ON THE STABILITY OF LINEAR FRACTIONAL DIFFERENCE SYSTEMS

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Abstract: A fractional linear system is defined by differential or difference equations of non-integer order. A well-known result about the stability of fractional differential systems will be extended to discrete-time systems defined by fractional difference equations. This will be accomplished using *time scales*, which permit to unify continuous and discrete-time systems.

Keywords: Linear dynamical systems, Stability, Fractional calculus, Time scales

1. INTRODUCTION

Leibniz and L'Hôpital, among others, where still developing the first rules of differential calculus as they began to wonder about the possible meaning of non-integer order derivatives. That mathematical curiosity grew steadily, since then, receiving plenty of answers, both of applied and theoretical nature.

Actually, due to their *long memory* transients or correlations (which, mathematically, are related to power laws instead of the usual exponential laws), mathematical models based on fractional order derivatives and differences have already been applied for more than fifty years to a wide range of areas, especially when dealing with physical phenomena – for instance, hydrological time series and medical images (Hosking, 1984; Lundahl *et al.*, 1986).

On the other hand, from a more mathematical point of view, various extensions of the integer order derivative or difference of a function were proposed, giving rise, so to speak, to more than one fractional calculus theory (Kilbas *et al.*, 2006).

Nevertheless, even if the properties of each derivative are somewhat different, they frequently exhibit common features. This is the case for this paper's starting point: stability of fractional differential systems.

Indeed, as it will be shown in Section 2, the stability of the fractional autonomous system $x^{(\alpha)}(t) = \lambda x(t)$, depends on λ and $\alpha \in \mathbb{R}$, i.e., the non-integer order of the differential equation, but not on the specific definition of the derivative.

However, to the authors' knowledge, no such result exists for discrete-time systems described by fractional difference equations (see Ortigueira (2000) for a different approach). The notion of time-scale will be introduced in Section 3 in order to establish a link between the continuous and discrete-time case, whose stability is investigated in Section 4.

Finally, after introducing the principal tool for the proof in Section 5, the main result will be obtained in Section 6, characterizing a class of asymptotically stable fractional discrete-time linear systems.

2. FRACTIONAL DIFFERENTIAL SYSTEMS

The main definitions given in Vettori (2010) will be recalled in this section.

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Usually, fractional derivatives and integrals are introduced by extending Cauchy's formula to non-integer values. Indeed, the linear operator

$$D_c^{-1} : f \mapsto \int_c^t f(\tau)\,\mathrm{d}\tau$$

which calculates a definite integral of f can be iterated obtaining the n-th integral

$$D_c^{-n} : f \mapsto \int_c^t \frac{(t-\tau)^{n-1}}{(n-1)!} f(\tau)\ \mathrm{d}\tau. \tag{2.1}$$

This, noting that $(n-1)! = \Gamma(n)$, defines the integral operator of non-integer order $-\alpha > 0$

$$D_c^{\alpha} : f \mapsto \int_c^t \frac{(t-\tau)^{-\alpha-1}}{\Gamma(-\alpha)} f(\tau)\,\mathrm{d}\tau. \tag{2.2}$$

The derivative, which would correspond to values $\alpha \geq 0$, cannot be calculated due to convergence issues. So, unfortunately, the fractional derivatives have to be defined in different ways. For instance, as the composition of a (normal) derivative and a fractional integral: if $m-1 < \alpha < m$, with $m \in \mathbb{N}$, then the Riemann-Liouville and the Caputo derivatives of order $\alpha > 0$ are, respectively, $\frac{\mathrm{d}^m}{\mathrm{d}t^m} D_c^{\alpha-m}$ and $D_c^{\alpha-m} \frac{\mathrm{d}^m}{\mathrm{d}t^m}$ (observe that the order of the composition is reversed). See Gorenflo and Mainardi (1997) for more details.

