Learning through patterns:
a powerful approach to algebraic thinking

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Abstract

We are engaged in a project named *Mathematics and patterns in elementary schools: perspectives and classroom experiences of students and teachers.* Our aim is to analyze the impact of an intervention centered on the study of patterns in the learning of mathematics concepts and on the development of communication and development of higher order thinking skills. In this paper we present part of an ongoing research with pre-service teachers concerning the development of teachers' algebraic thinking, in particular how they move through pattern tasks involving generalization. We will present some of the tasks used in the didactical experience and some preliminary conclusions of its implementation in the mathematics didactics classes of a mathematics elementary teachers' course of a School of Education.

Introduction

Many educators use patterns to promote generalization as a pre-algebraic activity (e.g. Mason, 1996). If algebra is a tool for expressing generalities, exploring patterns in the elementary levels lays the foundation for the algebraic reasoning. School teachers traditionally has a tendency to explore more the numerical than the visual patterns, which in many situations can be problematic to reach generalization, and consequently to get an algebraic expression or formula that generate any term of the sequence. Particularly counting tasks can be a way to develop some skills that provide students to translate visual patterns into numerical expressions and later to reach the far generalization of a pattern.

Number patterns, the relationship between variables and generalization, are considered important components of algebra curricula reform in many countries. Those curricula often use generalized number patterns as an introduction to algebra. In Portugal, these features are only now being considered in the elementary curricula development (ME, 2007). So, there is yet insufficient research with patterns at our country. As mathematics educators at training institutions that prepare pre-service teachers for K-6 grades it is important to develop such approach with these future teachers. In particular we are interested to develop some didactics strategies to enable future teachers/students to work with pattern tasks involving generalization grounded in visual/figurative contexts. In this paper we present some pattern tasks and pre-service teachers' performance where it is shown the relation between visual and numeric expressions.

Theoretical framework and methodology

As mathematics educators we believe that professional development programs focusing on helping teachers to understand both the mathematics of specific content domains and students' mathematical thinking in that domain have consistently been found to contribute to major changes in teachers' instructional practices that have resulted in significant gains in students' achievement.

We defend a constructivist perspective of mathematics learning where *Problem solving is in the heart of mathematics* (Halmos, 1980) and *Mathematics is the science of patterns* (Devlin, 2002). So, future teachers must be involved in the same kind of activities that we want them to propose their own students, providing them a mathematics teaching that allow them to know and recognize the essence and power of patterns in learning mathematics.

Patterns and problem solving

Instructional mathematics programs should enable students, from pre-kindergarten through grade 12, to engage in several tasks involving the understanding of problems, patterns, relations and functions (NCTM, 2000). Problem solving tasks challenge students and demand for high mathematical thinking skills that involve communication, conjecture, generalization, argumentation and proof. A pattern based methodology approach challenges students to use higher order thinking skills and emphases exploration, investigation, conjecture and generalization; look for a pattern is a powerful problem solving strategy.

Patterns and algebraic thinking

Patterns are an effective way to encourage students to explore important ideas in the study of algebra as conjecture and generalization (Yeats et al., 2004). If algebra is a tool for expressing generalities, exploring patterns in the elementary levels lays the foundation for the algebraic reasoning (Usiskin, 1999; Kaput, 2007) considering Algebra the generalized arithmetic (Usiskin, 1999; Kaput, 2007). Algebra is more then manipulation of variables and formulas and students must experienced generalization tasks to get formula construction for remaining flexible and creative for long periods in their search for solution methods in solving pattern problems. If teachers are not in the habit of getting students to work at expressing their own generalizations, then mathematical thinking is not taking place, in particular the algebraic thinking. Otherwise pattern tasks gave students the opportunity to observe and verbalize their own generalizations and translate them symbolically (English & Warren, 1998). As Mason et al. (1985) say *Generalization is the heartbeat of mathematics (pag.65).*

Algebraic reasoning is a process in which students generalize mathematical ideas by the observation of a set of evidences establishing those generalizations through representations and argumentations, expressing them more and more in a formal way according age (Blanton & Kaput, 2005).

