, $J(u)$ is bounded from below in $B_0(\rho_0)$ (the ball centered at 0 with radius ρ_0) and $J(u) \geq r > 0$ for $\|u\|_\lambda = \rho_0$. Let $\phi \in H_0^{'}(\Omega)$ such that $\|u\|_\lambda = 1$. Then, for $t > 0$, we have

$$J(t\phi) = \frac{1}{2}t^2 - \frac{\zeta t^q}{q}\int \phi^q - \frac{t^{p^*(s)}}{p^*(s)}\int \frac{\phi^{p^*(s)}}{|x|^{(s)}}$$

Thus, there is $t_0 \leq \rho_0$ such that $J(t\phi) < 0$ for $0 < t < t_0$. then

$$c_0, f = \inf_{u\in B_0(\rho_0)} J(\mu) < 0.$$

The above Lemma, implies that J can achieves its minimun $c_{0,f}$ at the function $\sigma = w_0$ i.e., $c_{0,f} = J(w_0)$.
Let w_0 be as before and define $w_1 = w_0 + v$ with $v > 0$ in $H_0^1(\Omega)$. We have

$$\begin{aligned} -\Delta v - \frac{\lambda v}{|x|^2} &= \zeta f(x)\,|w_0 + v|^{q-2}(w_0 + v) - \zeta f(x)\,|w_0|^{q-2} w_0 \\ &\quad + \frac{|w_0 + v|^{p^*(s)-2}(w_0 + v)}{|x|^s} - \frac{|w_0|^{p^*(s)-2} w_0}{|x|^s} \end{aligned}$$

Let us define the map $g_\zeta : \Omega \times \mathbb{R} \to \mathbb{R}$ for $\zeta > 0$ by

$$g_\zeta(x,t) = \begin{cases} \zeta f(x)\,|w_0 + t|^{q-2}(w_0 + t) - \zeta f(x)\,|w_0|^{q-2} w_0 & \\ \quad + \frac{|w_0+t|^{p^*(s)-2}(w_0+t)}{|x|^s} - \frac{|w_0|^{p^*(s)-2} w_0}{|x|^s}, & \text{if } t \geq 0; \\ 0, & \text{if } t < 0; \end{cases}$$

Define also

$$G_\zeta(x,v) \doteq \int_0^v g_\zeta(x,t)\,dt$$

and

$$\begin{aligned} \bar{J}(v) &\doteq \frac{1}{2}\|v\|_\lambda^2 - \int G_\zeta\left(x, v^+(x)\right) \\ &= \frac{1}{2}\int \left(|\nabla v|^2 - \int \frac{\lambda}{|x|^2} v^2\right) - \int G_\zeta\left(x, v^+(x)\right). \end{aligned}$$

Lemma 3.2.11. *The origin $v = 0$ is a local minimum of $\bar{J}$.*

Proof. Since w_0 is a local minimum of J, there exists $\varepsilon_1 > 0$, such that $J(w_0) \leq J(w_0 + v)$, for all $\|w_0 + v - w_0\| \leq \varepsilon_1$, $v \in H_0^1(\Omega)$. On the other hand $\|v^+\| \leq \|v\| \leq \varepsilon_1$ for all $v \in H_0^1(\Omega)$. Thus

$$\bar{J}(v) = J(w_0 + v^+) - J(w_0) \geq 0.$$

Therefore $0 = \bar{J}(0) \leq \bar{J}(v)$, for all v such that $\|v\| \leq \varepsilon_1$. Then simply choose $\varepsilon > 0$ such that $0 < \varepsilon \leq \varepsilon_1$ and we obtain $\bar{J}(0) = 0 \leq \bar{J}(v)$, for all v such that $\|0 - v\| \leq \varepsilon$. $\square$

Now, we prove the existence of a second solution of problem $P_2(\lambda, \zeta, q, s, f)$.

Lemma 3.2.12. *If $v \equiv 0$ is the only critical point of $\bar{J}$, then $\bar{J}$ satisfies the $(PS)_c$-condition for any $c < \frac{2-s}{2(N-s)} S_{\lambda,s}^{N-s/2-s}$.*

Proof. Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in $H_0^1(\Omega)$, such that

$$\begin{cases} \bar{J}(v_n) & \to c \quad \text{with } c < \frac{2-s}{2(N-s)} {S_{\lambda,s}}^{N-s/2-s}, \\ \bar{J}'(v_n) & \to 0 \quad \text{in } H^{-1}(\Omega) \text{ as } n \to \infty. \end{cases} \tag{3.20}$$

Hence, $(v_n)_{n\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$. Now, we prove that $v_n \to 0 \in H_0^1(\Omega)$. Since v_n is bounded in $H_0^1(\Omega)$, we can assume, if necessary passing to a subsequence, that

$$\begin{cases} v_n & \rightharpoonup v \quad \text{in } H_0^1(\Omega), \\ v_n & \to v \quad \text{a.e in } \Omega, \\ v_n & \to v \quad \text{in } L^t(\Omega), 1 < t < 2^* \\ v_n & \to v \quad \text{in } L^r(\Omega, |x|^{-s}dx), 2 \leq r < p^*(s) \end{cases} \tag{3.21}$$

From the assumptions, we have that $v \equiv 0$. On the other hand, from the definition of the functional $\bar{J}$, we have

$$\begin{aligned} \left\langle \bar{J}'(v_n), w_0 + v_n \right\rangle &= \int \nabla v_n \nabla (w_0 + v_n) - \int \frac{\lambda}{|x|^2} v_n (w_0 + v_n) + o(1) \\ &\quad -\zeta \int f(x) (w_0 + v)^{q-2} (w_0 + v_n) - \zeta \int f(x) w_0 (w_0 + v_n) \\ &\quad + \int \frac{1}{|x|^s} \left(w_0 + v^+\right)^{p^*(s)-2} (w_0 + v_n) - \int {w_0}^{p^*(s)-2} (w_0 + v_n). \\ &= \int \nabla v_n \nabla w_0 + \int \nabla v_n^2 - \int \frac{\lambda}{|x|^2} v_n w_0 - \int \frac{\lambda}{|x|^2} v_n^2 + o(1) \\ &\quad -\zeta \int f(x) \left(w_0 + v^+\right)^{q-2} (w_0 + v_n) - \zeta \int f(x) {w_0}^{q-2} w_0 \\ &\quad -\zeta \int f(x) {w_0}^{q-2} v_n - \int \frac{1}{|x|^s} \left(w_0 + v^+\right)^{p^*(s)-2} (w_0 + v_n) \\ &\quad -\int \frac{1}{|x|^s} (w_0)^{p^*(s)-2} w_0 - \int \frac{1}{|x|} (w_0)^{p^*(s)-2} v_n. \end{aligned} \tag{3.22}$$

Since w_0 is a solution and $v_n \in H_0^1(\Omega)$, we have

$$\int \nabla w_0 \nabla v_n - \int \frac{\lambda}{|x|^2} w_0 v_n - \int \zeta f(x) |w_0|^{q-2} w_0 v_n - \int \frac{|w_0|^{p^*(s)-2} w_0 v_n}{|x|^s} = 0.$$

Thus, since $v_n \to v \equiv 0$ in $L^r(\Omega)$ and $L^r\left(\Omega, |x|^{-s} dx\right)$ for $2 \leq r < p^*(s)$, then

$$\int \nabla w_0 \nabla v_n - \int \frac{\lambda}{|x|^2} w_0 v_n = \zeta \int f(x) |w_0|^{q-2} w_0 v_n - \int \frac{|w_0|^{p^*(s)-2} w_0 v}{|x|^s} \to 0. \tag{3.23}$$

Moreover $v_n \to v \equiv 0$ in $L^t(\Omega)$, $1 < t < 2^*$ then

$$\zeta \int f(x)(w_0+v_n)^{q-2}\left(w_0+v_n^+\right) = \zeta \int f(x)|w_0|^{q-2} w_0 + o(1) \tag{3.24}$$

and

$$\zeta \int f(x) w_0{}^{q-2}\left(w_0+v_n^+\right) = \zeta \int f(x) w_0{}^{q-2} w_0 + o(1).$$

Then substituting (3.23), (3.24) in (3.22) and using Ghoussoub-Yuan's relation

$$\int \frac{|w_0+v_n^+|^{2^*}}{|x|^s} - \int \frac{|w_0|^{p^*(s)}}{|x|^s} = \int \frac{|v_n^+|^{p^*(s)}}{|x|^s} + o(1), \tag{3.25}$$

we have

$$\left\langle \bar{J}'(v_n), w_0+v_n \right\rangle = \int \nabla v_n^2 - \int \frac{\lambda}{|x|^2} v_n^2 - \int_\Omega \frac{(v_n^+)^{p^*(s)}}{|x|^s} + o(1).$$

Thus $\left\langle \bar{J}'(v_n), w_0+v_n \right\rangle \to 0$ as $n \to \infty$. We can assume that exists $d \geq 0$ such that

$$\|v_n\|_\lambda^2 \to d \text{ and } \int \frac{|v_n^+|^{p^*(s)}}{|x|^s} \to d, \text{ when } n \to \infty.$$

If $d \neq 0$, by using the fact, that

$$S_{\lambda,s}\Big(\int \frac{|v_n^+|^{p^*(s)}}{|x|^s}\Big)^{\frac{2}{p^*(s)}} \leq \int \Big(|\nabla v_n|^2 - \frac{\lambda}{|x|^2}|v_n|^2\Big)$$

for all $v_n \in H_0^1(\Omega)$, we obtain that $d \geq S_{\lambda,s}^{N-s/(2-s)}$. Thus

$$\begin{aligned} c &= o(1) + \bar{J}(v_n) \\ &= \frac{1}{2}\|v_n\|_\lambda^2 - \frac{1}{p^*(s)} \int \frac{|v_n^+|^{p^*(s)}}{+|x|^s} + o(1) \\ &> 1/2d - \frac{1}{2^*} d \\ &\geq \frac{2-s}{2(N-s)} S_{\lambda,s}^{N-s/2-s}, \end{aligned}$$

which contradicts the assumption $c < \frac{2-s}{2(N-s)} S_{\lambda,s}^{N-s/(2-s)}$. Therefore $d = 0$ and the proof is complete. $\square$

Let

$$v_\varepsilon(x) \doteq \Psi(x) U_\varepsilon \left(\int \frac{|\Psi(x) U_\varepsilon|^{p^*(s)}}{|x|^s}\right)^{-1/p^*(s)}$$

with $0 \leq \Psi(x) \leq 1$, $\Psi(x) = 1$ for $|x| \leq \rho$, $\Psi(x) = 0$ for $|x| \geq 2\rho$, where ρ is chosen as in Proposition 3.2.7 and $\Psi(x) \in C_0^\infty(\Omega)$.

Lemma 3.2.13. *If (H_4) hold, we have* $\sup_{t\geq 0}\bar{J}(tv_\varepsilon) < \frac{2-s}{2(N-s)} S_{\lambda,s}^{\frac{N-s}{2-s}}$.

Proof. Recall the elementary inequality, for $p > 1$ and $a, b \geq 0$,

$$(a+b)^p \geq a^p + b^p + pa^{p-1}b.$$

Then, we have

$$\begin{aligned} g_\zeta(x, v_\varepsilon) &= \zeta f(x) |w_0 + v_\varepsilon|^{q-2} (w_0 + v_\varepsilon) - \zeta f(x) |w_0|^{q-2} w_0 \\ &\quad + \frac{|w_0 + v_\varepsilon|^{p^*(s)-2} (w_0 + v_\varepsilon)}{|x|^s} - \frac{|w_0|^{p^*(s)-2} w_0}{|x|^s} \\ &\geq \frac{v_\varepsilon^{+p^*(s)-1}}{|x|^s} + (p^*(s)-1) \frac{|w_0|^{p^*(s)-2}}{|x|^s} v_\varepsilon^+ . \end{aligned}$$

and

$$G_\zeta(x, tv_\varepsilon) \geq \frac{t^{p^*(s)}}{p^*(s)} \frac{v_\varepsilon^{+p^*(s)}}{|x|^s} + \frac{(p^*(s)-1)t^2}{2} \frac{|w_0|^{p^*(s)-2}}{|x|^s} \left(v_\varepsilon^+\right)^2 .$$

Since $w_0 \in H_0^1(\Omega)$ is a positive solution of problem $P_2(\lambda, \zeta, q, s, f)$, by the Proposition 3.2.7, we have

$$w_0(x) \geq K_1 |x|^{-\left(\sqrt{\Lambda}-\sqrt{\Lambda-\lambda}\right)}, \quad x \in B_{\rho(0)} \setminus \{0\},$$

for $\rho > 0$ sufficiently small and

$$(p^*(s)-1) \frac{|w_0|^{p^*(s)-2}}{|x|^s} \geq (p^*(s)-1) K_1 \frac{|x|^{-\left(\sqrt{\Lambda}-\sqrt{\Lambda-\lambda}\right)}}{|x|^s} \geq \bar{K_1} > 0.$$

Note that $\int \frac{v_\varepsilon^{p^*(s)}}{|x|^s} = 1$ and

$$\begin{aligned} \bar{J}(tv_\varepsilon) &= \frac{t^2}{2} \|v_\varepsilon\|_\lambda^2 - \int G_\zeta(x, tv_\varepsilon) \\ &\leq \frac{t^2}{2} \|v_\varepsilon\|_\lambda^2 - \int \frac{t^{p^*(s)}}{p^*(s)} \frac{v_\varepsilon^{p^*(s)}}{|x|^s} + \int \frac{(p^*(s)-1)t^2}{2} \frac{|w_0|^{p^*(s)-2} v_\varepsilon^+}{|x|^s} \\ &= \frac{t^2}{2} \|v_\varepsilon\|_\lambda^2 - \frac{t^{p^*(s)}}{p^*(s)} - \frac{\bar{K_1} t^2}{2} \int v_\varepsilon^2 . \end{aligned}$$

Let

$$\beta_\varepsilon(t) = \frac{t^2}{2} \|v_\varepsilon\|_\lambda^2 - \frac{t^{p^*(s)}}{p^*(s)} - \frac{\bar{K_1} t^2}{2} \int v_\varepsilon^2 .$$

Then

$$\beta_\varepsilon'(t) = t \left[\|v_\varepsilon\|_\lambda^2 - t^{p^*(s)-2} - \bar{K_1} \int v_\varepsilon^2 \right].$$

Thus, from $\beta_\varepsilon'(t) = 0$, we have

$$t_\varepsilon = \left[\|v_\varepsilon\|_\lambda^2 - \bar{K_1} \int v_\varepsilon^2 \right]^{\frac{1}{p^*(s)-2}}$$

with

$$\max_{t\geq 0}\beta(t)=\beta(t_\varepsilon).$$

So

$$\begin{aligned}
\bar{J}(tv_\varepsilon) &\leq \beta(t_\varepsilon)\\
&= \frac{1}{2}\left[\|v_\varepsilon\|_\lambda^2-\bar{K}_1\int v_\varepsilon^2\right]^{\frac{2}{p^*(s)-2}}\|v_\varepsilon\|_\lambda^2-\frac{1}{p^*(s)}\left[\|v_\varepsilon\|_\lambda^2-\bar{K}_1\int v_\varepsilon^2\right]^{\frac{p^*(s)}{p^*(s)-2}}\\
&\quad -\frac{1}{2}\left[\|v_\varepsilon\|_\lambda^2-\bar{K}_1\int v_\varepsilon^2\right]^{\frac{2}{p^*(s)-2}}\bar{K}_1\int v_\varepsilon^2\\
&= \frac{1}{2}\left[\|v_\varepsilon\|_\lambda^2-\bar{K}_1\int v_\varepsilon^2\right]^{\frac{2}{p^*(s)-2}}\left[\|v_\varepsilon\|_\lambda^2-\bar{K}_1\int v_\varepsilon^2\right]\\
&\quad -\frac{1}{p^*(s)}\left[\|v_\varepsilon\|_\lambda^2-\bar{K}_1\int v_\varepsilon^2\right]^{\frac{p^*(s)}{p^*(s)-2}}\\
&= \frac{1}{2}\left[\|v_\varepsilon\|_\lambda^2-\bar{K}_1 v_\varepsilon^2\right]^{\frac{p^*(s)}{p^*(s)-2}}-\frac{1}{p^*(s)}\left[\|v_\varepsilon\|_\lambda^2-\bar{K}_1\int v_\varepsilon^2\right]^{\frac{p^*(s)}{p^*(s)-2}}\\
&= \left(\frac{1}{2}-\frac{1}{p^*(s)}\right)\frac{1}{2}\left[\|v_\varepsilon\|_\lambda^2-\bar{K}_1\int v_\varepsilon^2\right]^{\frac{p^*(s)}{p^*(s)-2}}
\end{aligned}$$

and using the estimates from Proposition 3.2.8, we have

$$\begin{aligned}
\bar{J}(tv_\varepsilon) &= \left(\frac{1}{2}-\frac{1}{p^*(s)}\right)\frac{1}{2}\left[\|v_\varepsilon\|_\lambda^2-\bar{K}_1\int v_\varepsilon^2\right]^{\frac{p^*(s)}{p^*(s)-2}}\\
&= \left(\frac{1}{2}-\frac{1}{p^*(s)}\right)S^{\frac{N-s}{2-s}}+O(\varepsilon^{\frac{N-s}{2-s}})O(\varepsilon^{\frac{(N-q\sqrt{\Lambda})\sqrt{\Lambda}}{2-s\sqrt{\Lambda-\lambda}}}).
\end{aligned}$$

If $\frac{N}{\sqrt{\Lambda}+\sqrt{\Lambda-\lambda}}<q<2$, we get

$$\sup_{t\geq 0}\bar{J}(tv_\varepsilon)<\frac{2-s}{2(N-s)}S^{\frac{N-s}{2-s}}.$$

□

Consider the following minimax value

$$\bar{c}\doteq\inf_{\gamma\in\Gamma}\sup_{0\leq t\leq 1}\bar{J}(\gamma(t)),$$

where

$$\Gamma\doteq\left\{\gamma\in C([0,1],\ H_0^1(\Omega)):\gamma(0)=0,\ \gamma(1)=kv_\varepsilon\right\}$$

with suitable ε and k.