In this paper, a different approach is chosen: note that, when $\alpha < 0$, $D_0^{\alpha} f$ is a classical convolution of f and the function given by $\frac{1}{\Gamma(-\alpha)} t^{-\alpha-1}$ for $t > 0$. Since its Laplace transform, denoted by $\mathscr{L}$, is s^{α}, it turns out that

$$\mathscr{L}[D_0^{\alpha} f](s) = s^{\alpha} \mathscr{L}[f](s)$$

when α is negative, being this exactly the Laplace transform of integration in the integer case. However, when $\alpha > 0$, the classical transform of derivatives depends on the *initial conditions* and, actually, initial conditions appear, in different ways, in the Laplace transforms of the Riemann-Liouville and of the Caputo derivatives too (Gorenflo and Mainardi, 1997).

Here, initial conditions will be simply ignored: this is made possible by a suitable choice of the functions space (smooth functions with left compact support) and by defining the convolution on $\mathbb{R}$ and not just on $\mathbb{R}^+$. Anyway, the consequences are the following:

- it is not necessary to specify the initial point c of integration, so the operator is just D^{α};
- when $\alpha > 0$, D^{α} represents a convolution with a (well-defined) distribution, which is the (distributional) inverse of $\frac{1}{\Gamma(\alpha)} t^{\alpha-1}$;
- the exponent law $D^{\alpha} D^{\beta} = D^{\alpha+\beta}$ holds for any α and β, since there is correspondence of the application (or composition) of D^{α} with the multiplication by s^{α} in the Laplace transform domain for every $\alpha \in \mathbb{R}$.

Now, with three different definitions of fractional derivative, it seems hard to say anything about the stability of a dynamical system like

$$x^{(\alpha)}(t) = \lambda x(t), \tag{2.3}$$

where $x^{(\alpha)}$ is some fractional derivative of order α. Luckily, this is not the case. In fact, the following result holds (see also Figure 2).

Theorem 1. For any $0 < \alpha < 2$, the autonomous system (2.3) is asymptotically stable if and only if

$$|\arg \lambda| > \frac{\alpha\pi}{2}.$$

Theorem 1, which is here proposed in a simplified form, was first stated in Matignon (1996). In that paper, fractional derivatives are defined, as D^{α}, through convolution with distributions and, by *smoothing* them, Caputo derivatives are obtained. There, the proof of the theorem is only sketched, but many subsequent papers were devoted to prove the same result for different derivatives (Qian *et al.*, 2010; Zhang and Li, 2011).

For this reason, only the simplest definition of D^{α} will be extended to fractional difference operators in Section 6, in order to obtain a discrete-time equivalent of Theorem 1.

Remark 2. Both stability and asymptotic stability are characterized in Matignon (1996). Here, for the sake of simplicity, the term *stability* will be used instead of *asymptotic stability*, meaning that all the solutions of the dynamical system converge to zero.

Anyway, just a few technical details would be necessary to extend this paper's main result to analyze both stable and asymptotically stable systems.

3. TIME SCALES

The theory of time scales will not be used here in its entire generality. More details may be found, for example, in the books by Bohner and Peterson (2001) and Bohner and Peterson (2003).

The time scales theory aims at unifying continuous and discrete analysis by defining *differential* calculus on any closed set $\mathbb{T} \subseteq \mathbb{R}$, called *time scale*. In this paper, only the particular cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = h\mathbb{Z} = \{hn : n \in \mathbb{Z}\}$, with $h > 0$, will be considered, whereas a general time scale $\mathbb{T}$ may contain both intervals and isolated points.

The usual derivative can be extended in various ways to the discrete case and, therefore, to a general time scale. Indeed, the theory of time scales proposes the notion of α calculus, where α is a suitable function acting as a parameter in the definition of the derivative. [1]

Definition 3. (Bohner and Peterson, 2003, p. 12) The α *derivative* of $f : \mathbb{T} \to \mathbb{R}$ at $t \in \mathbb{T}$ is the number

[1] Unfortunately, the symbol α is used in this contribution, and in the literature, to denote the order of fractional derivatives. However, α as a function will be used only in this section to define the derivative on time scales that will be used afterwards.