"Seeing" a pattern

Patterns can suggest numerical, visual and mixed approaches (Orton, 1999, Stacey, 1989). Visualization has an important role on student reasoning (Dreyfus, 1990). The ability to develop and use visual representational forms is valuable enough that should became an integral part of mathematical learning (Stylianous and Silver, 2004). Children and young adults have been known to possess a strong intuitive, visual grasp of mathematical ideas and concepts. So, teaching of mathematics must capitalize this feature of learners (Rivera & Becker, 2005). Mathematics learning must include problems that compel students to think visually and they can develop this ability through experiences in situations that require such thinking (Tripathi, 2008). As Mason et al (1989) say, before the use of algebraic symbolism we must look to prior aspects of generalization. This is our main concern. We must pay attention to the visual/figurative features that can be related to generalization. "Seeing" is an important component of generalization that young students must explore. So teaching needs to propose challenge tasks that emphasis the figurative and numerical understanding of generalization (Rivera & Becker, 2005)

This study seeks to understand in what way a didactical experience to both elementary pre-service teachers and students of grade 1–6, grounded on figurative pattern tasks that involve generalization, can contribute to approach algebraic thinking. We adopted a qualitative exploratory approach. We followed, in a part of the study, a class with 11 elementary (grade 1–6) pre-service teachers of ESEVC of the 4th year of a mathematics elementary teachers' course of a School of Education during the mathematics didactic classes where it was implemented the didactical experience. The data was collected in a holistic, descriptive and interpretative way through observations, questionnaires and documents (e.g. worksheets, tests, individual works).

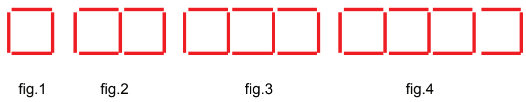
The didactical experience

The goal of the teaching experience was to promote a pattern approach to algebraic thinking through figurative tasks and explore the ways in which generalization tasks are related to figurative contexts and can be meaningfully to students. The key ideas are that patterns and algebra should be taught in combination with number concepts (Anderson & Gillard, 2004) and visual/ figurative patterns tasks can be a powerful tool to get comprehensible numerical expressions. More than develop students skills to get a formula it is important that they understand the meaning of that formula or rule and reasoning in a way to convince themselves and the others of the validity of the rule or formula they get through generalization using numerical or visual/figurative methods. The aim of this experiment was to explore growth patterns to get generalization within figurative or concrete contexts that will be translated in numerical expressions and to apply basic mathematics concepts that can be used with elementary students. We were also interested to find out how future teachers performed inductive reasoning on pattern tasks that involved arithmetical sequences of numbers or from figures. We propose three main categories: counting, sequences and problems tasks.

School teachers traditionally has a tendency to explore more the numerical than the visual patterns which, in many situations, can be too difficult to reach generalization and, consequently, to get an algebraic expression or formula that generate any term of the sequence. Seeing a pattern is a necessary first step in pattern exploration (Lee & Freiman, 2006). Observe the following well known example on figure 1.

Figure 1: A pattern task

These figures are made with toothpicks



1. How many toothpicks are needed for making the 10th figure?
2. How many toothpicks are needed for the 10th figure? Show or explain how you figured out.
3. Discover a rule for finding the number of toothpicks used in each figure. Explain with words or write a formula.

Traditionally, students as well teachers in their instruction translate the visual information into numerical information. We can look for sequence 4, 7, 10, 13, 16, … and use the inductive reasoning. Depending on the level of students, they can use a recursive method using the finite differences between consecutive numbers, as it is shown in the figure 2.

Figure 2: Numerical solution using finite differences

figure	# toothpicks		# toothpicks
1	4	+ 3	4
2	7		4 + 3
3	10	+ 3	7 + 3
4	13		10 + 3
5	16	+ 3	13 + 3
6	19		16+3
7	22		19+3

That can be translated in this form of "seeing"

Despite the recursive method it is a good strategy for near generalization it is not for distant generalization. So, we have to get other approaches. We can get other "seeings" of the pattern. There are many ways to do it. For instance, we can see in another way the pattern

That can be translated into the numerical chart

Figure 3: Another Numerical solution

figure	# toothpticks	# toothpticks
1	4	4
2	4+3	4 + 2 x 3
3	4+3+3	4 + 3 x 3
4	4+3+3+3	4 + 4 x 3
5	4+3+3+3+3	4 + 5 x 3
…		
n	4+3+3+3+3+…+3	4 + (n-1) x 3

And now we get a general rule that can be translated into a numerical sequence and an algebraic expression or formula that generate any term of the sequence in a easier way using basic mathematics concepts. Reciprocally they can understand and translate an expression into a visual sequence. They can get a rule explained with words or using a formula.

This example shows our thesis—an instruction that promote "seeing" the arrays in different ways we promote algebraic thinking and young students can get a numerical expression more easily. But, to promote this way of thinking, instructional programs must adopt it with (future) teachers.