Proposition 3.2.14. *If (H_4) hold, the minimum $c_{0,f}$ and $c_{1,f}$ are achieved by w_0 and w_1 respectively. Moreover, w_0 and w_1 are positive solutions of $P_2(\lambda, \zeta, q, s, f)$.*

Proof. From Lemma 3.2.9 J achieves its minimum $c_{0,f}$ at w_0 and from Remark 3.2.10, w_0 is a positive solution of $P_2(\lambda, \zeta, q, s, f)$. From Lemma 3.2.11, $v \equiv 0$ is a local minimizer of $\bar{J}$, then there exists a sufficiently small positive number $\overline{\rho}$ such that $\bar{J}(v) > 0$ for $\|v\|_\lambda = \overline{\rho}$.

Since $\bar{J}(tv_\varepsilon) \to -\infty$ as $t \to \infty$, then there exists $T > 0$ such that $\|Tv_\varepsilon\|_\lambda > \overline{\rho} > 0$ and $\bar{J}(tv_\varepsilon) < 0$. For $\bar{c} < \frac{2-s}{2(N-s)} S_{\lambda,s}^{\frac{N-s}{2-s}}$, $(PS)_c$-condition is satisfied by (3.2.12), then we conclude by (3.2.13) that

$$\bar{c} \le \sup_{t\ge0} \bar{J}(Tv_\varepsilon) \le \sup_{t\ge0} \bar{J}(tv_\varepsilon) < \frac{2-s}{2(N-s)} S_{\lambda,s}^{\frac{N-s}{2-s}}.$$

Hence applying the mountain pass theorem whenever $\bar{c} > 0$ and the Ghoussoub-Preiss version whenever $\bar{c} = 0$ (see Ghoussoub-Preiss [61]), we obtain a nontrivial critical point v of $\bar{J}$. Set $w_1 = w_0 + v^+$, then w_1 is a critical point of J and $w_1 > w_0 > 0$ in Ω. □

3.2.4 Existence of sign-changing solutions

We define two subsets of M^- as

$$M_1{}^- \doteq \{u \in M : u^+ \in M^-\} \quad \text{and} \quad M_2{}^- \doteq \{u \in M : -u^- \in M^-\},$$

where $u^+ \doteq \max\{0, u\}$, $u^- \doteq \max\{0, -u\}$ and $u = u^+ - u^-$. Set $M_*^- \doteq M_1^- \cap M_2^-$ and

$$c_2 \doteq \inf_{u \in M_*^-} J(u). \tag{3.26}$$

We prove that c_2 is achieved by some $w_2 \in M_*^-$ which must be a sign-changing solution of problem $P_2(\lambda, \zeta, q, s, f)$. Since the associated functional of this problem is odd with respect to u, we have that $-w_2$ is also a sign-changing solution. In order to solve the minimization problem (3.26), we combine some ideas from Tarantello [119] and the methods recently developed in Castro-Cossio-Neuberger [24], Chen-Rocha [42] and Hirano-Shioji [71].

Lemma 3.2.15. *If (H_4) hold,*

$$c_2 < c_{1,f} + \frac{2-s}{2(N-s)} S_{\lambda,s}^{\frac{N-s}{2-s}}.$$

Proof. In the first place, we prove that $M_*^- \ne \emptyset$. To see this it suffices to prove that there is r_0 and τ_0 such that

$$r_0(u_\varepsilon - r_0 w_1)^+ \in M^- \text{and} \quad -r_0(u_\varepsilon - r_0 w_1)^- \in M^-, \tag{3.27}$$

where w_1 is a positive solution of $P_2(\lambda, \zeta, q, s, f)$ with $J(w_1) = c_{1,f}$.

Denote

$$\tau_2 = \max_{\bar{\Omega}\setminus\{0\}} \frac{u_\varepsilon}{w_1} \quad \text{and} \quad \tau_1 = \min_{\bar{\Omega}\setminus\{0\}} \frac{u_\varepsilon}{w_1}. \tag{3.28}$$

Then, from Proposition 3.2.7, τ_1 and τ_2 are finite. For any given $\tau \in (\tau_1, \tau_2)$, we obtain from Lemma 3.2.3 that there are positive values $r_+(\tau)$ and $r_-(\tau)$ such that

$$r_+(\tau)(u_\varepsilon - \tau w_1)^+ \in M^- \quad \text{and} \quad -r_-(\tau)(u_\varepsilon - \tau w_1)^- \in M^-. \tag{3.29}$$

Note that r_+ is continuous with respect to τ and satisfies

$$lim_{\tau\to\tau_1^+} r_+(\tau) = t^+(u_\varepsilon - \tau_1 w_1)^+ < +\infty \quad \text{and} \quad lim_{\tau\to\tau_2^-} r_+(\tau) = +\infty. \tag{3.30}$$

Similarly, r_- is continuous with respect to τ,

$$lim_{\tau\to\tau_1^+} r_-(\tau) = +\infty \quad \text{and} \quad lim_{\tau\to\tau_2^-} r_-(\tau) = t^+(u_\varepsilon - \tau_2 w_1)^+ < +\infty. \tag{3.31}$$

The continuity of $r_\pm(\tau)$ imply that there is $\tau_0 \in (\tau_1, \tau_2)$ such that

$$r_+(\tau_0) = r_-(\tau_0) = \tau_0 > 0.$$

Therefore $M_*^- \neq \emptyset$. In the second place, we estimate c_2. From the previous proof, we only need to estimate $J(ru_\varepsilon - tw_1)$ for $r \geq 0$ and $t \in \mathbb{R}$. By the structure of J, we find $R_1 > 0$ large enough such that $J(ru_\varepsilon - tw_1) \leq c_1$ for all $r^2 + t^2 \geq R_1^2$. Thus it suffices to estimate $J(ru_\varepsilon - tw_1)$ for all $r^2 + t^2 \leq R_1^2$. Recalling the elementary inequality

$$|a_1 + a_2|^m \geq |a_1|^m + |a_2|^m - K(|a_1|^{m-1}|a_2| + |a_1||a_2|^{m-1}), \ \forall a_1, a_2 \in \mathbb{R}, \ m > 1$$

we have from Proposition 3.2.8 and the assumption on q that

$$\begin{aligned}
J(ru_\varepsilon - tw_1) \ &\leq \ J(ru_\varepsilon) + J(tw_1) - rt\zeta_1 \int w_1 u_\varepsilon^{q-1} - rt \int \frac{u_\varepsilon^{p^*(s)-1} w_1}{|x|^s} + \\
&\quad +K \int \frac{|ru_\varepsilon|^{p^*(s)-1}|tw_1|}{|x|^s} + K \int \frac{|ru_\varepsilon||tw_1|^{p^*(s)-1}}{|x|^s} + \\
&\quad +K \int |rw_1|^{q-1}|tu_\varepsilon| + K \int |tu_\varepsilon|^{q-1}|rw_1| \\
&\leq \ J(rw_1) + J(tu_\varepsilon) + O\left(\varepsilon^{\frac{\sqrt{\Lambda}}{2-s}}\right) + O\left(\varepsilon^{\frac{(q-1)\sqrt{\Lambda}}{2-s}}\right) \\
&= \ J(rw_1) + J(tu_\varepsilon) + O\left(\varepsilon^{\frac{(q-1)\sqrt{\Lambda}}{2-s}}\right).
\end{aligned}$$

Writing $\varphi(r) \doteq J(ru_\varepsilon) + \frac{\zeta}{q}\int f(x)|ru_\varepsilon|^q$, we have that

$$\varphi(r) = \frac{r^2}{2} \int \left(|\nabla u_\varepsilon|^2 - \frac{\lambda}{|x|^2} u_\varepsilon^2\right) dx - \frac{r^{p^*(s)}}{2^{p^*}(s)} \int \frac{u_\varepsilon^{p^*}(s)}{|x|^s} dx$$

attains its maximum at $T_{max} = \left(\|u_\varepsilon\|_\lambda^2 / \int \frac{u_\varepsilon^{p^*(s)}}{|x|^s} \right)^{\frac{1}{p^*(s)-2}}$ and there is $T^+ > T_{max}$ such that $\frac{\partial J}{\partial r}(T^+ u_\varepsilon) = 0$. It follows from (3.12) that

$$\begin{aligned} \max_{r>0} J(ru_\varepsilon) &\leq \varphi(T_{\max}) - \frac{T_{\max}^q}{q} K_4 \int |u_\varepsilon|^q \\ &\leq \frac{2-s}{2(N-s)} S_{\lambda,s}{}^{\frac{N-s}{2-s}} - K_4 \int |u_\varepsilon|^q \\ &\leq \frac{2-s}{2(N-s)} S_{\lambda,s}{}^{\frac{N-s}{2-s}} - O\left(\varepsilon^{\frac{(N-q\sqrt{\Lambda})\sqrt{\Lambda}}{(2-s)\sqrt{\Lambda-\lambda}}} \right). \end{aligned}$$

In here we have used the assumption on q and the integral estimates in Proposition 3.2.8 to compare the error order of ε. Thus we can say that, for $\varepsilon > 0$ sufficiently small,

$$\begin{aligned} &\max_{r>0, t\in\mathbb{R}} J(ru_\varepsilon - tw_1) \\ \leq\ & \max_{r>0} J(ru_\varepsilon) + \max_{t\in\mathbb{R}} J(tw_1) + O\left(\varepsilon^{\frac{(q-1)\sqrt{\Lambda}}{2-s}} \right) - O\left(\varepsilon^{\frac{(N-q\sqrt{\Lambda})\sqrt{\Lambda}}{(2-s)\sqrt{\Lambda-\lambda}}} \right) \\ \leq\ & c_{1,f} + \frac{2-s}{2(N-s)} S_{\lambda,s}{}^{\frac{N-s}{2-s}} \left(\text{since } \frac{N+\sqrt{\Lambda-\lambda}}{\sqrt{\Lambda}+\sqrt{\Lambda-\lambda}} < q < 2 \right). \end{aligned}$$

The proof is complete. □

Proposition 3.2.16. *If (H_4) hold and $\zeta \in (0, \Lambda^*)$, then there is $w_2 \in M_*^-$ such that $J(w_2) = c_2$ and w_2 is a sign-changing solution of problem $P_2(\lambda, \zeta, q, s, f)$.*

Proof. In the first step, we prove that there is $w_2 \in M_*^-$ such that $J(w_2) = c_2$. Let $(u_n)_{n\in\mathbb{N}} \subset M_*^-$ be such that $J(u_n) \to c_2$. Using the fact that $(u_n^+)_{n\in\mathbb{N}} \subset M^-$ and by the Hardy-Sobolev inequality, one has

$$0 < \inf \|u_n^+\|_\lambda \leq \sup \|u_n^+\|_\lambda < +\infty.$$

Similarly, we have $\|u_n^-\|_\lambda$ is bounded with respect to n. Going if necessary to a subsequence, we may assume that $u_n^+ \rightharpoonup u^+$ and $u_n^- \rightharpoonup u^-$ in $H_0^1(\Omega)$ and that $J(u_n^+) \to d_1$, $J(u_n^-) \to d_2$ with $c_2 = d_1 + d_2$.
We claim that $u^+ \not\equiv 0$ and $u^- \not\equiv 0$. By Proposition 3.2.5, we have that

(a) If $u^+ = 0$ and $u^- = 0$, then

$$d_1 \geq \frac{2-s}{2(N-s)} S_{\lambda,s}^{\frac{N-s}{2-s}},\ d_2 \geq \frac{2-s}{2(N-s)} S_{\lambda,s}^{\frac{N-s}{2-s}} \text{ and hence } c_2 \geq \frac{2-s}{2(N-s)} S_{\lambda,s}^{\frac{N-s}{2-s}};$$

(b) If $u^+ = 0$ and $u^- \neq 0$, then

$$d_1 \geq \frac{2-s}{2(N-s)} S_{\lambda,s}^{\frac{N-s}{2-s}},\ d_2 \geq c_{1,f} \text{ or } d_2 \geq c_{0,f} + \frac{2-s}{2(N-s)} S_{\lambda,s}^{\frac{N-s}{2-s}},$$

which implies that

$$c_2 \geq c_{1,f} + \frac{2-s}{2(N-s)} S_{\lambda,s}^{\frac{N-s}{2-s}} \text{ or } c_2 \geq c_{0,f} + \frac{2-s}{2(N-s)} S_{\lambda,s}^{\frac{N-s}{2-s}};$$

(c) If $u^+ \neq 0$ and $u^- = 0$, then

$$c_2 \geq c_{1,f} + \frac{2-s}{2(N-s)} S_{\lambda,s}^{\frac{N-s}{2-s}} \text{ or } c_2 \geq c_{0,f} + \frac{2-s}{2(N-s)} S_{\lambda,s}^{\frac{N-s}{2-s}}.$$

All the above three cases contradict (3.5) and the Lemma 3.2.15. Therefore $u^+ \not\equiv 0$ and $u^- \not\equiv 0$. According to (1) and (2) of Proposition 3.2.5, we have one of the following:

(i) $(u_n^+)_{n\in\mathbb{N}}$ converges strongly to u^+;

(ii) $d_1 > J(t_+(u^+)u^+)$;

(iii) $d_1 > J(t_-(u^+)u^+) + \frac{2-s}{2(N-s)} S_{\lambda,s}^{\frac{N-s}{2-s}}$;

and we also have one of the following:

(iv) $(u_n^-)_{n\in\mathbb{N}}$ converges strongly to u^-;

(v) $d_2 > J(-t_+(-u^-)u^-)$;

(vi) $d_2 > J(-t_-(-u^-)u^-) + \frac{2-s}{2(N-s)} S_{\lambda,s}^{\frac{N-s}{2-s}}$.