$f_\alpha(t)$, provided it exists, such that for any $\varepsilon > 0$ there is a neighborhood U of t such that for any $\tau \in U$,

$$|f\big(\alpha(t)\big) - f(\tau) - f_\alpha(t)\big(\alpha(t) - \tau\big)| \leq \varepsilon|\alpha(t) - \tau|.$$

The two principal *concretizations* of the α derivative, both presented in Bohner and Peterson (2003), are:

- Δ (delta) calculus — $\alpha(t) = \inf\{\tau \in \mathbb{T} : \tau > t\}$;
- ∇ (nabla) calculus — $\alpha(t) = \sup\{\tau \in \mathbb{T} : \tau < t\}$.

Observe that, when $\mathbb{T} = \mathbb{R}$, in both cases $\alpha(t) = t$ and it is not difficult to prove that $f_\alpha = f'$.

In this paper, the second type of derivative has been chosen, mainly for two reasons.

First of all, it is clear that the Δ derivative of f at t depends on $f(\tau)$ with $\tau \geq t$, i.e., on the *future*, while the ∇ derivative depends on $f(\tau)$ with $\tau \leq t$, i.e., on the *past*. Indeed, when the time scale is $\mathbb{T} = h\mathbb{Z}$, then

$$\nabla f(t) = f_\alpha(t) = \frac{f(t) - f(t-h)}{h}. \tag{3.1}$$

So, since when dealing with dynamical systems the *time's arrow* usually points towards the future, a past-dependent derivations is way more reasonable.

Secondly, calculations with the fractional Δ derivative would give somewhat strange results, as it becomes clear just comparing the examples presented in Atıcı and Eloe (2007) (Δ derivative) and Atıcı and Eloe (2009) (∇ derivative): in the first case the time scale has to be changed according to the fractional order of the derivative!

4. STABILITY OF DISCRETE SYSTEMS

In this section, the stability of the scalar difference equation

$$\nabla x(t) = \lambda x(t),\ t \in \mathbb{T} = h\mathbb{Z} \tag{4.1}$$

and its relation with the nabla Laplace transform will be investigated.

Note that equation (4.1) is not exactly the same as the difference equation $x(t+h) = \lambda x(t)$. Indeed, its solutions are

- $x(t) = 0$, if $h\lambda = 1$;
- $x(t) = K(1-h\lambda)^{-\frac{t}{h}}$, if $h\lambda \neq 1$, for any $K \in \mathbb{R}$ (remember that $\frac{t}{h} \in \mathbb{Z}$),

which can be easily verified, since

$$\begin{aligned}\nabla x(t) = \frac{x(t) - x(t-h)}{h} =& \lambda x(t) \Leftrightarrow \\ (1-h\lambda)x(t) =& x(t-h).\end{aligned}$$

Observe that the function $e_\lambda(t) = (1-h\lambda)^{-\frac{t}{h}}$ plays the role of the $e^{\lambda t}$ in the nabla calculus. As a matter of fact, besides being $e_0(t) = 1$, if we let h tend to zero, i.e., *turn* $h\mathbb{Z}$ into $\mathbb{R}$, we obtain that

$$\begin{aligned}\lim_{h\to 0} e_\lambda(t) &= \lim_{h\to 0}(1-h\lambda)^{-\frac{t}{h}} \\ &= \lim_{h\to 0}\left((1-h\lambda)^{\frac{1}{-h\lambda}}\right)^{\lambda t} = e^{\lambda t}.\end{aligned}$$