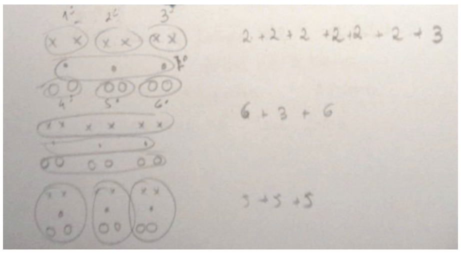
Some Results

We present some resolutions of some tasks of the different categories of the experiment.

Counting

A sound arithmetic foundation is required for the learning of algebra. So, patterns and Algebra should be taught in combination with number concepts (Anderson & Gillard, 2004). Numerical tasks of counting have their foundations on the recognition of patterns. Counting tasks—find different ways of counting for choosing the best one—can be a way to develop some skills that provide students to translate visual patterns into numerical expressions and later to reach the far generalization of a pattern. Visual patterns tasks are a tool to get comprehensible numerical expressions: writing and see equivalence.

Task: Find different ways (as many as you can) to count the elements of the figures. Record each way as a numerical sentence.



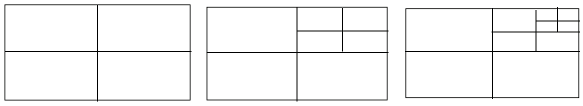
The written work of this student shows three of the most common ways of seeing. Besides, promotes mental computation these kind of tasks give students the foundation of algebraic thinking.

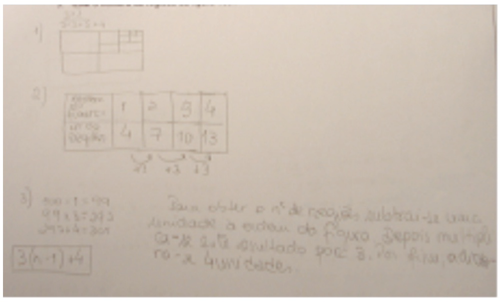
Sequences with figures

To look for patterns in sequences (concrete, numerical, figurative) to reach generalization through rules that students formulate using symbols allow algebra learning in a gradual manner and reach abstraction.

We expected that students reach near and far generalization and represent mathematical ideas in different ways. Different solutions can be obtained through different ways of seeing that correspond to different algebraic expressions and allow them to understand their equivalence. Near generalization gives students an opportunity to get recursive reasoning while far generalization is an opportunity to get functional reasoning

Task: ***Observe the sequence of rectangles***





In this example we note that the student uses recursive reasoning through finite difference to get near generalization. Nevertheless, to get a far generalization, or the formula that translate the rule, he goes through another approach, perhaps grounded in the draws of the sequence.

1. Construct next figure
2. How many regions have each figure?
3 How many regions are needed for the 100th figure? Show or explain how you figured out.

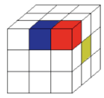
In this example we note that the student uses recursive reasoning through finite difference to get near generalization. Nevertheless, to get a far generalization, or the formula that translate the rule, he goes through another approach, perhaps grounded in the draws of the sequence.

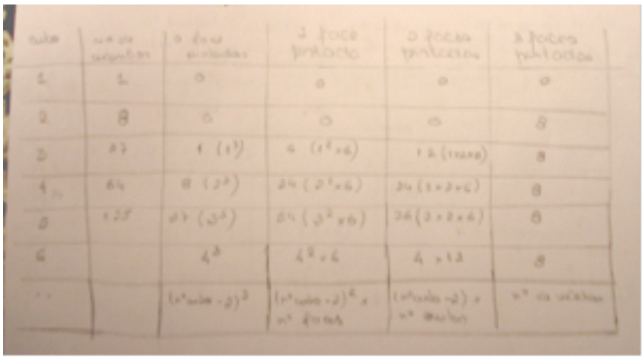
Problems

To look for a pattern is a powerful strategy of problem solving.

With these tasks, students have to construct their own sequences and discover the pattern for reach generalization and consequently get the solution. Far generalization can be reached through formulas or verbally depend on the level of the solver.

Task: ***The Osvald game***
Osvald was playing with a cube and decide to paint in red all its faces. As he needs more cubes, decided to cut it in small cubes with the same dimensions. How many small cubes get Osvald from the big one? How many faces were painted? Try with different number of cuts. Get expressions to indicate the number of small cubes that have painted faces.