We will prove that only cases (i) and (iv) hold. For example, in the situation $(ii)+(v)$, we have

$$t_+(u^+)u^+ - t_+(-u^-)u^- \in M_*^-$$

and, hence

$$\begin{aligned} c_2 &\leq J(t_+(u^+)u^+ - t_+(-u^-)u^-) \\ &= J(t_+(u^+)u^+) + J(-t_+(-u^-)u^-) \\ &\leq d_1 + d_2 = c_2. \end{aligned}$$

which is a contradiction. Case $(iii) + (vi)$, we have $t_-(u^+)u^+ - t_-(-u^-)u^- \in M^+$ and hence

$$\begin{aligned} c_{1,f} + \frac{2-s}{2(N-s)} S_{\lambda,s}{}^{\frac{N-s}{2-s}} &< c_{0,f} + \frac{2-s}{2(N-s)} S_{\lambda,s}{}^{\frac{N-s}{2-s}} \\ &\leq J(t_-(u^+)u^+ - t_-(-u^-)u^-) + \frac{2-s}{2(N-s)} S_{\lambda,s}{}^{\frac{N-s}{2-s}} \\ &= J(t_-(u^+)u^+) + J(t_-(u^-)u^-) + \frac{2-s}{2(N-s)} S_{\lambda,s}{}^{\frac{N-s}{2-s}} \\ &\leq d_1 + d_2 = c_2, \end{aligned}$$

which contradicts Lemma 3.2.15. Case $(ii) + (vi)$, we have
$t_+(u^+)u^+ - t_-(-u^-)u^- \in M^-$ and hence

$$c_{1,f} + \frac{2-s}{2(N-s)} S^{\frac{N-s}{2-s}} \leq J(t_+(u^+)u^+ + t_-(u^-)u^-) + \frac{2-s}{2(N-s)} S_{\lambda,s}{}^{\frac{N-s}{2-s}} < d_1 + d_2 = c_2,$$

which again contradicts Lemma 3.2.15. If (i) and (v) hold, then $u^+ - t_+(-u^-)u^- \in M_*^-$ and hence

$$c_2 \leq J(u^+ - t_+(-u^-)u^-) < d_1 + d_2 = c_2,$$

which is also a contradiction. For other situations (i)+(vi), (ii)+(iv), (iii)+(v), (iii)+(iv), we can get a contradiction by a similar argument. Therefore we proved that only $(i)+(iv)$ hold. Hence both $(u_n^+)_{n\in\mathbb{N}}$ and $(u_n^-)_{n\in\mathbb{N}}$ converge strongly to u^+ and u^-, respectively and $u^+,\ u^- \in M^-$. Denote $w_2 = u^+ - u^-$. Therefore, $J(w_2) = c_2$.
Next we show that w_2 is a critical point of J. Suppose that w_2 is not a critical point of J, i.e. $\nabla J(w_2) \neq 0$. Denote

$$Q(u) \doteq \|u\|_\lambda^2 - \zeta \int f(x)\,|u|^q - \int \frac{u^{p^*(s)}}{|x|^s}.$$

Note that for $u \in M^-$, we have

$$\langle \nabla Q(u), u\rangle = (2-q)\,\|u\|_\lambda^2 - (p^*(s) - q) \int \frac{u^{p^*(s)}}{|x|^s} < 0.$$

Hence, we can define

$$V(u) \doteq \nabla J(u) - \left\langle \nabla J(u), \frac{\nabla Q(u)}{\|\nabla Q(u)\|_\lambda} \right\rangle \frac{\nabla Q(u)}{\|\nabla Q(u)\|_\lambda}, \qquad u \in M^-,$$

Choose $\delta \in (0, \frac{1}{3}\min\{\|u^+\|_\lambda, \|u^-\|_\lambda\})$ such that $\|V(v) - V(w_2)\|_\lambda \leq \frac{1}{2}\|V(w_2)\|_\lambda$ for each $v \in M^-$ with $\|v - w_2\|_\lambda \leq 2\delta$. Let $\psi :\ M^- \to [0,1]$ be a Lipschitz mapping such that

$$\psi(v) = \begin{cases} 1 & \text{for } v \in M^- \quad \text{with} \quad \|v - w_2\|_\lambda \leq \delta, \\ 0 & \text{for } v \in M^- \quad \text{with} \quad \|v - w_2\|_\lambda \geq 2\delta, \end{cases}$$

Let $\eta : [0, s_0] \times M^- \to \mathbb{R}$ be the solution of the differential equation Cauchy problem

$$\eta(0, v) = v, \quad \frac{d}{ds}\eta(s, v) = -\psi(\eta(s, v)))V(\eta(s, v)) \tag{3.32}$$

for some positive number s_0 and $(s, v) \in [0, s_0] \times M^-$. We set

$$\chi(t) \doteq t_+((1-t)u^+ - tu^-)((1-t)u^+ - tu^-) \quad \text{and} \quad \xi(t) \doteq \eta(s_0, \chi(t)),$$

for $0 \leq t \leq 1$. Keep the definition of $u^+ = max\{u, 0\}$ and $u^- = max\{-u, 0\}$ in mind. We have that if $t \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ then

$$\begin{aligned} J(\xi(t)) &= J(\eta(s_0, \chi(t))) \\ &\leq J(\eta(0, \chi(t))) \\ &= J(\chi(t)) = J(\chi(t)^+) + J(\chi(t)^-) < J(u^+) + J(u^-) = J(w_2). \end{aligned}$$

and

$$J\left(\xi(\frac{1}{2})\right) < J\left(\chi(\frac{1}{2})) = J(w_2)\right).$$

Therefore $J(\xi(t)) < J(w_2)$ for $t \in (0,1)$.
Since

$$t_+(\xi(t)^+) - t_+(-\xi(t)^-) = \eta(s_0, t_+(\chi(t)^+) - t_+(-\chi(t)^-)) \to -\infty$$

as $t \to 0$ from the right hand side and

$$t_+(\xi(t)^+) - t_+(-\xi(t)^-) = \eta(s_0, t_+(\chi(t)^+) - t_+(-\chi(t)^-)) \to \infty$$

as $t \to 1-0$, we get a $t_1 \in (0,1)$ such that $t_+(\xi(t_1)^+) = t_+(-\xi(t_1)^-)$. So

$\xi(t_1) = \xi(t_1)^+ - \xi(t_1)^- \in M_*^-$ and $J(\xi(t_1)) < J(w_2)$, which is a contradiction. Hence, it is true that $\nabla J(w_2) = 0$. □

We are now ready for the multiplicity theorem of problem $P_2(\lambda, \zeta, q, s, f)$.

Theorem 3.2.17. *If (H_4) hold, then $P_2(\lambda, \zeta, q, s, f)$ has at least two positive solutions and at least one pair of sign-changing solutions in $H_0^1(\Omega)$ for $\zeta \in (0, \Lambda^*)$.*

Proof. By Proposition 3.2.14, we know that problem $P_2(\lambda, \zeta, q, s, f)$ has two positive solutions w_0 and w_1. It is deduced from Proposition 3.2.16 that $P_2(\lambda, \zeta, q, s, f)$ possesses a sign-changing solution w_2. Since $P_2(\lambda, \zeta, q, s, f)$ is odd with respect to u, we know that $-w_2$ is an additional sign-changing solution of $P_2(\lambda, \zeta, q, s, f)$. □

Chapter 4

Existence of solutions for a class of singular equations in Lorentz space

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and satisfying the uniform exterior sphere condition. In this chapter, we study the existence of solutions $u \in W_0^{1,p}(\Omega)$ for the Dirichlet nonlinear problem $P_3(\psi, a, f)$:

$$\begin{cases} -div(\psi(x, u(x), \nabla u(x))) + a(x)u(x) &= f(x) \quad \text{in} \quad \Omega, \\ u(x) &= 0 \quad\quad \text{on} \quad \partial\Omega, \end{cases} \tag{4.1}$$

where $2 \leq p < N$, $\Psi(u) = -div(\psi(x, u(x), \nabla u(x)))$ is a Leray-Lions operator, $a \in L^\infty_{loc}(\Omega)$ with $a(x) \geq 0$ for all $x \in \Omega$, and $f \in L^{q,q_1}(\Omega)$ is a function in a Lorentz space with suitable exponents q and q_1.

As we have mention before, Problem $P_3(\psi, a, f)$ has a difficulty since $a \in L^\infty_{loc}(\Omega)$, so the standard definition of weak solution may not make sense (i.e. with test functions in $W_0^{1,p}(\Omega)$). Therefore, it is necessary to introduce a special notion of weak solution involving open subsets of Ω. Moreover, we study an approximation problem $P(\Omega_n)$, where $\Omega_n \subset \Omega$ is suitably defined for each $n \in \mathbb{N}$. Then, we prove that there exists a solution of $P(\Omega_n)$ for each $n \in \mathbb{N}$ and that the sequence of solutions converges to the solution of problem $P_3(\psi, a, f)$.

Our approach combines a surjectivity result for monotone, coercive and radially continuous operators with special properties of Leray-Lions operators. In this chapter, we prove that if $f \in L^{q,q_1}(\Omega)$ $(q < q_1)$, then there exists (at least) one solution u in the space $W_0^{1,p}(\Omega) \cap L^{r,s}(\Omega)$ with suitable exponents r and s. Moreover we find an estimate for the solution. We also prove the uniqueness of the solution under some conditions.

The structure of this chapter is the following: In the Section 4.1, we present relevant results of particular cases of our problem. In the Section 4.2, we prove the existence of solution for the problem with a linear operator. In the Section 4.3, we prove an existence result for the nonlinear case, in three steps: existence, uniqueness and estimate for the solution. The results obtained in this chapter are related to the work of Huang-Murillo-Rocha in [73].

4.1 Previous results

The commom framework for elliptic problems are Sobolev spaces. Problems with terms defined in Lorentz spaces are considerably less common, mainly because the use of non-increasing rearrangements in their definition limits the application of several standard techniques. However the embedding of the Sobolev space $W_0^{1,p}(\Omega)$ into a Lorentz space improves the standard Sobolev embedding into a Lebesgue space. So in some sense, the results on elliptic equations and system may be improved using Lorentz spaces.

To continuation, we present some interesting results in Lorentz spaces. Consider the degenerate linear version of problem $P_3(\psi, a, f)$ without singularity, i.e. $a \equiv 0$ and the Leray-Lions operator $\psi(x,\xi) = M(x)\xi$, where M is a symmetric matrix in $L^\infty(\Omega)^{N\times N}$ satisfying the ellipticity condition, i.e. there exists $\alpha > 0$ such that for $x \in \Omega$ and $\xi \in \mathbb{R}^N$

$$M(x)\xi \cdot \xi \geq \alpha|\xi|^2.$$

Let $\Omega \subset \mathbb{R}^N$ $(N > 2)$ be a bounded domain with smooth boundary. Napoli and Mariani [91] proved the existence of a unique solution in $H_0^1(\Omega) \cap L^{r,s}(\Omega)$ of the problem

$$\begin{cases} -div(\psi(x, \nabla u(x))) &= divF(x) \quad \text{in} \quad \Omega, \\ u(x) &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{4.2}$$

with suitable exponents r and s. If $F \in (L^2(\Omega))^N$, using the Lax-Milgram lemma they obtained the existence of a unique solution in $H_0^1(\Omega)$ for the problem (4.2). Moreover for $F \in L^q(\Omega)$, with $q > 2$ used the Stampacchia argument (Theorem 4.2 in Stampacchia [112]) to improve summability.

Consider the nonlinear problem

$$\begin{cases} -div(\psi(x, u(x), \nabla u(x))) &= divF(x) \quad \text{in} \quad \Omega, \\ u(x) &= 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{4.3}$$

For $F \in L^{q,q^\sharp}(\Omega)$ for some q and $q^\sharp$, Napoli and Mariani [91], proved that there exists a unique solution of the problem (4.3) in $W_0^{1,p}(\Omega) \cap L^{\bar{r},\bar{s}}(\Omega)$ for suitable exponents $\bar{r}$ and $\bar{s}$.

For the particular set of equations

$$(a_{i,j}(x)u_{x_i})_{x_j} = (f_i)_{x_i}, \text{ in } \Omega$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is an open set and the functions $a_{i,j}(x)$ ($i, j = 1, ..., n$) are bounded measurable, and satisfy the ellipticity condition, Karch-Ricciardi [79], showed that weak solutions are differentiable almost everywhere when $u \in H^1_{loc}(\Omega)$ and $\frac{\partial}{\partial x_i} f \in L^{n,1}_{loc}(\Omega)$. The space $L^{n,1}_{loc}(\Omega)$ is the local version of Lorentz space, consisting of all measurable functions $g \in \Omega$ such that $g_{\chi_A} \in L^{p,q}(\Omega)$ for each compact set $A \in \Omega$.

For other type of singularities, we mention the work of Giachetti-Segura de Leon [63] in Sobolev spaces, in which they obtained for a problem involving a Leray-Lions operator plus the term

$$\frac{sig(u-1)}{|u-1|^K}|\nabla u|^2 - f,$$

the existence of a weak solution $u \in H^1_0(\Omega)$ when $f \in L^m(\Omega)$, with $m \geq \frac{2N}{N+2}$. By using Stampacchia theorem, Giachetti-Segura de Leon showed that the gradient of u goes to zero faster than $|u-1|^K$ so, in fact, the term does not blow-up.

4.2 The linear case

In this section, we study the problem $P_3(\psi, a, f)$ considering a linear operator instead of the Leray-Lions operator Ψ. First, we introduce a geometric condition on Ω.

Definition 4.2.1. *We say that $\Omega \subset \mathbb{R}^N$ satisfy the uniform exterior sphere condition, if there exists a real number $r > 0$, such that for each $z \in \partial\Omega$ there exists a close ball $\bar{B}$ of radius r with $\bar{B} \cap \bar{\Omega} = \{z\}$.*

Remark 4.2.2. *Any open bounded set C^2 contained in $\mathbb{R}^N$, satisfies the uniform exterior sphere condition.*

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and satisfying the uniform exterior sphere condition. Here, we study the existence of solutions $u \in H^1_0(\Omega)$ that satisfies the problem $P_3(M, a, f)$:

$$\begin{cases} -div(M(x)\nabla u(x)) + a(x)u(x) &= f(x) \quad \text{in} \quad \Omega, \\ u(x) &= 0 \quad\quad \text{on} \quad \partial\Omega, \end{cases} \tag{4.4}$$

where $a \in L^\infty_{loc}(\Omega)$ is such that $a(x) \geq 0$ for all $x \in \Omega$ and $M(x)$ is a symmetric matrix in $L^\infty(\Omega)^{N\times N}$ satisfying the ellipticity condition

$$M(x)\xi \cdot \xi \geq \alpha|\xi|^2,$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^N (\alpha > 0)$.

As described before, since $a \in L^\infty_{loc}(\Omega)$, we need a special notion of solution of the problem.

Lemma 4.2.3. *Let $\Omega \subset \mathbb{R}^N$ be a open bounded set that satisfies the uniform exterior sphere condition, then there exists $(\Omega_m)_{m\in\mathbb{N}}$ of open sets such that $\overline{\Omega}_m \subseteq \Omega_{m+1} \subseteq \Omega$, $\Omega = \bigcup_{m=1}^{\infty} \Omega_m$ and the boundary $\partial\Omega$ is a smooth subvariety C^∞ of dimension $N-1$ for $m \geq 1$.*

From the Lemma 4.2.3, we can consider $(\Omega_n)_{n\in\mathbb{N}}$ an increasing sequence of open subsets of Ω, such that

$$\overline{\Omega}_n \subseteq \Omega_{n+1} \text{ and } \Omega = \bigcup_{n=1}^{\infty} \Omega_n.$$

Definition 4.2.4. *The weak formulation of problem (4.4) is: find $u \in H^1_0(\Omega)$ with*

$$\int_\Omega M(x)\nabla u \nabla \varphi + a(x)u\varphi dx = \int_\Omega f\varphi dx, \tag{4.5}$$

for all $\varphi \in \bigcup_{n=1}^{\infty} H^1_0(\Omega_n)$.

Remark 4.2.5. *The first integral in (4.5) has sense, since $u, \varphi \in H^1_0(\Omega)$ and the second integral has sense when $f \in L^2(\Omega)$. Note that, if $2 < q < \frac{N}{2}$ and $q < q_1$, from the Lorentz scale (Lemma A.2.2), we have $L^{q,q_1}(\Omega) \subset L^2(\Omega)$.*

4.2.1 Existence of the solution

To prove the existence of solution of problem $P_3(M, a, f)$, we apply the Lemma 1.2.11 due to An et al. [8].