Using the nabla exponential function $e_\lambda(t)$, it is possible to define the nabla Laplace transform, which, according to Bohner and Peterson (2003, ch. 3) and Akin-Bohner and Bohner (2004), is

$$\begin{aligned}\mathscr{L}^\nabla[x](s) &= \int_0^\infty x(t)e_{\ominus s}(t-h)\nabla t \\ &= h\sum_{i=1}^\infty x(ih)e_{\ominus s}(ih-h) \\ &= h\sum_{i=0}^\infty x(ih+h)e_{\ominus s}(ih),\end{aligned}$$

where $\ominus s = -\frac{s}{1-hs}$. Actually, as one would expect,

$$e_{\ominus s}(ih) = \left(1 - h\frac{-s}{1-hs}\right)^{-i} = (1-hs)^i$$

is the inverse of $e_s(ih) = (1-hs)^{-i}$.

We can now calculate the nabla Laplace transform of $e_\lambda(t)$ that, neglecting convergence issues which may be found in Davis *et al.* (2010), is

$$\begin{aligned}\mathscr{L}^\nabla[e_\lambda(t)](s) =& h\sum_{i=0}^\infty (1-h\lambda)^{-i-1}(1-hs)^i \\ =& \frac{h}{1-h\lambda}\sum_{i=0}^\infty\left(\frac{1-hs}{1-h\lambda}\right)^i \\ =& \frac{h}{1-h\lambda}\cdot\frac{1}{1-\frac{1-hs}{1-h\lambda}} \\ =& \frac{1}{s-\lambda}.\end{aligned}$$

Formally, this is exactly the classic Laplace transform of $e^{\lambda t}$, related to the (asymptotic) stability of $x'(t) = \lambda x(t)$ through its pole λ: for the system to be stable, it must lie in the open left half plane, i.e., satisfy the well-known condition $\operatorname{Re}\lambda < 0$.

However, even if λ is a pole in any time scale we are considering, what is the stability condition if $\mathbb{T} = h\mathbb{Z}$? The answer is given by the expression of the nabla exponential: $e_\lambda(t) = (1-h\lambda)^{-\frac{t}{h}}$ converges if and only if $|1-h\lambda| > 1$. Thus, we may conclude that:

Proposition 4. The dynamical system defined by the equation $\nabla x(t) = \lambda x(t)$, $t \in \mathbb{T} = h\mathbb{Z}$, is asymptotically stable if and only if $\lambda \in \mathbb{C}$ lies outside the closed disk centered at $\frac{1}{h}$ and passing through the origin.

In the limit, as $h \to 0$, i.e., $\mathbb{T} = \mathbb{R}$ and $\nabla x = x'$, the condition becomes $\operatorname{Re}\lambda < 0$.

Proof. Note that $|1-h\lambda| = 1 \Leftrightarrow |\frac{1}{h} - \lambda| = \frac{1}{h}$, which is the equation of a circle on the complex plane with center and radius $\frac{1}{h}$, thus the stability condition is verified. To check the limit condition, let $\lambda = a + ib$:

$$\begin{aligned}|\tfrac{1}{h} - \lambda| > \tfrac{1}{h} \Leftrightarrow |\tfrac{1}{h} - a - ib|^2 = (\tfrac{1}{h} - a)^2 + b^2 > \tfrac{1}{h^2} \\ \Leftrightarrow -\tfrac{2}{h}a + a^2 + b^2 > 0 \Leftrightarrow \operatorname{Re}\lambda < \tfrac{h|\lambda|^2}{2}\end{aligned}$$

and, as $h \to 0$, $\operatorname{Re}\lambda < 0$. □

Remark 5. Straightforward calculations show that to check the stability of $\nabla x(t) = Ax(t)$, where A is a square matrix, it is sufficient to verify if each eigenvalue of A satisfies the condition of Proposition 4.

5. A FAMILY OF MÖBIUS TRANSFORMATIONS

In this section, the result of Proposition 4 will be further analyzed, in order to find the convergence region for fractional nabla equations.

