GLOBAL REACHABILITY OF 2D STRUCTURED SYSTEMS

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Abstract: In this paper the new concept of 2D structured system is defined and a characterization of global reachability is obtained. This extends a well known result for 1D structured systems, according to which (A_λ, B_λ) is (generically) reachable if and only if the matrix $\begin{bmatrix} A_\lambda & B_\lambda \end{bmatrix}$ is full generically row rank and irreducible.

Keywords: Structured System, 2D Discrete System, Global Reachability

FOREWORD

This paper is dedicated to Professor Fátima Leite, to whom we wish to pay our tribute for the outstanding role in the development of Mathematical Systems and Control Theory in Portugal. The second author is particularly thankful for the encouragement and help she received from her at several stages of her (academic) life.

1. INTRODUCTION

Structured systems have been introduced in (Lin, 1974) in order to model phenomena where the only available information is the existence or absence of relations between the relevant variables. Since then a vast literature has been produced on this subject, in particular for structured linear systems in state space form (Glover and Silverman, 1976; Shields and Pearson, 1976; Mayeda, 1981; Dion *et al.*, 2003) In such systems, the system matrices are supposed to have entries that are zero or then assume arbitrary values, and each nonzero entry is identified with a parameter. In this setting, several system theoretic properties have been defined in a generic way, i.e., as holding for almost all the concretizations of the values of the parameters. One of these properties is reachability, that is defined as the possibility of attaining an arbitrary state starting from the origin, by using a suitable control sequence.

In this paper we study the property of reachability for 2D structured systems. Whereas dynamical systems evolve over time (a one-dimensional variable), 2D systems evolve over a two-dimensional domain (for instance (1D) space-time or 2D space) (Fornasini and Marchesini, 1978). To our knowledge up to now no research has been done on 2D structured systems. The results of this paper concern the characterization of global reachability for 2D state space systems described by a Fornasini-Marchesini model (Fornasini and Marchesini, 1978) and constitute a first step to build a full theory of 2D structured systems.

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2. STRUCTURED SYSTEMS

A matrix $M \in \mathbb{R}^{n\times m}$ is said to be a *structured matrix* if its entries are either fixed zeros or independent parameters, in which case they are referred to as the nonzero entries. In this paper, we assume that the actual value of each of the nonzero entries is unknown, but can take any real value (including zero). Therefore a structured matrix M having r nonzero entries can be parameterized by means of a parameter vector $\lambda \in \mathbb{R}^r$ and is denoted by M_λ.

Example 1. Let λ_i, $i = 1, 2, 3$, be free parameters. The matrix

$$M_\lambda = \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & 0 \end{bmatrix}$$

is a structured matrix where $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$.

However, neither

$$\begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_1 & 0 \end{bmatrix} \text{ nor } \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 0 \end{bmatrix}$$

are structured matrices.

Let us consider a discrete time-invariant system of the form

$$x(t+1) = Ax(t) + Bu(t), \qquad (2.1)$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $x(\cdot) \in \mathbb{R}^n$ denotes the state of the system and $u(\cdot) \in \mathbb{R}^m$ the input.

If in the system (2.1) we assume that the matrices A and B are structured matrices having together r nonzero entries, the system can be parameterized by means of a parameter vector $\lambda \in \mathbb{R}^r$. The set of parameterized systems thus obtained is called a *structured system* and is denoted by

$$x(t+1) = A_\lambda x(t) + B_\lambda u(t) \qquad (2.2)$$

with $\lambda \in \mathbb{R}^r$, or simply by (A_λ, B_λ).

By choosing λ, system (2.2) becomes completely known and can be written as a system of the form (2.1). Thus, for each value of λ, its system theoretic properties can be studied in the usual way. It is clear that these properties may depend on the parameter values and hold for some of them while for others not. In this context, for structured systems, the relevant issue is not whether a property holds for some particular parameter values, but rather whether it is a *generic property*, in the sense that it holds "for almost all parameter values", i.e., it holds for all parameter values except for those in some proper algebraic variety in the parameter space (which is a set with Lebesgue measure zero) (Davison and Wang, 1973). Hence we shall say that the structured system (2.2) has a certain property P if P is a generic property of the system.