Proposition 4.2.6. *Let $N > 4$, $2 \leq q < \frac{N}{2}$, $\sigma = (N-2q)^{-1}$, $\mu_2 = \sigma(N-2)q$ and $a \in L^\infty_{loc}(\Omega; \mathbb{R}^+_0)$. If $f \in L^{q,\mu_2}(\Omega)$ then there exists (at least) one solution $u \in H^1_0(\Omega)$ for the problem $P_3(M, a, f)$.*

Proof. Let $(\Omega_n)_{n\in\mathbb{N}}$ be an increasing sequence of open subsets of Ω, such that $\overline{\Omega}_n \subseteq \Omega_{n+1}$ and $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$. We consider

$$X \doteq H^1_0(\Omega) \text{ and } X_n \doteq H^1_0(\Omega_n).$$

Note that we can consider each X_n as a closed subspace of X by extending its elements by zero outside Ω_n. Let $V \doteq \bigcup_{n=1}^{\infty} X_n$.
Let $A : H^1_0(\Omega) \times V \to \mathbb{R}$ be the bilinear map defined by

$$A(u, \varphi) \doteq \int_\Omega M(x)\nabla u \nabla \varphi + a(x)u\varphi dx,$$

for all $u \in H_0^1(\Omega)$ and $\varphi \in V$.
Let $A_n = X_n \times X_n \to \mathbb{R}$ be defined by

$$A_n(u,\varphi) = A|_{X_n \times X_n}(u,\varphi).$$

Then we have:

(a) A_n is a bounded bilinear form for all $n \in \mathbb{N}$

In fact, note firstly that, since $a \in L^\infty_{loc}(\Omega)$ it follows that $a \in L^\infty(\Omega_n)$ and there exists a constant $\bar{c_a}$ such that

$$\underset{x\in\Omega_n}{ess\ sup}\, |a(x)| \leq \bar{c_a}.$$

Thus

$$\begin{aligned}
\int_{\Omega_n} (M(x)\nabla u \nabla\varphi + a(x)u\varphi)dx &\leq \int_{\Omega_n} |(M(x)\nabla u\nabla\varphi + a(x)u\varphi)|dx \\
&\leq c_M \int_{\Omega_n} |\nabla u||\nabla\varphi|dx + \int_{\Omega_n} |a(x)||u\varphi|dx \\
&\leq c_M \left(\int_{\Omega_n} |\nabla u|^2 dx\right)^{1/2} \left(\int_{\Omega_n} |\nabla\varphi|^2 dx\right)^{1/2} + \bar{c_a}\int_{\Omega_n} |u\varphi|dx \\
&\leq c_M \|\nabla u\|_{L^2(\Omega_n)} \|\nabla\varphi\|_{L^2(\Omega_n)} + \bar{c_a}\|u\|\|\varphi\|.
\end{aligned}$$

By the Poincaré inequality, $\|\nabla u\|_{L^2(\Omega_n)}$ is equivalent to the norm of $H_0^1(\Omega_n)$. Then

$$\int_{\Omega_n} M(x)\nabla u\nabla\varphi + a(x)u\varphi dx \leq \bar{c}_M \|u\|\|\varphi\|,$$

where $\bar{c}_M = c_M + \bar{c_a}$.

(b) $A(\cdot,\varphi)$ is a bounded linear functional on X, for all $\varphi \in V$.

For any $\varphi \in V$, there exists some $n_0 \in \mathbb{N}$ such that $\varphi \in X_{n_0} \equiv H_0^1(\Omega_{n_0})$ and

$$\begin{aligned}
A(u,\varphi) &= \int_\Omega M(x)\nabla u\nabla\varphi + a(x)u\varphi dx \\
&= \int_{\Omega_{n_0}} M(x)\nabla u\nabla\varphi + a(x)u\varphi dx.
\end{aligned}$$

Using the idea in (a), we can get that $A(\cdot,\varphi)$ is a bounded linear functional on X.

(c) A is coercive. In fact,

$$A(u,u) = \int_\Omega M(x)\nabla u\cdot\nabla u + a(x)u^2 dx \geq \alpha \int_\Omega |\nabla u|^2 dx \geq \alpha\|u\|^2.$$

We have verified all the hypotheses of Theorem 1.2.11, so if $F \in V^*$ is defined by $F(\varphi) = \int_\Omega f\varphi dx$, there exists $u \in H_0^1(\Omega)$ such that

$$A(u,\varphi) = F(\varphi)$$

for all $\varphi \in V$. Thus u satisfies the problem $P_3(M, a, f)$. □

4.3 The nonlinear case

The hypotheses (H_5^{loc}) that we consider here are:

(i) $\Psi(u) = -div(\psi(x, u(x), \nabla u(x)))$ is a Leray-Lions operator;

(ii) $a \in L^\infty_{loc}(\Omega; \mathbb{R}_0^+)$;

(iii) f is a function defined in the Lorentz space $L^{q,q_1}(\Omega)$, where $p' \leq q < \frac{N}{p}$ and $q_1 = \sigma(N-p)q$, with $\sigma = (N-pq)^{-1}$, $2 \leq p < N$ and $p' = \frac{p}{p-1}$.

The hypotheses (H_5) are the same as (H_5^{loc}) when, in *(ii)*, we replace $a \in L^\infty_{loc}(\Omega; \mathbb{R}_0^+)$ by $a \in L^\infty(\Omega; \mathbb{R}_0^+)$.

Problem $P_3(\psi, a, f)$ is well defined since (by standard arguments) the left-hand side of $P_3(\psi, a, f)$ is in $W^{-1,p'}(\Omega)$ and, for $f \in L^{q,q_1}(\Omega)$, $f \in W^{-1,p'}(\Omega)$. In fact, since $p' \leq q \leq q_1$, from the Lorentz spaces scale (see Lemma A.2.2), we have $L^{q,q_1}(\Omega) \subset L^{\bar{q},p'}(\Omega)$, for any $\bar{q} < q$, so for $\bar{q} = p'$ we get

$$f \in L^{q,q_1}(\Omega) \subset L^{p',p'}(\Omega) \equiv L^{p'}(\Omega) \equiv (L^p(\Omega))^* \subset \left(W_0^{1,p}(\Omega)\right)^* \equiv W^{-1,p'}(\Omega).$$

Definition 4.3.1. *(weak solution) We use the following notion of solution.*
We say that $u \in W_0^{1,p}(\Omega)$ is a weak solution *of problem $P_3(\psi, a, f)$ if satisfies*

$$\int_\Omega (\psi(x, u(x), \nabla u(x)) \nabla\varphi + a(x) u\varphi)\, dx = \int_\Omega f\varphi dx \tag{4.6}$$

for all $\varphi \in \bigcup_{n=1}^{\infty} W_0^{1,p}(\Omega_n)$.

Recall that by problem (P_{Ω_n}) we mean the same problem as $P_3(\psi, a, f)$ but defined on Ω_n.

4.3.1 Existence of solution

The following proof is motivated by the Lemma 1.2.12 due to Drivaliaris-Yannakakis [52].

Proposition 4.3.2. *If (H_5^{loc}) hold then there exists (at least) one solution $u \in W_0^{1,p}(\Omega)$ of problem $P_3(\psi, a, f)$.*

Proof. We do the proof by steps, showing several claims. We define

$$X \doteq W_0^{1,p}(\Omega) \text{ and } X_n \doteq W_0^{1,p}(\Omega_n)$$

for each $n \in \mathbb{N}$. Each X_n is a closed subspace of X by extending its elements by zero outside Ω_n. Define $V \doteq \bigcup_{n=1}^{\infty} X_n$ and the map $T : X \to X^*$ by

$$T(u)(x) \doteq \Psi(u)(x) + a(x)u(x) = -div(\psi(x, u(x), \nabla u(x)) + a(x)u(x),$$

for all $x \in \Omega$ and $u \in X$, and the operator $A : X \times V \to \mathbb{R}$ by

$$A(u, v) \doteq \langle T(u), v\rangle_{X^*,X} = \int_\Omega \psi(x, u(x), \nabla u(x))\nabla v(x) + a(x)u(x)v(x)\,dx$$

for all $u \in X$ and $v \in V$. We also consider the operators $A_n : X_n \times X_n \to \mathbb{R}$, with $n \in \mathbb{N}$, defined by

$$A_n(u, v) \doteq \langle T(u), v\rangle_{X_n{}^*,X_n} = \int_{\Omega_n} \psi(x, u(x), \nabla u(x))\nabla v(x) + a(x)u(x)v(x)\,dx$$

for all $u, v \in X_n$. Note that $A : X \times V \to \mathbb{R}$ is well defined. In fact, for any $v \in V$ there exists a $k \in \mathbb{N}$ such that $v \in X_k$ and

$$A(u, v) = \int_{\Omega_k} \psi(x, u(x), \nabla u(x))\nabla v(x) + a(x)u(x)v(x)\,dx < \infty.$$

Claim 4.3.3. *The operators A_n are coercive for any $n \in \mathbb{N}$.*

From the elliptic condition, we have for $u \in X_n$,

$$A_n(u, u) \;\geq\; \alpha \int_{\Omega_n} |\nabla u|^p\,dx + \int_{\Omega_n} a\,u^2\,dx \geq \alpha\,\|u\|^p + \int_{\Omega_n} a\,u^2\,dx.$$

Thus, since $a \geq 0$, we get

$$\lim_{\|u\|\to\infty} \frac{A_n(u, u)}{\|u\|} \geq \lim_{\|u\|\to\infty} \alpha\,\|u\|^{p-1} = \infty.$$

Claim 4.3.4. *We have $A_n(u, \cdot) \in X_n{}^*$ for all $n \in \mathbb{N}$ and $u \in X_n$.*

From $a \in L^\infty(\Omega_n)$, there exists a constant $c_a \geq 0$ such that

$$A_n(u, v) \leq \int_{\Omega_n} |\psi(x, u(x), \nabla u(x))\nabla v|dx + c_a \int_{\Omega_n} |uv|dx.$$

Since $\psi(x, u(x), \nabla u(x)) \in L^{p'}(\Omega_n)$ and $\nabla v \in L^p(\Omega_n)$, we can apply Hölder inequality in the first term. For the second term, since $p' = \frac{p}{p-1}$, $p \geq 2$, and $p' < p$, we use the embedding $L^p(\Omega_{n_0}) \subset L^{p'}(\Omega_{n_0})$, i.e. if $u \in L^p(\Omega_{n_0})$, we can apply the Hölder inequality with $u \in L^{p'}(\Omega_n)$ and $v \in L^p(\Omega_n)$. Additionally from the Poincaré inequality, we obtain

$$\begin{aligned}
A_n(u,v) &\leq \|\psi(x,u(x),\nabla u(x))\|_{L^{p'}(\Omega_{n_0})}\|v\|_{L^p(\Omega_{n_0})} + c_a\|u\|_{L^{p'}(\Omega_{n_0})}\|v\|_{L^p(\Omega_{n_0})} \\
&= \left(\|\psi(x,u(x),\nabla u(x))\|_{L^{p'}(\Omega_{n_0})} + c_a\|u\|_{L^{p'}(\Omega_{n_0})}\right)\|v\|_{L^p(\Omega_{n_0})} \\
&\leq c_1\left(\|\psi(x,u(x),\nabla u(x))\|_{L^{p'}(\Omega_{n_0})} + c_a\|u\|_{L^{p'}(\Omega_{n_0})}\right)\|\nabla v\|_{L^p(\Omega_{n_0})} \\
&\leq c_2\|v\|_{W_0^{1,p}(\Omega_{n_0})}
\end{aligned}$$

for some $c_1, c_2 \geq 0$. Hence, we get that $A_n(u,\cdot)$ are bounded linear functionals on ${X_n}^*$. Since each $A_n(u,\cdot)$ is a bounded linear functional on ${X_n}^*$, the operators $T|_{X_n}$ are well defined for all $n \in \mathbb{N}$.

Claim 4.3.5. *The operators $T|_{X_n}$ are monotone and hemicontinuous for all $n \in \mathbb{N}$.*

This is clear from the fact that $T|_{X_n}$ are the sum between a Leray-Lions operator and a linear operator, i.e. each one is monotone and hemicontinuous.

Claim 4.3.6. *There exists a solution $u_n \in X_n$ for each problem (P_{Ω_n}).*

Since any hemicontinuous operator is radially continuous and $T|_{X_n}$ are monotone and coercive operators, $T|_{X_n}$ satisfy the conditions of the Browder-Minty theorem, so

$$\exists u_n \in X_n \quad \text{s.t.} \quad \langle T(u_n), v\rangle_{X_n{}^* \times V} = \langle f^*, v\rangle_{X_n{}^* \times V} \quad \text{for all } v \in V. \tag{4.7}$$

Recall that for any $v \in V \doteq \cup_{n=1}^{\infty} X_n$, there exists $\bar{n} \in \mathbb{N}$ such that $v \in X_{\bar{n}}$. By the definition of X_n, we know that $(X_n)_{n\in\mathbb{N}}$ is an upwards direct family of closed subspaces of X and hence for any $n \geq \bar{n}$, $v \in X_n$. Hence, by (4.7), we consecutively have

$$A(u_n, v) \to \langle f^*, v\rangle \quad \text{for all } n \geq \bar{n}, \tag{4.8}$$

$$A(u_n, v) \to \langle f^*, v\rangle \quad \text{for all } v \in V, \tag{4.9}$$

$$A(u_n, w) \to \langle f^*, w\rangle \quad \text{for all } w \in X, \tag{4.10}$$

since V is dense in X and $\int_\Omega avw\,dx = \int_{\Omega_n} avw\,dx$ for all $v \in X_n$ and $w \in X$.

Claim 4.3.7. *The sequence of solutions u_n of (P_{Ω_n}) converges weakly in X, i.e. exists $u \in X$ such $u_n \rightharpoonup u$.*

From equation (4.7), setting $v = u_n$, we have $\langle T(u_n), u_n\rangle = \langle f^*, u_n\rangle$, which together with the coercivity of the operator $T|_{X_n}$ gives that the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded. If not, suppose that $\|u_n\| \to \infty$ then

$$\lim_{\|u_n\|\to\infty} \frac{\langle T(u_n), u_n\rangle}{\|u_n\|} \leq \lim_{\|u_n\|\to\infty} \frac{\|f^*\|\|u_n\|}{\|u_n\|} = \|f^*\| < \infty,$$

which is a contradiction with the fact that the operator is coercive. Hence, since $X \equiv W_0^{1,p}(\Omega)$ is a separable reflexive Banach space, using Alaouglus lemma we have that

$(u_n)_{n\in\mathbb{N}}$ bounded implies $u_n \rightharpoonup u \in W_0^{1,p}(\Omega)$.

Claim 4.3.8. *The sequence $(u_n)_{n\in\mathbb{N}}$ converges strongly in X.*

By the weak convergence of the sequence $(u_n)_{n\in\mathbb{N}}$ to $u \in X$, we have $\langle f^*, u_n\rangle \to \langle f^*, u\rangle$. So, using (4.9) with $v = u_n$, we have

$$A(u_n, u_n) \to \langle f^*, u_n\rangle \to \langle f^*, u\rangle. \tag{4.11}$$

Using the compactness of the embedding $X \equiv W_0^{1,p}(\Omega)$ in $L^{p'}(\Omega)$, we conclude the strong convergence of $u_n \to u$ in X.

Alternatively the strong convergence of the sequence $(u_n)_{n\in\mathbb{N}} \subset X$ is ensured by the fact that the Leray-Lions operator Ψ is a (S_+)–type operator, $u_n \rightharpoonup u$ and, by (4.10), we have

$$\limsup_{n\to\infty} \langle T(u_n), u_n - u\rangle = \langle f^*, u_n - u\rangle = 0. \tag{4.12}$$

Note also that, since the Leray-Lions operator Ψ is a pseudo-monotone operator and (4.12), then $\Psi(u_n) \rightharpoonup \Psi(u)$ when $u_n \rightharpoonup u$.

Claim 4.3.9. *The map $A : X \times V \to \mathbb{R}$ defined by $A(u, v) \doteq \langle T(u), v\rangle_{X^*\times X}$ is M-type with respect to V.*

Let $(v_\lambda)_{\lambda\in\Lambda} \subset V$, $w \in X$ and $v^* \in V^*$. Assume the conditions (a)-(c) of Definition 1.2.5, then

$$A(v_\lambda, v) = \langle \Psi(v_\lambda), v\rangle + \langle a\, v, v_\lambda\rangle \to \langle \Psi(w), v\rangle + \langle a\, v, w\rangle = A(w, v)$$

since $v_\lambda \rightharpoonup w$ and $\Psi(v_\lambda) \rightharpoonup \Psi(w)$, which merging with (b) gives $A(w, v) = \langle v^*, v\rangle$ for all $v \in V$.