For notational convenience, let us use the complex variable s for the Laplace transforms associated with the continuous-time system $x'(t) = \lambda x(t)$, $t \in \mathbb{R}$, and the complex variable z for the nabla Laplace transform used in the study of $\nabla x(t) = \lambda x(t)$, $t \in \mathbb{T}$.

As we saw, the regions of stability (or instability) of these systems are defined by lines and circles, which can be mapped one into the other by a well-known invertible function, called Möbius transformation.

In this case, as it will be verified below, the family of transformations, depending on the parameter h, are given by the following formulas:

$$z = m_h(s) = \frac{2s}{2+hs}, \quad s = m_h^{-1}(z) = \frac{2z}{2-hz}. \tag{5.1}$$

First, observe that m_h are bijective maps of the extended complex plane $\mathbb{C} \cup \{\infty\}$, establishing the following correspondences:

$$z \leftrightarrow s: \ 0 \leftrightarrow 0, \ \tfrac{1}{h} \leftrightarrow \tfrac{2}{h}, \ \tfrac{2}{h} \leftrightarrow \infty, \ \infty \leftrightarrow -\tfrac{2}{h}. \tag{5.2}$$

Indeed, lines in the s plane, which *contain* ∞, are mapped to circles or lines passing through $z = \frac{2}{h}$.

In particular, the imaginary axis on the s plane (the border of the stable region) is associated with the circle $|z - \frac{1}{h}| = \frac{1}{h} \Leftrightarrow |1 - hz| = 1$. Indeed, if $s = iy$,

$$|1 - hz| = \left|1 - h\frac{i2y}{2+ihy}\right| = \left|\frac{2-ihy}{2+ihy}\right| = 1,$$

since numerator and denominator are conjugated. Besides, it is easy to check that if $y > 0$, also $\operatorname{Im} z > 0$.

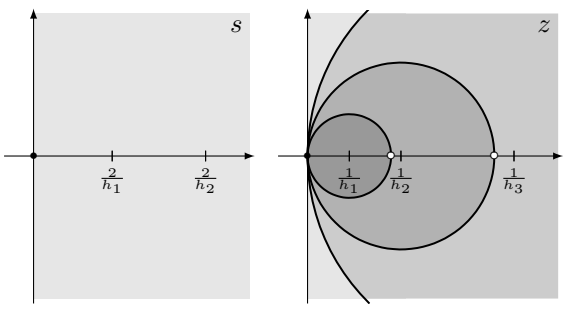

Fig. 1. Instability regions in the s and in the z planes

So, to conclude that the transformations m_h map bijectively stable regions onto stable regions, it is sufficient to observe that, by (5.2), the center of the circle $z = \frac{1}{h}$ and, therefore, all unstable points, are mapped to the unstable region *around* $s = \frac{2}{h}$.

The regions of instability of these systems are showed in Figure 1, where $h_1 > h_2 > h_3$.

Remark 6. These, and the following figures, highlight the instable instead of the stable regions for a solely graphical reason, since it is simpler to draw and understand pictures representing the former ones.

As a last observation, note that, when $h \to 0$, the transformations m_h tend to the identity, as expected, because the s and the z plane are associated with the same equation when $\mathbb{T} = \mathbb{R}$.

6. STABILITY OF DISCRETE FRACTIONAL SYSTEMS

As shown in Vettori (2010) (where only the time scale $\mathbb{T} = \mathbb{Z}$ was considered, i.e., the case $h = 1$), the ideas presented in Section 2 may be applied to the operator nabla, instead of D, leading in this way to a (possible) definition for the fractional nabla derivative.

To our purposes, out of the results of Vettori (2010) the following facts must be recalled (which, *mutatis mutandis*, are easily extended to $h\mathbb{Z}$).

(1) The fractional nabla derivative ∇^α, with order $\alpha \in \mathbb{R}$, may be defined as a convolution with a suitable function — in the discrete-time case there is no need to use distribution theory.
(2) The calculus of the nabla derivative of order α corresponds, in the nabla Laplace transform domain, to multiplication by s^α.