In this paper we shall focus in the study of reachability. As is well-known, the system (2.1) is said to be *reachable* if for every $x^* \in \mathbb{R}^n$ there exist $t^* > 0$ and an input sequence $u(t)$, $t = 0, 1, \ldots, t^* - 1$, that steers the state from $x(0) = 0$ to $x(t^*) = x^*$.

Characterizations of reachability for completely specified systems of type (2.1) are given by the following results (Kučera, 1992).

Theorem 2. The system (2.1) is reachable if and only if $\operatorname{rank} \mathcal{R}^n = n$, where $\mathcal{R}^n$ is the reachability matrix of the system, i.e.,

$$\mathcal{R}^n := \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}.$$

Theorem 3. (PBH test). The system (2.1) is reachable if and only if

$$\operatorname{rank} \begin{bmatrix} zI - A \mid B \end{bmatrix} = n, \forall z \in \mathbb{C}.$$

By Theorem 2, system (2.2) is reachable if and only if the reachability matrix

$$\mathcal{R}^n = \begin{bmatrix} B_\lambda & A_\lambda B_\lambda & \cdots & A_\lambda^{n-1} B_\lambda \end{bmatrix}$$

has rank n for almost all $\lambda \in \mathbb{R}^r$. But, noting that $\mathcal{R}^n$ is a polynomial matrix in r indeterminates, we can show that this is equivalent to say that $\operatorname{rank} \mathcal{R}^n = n$, for some $\lambda^* \in \mathbb{R}$. This means that the structured system (2.2) is reachable if and only if it is reachable for one choice of λ. However, neither this characterization nor, equivalently, the study of the rank of the polynomial matrix $\mathcal{R}^n$ yield useful tests. This suggests that a different approach should be adopted, for instance based only on the structure of the relevant matrices.

Unfortunately, given two structured matrices A_λ and B_λ the reachability matrix $\mathcal{R}^n$ is not necessarily structured, as is illustrated by the next example.

Example 4. Let

$$A_\lambda = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 0 & \lambda_3 \end{bmatrix} \text{ and } B_\lambda = \begin{bmatrix} \lambda_4 \\ 0 \end{bmatrix}$$

be two structured matrices. Then

$$\mathcal{R}^2 = \begin{bmatrix} B_\lambda & A_\lambda B_\lambda \end{bmatrix} = \begin{bmatrix} \lambda_4 & \lambda_1\lambda_4 \\ 0 & 0 \end{bmatrix}$$

is not a structured matrix since its nonzero entries are not independent.

Moreover, define a structured polynomial matrix $M_\lambda(z)$ as

$$M_\lambda(z) = M_N^\lambda z^N + \cdots + M_1^\lambda z + M_0^\lambda$$

for some nonnegative integer N, where the coefficients M_j^λ, $j = 0, 1, \ldots, N$, are structured matrices. Then the matrix

$$\begin{bmatrix} zI - A_\lambda \mid B_\lambda \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix} z + \begin{bmatrix} -A_\lambda & B_\lambda \end{bmatrix}$$

associated to the pair (A_λ, B_λ) of structured matrices is also not structured.

Thus, if we wish to make a study of reachability of a structured system by analyzing structured matrices we must use different tools.

The next two concepts are fundamental for this study (Dion *et al.*, 2003)

Let $A_\lambda \in \mathbb{R}^{n\times n}$ and $B_\lambda \in \mathbb{R}^{n\times m}$ be structured matrices. The pair (A_λ, B_λ) is said to be:

- *reducible*, or to ***be in form I***, if there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ such that

$$P^{-1}A_\lambda P = \begin{bmatrix} A_{11}^\lambda & 0 \\ A_{21}^\lambda & A_{22}^\lambda \end{bmatrix} \quad \text{and} \quad PB_\lambda = \begin{bmatrix} 0 \\ B_2^\lambda \end{bmatrix}$$

  where A_{ij}^λ is an $n_i \times n_j$ structured matrix for $i, j = 1, 2$, with $0 < n_1 \le n$ and $n_1 + n_2 = n$, and where B_2^λ is an $n_2 \times m$ structured matrix.
- *not of full generic row rank*, or to ***be in form II***, if the generic rank of $\begin{bmatrix} A_\lambda & B_\lambda \end{bmatrix}$ is less than n.