Therefore, the existence of a solution $u \in X$ is a direct consequence of Claim 4.3.7, (4.9), (4.11), and Claim 4.3.9. □

4.3.2 Uniqueness of the solution

In this subsection, we establish a sufficient condition for the solution of problem $P_3(\psi, a, f)$ to be unique. Since $a \in L^\infty_{loc}(\Omega; \mathbb{R}_0^+)$, we have $a \in L^\infty(\Omega_n; \mathbb{R}_0^+)$ for each compact $\Omega_n \subset \Omega, n \in N$. Here we modify this condition and suppose $a \in L^\infty(\Omega; \mathbb{R}_0^+)$, i.e. we use hypotheses (H_5) for obtaining uniqueness.

Proposition 4.3.10. *Suppose (H_5) hold, then there exists a unique solution u of problem $P_3(\psi, a, f)$ and $\int_\Omega f u\, dx \geq 0$.*

Proof. Let $J : W_0^{1,p}(\Omega) \to \mathbb{R}$ be defined by $J(u) \doteq \Psi(u) + a(x)u(x) - f(x)$. Suppose $u_1, u_2 \in W_0^{1,p}(\Omega)$ are two solutions of problem $P_3(\psi, a, f)$. Thus $\langle J(u_1), v\rangle = \langle J(u_2), v\rangle = 0$ for all $v \in W_0^{1,p}(\Omega)$. In particular, we have

$$\begin{aligned} & \langle J(u_2) - J(u_1), u_2 - u_1\rangle = 0 \\ \Leftrightarrow\quad & \langle \Psi(u_2) - \Psi(u_1), u_2 - u_1\rangle + \langle au_2 - au_1, u_2 - u_1\rangle = 0, \end{aligned}$$

and Ψ is a monotone operator, i.e. $\langle \Psi(u_2) - \Psi(u_1), u_2 - u_1 \rangle \geq 0$, which implies

$$\langle a\,(u_2 - u_1), u_2 - u_1 \rangle \leq 0,$$

, hence $u_2 = u_1$. Moreover, by the ellipticity condition, if u is the solution of problem $P_3(\psi, a, f)$ we have

$$\begin{aligned} \langle J(u), u \rangle = 0 \quad &\Leftrightarrow \quad \int_\Omega fu - au^2\, dx = \int_\Omega \psi(x, u, \nabla u)\nabla u\, dx \\ &\Rightarrow \quad \int_\Omega fu - au^2\, dx \geq \alpha \int_\Omega |\nabla u|^p\, dx \geq 0. \end{aligned}$$

So $a \geq 0$ implies $\int_\Omega fu\, dx \geq 0$. Therefore $f > 0$ implies u cannot be a negative solution. $\square$

4.3.3 Estimate for the solution

In this subsection, we will study an estimate for the solution of problem $P_3(\psi, a, f)$. To obtain the *apriori* estimate for the solution, we use truncation functions as the main tool. For $k > 0$ and $x \in \mathbb{R}$ define the *truncating function* $\mathrm{T}_k : \mathbb{R} \to \mathbb{R}$ by

$$\mathrm{T}_k\,(x) \doteq \begin{cases} -k & \text{if } x < -k, \\ x & \text{if } -k \leq x \leq k, \\ k & \text{if } x > k. \end{cases}$$

For any $u : \Omega \to \mathbb{R}$, by $\mathrm{T}_k(u)$ we mean the map $\mathrm{T}_k(u) : \Omega \to \mathbb{R}$ defined by $x \mapsto \mathrm{T}_k(u(x))$.

Lemma 4.3.11. *(see Napoli-Mariani [91]) The truncation function T_k satisfies*

(i) For any $k > 0$, we have $x\mathrm{T}_k(x) \geq 0$;

(ii) If $\|\mathrm{T}_k\,(u)\|_{L^{p,q}(\Omega)} \leq C$ for any $k > 0$, then $u \in L^{p,q}\,(\Omega)$ and $\|u\|_{L^{p,q}(\Omega)} \leq C$.

Proposition 4.3.12. *If (H_5^{loc}) hold, then the solution of problem $P_3(\psi, a, f)$ satisfies the apriori estimate*

$$\|u\|_{L^{r,s}(\Omega)} \leq C \|f\|_{L^{q,q_1}(\Omega)}^{p'/p} \quad \textit{(for some } C > 0\textit{)}, \tag{4.13}$$

where $r = \sigma N(q-1)q$ and $s = \sigma(N-p)(p-1)q$.

Proof. Let u be a solution of problem $P_3(\psi, a, f)$ and set

$$\varphi_n \doteq \frac{1}{pm+1}\, |\mathrm{T}_k(u)|^{pm}\, \mathrm{T}_k(u)\chi_{\Omega_n}, \quad \text{for any } n \in \mathbb{N} \quad \text{and some} \quad m \in \mathbb{N}, \tag{4.14}$$

where $\chi_{\mathcal{S}}$ is the characteristic function of the subset $\mathcal{S} \subset \mathbb{R}^N$, i.e.

$$\chi_{\mathcal{S}} = \begin{cases} 1, & x \in \mathcal{S}, \\ 0, & x \in \mathbb{R}^N \backslash \mathcal{S}. \end{cases}$$

From the definition of $\mathrm{T}_k(u)$ and $u \in W_0^{1,p}(\Omega)$, we get that $\varphi_n \in V \doteq \bigcup_{n=1}^{\infty} X_n$ and

$$\nabla\varphi_n = |\mathrm{T}_k(u)|^{pm}\,\nabla\left(\mathrm{T}_k(u)\right)\chi_{\Omega_n} \quad \text{a.e. in } \Omega.$$

It follows from (4.6) that

$$\begin{aligned} &\int_{\Omega_n} \left(\psi(x,u(x),\nabla u(x)\right)|\mathrm{T}_k(u)|^{pm}\,\nabla\left(\mathrm{T}_k(u)\right)dx \\ &+\frac{1}{pm+1}\int_{\Omega_n} a(x)u|\mathrm{T}_k(u)|^{pm}\mathrm{T}_k(u)\chi_{\Omega_n}dx \\ = &\ \frac{1}{pm+1}\int_{\Omega_n} f(x)\,|\mathrm{T}_k(u)|^{pm}\,\mathrm{T}_k(u)dx. \end{aligned} \tag{4.15}$$

We denote the first, second and last term in (4.15), respectively, by I_1, I_2 and I_3. Again from the definition of $\mathrm{T}_k(u)$ and the ellipticity condition, we have

$$I_1 = \int_{\Omega_n} \psi(x,u(x),\nabla\mathrm{T}_k(u))\,|\mathrm{T}_k(u)|^{pm}\,\nabla\mathrm{T}_k(u)dx \geq \alpha\int\limits_{\Omega} |\nabla\mathrm{T}_k(u)|^p\,|\mathrm{T}_k(u)|^{pm}\,dx.$$

Note that

$$|\nabla\left(\mathrm{T}_k(u)\right)|^p\,|\mathrm{T}_k(u)|^{pm} = \left|\nabla\left(\frac{|\mathrm{T}_k(u)|^m\,\mathrm{T}_k(u)}{m+1}\right)\right|^p,$$

and so

$$I_1 \geq \alpha\int_{\Omega_n}\left|\nabla\left(\frac{|\mathrm{T}_k(u)|^m\,\mathrm{T}_k(u)}{m+1}\right)\right|^p dx = \alpha\left\|\frac{|\mathrm{T}_k(u)|^m\,\mathrm{T}_k(u)}{m+1}\right\|^p_{W_0^{1,p}(\Omega_n)}.$$

From the embedding of Sobolev spaces into Lorentz spaces (see Lemma A.2.7), we obtain

$$\left\|\frac{|\mathrm{T}_k(u)|^m\,\mathrm{T}_k(u)}{m+1}\right\|^p_{L^{p^*,p}(\Omega_n)} \leq C\left\|\frac{|\mathrm{T}_k(u)|^m\,\mathrm{T}_k(u)}{m+1}\right\|^p_{W_0^{1,p}(\Omega_n)}.$$

Thus

$$I_1 \geq \widetilde{\alpha}\left\|\frac{|\mathrm{T}_k(u)|^m\,\mathrm{T}_k(u)}{m+1}\right\|^p_{L^{p^*,p}(\Omega_n)} = \frac{\widetilde{\alpha}}{(m+1)^p}\,\|\mathrm{T}_k(u)\|^{p(m+1)}_{L^{p^*(m+1),p(m+1)}(\Omega_n)}. \tag{4.16}$$

It follows from Remark 4.3.11 that

$$I_2 = \frac{1}{pm+1}\int_{\Omega_n} a(x)u\,|\mathrm{T}_k(u)|^{pm}\,\mathrm{T}_k(u)dx \geq 0.$$

Using (d) and (e) of Lemma A.2.2, we have

$$\begin{aligned} I_3 &\leq \frac{1}{pm+1}\int_{\Omega_n}|f|\,|\mathrm{T}_k(u)|^{pm+1}\,dx \\ &\leq \frac{1}{pm+1}\|f\|_{L^{q,q_1}(\Omega_n)}\left\||\mathrm{T}_k(u)|^{pm+1}\right\|_{L^{q',q'_1}(\Omega_n)} \\ &= \frac{1}{pm+1}\|f\|_{L^{q,q_1}(\Omega_n)}\|\mathrm{T}_k(u)\|_{L^{q'(pm+1),q'_1(pm+1)}{}^{pm+1}(\Omega_n)}. \end{aligned} \tag{4.17}$$

Combining (4.15)-(4.17) we arrive at

$$\frac{\widetilde{\alpha}}{(m+1)^p}\|\mathrm{T}_k(u)\|^{p(m+1)}_{L^{p^*(m+1),p(m+1)}(\Omega_n)} \leq \frac{1}{pm+1}\|f\|_{L^{q,q_1}(\Omega_n)}\|\mathrm{T}_k(u)\|^{pm+1}_{L^{r,s}(\Omega_n)}. \tag{4.18}$$

Let $p^*(m+1)$ and $p(m+1)$. Now we choose the exponents q_1, r and s such that:
(i) $r = q'\,(pm+1)$;
(ii) $s = q'_1(pm+1)$.
Since $p^* = \frac{pN}{N-p}$ and $q' = \frac{q}{q-1}$, from (i) we obtain

$$m = [Np(q-1) - q(N-p)]p^{-1}\sigma.$$

where $\sigma = (N-pq)^{-1}$. Replacing the value of m, we obtain

$$r = \sigma N(q-1)q \quad \text{and} \quad s = \sigma(N-p)(p-1)q.$$

From (ii), we have

$$q_1 = \sigma(N-p)q.$$

Therefore, from (4.18), we get

$$\frac{\widetilde{\alpha}}{(m+1)^p}\|\mathrm{T}_k(u)\|^{p(m+1)}_{L^{r,s}(\Omega_n)} \leq \frac{1}{pm+1}\|f\|_{L^{q,q_1}(\Omega_n)}\|\mathrm{T}_k(u)\|^{pm+1}_{L^{r,s}(\Omega_n)},$$

then

$$\|\mathrm{T}_k(u)\|^{p-1}_{L^{r,s}(\Omega_n)} \leq C\,\|f\|_{L^{q,q_1}(\Omega_n)}.$$

By Remark 4.3.11, we have $u \in L^{r,s}(\Omega_n)$ and

$$\|u\|_{L^{r,s}(\Omega_n)} \leq C\,\|f\|^{p'/p}_{L^{q,q_1}(\Omega_n)}. \tag{4.19}$$

Now, for fixed $s \geq 0$, by the monotone convergence properties of measures, we obtain

$$d^u_{\Omega_n}(s) \text{ increasingly converges to } d^u_{\Omega}(s) \text{ as } n \to \infty.$$

Therefore

$$u^{**}_{\Omega_n}(s) \text{ increasingly converges to } u^{**}_{\Omega}(s) \text{ as } n \to \infty.$$

Thus it follows from Levi theorem (see Bartle [13]) that

$$
\begin{aligned}
\lim_{n\to\infty}\|u\|_{L^{r,s}(\Omega_n)} &= \lim_{n\to\infty}\left(\int_0^\infty \left(s^{1/p}u^{**}_{\Omega_n}(s)\right)^q \frac{ds}{s}\right)^{1/q}\\
&= \left(\int_0^\infty \left(s^{1/p}\lim_{n\to\infty}u^{**}_{\Omega_n}(s)\right)^q \frac{ds}{s}\right)^{1/q}\\
&= \left(\int_0^\infty \left(s^{1/p}u^{**}_{\Omega}(s)\right)^q \frac{ds}{s}\right)^{1/q}\\
&= \|u\|_{L^{r,s}(\Omega)}.
\end{aligned}
\tag{4.20}
$$

From (4.19) and (4.20) we get

$$\|u\|_{L^{r,s}(\Omega)} \le C\|f\|^{p'/p}_{L^{q,q_1}(\Omega)}$$

□

We are now ready for the multiplicity theorem for problem $P_3(\psi, a, f)$.

Theorem 4.3.13. *If hypotheses (H_5^{loc}) hold, then there exists (at least) one solution $u \in W_0^{1,p}(\Omega) \cap L^{r,s}(\Omega)$ of problem $P_3(\psi, a, f)$ and the solution satisfies the apriori estimate*

$$\|u\|_{L^{r,s}(\Omega)} \le C\|f\|^{p'/p}_{L^{q,q_1}(\Omega)} \quad \textit{(for some } C > 0\textit{)}, \tag{4.21}$$

where $r = \sigma N(q-1)q$ and $s = \sigma(N-p)(p-1)q$. Suppose (H_5) hold, then the solution u is unique and $\int fu \ge 0$.

Proof. From the Proposition 4.3.2, we have the existence of a solution for the problem $P_3(\psi, a, f)$, the uniqueness from Proposition 4.3.10 and the estimate for the solution from Proposition 4.3.12. □

Additionally, we have the following result for problems $P(\Omega_n)$.

Lemma 4.3.14 (Aproximation of the solution)**.** *Let $u \in W_0^{1,p}(\Omega)$ be the solution given in Theorem 4.3.13. For any $n \in \mathbb{N}$, the problem $P(\Omega_n)$ has a unique solution $u_n \in W_0^{1,p}(\Omega_n)$ and the sequence $(u_n)_{n\in\mathbb{N}} \subset W_0^{1,p}(\Omega)$ converges strongly to u.*

Proof. The statements are precisely Claims 4.3.6 and 4.3.8. □

Remark 4.3.15. *Using the same idea of the subsection 4.3.2 and 4.3.3, we obtain uniqueness for the linear case $P_3(M, a, f)$ and an apriori estimate for the solution. Thus, we have the following similar result.*

Theorem 4.3.16 (Linear case)**.** *Let $N > 4$, $2 \le q < \frac{N}{2}$, $\sigma = (N-2q)^{-1}$, $\mu_1 = \sigma Nq$, $\mu_2 = \sigma(N-2)q$ and $a \in L^\infty_{loc}(\Omega; \mathbb{R}_0^+)$. If $f \in L^{q,\mu_2}(\Omega)$ then there exists (at least) one solution $u \in H_0^1(\Omega) \cap L^{\mu_1,\mu_2}(\Omega)$ for the problem $P_3(M, a, f)$, which satisfies the apriori estimate*

$$\|u\|_{L^{\mu_1,\mu_2}(\Omega)} \le C\|f\|_{L^{q,\mu_2}(\Omega)} \quad \textit{(for some } C > 0\textit{)}.$$

Chapter 5

Some considerations and future research

In this last chapter, we present some final comments about problems under study and we give some final remarks regarding the problems studied and discuss some possible directions of future research.

5.1 Some considerations

5.1.1 Problems $P_1(\lambda, \mu, \alpha, f, \gamma)$ and $P_2(\lambda, \zeta, q, s, f)$

(i) As we mentioned in Chapter 1, from the Hardy inequality, the linear elliptic operator $-\Delta u - \frac{\lambda}{|x|^2} u$ is positive and has discrete spectrum if $\lambda < \Lambda = \left(\frac{N-2}{2}\right)^2$. This condition was considered in the problem $P_1(\lambda, \mu, \alpha, f, \gamma)$. For the problem $P_2(\lambda, \zeta, q, s, f)$, the condition $0 \leq \lambda < \Lambda - 4$ was considered because we were dealing with the critical nonlinear term $|x|^{-s}|u(x)|^{\frac{4-2s}{N-2}} u(x)$ where $N > 6$.