By the first fact, many difficulties which rise in the continuous case, due to the use of distributions, vanish in the discrete case. However, the main tool that was used to prove Theorem 1 (i.e., special functions which solve equation (2.3), thus generalizing the exponential function to the fractional case), have been deeply studied for more than a century while, on the other hand, the solutions of the time scales equation

$$\nabla^\alpha x(t) = \lambda x(t), \ t \in \mathbb{T}, \tag{6.1}$$

(i.e., *generalizations* to the fractional case of *generalizations* to the time scales case of the exponential) have not been defined yet, as far as the authors know.

Nevertheless, thanks to the second fact, it will be shown that the stability of equation (6.1) can be analyzed for time scales $\mathbb{T} = h\mathbb{Z}$ by using, indirectly, Theorem 1.

Actually, solving the equation $\nabla x(t) = \lambda x(t)$ means finding the kernel of the operator $\nabla - \lambda$, whose nabla Laplace transform is the *characteristic polynomial* $z - \lambda$: the same in every time scale. On the contrary, checking stability is equivalent to knowing if λ belongs or not to the stability region: this *depends* on the time scale, i.e., on h. However, as we saw in Section 5, the stability region can be deduced, for any h, from the continuous case region $\operatorname{Re} s < 0$, just applying the Möbius transformation m_h (5.1).

The same happens in the fractional case: the equation $\nabla^\alpha x(t) = \lambda x(t)$ has characteristic *function* $z^\alpha - \lambda$ in any time scale, but the solutions and their stability vary depending on h. Again, once α is given, the stability region associated with the time scale $h\mathbb{Z}$ can be obtained from the corresponding continuous time region $|\arg s| > \frac{\alpha\pi}{2}$, leading to the following result.

Theorem 7. Let $0 < \alpha < 2$ and define $c = \cot\frac{\alpha\pi}{2}$. Then system (6.1), with $\mathbb{T} = h\mathbb{Z}$, is asymptotically stable if and only if

$$\operatorname{Re}\lambda < \frac{h|\lambda|^2}{2} + c|\operatorname{Im}\lambda|. \tag{6.2}$$

Remark 8. Before proving the theorem, observe that the constant c is well defined for every α, being their relation bijective, since $\alpha = 1 - \frac{2}{\pi}\arctan c$.

Furthermore, the theorem extends the results that have been shown so far, because

- if $\alpha = 1$ then $c = 0$ and condition (6.2) becomes $\operatorname{Re}\lambda < \frac{h}{2}|\lambda|^2$, which is exactly the formula obtained in the proof of Proposition 4 and
- as $h \to 0$, condition (6.2) becomes $\operatorname{Re}\lambda < c|\operatorname{Im}\lambda|$, which is equivalent to the condition $|\arg\lambda| > \frac{\alpha\pi}{2}$ of Theorem 1, as it will be shown in the first part of the proof in equation 6.3.

Finally, similarly to what was stated in Remark 5 about Proposition 4, also Theorem 7 has a direct generalization to the stability analysis of the system

$$\nabla^\alpha x(t) = Ax(t),$$

where A is a square matrix, by checking condition (6.2) on its eigenvalues.

Proof. First of all, let $s = x + iy$ and note that equation $\arg s = \frac{\alpha\pi}{2}$ represents the half-lines leaving the origin with angle $\frac{\alpha\pi}{2}$, i.e., is equivalent to equation $x = cy$ for $y \geq 0$, where $c = \cot\frac{\alpha\pi}{2}$. Therefore, with the *graphical* help of Figure 2, where the points satisfying $x = c|y|$ are highlighted, the region of stability of Theorem 1 is defined by