Recall that the *generic rank* of a structured matrix M_λ is ρ if it is equal to ρ for almost all $\lambda \in \mathbb{R}^r$. This coincides with the maximal rank that M_λ achieves as a function of the parameter λ.

A necessary and sufficient condition for a pair (A_λ, B_λ) to be in form II is that $\begin{bmatrix} A_\lambda & B_\lambda \end{bmatrix}$ has a zero submatrix of order $k \times l$ where $k + l \ge n + m + 1$ (Shields and Pearson, 1976).

For structured systems of type (2.2), the following result has been proved (see (Glover and Silverman, 1976; Lin, 1974; Shields and Pearson, 1976)).

Theorem 5. The structured system (2.2) is (generically) reachable if and only if the pair (A_λ, B_λ) is neither in form I nor in form II.

The main goal of this paper is to generalize this theorem for 2D structured systems.

3. 2D SYSTEMS

One of the most frequent representations of 2D systems is the Fornasini-Marchesini state space model (Fornasini and Marchesini, 1978) which is described by the following 2D first order state updating equation

$$\begin{aligned} x(i+1, j+1) &= A_1 x(i, j+1) + A_2 x(i+1, j) \\ &\quad + B_1 u(i, j+1) + B_2 u(i+1, j), \end{aligned} \tag{3.1}$$

with *local states* $x(\cdot,\cdot) \in \mathbb{R}^n$, inputs $u(\cdot,\cdot) \in \mathbb{R}^m$, state matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$ and input matrices $B_1, B_2 \in \mathbb{R}^{n\times m}$. In the sequel this 2D system will be denoted by (A_1, A_2, B_1, B_2). The corresponding updating scheme is illustrated in Figure 1.

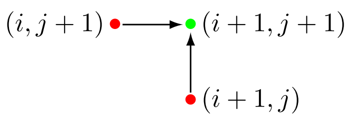

Figure 1

Introducing the *shift operators*

$$\begin{aligned} \sigma_1 x(i,j) &:= x(i+1, j), \\ \sigma_2 x(i,j) &:= x(i, j+1), \end{aligned}$$

the equation (3.1) can be written as

$$\sigma_1\sigma_2 x = A_1\sigma_2 x + A_2\sigma_1 x + B_1\sigma_2 u + B_2\sigma_1 u,$$

or, equivalently,

$$\sigma_1 x = \left(A_1 + A_2\sigma_1\sigma_2^{-1}\right) x + \left(B_1 + B_2\sigma_1\sigma_2^{-1}\right) u.$$

Defining a new operator $\sigma := \sigma_1\sigma_2^{-1}$ equation (3.1) can be written as

$$\sigma_1 x = (A_1 + A_2\sigma)\, x + (B_1 + B_2\sigma)\, u. \tag{3.2}$$

The initial conditions for this equation may be assigned by specifying the values of the state on the separation set $\mathcal{C}_0$, where

$$\mathcal{C}_k = \{(i,j) \in \mathbb{Z}^2 : i + j = k\},$$

see Figure 2.