(ii) The parameter α, which corresponds to a subcritical term in $P_1(\lambda, \mu, \alpha, f, \gamma)$, has a direct relation with the values of λ and μ, in fact we consider $0 < \alpha < \sqrt{\Lambda - \lambda}$ to ensure that the functional T has a good behavior and we can use the estimate of local behavior of the solution. The condition on α is used explicitly for proving that the two nontrivial solutions w_0 and w_1 are different.

(iii) In obtaining solutions to the problems $P_1(\lambda, \mu, \alpha, f, \gamma)$ and $P_2(\lambda, \zeta, q, s, f)$, the Lemmas 2.2.13 and 3.2.3 respectively, play an important role. To continuation, we describe in detail how these Lemmas were applied. The Lemma 2.2.13 was applied to find each solution in $P_1(\lambda, \mu, \alpha, f, \gamma)$ as follows.

• For the first solution $w_0 \in M^+$, we say that for all $u \in M$, there exists $t_-(u) \in \mathbb{R}$ such that $t_-(u) < t_{\max}(u)$, $t_-(u)u \in M^+$ and (2.28), we have

$$I\left(t_-(u)u\right) \leq I(\xi u), \quad \text{for all } 0 < \xi < t_{\max}(u).$$

Moreover from Lemma 2.2.13, for $|w_0| \in H_0^1(\Omega)$, there exists a unique value $t_-(|w_0|) \in \mathbb{R}$ such that $t_-(|w_0|)\,|w_0| \in M^+$, $t_-(|w_0|) < t_{\max}(|w_0|) = t_{\max}(w_0)$ and

$$I(t_-(|w_0|)\,|w_0|) = \min_{-t_{\max} \leq t \leq t_{\max}} I(t(|w_0|)\,|w_0|).$$

• For the second solution $w_1 \in M^-$, we say that for all $u \in M$, there exists $t_+ = t_+(u) > 0$ such that $t_+(u)u \in M^-$ and there exists $t_+ \in R$, such that $s_f\, t_+(|w_1|) > 0$, $t_+(|w_1|)\,|w_1| \in M^-$, $s_f\, t_+(|w_1|) > t_{\max}(|w_1|) = t_{\max}(w_1)$ and $I(t_+(|w_1|)\,|w_1|) = \max_{s_f t \geq 0} I(t(|w_1|)\,|w_1|)$.
• For the third solution $w_2 \in M_*^- \doteq M_1^- \cap M_2^-$, where $M_1^- \doteq \{u \in M; \quad u^+ \in M^-\}$ and $M_2^- \doteq \{u \in M; \quad -u^- \in M^-\}$, the values t_+ and t_- are used explicitly in the Proposition 2.2.18.
• For the fourth solution $\bar{w_1} \in H_0^1(\Omega)$ we don't use explicitly the values t_- or t_+, we use $w_0 \in M^+$ and $-w_2^- \in M^-$ to prove that $w_{1,1} > 0$ and $w_2 \neq w_{1,1}$.

In the case of problem $P_2(\lambda, \zeta, q, s, f)$, the values t_- and t_+ from Lemma 3.2.3 have the following characterization

$$t_-(u)\,u \in M^+ \text{ and } J(t_-(u)\,u) = \min_{0 \leq t \leq t_{\max}} J(tu)\,,$$

$$t_+(u)\,u \in M^- \text{ and } J(t_+(u)\,u) = \max_{t \geq t_{\max}} J(tu)\,.$$

These values are only used for obtaining the sign-changing-solution w_2. In fact, this solution is obtained as result of the Proposition 3.2.5 and the Lemma 3.2.15, in which the values t_- and t_+ are vital. Specifically, in Proposition 3.2.5 we proved under some considerations on the function J and the value c that: (1) $c > J(t_+(u)\,u)$ in the case $u \neq 0$ and $t_+(u) \leq 1$; and (2) $c \geq J(t_-(u)\,u) + \left(\frac{2-s}{2(N-s)}\right) S_{\lambda,s}{}^{\frac{N-s}{2-s}}$ in the case $u \neq 0$ and $t_+(u) > 1$. The Lemma 3.2.15 is obtained, because for any given $\tau \in (\tau_1, \tau_2)$, we obtain from Lemma 3.2.3 that there are positive values $r_+(\tau)$ and $r_-(\tau)$ such that

$$r_+(\tau)(u_\varepsilon - \tau w_1)^+ \in M^-, \text{ and } \quad -r_-(\tau)(u_\varepsilon - \tau w_1)^- \in M^-. \tag{5.1}$$

5.1.2 Problem $P_3(\psi, a, f)$

(i) We use the idea of Lemma 1.2.12 due to Drivaliaris-Yannakakis [52], because its necessary to guarantee that the definition of the function A makes sense, i.e. its vanishing on the boundary and hence permits to overcome the difficulty of the singularity. Note that $A : X \times X \longrightarrow R$ has not sense, but

$$A : X \times V \longrightarrow \mathbb{R},\ A : V \times V \longrightarrow \mathbb{R} \text{ and } A : V \times X \longrightarrow \mathbb{R}$$

has sense with $W_0^{1,p}(\Omega)$ and $V \doteq \bigcup_{n=1}^{\infty} X_n$.

(iii) The method for estimating the solution used in the Chapter 3, cannot be directly applied to elliptic systems, since it is difficult to find suitable functions φ_n (see 4.14).

(iv) Since we need to apply Hölder inequality to show that A_n is bounded linear functional on X_n^*, we consider $p \geq 2$.

(v) We use a Lorentz scale argument to obtain $f \in L^{q,q_1}(\Omega) \subset L^{p'}(\Omega)$ with $p' \leq q$ (see 4.3.12). For that, we consider the test function

$$\varphi_n \doteq \frac{1}{pm+1} |\mathrm{T}_k(u)|^{pm} \mathrm{T}_k(u)\chi_{\Omega_n}, \quad \text{for any } n \in \mathbb{N} \quad \text{and some} \quad m \in \mathbb{N}, \tag{5.2}$$

and posteriori we find the suitable value of m

$$m = \frac{Np(q-1) - q(N-p)}{p(N-pq)}.$$

So, we need $p(N-pq) > 0$ and thus we have $q < \frac{N}{p}$. Note that, when $q = \frac{N}{p}$ the test function ceases to exist.

5.2 Some directions of future research

The classes of elliptic problems, studied in this work, are quite rich in the research point of view. We now describe some possible directions of future investigation, which turn to be some kind of generalization of already obtained results, situations not already considered, or adjacent problems which interest was increased during our current research.

5.2.1 Problem $P_1(\lambda, \mu, \alpha, f, \gamma)$ with more general conditions

Note that the functionals $T(u), U(u), Q(u)$ and $J(u)$, defined in the Chapter 2, satisfy more general conditions (H_1):

- (i) $T : H_0^1(\Omega) \to R_0^+$ positive away from zero; exists $\alpha > 0$ such that $T(su) = |s|^\alpha T(u)$ for any $s \in R$ and $u \in H_0^1(\Omega)$, and there exist $K_1^T, K_2^T > 0$ such that $K_1^T \|u\|^\alpha_{H_0^1(\Omega)} \leq T(u) \leq K_2^T \|u\|^\alpha_{H_0^1(\Omega)}$;
- (ii) $U : H_0^1(\Omega) \to R_0^+$ positive away from zero; exists $\beta > 0$ such that $U(su) = |s|^\beta U(u)$ for any $s \in R$ and $u \in H_0^1(\Omega)$, and exists $K^U > 0$ such that $U(u) \leq K^U \|u\|^\beta_{H_0^1(\Omega)}$;
- (iii) $F : H_0^1(\Omega) \to R$ with $F(0) = 0$; exists $\gamma > 0$ such that $F(su) = s|s|^\gamma F(u)$ for any $s \in R$ and $u \in H_0^1(\Omega)$, exists $K^F > 0$ such that $F(u) \leq K^F \|u\|^{\gamma+1}_{H_0^1(\Omega)}$;
- (iv) $\alpha < \beta$ and $\gamma < \beta - 1$.

Our situation, in Chapter 2, is the particular case $\alpha = 2$, $\beta = 2^*$ and $0 \leq \gamma < 1$. The inequalities in $(H_1)(i)$ are valid by Lemma 2.2.1, inequality $(H_1)(ii)$ is valid by Gagliardo-Nirenberg-Sobolev inequality (Theorem A.1.6), and inequality $(H_1)(iii)$ is valid by (2.8).

It may be interesting to study the possibility of existence of solution for our problem, when only these general conditions are consider.

5.2.2 Problem $P_1(\lambda, \mu, \alpha, f, \gamma)$ with even nonlinearity

A variant of $P_1(\lambda, \mu, \alpha, f, \gamma)$ is changing the term $f|u|^\gamma$ to $f|u|^{\gamma-1}u$. In this case, the associated functional will be even, meaning that if u is a solution, $-u$ is also a solution. It would be interesting to study this problem and also if an infinite number of solutions exist, which is a typical situation for some even functionals.

5.2.3 Problem $P_1(\lambda, \mu, \alpha, f, \gamma)$ with different conditions on γ and f

We have considered $0 \leq \gamma < 1$. It may be of some interest to study also $\gamma > 1$ for which the behavior of ϕ_u is represented in Figure 2.3. Another direction is to study if the problem without the condition $\widetilde{\mu}_f$ (see 2.2) still to have the existence of one solution since t_+ and t_0 can be used (see Figure 2.2)

5.2.4 Problem $P_1(\lambda, \mu, \alpha, f, \gamma)$ and $P_2(\lambda, \zeta, q, s, f)$ considering others values of t

Note that, we never use in problem $P_1(\lambda, \mu, \alpha, f, \gamma)$ the value t_0, obtained from Lemma 2.2.13 (see 5.1.1). One interesting future research is to analyze if the value t_0 allows to obtain another solution. On the other hand, since the problem $P_2(\lambda, \zeta, q, s, f)$ is odd, we can study the existence of other values $t < 0$ (see Lemma 3.2.3) in order to find other solutions (see 5.1.1).

5.2.5 Problem $P_3(\phi, a, f)$ with f defined in a weighted Lorentz space

One natural generalization of problem $P_3(\phi, a, f)$ is to consider f defined in a weight Lorentz space. In other words, study the problem $P_3(\phi, a, f)$ when $w(s) \neq s^{q/p-1}$ so $\Lambda^q(w) \neq L^{p,q}$.

5.2.6 Supercritical exponent in a Lorentz setting

We know that, the solvability of problem

$$-\Delta u(x) = |u(x)|^{p-2}u \ \ in \ \ \Omega,$$

when $p \geq 2^*$ and u is defined in the Sobolev space $H_0^1(\Omega)$, depends on the shape of Ω. However, from Brezis-Nirenberg [16], some perturbations of this problem by lower order terms can guarantee the existence of positive solutions independently of the shape of Ω. The main idea is to consider u in a suitable Lorentz space and to investigate which is the critical hyperbola under different conditions on the nonlinearity f.

We recall that the embedding of $H_0^1(\Omega)$ into a Lorentz space is some how more fit than into a Lebesgue space. In fact, this turn to be more relevant to elliptic systems than to elliptic equations. The key point then is, by using a Lorentz space setting where the functional is defined on the cartesian product of Sobolev spaces over different Lorentz spaces, to determine the properties of the critical hyperbola and establish the true maximal admissible growth for some classes of systems.

Appendix A

Spaces of functions

Here we state some notions which are standard but help to clarify the reader. We start by defining the spaces where we will work, i.e. Sobolev spaces and Lorentz spaces with special emphasis on the embedding.

A.1 Sobolev spaces

Let $\Omega \subset \mathbb{R}^N$ be an open set and $f : \Omega \longrightarrow \mathbb{R}$ a continuous function. The support of f is denoted by supp(f), i.e. the closure in Ω of the set $\{x \in \Omega;\ f(x) \neq 0\}$.

A vector of nonnegative integers $\alpha = (\alpha_1, ..., \alpha_n)$ is called a multi-index and its order is defined by $|\alpha| = \alpha_1 + ... + \alpha_n$.

Denote by D^α the operator of derivation of order $|\alpha|$, that is,

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}.$$

For $\alpha = (0, 0, ..., 0)$, set $D^0 u = u$, for all function u.

By $C_0^\infty(\Omega)$ we mean the space of infinitely differentiable functions with compact support in Ω.

Definition A.1.1. *We say that a sequence* $(\varphi_n)_{n \in \mathbb{N}}$ *in* $C_0^\infty(\Omega)$ *converge to* φ *in* $C_0^\infty(\Omega)$, *when the following conditions hold:*

(i) There exists a compact K of Ω such that supp$(\varphi) \subset K$ and supp$(\varphi_n) \subset K$, $\forall\, n \in \mathbb{N}$,

(ii) $D^\alpha \varphi_n \to D^\alpha \varphi$ uniformly in K, for all multi-índices α.

The space $C_0^\infty(\Omega)$, provided with the notion of convergence above defined, will be denoted by $\mathcal{D}(\Omega)$ and called space of test functions.

A distribution (scalar) on Ω is a linear continuous, functional on $\mathcal{D}(\Omega)$.

We denote the value of a distribution T in φ by $\langle T, \varphi \rangle$. The set of all distributions on Ω, with the usual operations, is a vectorial space, which is represented by $\mathcal{D}'(\Omega)$.

Definition A.1.2. *We say that a sequence* $(T_n)_{n\in\mathbb{N}}$ *in* $\mathcal{D}'(\Omega)$ *converge to* T *in* $\mathcal{D}'(\Omega)$, *when the numerical sequence* $(\langle T_n,\varphi\rangle)_{n\in\mathbb{N}}$ *converge to* $\langle T,\varphi\rangle$ *in* $\mathbb{R}$, *for all* $\varphi\in\mathcal{D}(\Omega)$.

Definition A.1.3. *Let* T *be a distribution on* Ω *and* α *be a multi-index. The derivative* $D^\alpha T$ *(in the sense of distributions) of order* $|\alpha|$ *of* T *is the functional defined in* $\mathcal{D}(\Omega)$ *by*

$$\langle D^\alpha T,\varphi\rangle=(-1)^{|\alpha|}\langle T,D^\alpha\varphi\rangle,\quad \forall\varphi\in\mathcal{D}(\Omega).$$

Given an integer $m>0$, by $W^{m,p}(\Omega)$ for $1\le p\le\infty$, we represent the Sobolev space of order m on Ω, that is the space of all functions $u\in L^p(\Omega)$ such that $D^\alpha u\in L^p(\Omega)$, for all multi-index α with $|\alpha|\le m$.

The space $W^{m,p}(\Omega)$ provided with the norm

$$\|u\|_{W^{m,p}(\Omega)}=\left(\sum_{|\alpha|\le m}\int_\Omega|D^\alpha u(x)|^p\,dx\right)^{\frac{1}{p}},\text{ for }1\le p<\infty$$

and

$$\|u\|_{W^{m,\infty}(\Omega)}=\sum_{|\alpha|\le m}\sup_{x\in\Omega} ess\,|D^\alpha u(x)|,\text{ for }p=\infty,$$

is a Banach space.

Now, we summarize some basic properties of Sobolev spaces stated in the next theorem.

Theorem A.1.4. *Let* $\Omega\subset\mathbb{R}^N$ *be a bounded domain with* $N\ge 1$, *then we have the following:*
(i) $W^{m,p}(\Omega)$ *is separable for* $1\le p<\infty$;
(ii) $W^{m,p}(\Omega)$ *is reflexive for* $1<p<\infty$;
(iii) Let $1\le p<\infty$, *then* $C^\infty(\Omega)\cap W^{m,p}(\Omega)$ *is dense in* $W^{m,p}(\Omega)$, *where* $C^\infty(\Omega)$ *is the spaces of infinitely differentiable functions in* Ω.

The space $W_0^{m,p}$ denotes the closure of $\mathcal{D}(\Omega)$ with the norm of $W^{m,p}(\Omega)$.

Heuristically, the space $W_0^{m,p}(\Omega)$ consists of all functions in $W^{m,p}(\Omega)$ that "vanish" on the boundary $\partial\Omega$ together with all their derivatives up to order $m-1$.