$$|\arg s| > \frac{\alpha\pi}{2} \Leftrightarrow x < c|y|. \tag{6.3}$$

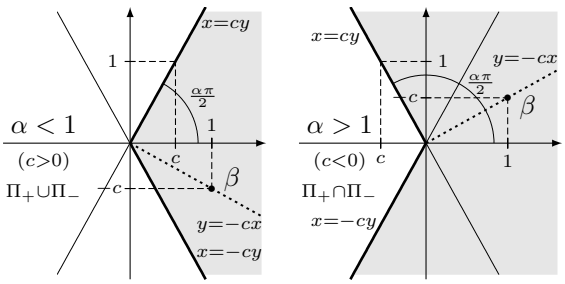

Fig. 2. Instability regions for $\alpha<1$ and $\alpha>1$ (s plane)

By looking at Figure 2, it should be clear that it is possible to write condition 6.3 without using the absolute value. Indeed, consider the following half planes

$$\Pi_+ = \{s : x < cy\} \text{ and } \Pi_- = \{s : x < -cy\}$$

(which contain the negative real axis) and observe that the stability region is $\Pi_+ \cup \Pi_-$ when $c > 0$ and $\Pi_+ \cap \Pi_-$ when $c < 0$ (being equal to $\Pi_+ = \Pi_-$ in the non-fractional case $c = 0$).

In spite of the apparent complication, it is much simpler to determine the corresponding regions

$$\mathcal{C}_+ = m_h(\Pi_+) \text{ and } \mathcal{C}_- = m_h(\Pi_-)$$

of the z plane. Therefore, the stability region of system 6.1 will be obtained by calculating the union, when $c > 0$, or the intersection, when $c < 0$, of $\mathcal{C}_+$ and $\mathcal{C}_-$.

So, in order to characterize $\mathcal{C}_+$ ($\mathcal{C}_-$ will be deduced by symmetry), it is sufficient to *transform* through m_h the defining condition of Π_+, i.e., $x < cy \Leftrightarrow x - cy < 0$.

However, to easier *apply* the transformation m_h to this condition, let us write it using the complex variable s: first, define $\beta = 1 - ic$ that belongs to line $y = -cx$, which is orthogonal (at the origin) to $x = cy$, as shown in Figure 2. Then note that

$$s\bar\beta + \bar s\beta = 2\operatorname{Re}\big((x+iy)(1+ic)\big) = 2(x-cy) < 0.$$

Hence, by (5.1) and being $\beta + \bar\beta = 2$, it follows that

$$\begin{aligned} s\bar\beta + \bar s\beta =& \tfrac{2z\bar\beta}{2-hz} + \tfrac{2\bar z\beta}{2-h\bar z} \\ =& 2\tfrac{z\bar\beta(2-h\bar z)+\bar z\beta(2-hz)}{|2-hz|^2} < 0 \\ \Leftrightarrow& z\bar\beta(2-h\bar z) + \bar z\beta(2-hz) < 0 \\ \Leftrightarrow& 2z\bar\beta - hz\bar z\bar\beta + 2\bar z\beta - hz\bar z\beta < 0 \\ \Leftrightarrow& -2h\big(z\bar z - z\tfrac{\bar\beta}{h} - \bar z\tfrac{\beta}{h}\big) < 0 \\ \Leftrightarrow& z\bar z - z\tfrac{\bar\beta}{h} - \bar z\tfrac{\beta}{h} > 0 \\ \Leftrightarrow& \big(z - \tfrac{\beta}{h}\big)\big(\bar z - \tfrac{\bar\beta}{h}\big) > \tfrac{\beta\bar\beta}{h^2} \\ \Leftrightarrow& \big|z - \tfrac{\beta}{h}\big| > \tfrac{|\beta|}{h}. \end{aligned} \tag{6.4}$$

So, in the end, $\mathcal{C}_+$ ($\mathcal{C}_-$) is the exterior of the circle with center $\frac{\beta}{h}$ ($\frac{\bar\beta}{h}$) and passing through the origin where, since the center lies on the line containing β ($\bar\beta$), it is tangent to $x = cy$ ($x = -cy$), as is shown in Figure 3.