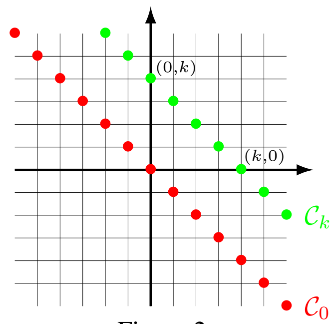

Figure 2

Defining the *global state* on the separation set $\mathcal{C}_k$ as

$$\mathcal{X}_k(t) := \big(x(k+t, -t)\big)_{t\in\mathbb{Z}}$$

and the global input as

$$\mathcal{U}_k(t) := \big(u(k+t, -t)\big)_{t\in\mathbb{Z}},$$

by (3.2), the global state evolution is given by

$$\mathcal{X}_{k+1} = A(\sigma)\mathcal{X}_k + B(\sigma)\mathcal{U}_k, \tag{3.3}$$

where $A(\sigma) = A_1 + A_2\sigma$, $B(\sigma) = B_1 + B_2\sigma$, and the action of σ on $\mathcal{X}_k$ is given by

$$\begin{aligned} \sigma\mathcal{X}_k(t) &= \big(x(k + (t+1), -(t+1))\big)_{t\in\mathbb{Z}} \\ &= \mathcal{X}_k(t+1). \end{aligned}$$

The action of σ on $\mathcal{U}_k$ is analogous.

Denote by $\mathbb{R}\big[\big[z, z^{-1}\big]\big]$ the set of bilateral Laurent formal power series in the indeterminate z with coefficients in $\mathbb{R}$ and define the $z-$transform $\mathcal{Z} : (\mathbb{R})^{\mathbb{Z}} \to \mathbb{R}\big[\big[z, z^{-1}\big]\big]$ by

$$\mathcal{Z}[\mathcal{W}_k] := \sum_{t=-\infty}^{+\infty} \mathcal{W}_k(t) z^{-t}$$

which will be denotet by $W_k(z)$, with $k \in \mathbb{Z}$. For vector signals in $\left(\mathbb{R}^l\right)^{\mathbb{Z}}$ the $z-$transform is defined componentwise.

Then

$$\begin{aligned}\mathcal{Z}[\sigma\mathcal{X}_k] &= \sum_{t=-\infty}^{+\infty} \sigma\mathcal{X}_k(t)z^{-t} \\ &= \sum_{t=-\infty}^{+\infty} \mathcal{X}_k(t+1)z^{-t} \\ &= z\sum_{t=-\infty}^{+\infty} \mathcal{X}_k(t+1)z^{-(t+1)} \\ &= z\sum_{t=-\infty}^{+\infty} \mathcal{X}_k(t)z^{-t} \\ &= zX_k(t)\end{aligned}$$

and hence, by (3.3), we obtain

$$X_{k+1}(z) = A(z)X_k(z) + B(z)U_k(z), \qquad (3.4)$$

where $A(z) = A_1 + A_2 z$, $B(z) = B_1 + B_2 z$ and $U_k(z) := \mathcal{Z}[\mathcal{U}_k]$.

4. LOCAL AND GLOBAL REACHABILITY

When dealing with 2D systems, the concept of reachability is naturally introduced in two different forms: a weak (local) and a strong (global) form which refer, respectively, to single local states and to global states. These notions are defined next as in (Fornasini and Marchesini, 1978).

Definition 6. The 2D state space model (3.1) is

- ***locally reachable*** if, upon assuming $\mathcal{X}_0 \equiv 0$, for every $x^* \in \mathbb{R}^n$ there exists $(i, j) \in \mathbb{Z}^2$, with $i + j > 0$, and an input sequence $u(\cdot, \cdot)$ such that $x(i, j) = x^*$. In this case, we say that x^* is reachable in $i + j$ steps.
- ***globally reachable*** if, upon assuming $\mathcal{X}_0 \equiv 0$, for every global state sequence $\mathcal{X}^*$ with values in $\mathbb{R}^n$ there exists $k \in \mathbb{Z}_+$ and an input sequence $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_{k-1}$ such that the global state $\mathcal{X}_k$ coincides with $\mathcal{X}^*$. In this case, we say that $\mathcal{X}^*$ is reachable in k steps.

Clearly, global reachability implies local reachability. In this paper we shall focus on this global property.