Remark A.1.5. *When* $p=2$, *the space* $W^{m,p}(\Omega)$ *will be denoted by* $H^m(\Omega)$, *provided with inner product*

$$(u,v)_{H^m(\Omega)}=\sum_{j=0}^{m}\left(u^{(j)},v^{(j)}\right)_{L^2(\Omega)}$$

is a Hilbert space. Denote by $H_0^m(\Omega)$ *the closure, in* $H^m(\Omega)$, *of* $\mathcal{D}(\Omega)$ *and by* $H^{-m}(\Omega)$ *the topological dual of* $H_0^m(\Omega)$.

A.1.1 Sobolev embedding

Lemma A.1.6. *[Gagliardo-Nirenberg-Sobolev inequality, see Evans [56]] Let p such that $1 \leq p < N$. There is a constant $c > 0$ that depends only on p and N such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^N)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^N)}$$

for all $u \in C_0^1(\mathbb{R}^N)$. Here

$$p^* = \frac{pN}{N-p}$$

is the critical Sobolev exponent.

The following is the known Rellich-Kondrachov Lemma (see Struwe [113]).

Lemma A.1.7. *Let Ω be a bounded domain with smooth boundary, the*

(i) If $N > pm$, where $W^{m,p}(\Omega) \overset{c}{\hookrightarrow} L^q(\Omega)$, where $q \in \left[1, \dfrac{2N}{N-2m}\right)$.

(ii) If $N = pm$, where $W^{m,p}(\Omega) \overset{c}{\hookrightarrow} L^q(\Omega)$, where $q \in [1, +\infty)$.

(iii) If $pm > N$ where $W^{m,p}(\Omega) \overset{c}{\hookrightarrow} C^k(\overline{\Omega})$, where $k < m - (n/p) \leq k+1$.

Remark A.1.8. *When $m = 1$, from Sobolev embedding theorem (see Ambrosetti-Malchiodi [7]), we have:*
(i) $H_0^1(\Omega) \subset L^{2^*}(\Omega)$;
(ii) $\|u\|_{L^{2^*}(\Omega)} \leq c\|u\|_{H_0^1(\Omega)}$ for some $c > 0$,
(iii) There are bounded sequences in $H_0^1(\Omega)$ that are not precompact in $L^{2^*}(\Omega)$; i.e. the inclusion $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is continuous but is not compact.

A.1.2 The best Sobolev constant

Definition A.1.9. *Set $\mathcal{D}^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$. The best Sobolev constant for the embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is defined by*

$$S = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{\frac{2}{2^*}}} > 0.$$

It is well known that S is independent of $\Omega \subset \mathbb{R}^N$ in the sense that if

$$S(\Omega) = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2 dx}{\left(\int_{\Omega} |u|^{2^*} dx\right)^{\frac{2}{2^*}}} > 0,$$

then $S(\Omega) = S(\mathbb{R}^N) = S$ (see Ferrero-Gazzola [57]).

A.2 Lorentz spaces

Lorentz spaces $L^{p,q}(\Omega)$ were introduced by George G. Lorentz [89] in 1950. These spaces are relevant examples of rearrangement invariant function spaces and are a generalization of Lebesgue spaces.

In the last years a line of development for treating nonlinear elliptic problems is to employ the Lorentz spaces, in place of the standard Lebesgue spaces $L^p(\Omega)$.

Let (Ω, S, μ) be a σ-finite measure space. For a measurable function $f : \Omega \to \mathbb{R}$, we define the *distribution function* $d^f_\Omega(t) : [0,\infty) \longrightarrow [0,\infty)$ as

$$d^f_\Omega(t) \doteq \mu(\{x \in \Omega : |f(x)| > t\}).$$

The distribution function satisfies the following properties (see Talenti [117]):

- d^f_Ω is a non-increasing, right continuous function;
- $d^f_\Omega(0) = \mu(supp(f))$;
- $d^f_\Omega(+\infty) = 0$.

The *non-increasing rearrangement* of f is defined by

$$f^*_\Omega(s) \doteq \sup\left\{t > 0 : d^f_\Omega(t) > s\right\} = \inf\left\{t > 0 : d^f_\Omega(t) \le s\right\}$$

with $0 \le s \le |\Omega|$, and satisfies the following properties (see Talenti [117]):

- $f^*_\Omega(s)$ is right continuous;
- $f^*_\Omega(0) = \sup ess|f|$;
- $f^*_\Omega(+\infty) = 0$;
- $t < f^*_\Omega(s)$ if and only if $s < d^f_\Omega(t)$;
- $d^{f^*_\Omega}_\Omega(t) = d^f_\Omega(t)$;
- $f^*_\Omega = d^{d^f_\Omega}_\Omega(t)$;
- $\int_0^\infty f^*_\Omega(s)ds = \int_0^\infty d^f_\Omega(t)dt = \int_0^\infty |f|d\mu$.

Lemma A.2.1. *[Hardy-Littlewood-Pólya inequality]*

$$\int_0^\infty |fg|d\mu \le \int_0^\infty f^*_\Omega(s)g^*_\Omega(s)ds.$$

The Lorentz space $L^{p,q}(\Omega)$ is the collection of all measurable functions f on Ω such that $\|f\|_{L^{p,q}(\Omega)} < \infty$, where the norm is given by

$$\|f\|_{L^{p,q}(\Omega)} = \begin{cases} \left(\int_0^\infty \left(s^{1/p} f^{**}(s)\right)^q \frac{ds}{s}\right)^{\frac{1}{q}} & \text{if } 1 \le q < \infty, \\ \sup_{s>0} \left\{ s^{\frac{1}{p}} f^{**}(s) \right\} & \text{if } q = \infty; \end{cases}$$

with $1 \le p < \infty$ and $f^{**}(s) = \frac{1}{s}\int_0^s f_\Omega^*(t)dt$.

The following lemma presents the main properties of these spaces.

Lemma A.2.2. *(Hunt [74], Talenti[117]) For Lorentz spaces, we have the following results:*

(a) *$L^{p,p}(\Omega)$ coincides with the Lebesgue space $L^p(\Omega)$ and $\|u\|_{L^{p,p}(\Omega)} = \|u\|_{L^p(\Omega)}$ for $u \in L^{p,p}(\Omega)$;*

(b) *(Duality) Let $1 < p < q < \infty$ then*

$$(L^{p,q}(\Omega))^* = L^{p',q'}(\Omega),$$

where $(L^{p,q}(\Omega))^*$ denotes the space of all bounded linear functionals on $L^{p,q}(\Omega)$;

(c) *Let $1 \le q_1 < p < q_2 < \infty$ and $p_1 < p$, then the following inclusions hold*

$$L^{p,q_1}(\Omega) \subsetneq L^{p,p}(\Omega) \equiv L^p(\Omega) \subsetneq L^{p,q_2}(\Omega) \subsetneq L^{p,\infty}(\Omega) \subsetneq L^{p_1,q_1}(\Omega);$$

(d) *The following (Hölder type) inequality*

$$\|fg\|_{L^{p,q}(\Omega)} \le \|f\|_{L^{p_1,q_1}(\Omega)} \|g\|_{L^{p_2,q_2}(\Omega)},$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$.

(e) *If $f \in L^{pm,qm}(\Omega)$ with $m > 0$, then $|f|^m \in L^{p,q}(\Omega)$ and*

$$\||f|^m\|_{L^{p,q}(\Omega)} = \|f\|^m_{L^{pm,qm}(\Omega)} .$$

Lemma A.2.3. *Suppose that E_j are pairwise disjoint measurable subsets of Ω and $f \in L^{m,q}(\Omega)$, $1 \le q \le m$. Then*

$$\sum_j \|f\chi_{E_j}\|^m_{L^{m,q}} \le \|f\|^m_{L^{m,q}} .$$

Definition A.2.4. *For $0 < q < \infty$, the weighted Lorentz Space $\Lambda^q(w)$ is defined as the set of all measurable functions f such that*

$$\|f\|_{\Lambda^q(w)} = \left(\int_0^\infty w(s)\left(f_\Omega^*(s)\right)^q \frac{ds}{s}\right)^{\frac{1}{q}} < \infty,$$

where $f_\Omega^*(s)$ *denotes the non.increasing rearrangement of* f *and* w *is a weight in* $\mathbb{R}^+$.

Remark A.2.5. *The weights for which* $\Lambda^q(w)$ *is a Banach space were first characterized by Arino-Muckenhoupt [10], and it is known as the Bp-condition: there exists* $C > 0$ *such that, for all* $r > 0$,

$$r^q \int_r^\infty \frac{w(x)}{x^q} dx \le C \int_0^r w(x) dx.$$

A.2.1 Inclusions into Lorentz spaces

Lemma A.2.6. *Suppose* $1 \le m, q, M, Q \le \infty$.
(a) *If* $q < Q$, *then* $\|f\|_{L^{m,Q}} \le C \|f\|_{L^{m,q}}$;
(b) *If* $m < M$, *then* $(\mu(\Omega))^{-\frac{1}{m}} \|f\|_{L^{m,q}} \le C \left((\mu(\Omega))^{-\frac{1}{M}}\right) \|f\|_{L^{M,Q}}$.

The following result improves the classical result of the Sobolev embedding and it is relevant when, working with critical cases.

Lemma A.2.7. *(Talenti [117]) Let* $1 \le p < N$ *then* $W^{1,p}(R^N) \subset L^{p^*,p}(R^N)$ *with continuous embedding,* $W_0^{1,p}(\Omega) \subset L^{p^*,p}(\Omega)$ *with continuous embedding, and when* $\partial\Omega \in C^1(\Omega)$ *the same result applies to* $W^{1,p}(\Omega)$.

Remark A.2.8. *Note that if* $u \in W^{1,p}(\Omega)$ *by the Sobolev embedding we have*

$$u \in L^{p^*}(\Omega) \equiv L^{p^*,p^*}(\Omega);$$

but by the embedding to Lorentz spaces we have $u \in L^{p^*,q}(\Omega)$ *with* $p \le q \le p^*$. *Hence there is an improvement in using this embedding.*

Appendix B

Some integral estimates

In this chapter, we calculate some integral estimates, that will allow us to guarantee that the solutions of problem $P_1(\lambda, \mu, \alpha, f, \gamma)$ are different.

Proposition B.0.9. *For ε small enough, we have*

$$\int v_\varepsilon^{\gamma+1} = \begin{cases} O(\varepsilon^{\frac{N-2}{4}(\gamma+1)}), & 1 < 1+\gamma < \frac{N}{\sqrt{\Lambda}+\sqrt{\Lambda-\lambda}}, \\ O(\varepsilon^{\frac{N-2}{4}(\gamma+1)}|\ln\varepsilon|), & 1+\gamma = \frac{N}{\sqrt{\Lambda}+\sqrt{\Lambda-\lambda}}, \\ O\left(\varepsilon^{\frac{[N-(\gamma+1)\sqrt{\Lambda}]\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}}\right), & \frac{N}{\sqrt{\Lambda}+\sqrt{\Lambda-\lambda}} < 1+\gamma < 2. \end{cases} \tag{B.1}$$

Proof. We recall the definition of $v_\varepsilon(x) = \phi(x)U_\varepsilon(x)$, where

$$U_\varepsilon(x) = \frac{[4\varepsilon(\Lambda-\lambda)N/(N-2)]^{\frac{N-2}{4}}}{[\varepsilon|x|^{\gamma_1/\sqrt{\Lambda}} + |x|^{\gamma_2/\sqrt{\Lambda}}]^{\frac{N-2}{2}}} \quad \text{for } \varepsilon > 0,$$

with $\gamma_1 = \sqrt{\Lambda} - \sqrt{\Lambda-\lambda}$, $\gamma_2 = \sqrt{\Lambda} + \sqrt{\Lambda-\lambda}$, and ϕ is such that $\phi(x) = 1$ if $|x| \leq \delta$, $\phi(x) = 0$ if $|x| \geq 2\delta$, $\phi(x) \in C_0^2(\Omega)$ and $|\phi(x)| \leq 1$, $|\nabla\phi(x)| \leq C$ for some positive constant C.

From estimate (1.12) (Proposition 2.2.9), we have that

$$\int v_\varepsilon^{\gamma+1} = \int_{\Omega\setminus B(0,\delta)} v_\varepsilon^{\gamma+1} + \int_{B(0,\delta)} U_\varepsilon^{\gamma+1} = O(\varepsilon^{\frac{N-2}{4}(\gamma+1)}) + \int_{B(0,\delta)} U_\varepsilon^{\gamma+1}.$$

Now, we estimate the last integral. Let ρ and $\theta \in S^{N-1}$ being the polar coordinates, where S^{N-1} is the unit sphere in R^N. For $x = (x_1, ..., x_N) \in R^N$ and $(\rho, \theta_1, ..., \theta_N) \in (0,\infty) \times (0,\phi) \times ... \times (0,\phi) \times (0, 2\phi)$ we have

$$\begin{aligned} x_1 &= \rho cos(\theta_1) \\ x_2 &= \rho sen(\theta_1)cos(\theta_2) \\ x_{N-1} &= \rho sen(\theta_1)sen(\theta_2)...sen(\theta_{N-2})cos(\theta_{N-1}) \\ x_N &= \rho sen(\theta_1)sen(\theta_2)...sen(\theta_{N-1}). \end{aligned}$$

Thus $dx = \rho^{N-1}(sen(\theta_1))^{N-2}(sen(\theta_2))^{N-3}...sen(\theta_{N-2})d\rho d\theta$. Briefly, we write $x = \rho w$ and $w = (w_1, ..., w_N)$ then $|w| = 1$, which means that w belong to the unit sphere S^{N-1} in R^N and $dx = \rho^{N-1}d\rho dw$, where dw is the measure on S^{N-1}. Therefore

$$\begin{aligned}\int_{B(0,\delta)} U_\varepsilon^{\gamma+1}dx &= \int_{B(0,\delta)} \frac{[4\varepsilon(\Lambda-\lambda)N/(N-2)]^{\frac{N-2}{4}\gamma+1}}{[\varepsilon|x|^{\gamma_1/\sqrt{\Lambda}} + |x|^{\gamma_2/\sqrt{\Lambda}}]^{\frac{N-2}{2}\gamma+1}}dx \\ &= \int_0^\delta \int_{S^{N-1}} \frac{[4\varepsilon(\Lambda-\lambda)N/(N-2)]^{\frac{N-2}{4}\gamma+1}}{[\varepsilon|\rho w|^{\gamma_1/\sqrt{\Lambda}} + |\rho w|^{\gamma_2/\sqrt{\Lambda}}]^{\frac{N-2}{2}\gamma+1}}\rho^{N-1}d\rho dw \\ &= \int_0^\delta \frac{[4\varepsilon(\Lambda-\lambda)N/(N-2)]^{\frac{N-2}{4}\gamma+1}}{[\varepsilon|\rho|^{\gamma_1/\sqrt{\Lambda}} + |\rho|^{\gamma_2/\sqrt{\Lambda}}]^{\frac{N-2}{2}\gamma+1}}\rho^{N-1}d\rho \int_{S^{N-1}} dw.\end{aligned}$$

If we set, w_n as the surface area of the $(N-1)$-sphere S^{N-1}, then

$$\begin{aligned}&\int_{B(0,\delta)} U_\varepsilon^{\gamma+1}dx \\ &= w_n[4\varepsilon(\Lambda-\lambda)N/(N-2)]^{\frac{N-2}{4}\gamma+1}\int_0^\delta \frac{\rho^{N-1}}{[\varepsilon|\rho|^{\gamma_1/\sqrt{\Lambda}} + |\rho|^{\gamma_2/\sqrt{\Lambda}}]^{\frac{N-2}{2}\gamma+1}}d\rho \\ &= K\cdot\varepsilon^{\frac{(N-2)}{4}(\gamma+1)}\int_0^\delta \frac{\rho^{N-1}d\rho}{[\varepsilon\rho^{\gamma_1/\sqrt{\Lambda}} + \rho^{\gamma_2/\sqrt{\Lambda}}]^{(N-2)(\frac{\gamma+1}{2})}}.\end{aligned}$$