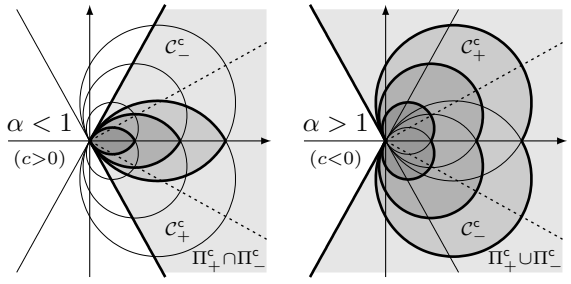

Fig. 3. Instability regions for $\alpha<1$ and $\alpha>1$ (z plane)

Observe that the instability regions depicted in Figure 3, being the complements of the stability regions,

are $\mathcal{C}_+^{\mathsf{c}} \cap \mathcal{C}_-^{\mathsf{c}}$ when $c > 0$ and $\mathcal{C}_+^{\mathsf{c}} \cup \mathcal{C}_-^{\mathsf{c}}$ when $c < 0$ (where the small c denotes the complement). Graphically, due to the mentioned tangential condition, it is clear that as $h \to 0$, i.e., as the circles get bigger, $\mathcal{C}_\pm \to \Pi_\pm$ (and the same happens with unions, intersections and complements).

Finally, in order to characterize the stability region analytically and obtain condition (6.2), notice that (6.4) is equivalent to $|z|^2 - \frac{2}{h}\operatorname{Re}(z\bar{\beta}) > 0 \Leftrightarrow \operatorname{Re}(z\bar{\beta}) - \frac{h}{2}|z|^2 > 0$. So, by letting $z = a + ib$, the condition which defines $\mathcal{C}_+$ becomes

$$\begin{aligned}\operatorname{Re}\big((a+ib)(1+ic)\big) - \tfrac{h}{2}|z|^2 =& a - bc - \tfrac{h}{2}|z|^2 < 0\\ \Leftrightarrow& a - \tfrac{h}{2}|z|^2 < bc.\end{aligned}$$

Analogously, $a - \frac{h}{2}|z|^2 < -bc$ defines $\mathcal{C}_-$. Thus, the conditions which characterize both $\mathcal{C}_+$ and $\mathcal{C}_-$ are

$$z = a + ib \in \mathcal{C}_\pm \Leftrightarrow a - \tfrac{h}{2}|z|^2 < \pm bc.$$

Now, observe the following.

- At least one of the two conditions is satisfied (union) if and only if
$$a - \tfrac{h}{2}|z|^2 < |bc|.$$
However, $\mathcal{C}_+ \cup \mathcal{C}_-$ is the stability region only when $c > 0$, whence the condition becomes, in this case, $a - \frac{h}{2}|z|^2 < |b|c$.
- On the other hand, both conditions are satisfied (intersection) if and only if
$$a - \tfrac{h}{2}|z|^2 < -|bc|.$$
Since the stability region is equal to $\mathcal{C}_+ \cap \mathcal{C}_-$ only when $c < 0$, the equivalent condition is $a - \frac{h}{2}|z|^2 < -|b|(-c) = |b|c$.

Therefore, in both cases, $\operatorname{Re} z - \frac{h}{2}|z|^2 < c|\operatorname{Im} z|$, which concludes the proof. □

7. CONCLUSIONS

In this paper a necessary and sufficient condition for the asymptotic stability of linear fractional discrete-time autonomous systems has been established. Although the system was considered in the scalar case, the vector case is a straightforward generalization.

The result has not been obtained analyzing the convergence of the system's trajectories, as was done before in the analogous continuous-time system, but transforming the stability region of the differential case through suitable Möbius transformations.

Nevertheless, the explicit solutions of the equation which defines the system shall be the subject of further research.

8. REFERENCES

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