Bearing in mind that, if $\mathcal{X}_0 \equiv 0$, then

$$X_k(z) = \sum_{l=0}^{k-1} A^{k-1-l}(z)B(z)U_l(z)$$

$$= \begin{bmatrix} B(z) & A(z)B(z) \cdots A^{k-1}(z)B(z) \end{bmatrix} \begin{bmatrix} U_{k-1}(z) \\ U_{k-2}(z) \\ \vdots \\ U_0(z) \end{bmatrix},$$

it is easy to see that the global state $\mathcal{X}^*$ is reachable in k steps if and only if

$$X^*(z) = \mathcal{Z}[\mathcal{X}^*] \in \operatorname{Im} \mathcal{R}^k(z)$$

with $\mathcal{R}^k(z) := \begin{bmatrix} B(z) & A(z)B(z) \cdots A^{k-1}(z)B(z) \end{bmatrix}$.

The matrix

$$\mathcal{R}^n(z) = \begin{bmatrix} B(z) & A(z)B(z) \cdots A^{n-1}(z)B(z) \end{bmatrix},$$

where n is the dimension of the local state and the polynomial matrices $A(z)$ and $B(z)$ are defined as in (3.4), is called ***global reachability matrix*** of the 2D system (A_1, A_2, B_1, B_2).

In the following theorem (Fornasini and Marchesini, 1978), global reachability is characterized in terms of the global reachability matrix.

Theorem 7. The 2D system (A_1, A_2, B_1, B_2) is global reachable if and only if the polynomial matrix $R^n(z)$ has rank n, i.e., $\operatorname{rank} \mathcal{R}^n(z) = n$.

5. 2D STRUCTURED SYSTEMS

In the sequel we consider 2D systems of the form (3.1), where the matrices A_1, A_2, B_1 and B_2 are structured, i.e., their entries are either fixed zeros or independent free parameters. In this case, the polynomial matrices $A_\lambda(z) = A_1^\lambda + A_2^\lambda z$ and $B_\lambda(z) = B_1^\lambda + B_2^\lambda z$ are structured matrices too. Moreover, their evaluations for any $\nu^* \in \mathbb{C}$, yield matrices $A_\lambda(\nu^*)$ and $B_\lambda(\nu^*)$ that are also structured.

Similar to the 1D case, we say that a 2D structured system $(A_1^\lambda, A_2^\lambda, B_1^\lambda, B_2^\lambda)$ is (*globally / locally*) *reachable* if it is generically (globally / locally) reachable, i.e., if it is reachable for almost all $\lambda \in \mathbb{R}^r$. Again this is equivalent to say that $(A_1^{\lambda^*}, A_2^{\lambda^*}, B_1^{\lambda^*}, B_2^{\lambda^*})$ is reachable for at least one value $\lambda^* \in \mathbb{R}^r$.

As in the 1D case, the notions of matrix pairs in *form I* and in *form II* play an important role in the characterization of 2D reachability, now applied to the polynomial matrix pair $(A_\lambda(z), B_\lambda(z))$. In the polynomial case the definitions remain the same as in the constant case, with the difference that the (generic) rank of $\begin{bmatrix} A_\lambda(z) & B_\lambda(z) \end{bmatrix}$ is to be understood as its rank as a polynomial matrix.

Lemma 8. Let $\nu^* \in \mathbb{C} \setminus \{0\}$. Then the pair of structured matrices $(A_\lambda(z), B_\lambda(z))$ is neither in form I nor in form II if and only if the pair of structured matrices $(A_\lambda(\nu^*), B_\lambda(\nu^*))$ is not in form I nor in form II, where $A_\lambda(z) = A_1^\lambda + A_2^\lambda z$ and $B_\lambda(z) = B_1^\lambda + B_2^\lambda z$.