Since

$$(\gamma_1\sqrt{\Lambda})(N-2)\left(\frac{\gamma+1}{2}\right) = (N-2)\left(\frac{\gamma+1}{2}\right)\left(\frac{2\sqrt{\Lambda-\lambda}}{\sqrt{\Lambda}}\right)$$

we have

$$\begin{aligned}&[\varepsilon\rho^{\gamma_1/\sqrt{\Lambda}} + \rho^{\gamma_2/\sqrt{\Lambda}}]^{(N-2)(\frac{\gamma+1}{2})} \\ &= (\varepsilon\rho^{\gamma_1/\sqrt{\Lambda}})^{(N-2)(\frac{\gamma+1}{2})}[1 + ... + (\varepsilon\rho^{\gamma_1/\sqrt{\Lambda}})^{-(N-2)(\frac{\gamma+1}{2})}(\rho^{\gamma_2/\sqrt{\Lambda}})] \\ &= (\varepsilon\rho^{\gamma_1/\sqrt{\Lambda}})^{(N-2)(\frac{\gamma+1}{2})}[1 + \varepsilon^{-1}\rho^{2\sqrt{\Lambda-\lambda}/\sqrt{\Lambda}}]^{(N-2)(\frac{\gamma+1}{2})}.\end{aligned}$$

Thus,

$$\begin{aligned}&\int_{B(0,\delta)} U_\varepsilon^{\gamma+1}dx \\ &= K\cdot\varepsilon^{\frac{(N-2)}{4}(\gamma+1)}\int_0^\delta \frac{\rho^{N-1}d\rho}{\varepsilon^{(N-2)(\frac{\gamma+1}{2})}\rho^{(\gamma_1/\sqrt{\Lambda})(N-2)(\frac{\gamma+1}{2})}[1+\varepsilon^{-1}\rho^{2\sqrt{\Lambda-\lambda}/\sqrt{\Lambda}}]^{(N-2)(\frac{\gamma+1}{2})}}.\end{aligned}$$

In general $\int_0^\delta x^m dx = \int_0^{\delta a} \frac{x^m}{a^{m+1}}dx$, hence

$$\begin{aligned}&\int_{B(0,\delta)} U_\varepsilon^{\gamma+1}dx = K\cdot\varepsilon^{\frac{(N-2)}{4}(\gamma+1)}. \\ &\int_0^{\delta\varepsilon^{-\frac{\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}}} \frac{\varepsilon^{\frac{N\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}}\rho^{N-1}d\rho}{\varepsilon^{\frac{(N-2)(\gamma+1)}{2}}\varepsilon^{\frac{\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}\frac{\gamma_1(N-2)(\gamma+1)}{2\sqrt{\Lambda}}}\rho^{(\gamma_1/\sqrt{\Lambda})(N-2)(\frac{\gamma+1}{2})}[1+\rho^{2\sqrt{\Lambda-\lambda}/\sqrt{\Lambda}}]^{(N-2)(\frac{\gamma+1}{2})}}.\end{aligned}$$

Now we consider the different possibilities for $1+\gamma$.

(i) If $1+\gamma = \frac{N}{\sqrt{\Lambda}+\sqrt{\Lambda-\lambda}}$, since the order of ρ in the integrand is

$$\begin{aligned}
&N-1-(\gamma_1/\sqrt{\Lambda})(N-2)(\tfrac{\gamma+1}{2})-2(\sqrt{\Lambda-\lambda}/\sqrt{\Lambda})(N-2)(\tfrac{\gamma+1}{2})\\
&=N-1-((\sqrt{\Lambda}-\sqrt{\Lambda-\lambda})/\sqrt{\Lambda})(N-2)(\tfrac{\gamma+1}{2})-2(\sqrt{\Lambda-\lambda}/\sqrt{\Lambda})(N-2)(\tfrac{\gamma+1}{2})\\
&=-1
\end{aligned}$$

and the order of ε is

$$\begin{aligned}
&\tfrac{N\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}-(N-2)(\tfrac{\gamma+1}{2})-\tfrac{\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}(\gamma_1/\sqrt{\Lambda})(N-2)(\tfrac{\gamma+1}{2})\\
&=\tfrac{N\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}-(N-2)(\tfrac{\gamma+1}{2})-\tfrac{\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}((\sqrt{\Lambda}-\sqrt{\Lambda-\lambda})/\sqrt{\Lambda})(N-2)(\tfrac{\gamma+1}{2})\\
&=\tfrac{N\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}-\tfrac{\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}(N-2)(\tfrac{\gamma+1}{2})-(N-2)(\tfrac{\gamma+1}{4})\\
&=(\tfrac{N-2}{2})[\tfrac{N}{2\sqrt{\Lambda-\lambda}}-\tfrac{(\sqrt{\Lambda-\lambda}+\sqrt{\Lambda})(\gamma+1)}{2\sqrt{\Lambda-\lambda}}]\\
&=0
\end{aligned}$$

we have $\int_{B(0,\delta)} U_\varepsilon^{\gamma+1}dx = K\cdot\varepsilon^{\frac{(N-2)}{4}(\gamma+1)}\int_0^{\delta\varepsilon^{-\frac{\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}}}\frac{1}{\rho}d\rho$. Then for ε small enough

$$\int v_\varepsilon^{\gamma+1} = O\Big(\varepsilon^{\frac{N-2}{4}(\gamma+1)}|\ln\varepsilon|\Big).$$

Now, if $1+\gamma \neq \frac{N}{\sqrt{\Lambda}+\sqrt{\Lambda-\lambda}}$, the order of ρ in the integrals is

$$\begin{aligned}
&N-1-(\gamma_1/\sqrt{\Lambda})(N-2)(\tfrac{\gamma+1}{2})-2(\sqrt{\Lambda-\lambda}/\sqrt{\Lambda})(N-2)(\tfrac{\gamma+1}{2})\\
&=N-1-((\sqrt{\Lambda}-\sqrt{\Lambda-\lambda})/\sqrt{\Lambda})(N-2)(\tfrac{\gamma+1}{2})-2(\sqrt{\Lambda-\lambda}/\sqrt{\Lambda})(N-2)(\tfrac{\gamma+1}{2})\\
&<-1
\end{aligned}$$

and the order of ε is

$$\begin{aligned}
&\tfrac{N\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}-(N-2)(\tfrac{\gamma+1}{2})-\tfrac{\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}(\gamma_1/\sqrt{\Lambda})(N-2)(\tfrac{\gamma+1}{2})\\
&=\tfrac{N\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}-(N-2)(\tfrac{\gamma+1}{2})-\tfrac{\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}((\sqrt{\Lambda}-\sqrt{\Lambda-\lambda})/\sqrt{\Lambda})(N-2)(\tfrac{\gamma+1}{2})\\
&=(\tfrac{N-2}{2})[\tfrac{N}{2\sqrt{\Lambda-\lambda}}-\tfrac{(\sqrt{\Lambda-\lambda}+\sqrt{\Lambda})(\gamma+1)}{2\sqrt{\Lambda-\lambda}}]\\
&<0.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\int_{B(0,\delta)} U_\varepsilon^{\gamma+1}dx\\
&=K\cdot\varepsilon^{\frac{N-2}{4}(\gamma+1)+(\frac{N-2}{2})[\frac{N}{2\sqrt{\Lambda-\lambda}}-\frac{(\sqrt{\Lambda-\lambda}+\sqrt{\Lambda})(\gamma+1)}{2\sqrt{\Lambda-\lambda}}]}\cdot\int_0^\infty \frac{\rho^{N-1}d\rho}{\rho^{\frac{\gamma_1(N-2)(\gamma+1)}{2\sqrt{\Lambda}}}[1+\rho^{2\sqrt{\Lambda-\lambda}/\sqrt{\Lambda}}]^{(N-2)(\frac{\gamma+1}{2})}}\\
&=O\Big(\varepsilon^{\frac{N-2}{4}(\gamma+1)+(\frac{N-2}{2})[\frac{N}{2\sqrt{\Lambda-\lambda}}-\frac{(\sqrt{\Lambda-\lambda}+\sqrt{\Lambda})(\gamma+1)}{2\sqrt{\Lambda-\lambda}}]}\Big)\\
&=O\Big(\varepsilon^{\frac{[N-(\gamma+1)\sqrt{\Lambda}](\frac{N-2}{2})}{2\sqrt{\Lambda-\lambda}}}\Big)\\
&=O\Big(\varepsilon^{\frac{[N-(\gamma+1)\sqrt{\Lambda}]\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}}\Big)
\end{aligned}$$

and since

$$\int_{B(0,\delta)} v_\varepsilon^{\gamma+1} dx = K \cdot \varepsilon^{\frac{[N-(\gamma+1)\sqrt{\Lambda}]\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}},$$

we get that

$$\begin{aligned}\int v_\varepsilon^{\gamma+1} &= \int_{\Omega\setminus B(0,\delta)} v_\varepsilon^{\gamma+1} + \int_{B(0,\delta)} U_\varepsilon^{\gamma+1} \\ &= O\Big(\varepsilon^{\frac{N-2}{4}(\gamma+1)}\Big) + O\Big(\varepsilon^{\frac{[N-(\gamma+1)\sqrt{\Lambda}]\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}}\Big).\end{aligned}$$

Thus,

(ii) If $1 < 1+\gamma < \frac{N}{\sqrt{\Lambda}+\sqrt{\Lambda-\lambda}}$, we have $N > (1+\gamma)\sqrt{\Lambda} + (1+\gamma)\sqrt{\Lambda-\lambda}$ and

$$\frac{N-(\gamma+1)\sqrt{\Lambda}\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}} > \frac{\Big[((1+\gamma)\sqrt{\Lambda}+(1+\gamma)\sqrt{\Lambda-\lambda}\Big)\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}} = (\gamma+1)\frac{\sqrt{\Lambda}}{2} = (\gamma+1)\frac{N-2}{4}.$$

Then for ε small enough

$$\int v_\varepsilon^{\gamma+1} = O\Big(\varepsilon^{\frac{N-2}{4}(\gamma+1)}\Big)$$

(iii) If $\frac{N}{\sqrt{\Lambda}+\sqrt{\Lambda-\lambda}} < 1+\gamma < 2$, we have $N < (1+\gamma)\sqrt{\Lambda} + (1+\gamma)\sqrt{\Lambda-\lambda}$ and

$$\frac{[N-(\gamma+1)\sqrt{\Lambda}\sqrt{\Lambda}]}{2\sqrt{\Lambda-\lambda}} < \frac{\Big[(1+\gamma)\sqrt{\Lambda}+(1+\gamma)\sqrt{\Lambda-\lambda}\Big]\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}} = (\gamma+1)\frac{\sqrt{\Lambda}}{2} = (\gamma+1)\frac{N-2}{4}$$

Then for ε small enough

$$\int v_\varepsilon^{\gamma+1} = O\left(\varepsilon^{\frac{[N-(\gamma+1)\sqrt{\Lambda}]\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}}\right).$$

□

Proposition B.0.10. *We have*

$$\frac{1}{N}T(v_\varepsilon)^{\frac{N}{2}}\Big(U(v_\varepsilon)\Big)^{1-\frac{N}{2}} < \frac{1}{N}S_\lambda^{\frac{N}{2}} - O(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}}).$$

Proof. From Proposition 2.2.9, we obtain

$$\begin{aligned}&\frac{1}{N}T(v_\varepsilon)^{\frac{N}{2}}\Big(U(v_\varepsilon)\Big)^{1-\frac{N}{2}} \\ &= \tfrac{1}{N}\Big(S_\lambda^{\frac{N}{2}} + O(\varepsilon^{\frac{N}{2}}) + O(\varepsilon^{\frac{N-2}{2}}) - O(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}})\Big)^{\frac{N}{2}}\Big(S_\lambda^{\frac{N}{2}} - O(\varepsilon^{\frac{N}{2}})\Big)^{1-\frac{N}{2}}.\end{aligned}$$

Since $(N-2)/2 < N/2$, we have $\varepsilon^{(N-2)/2} + \varepsilon^{N/2}\ \varepsilon^{(N-2)/2}$ and, by $(N-2)/2 > \frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}$, we have

$$O(\varepsilon^{(N-2)/2}) - O(\varepsilon^{\frac{N}{2}})^{1-\frac{N}{2}} < -O(\varepsilon^{\frac{N}{2}})^{1-\frac{N}{2}}.$$

Thus

$$
\begin{aligned}
&\frac{1}{N}\left(S_\lambda^{\frac{N}{2}}+O(\varepsilon^{\frac{N}{2}})+O(\varepsilon^{\frac{N-2}{2}})-O(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}})\right)^{\frac{N}{2}}\left(S_\lambda^{\frac{N}{2}}-O(\varepsilon^{\frac{N}{2}})\right)^{1-\frac{N}{2}}\\
&<\frac{1}{N}\left(S_\lambda^{\frac{N}{2}}+O(\varepsilon^{\frac{N-2}{2}})-O(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}})\right)^{\frac{N}{2}}\left(S_\lambda^{\frac{N}{2}}-O(\varepsilon^{\frac{N}{2}})\right)^{1-\frac{N}{2}}\\
&<\frac{1}{N}\left(S_\lambda^{\frac{N}{2}}-O(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}})\right)^{\frac{N}{2}}\left(S_\lambda^{\frac{N}{2}}-O(\varepsilon^{\frac{N}{2}})\right)^{1-\frac{N}{2}}\\
&=\frac{1}{N}\left(S_\lambda^{\frac{N}{2}}\right)^{\frac{N}{2}}\left(1-\frac{O(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}})}{S_\lambda^{\frac{N}{2}}}\right)^{\frac{N}{2}}\left(S_\lambda^{\frac{N}{2}}\right)^{1-\frac{N}{2}}\left(1-\frac{O(\varepsilon^{\frac{N}{2}})}{S_\lambda^{\frac{N}{2}}}\right)^{1-\frac{N}{2}}\\
&=\frac{1}{N}S_\lambda^{\frac{N^2}{4}+\frac{N}{2}(1-\frac{N}{2})}\left(1-O(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}})\right)^{\frac{N}{2}}\left(1-O(\varepsilon^{\frac{N}{2}})\right)^{1-\frac{N}{2}}\\
&=\frac{1}{N}S_\lambda^{\frac{N}{2}}\left(1-O(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}})\right)^{\frac{N}{2}}\left(1-O(\varepsilon^{\frac{N}{2}})\right)^{1-\frac{N}{2}}.
\end{aligned}
$$

Since $(1-a)^m=\sum_{i=0}^{m}\binom{m}{i}1^{m-i}(-a)^i$, $O(\varepsilon^a)+O(\varepsilon^a)=O(max(\varepsilon^a,\varepsilon^b))$, and $\varepsilon^b>\varepsilon^a$, if $b<a$, we have

$$
\left(1-O(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}})\right)^{\frac{N}{2}}=\left(1-\frac{N}{2}(O(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}})\right)
$$

as $\varepsilon\to 0$ and

$$
\left(1-O(\varepsilon^{\frac{N}{2}})\right)^{1-\frac{N}{2}}=\left(1-(1-\frac{N}{2})O(\varepsilon^{\frac{N}{2}})\right).
$$

Thus, since $kO(\varepsilon^a)=O(\varepsilon^a)$ for a constant k and $O(\varepsilon^a)O(\varepsilon^b)=O(\varepsilon^{a+b})$

$$
\frac{1}{N}T(v_\varepsilon)^{\frac{N}{2}}\left(U(v_\varepsilon)\right)^{1-\frac{N}{2}}<\tfrac{1}{N}S_\lambda^{\frac{N}{2}}\left(1-\tfrac{N}{2}O(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}})\right)\left(1-\left(1-\tfrac{N}{2}\right)O(\varepsilon^{\frac{N}{2}})\right).
$$

Then

$$
\begin{aligned}
\frac{1}{N}T(v_\varepsilon)^{\frac{N}{2}}\left(U(v_\varepsilon)\right)^{1-\frac{N}{2}}&<\tfrac{1}{N}S_\lambda^{\frac{N}{2}}\left(1-\tfrac{N}{2}O(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}})\right)\left(1-\left(1-\tfrac{N}{2}\right)O(\varepsilon^{\frac{N}{2}})\right)\\
&<\tfrac{1}{N}S_\lambda^{\frac{N}{2}}\left(1-O(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}})\right)\\
&=\tfrac{1}{N}S_\lambda^{\frac{N}{2}}-\tfrac{1}{N}S_\lambda^{\frac{N}{2}}O(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}})\\
&=\tfrac{1}{N}S_\lambda^{\frac{N}{2}}-O(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}}).
\end{aligned}
$$

□

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