This resolution shows that student reduce the initial problem to a simple one, used a table to record all the elements that he gets from the text and then look for a pattern to reach the solution.

He had transformed the numbers he gets in each column in order to reach easily the far generalization.

Preliminary conclusions

Teachers must re-learn their algebraic concepts in a way that can help students to reasoning algebraic meaningfully by making connections between figurative and numerical strategies. This instructional process promote students to use different representations for generate a rule or formula. Then, they will be aware to get several different formulas by different ways of seeing to reach the solution.

We haven't yet analyse all the data, but we can say that these kind of tasks allowed: to motivate students; to develop mathematical communication skills; to establish connections, namely between numbers and geometry; to give some comprehension about expressions and relationship with visual representation; to experience new situations promoting different strategies for counting; to look for different 'seeings', when they were working with numerical sequences and problems. That way, they develop communication and problem solving strategies and skills.

References

Alvarenga, D. e Vale, I. (prelo). *A exploração de problemas de padrão: um contributo para o desenvolvimento do pensamento algébrico*, *Quadrante.*

Arcavi, A. (2006). El desarrolo y el uso del sentido de los números. Em Vale, I. et al. (org.), *Números e álgebra* (pp.29–48). Lisboa: SPCE.

Blanton, M., & Kaput, J. (2000). Generalizing and progressively formalizing in a third grade mathematics classroom: conversations about even and odd numbers. In M. Fernández (Ed.), *Proceedings of the 20th annual meeting of the psychology of mathematics education*, pp. 115. Columbus: ERIC Clearinghouse (ED446945).

Carraher, D., Martinez, M., & Schliemann, A. (2008). Early algebra and mathematical genralization, *ZDM Mathematics Education*, 40, 3–22.

Devlin, K. (2002). *Matemática: a ciência dos padrões*. Porto: Porto Editora.

Dreyfus, T. (1990). Advanced Mathematical Thinking. *Mathematics and Cognition: A Research Synthesis by the International Group for the Psychology of Mathematics Education.* Cambridge: Cambridge University Press.

Kaput, J., Carraher, D. W., & Blanton, M. (Eds.). (2007). *Algebra in the early grades.* Hillsdale/Reston: Erlbaum/NCTM.

Mason, J. (1996), Expressing Generality and Roots of Algebra. In N. Bednarz, C. Kieran and L. Lee (eds.), *Approaches to Algebra, Perspectives for Research and Teaching* (pp. 65–86). Dordrecht: Kluwer Academic Publishers.

Mason, J., Burton, L., & Stacey, K. (1985). *Thinking Mathematically*. Bristol: Adiisson-Wesley.

Mason, J., Graham, A., Pimm, D., & Gowar, N.: 1985, *Routes to/roots of algebra*. Milton Keynes, UK: Open University Press.

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra,* (pp. 65–86). Dordrecht: Kluwer.

ME-DGIDC (2007). *Programa de Matemática do Ensino Básico*. Lisboa: Ministério da Educação, Departamento de Educação Básica.
NCTM (2000). *Principles and Standards for School Mathematics*. Reston: NCTM.
Noss, R., Healy, L., & Hoyles, C. (1997). The Construction of Mathematical Meanings: Connecting the Visual with the Symbolic. *Educational Studies in Mathematics 33*(2), pp. 203–33.
Orton, A. (1999). *Pattern in the teaching and learning of mathematics.* London: Cassel.
Pólya, G. (1973). *How to Solve it.* Princeton: Princeton University Press.
Rivera, F., & Becker, J. (2005). *Figural and numerical modes of generalization in Algebra, Mathematics Teaching in the middle school*, pp.198–203
Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition and sense making in mathematics. In D. A. Grouws (ed.), *Handbook of research on mathematics teaching and learning*. New York: Macmillan.
Smith, E. (2003). *Stasis and change: Integrating paterns functions and algebra throughout the K-12 curriculum*. Reston:NCTM
Stacey, K. (1989). Finding and Using Patterns in Linear Generalising Problems. *Educational Studies in Mathematics 20*(2), pp. 147–164.
Steen, L. A. (1988). The Science of Patterns, *Science*, 240, 611–616.
Sword, L. (2000). I Think in Pictures, You Teach in Words: The Gifted Visual Spatial Learner. *Gifted*. #114, pp.1 and 27–30.
Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In A. Coxford, & A. P. Schulte (Eds.), *Ideas of algebra*, K-12: 1988 Yearbook (pp. 8–19). Reston, VA:

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