PROOF. If $\nu^* \in \mathbb{C}\setminus\{0\}$ both implications are obvious since the pairs of structured matrices $(A_\lambda(z), B_\lambda(z))$ and $(A_\lambda(\nu^*), B_\lambda(\nu^*))$ have the same zero structure. □

Remark 9. The "if" part also holds for $\nu^* = 0$. In fact, if $\nu^* = 0$ then $A_\lambda(0) = A_1^\lambda$ and $B_\lambda(0) = B_1^\lambda$. Since all the zero entries of the matrix $A_\lambda(z)$ are also zero in A_1^λ and the same happens between $B_\lambda(z)$ and B_1^λ, the set of zero entries for the pair $(A_\lambda(z), B_\lambda(z))$ is contained in the set of zero entries of $(A_\lambda(0), B_\lambda(0)) = (A_1^\lambda, B_1^\lambda)$. The result is easily obtained by the definition of form I and the characterization of form II.

The following example shows that the "only if" part does not hold for $\nu^* = 0$.

Example 10. Let $A_1^\lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & 0 \end{bmatrix}$, $A_2^\lambda = \begin{bmatrix} 0 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{bmatrix}$, $B_1^\lambda = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $B_2^\lambda = \begin{bmatrix} \lambda_5 \\ \lambda_6 \end{bmatrix}$.

Since all the entries of $(A_1^\lambda + A_2^\lambda, B_1^\lambda + B_2^\lambda)$ are free, this pair is neither in form I nor in form II and by Lemma 8 the same holds for the pair $(A_\lambda(z), B_\lambda(z))$. However, its clear that the pair $(A_\lambda(0), B_\lambda(0)) = \left(A_1^\lambda, B_1^\lambda\right)$ is in form I and II.

The next theorem characterizes the global reachability of 2D structured systems

Theorem 11. A 2D structured system $(A_1^\lambda, A_2^\lambda, B_1^\lambda, B_2^\lambda)$ is globally reachable if and only if the pair of structured matrices $(A_\lambda(z), B_\lambda(z))$ is neither in form I nor in form II, where $A_\lambda(z) = A_1^\lambda + A_2^\lambda z$ and $B_\lambda(z) = B_1^\lambda + B_2^\lambda z$.

PROOF. By definition, the 2D structured system $(A_1^\lambda, A_2^\lambda, B_1^\lambda, B_2^\lambda)$ is globally reachable if there exists $\lambda^* \in \mathbb{R}^r$ such that the 2D system $(A_1^{\lambda^*}, A_2^{\lambda^*}, B_1^{\lambda^*}, B_2^{\lambda^*})$ is globally reachable.

Then, by Theorem 7, $\operatorname{rank} \mathcal{R}_{\lambda^*}^n(z) = n$, where $\mathcal{R}_{\lambda^*}^n(z)$ is the global reachability matrix of the 2D system $(A_1^{\lambda^*}, A_2^{\lambda^*}, B_1^{\lambda^*}, B_2^{\lambda^*})$. Note that, in this case, the set

$$\mathcal{L} := \{\eta \in \mathbb{C} : \operatorname{rank} \mathcal{R}_{\lambda^*}^n(\eta) < \operatorname{rank} \mathcal{R}_{\lambda^*}^n(z)\}$$

corresponds to the common zeros of the $n \times n$ minors of $\mathcal{R}_{\lambda^*}^n(z)$, and is hence a finite set. Thus $\operatorname{rank} \mathcal{R}_{\lambda^*}^n(z) = n$ means that there exist $\nu^* \in \mathbb{C} \setminus \mathcal{L}$ such that

$$\operatorname{rank} \mathcal{R}_{\lambda^*}^n(\nu^*) = n.$$

By Theorem 2, the system corresponding to the pair $(A_{\lambda^*}(\nu^*), B_{\lambda^*}(\nu^*))$ is reachable, for all $\nu^* \in \mathbb{C} \setminus \mathcal{L}$.

Thus, by definition, $(A_\lambda(\nu^*), B_\lambda(\nu^*))$ is a structured system which is reachable.

By Theorem 5 we have that the pair of structured matrices $(A_\lambda(\nu^*), B_\lambda(\nu^*))$ is neither in form I nor in form II and, by Lemma 8, $(A_\lambda(z), B_\lambda(z))$ is neither in form I nor in form II. The converse implication is analogous. □

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