RESORTING TO NON EUCLIDEAN PLANE GEOMETRIES TO DEVELOP DEDUCTIVE REASONING AN ONTO-SEMIOTIC APPROCH

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The main idea behind this work is the study of the potential of resorting to other models of Plane Geometry (e.g. Hyperbolic Geometry, Taxicab Geometry) to help students to progress towards a proper/better understanding of what a mathematical proof is about. A teaching experiment carried out with students of 15 to 17 years-old attending the 10th and11thgrade (the two first years of secondary school) of a Portuguese school. The experience started in the10th grade and lasted in the 11thgrade. Our main focus is the analysis of primary and secondary relationships of geometric objects involved in argumentation and proof (in the sense of Godino et al. and Gutiérrez et al.) activated by the students during production of arguments.

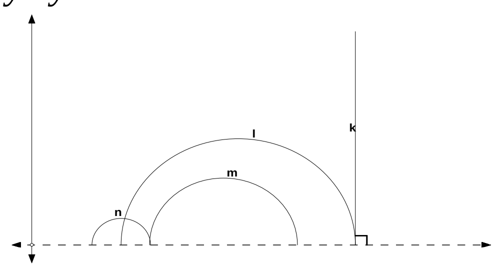
Recent research in the onto-semiotic approach to mathematics knowledge and instruction has highlighted that the systems of practices and its configurations are proposed as theoretical tools to describe mathematical knowledge, in its double version: personal and institutional (see Godino et al., 2007). Following these ideas, this researchers refer that for a finer analysis of the mathematical activity it is necessary to take into account six types of primary entities: Problem situation; Language (e.g., terms, expressions, notations, graphs) in its various registers (e.g., written, oral, sign language); Concepts (approached through definitions or descriptions); Propositions (statements on concepts); Procedures (e.g., algorithms, operations, calculation techniques); Arguments (statements used to validate or explain the propositions and procedures, of deductive nature or another type). These six objects relate to each other by epistemic (networks of institutional objects) and cognitive configurations (networks of personal objects). Considering an entity as being primary is not an absolute question but rather a relative one, since we are dealing with functional entities in contexts of use. The contextual attributes signaled by these researchers are: Personal/institutional - The personal cognition is the result of thought and action of the individual subject confronted by a class of problems, whereas institutional cognition is the result of dialogue, understanding and regulation within a group of individuals who make up a community of practices; Ostensive / non-ostensive – The ostensive attribute refers to the representation of a non-ostensive object, that is to say, of an object that cannot be shown to another. The classification between ostensive and non-ostensive depends on the contexts of use. Diagram, graphics and symbols are examples of objects with ostensive attributes, perforated cubes and plane sections are examples of objects with non-ostensive attributes; Expression / content (antecedent and consequent of any semiotic function) - The relationship is established by means of semiotic functions, understood as a relationship between an antecedent (expression, signifier) and a consequent (content, signified or meaning) established by a subject (person or institution) according to a specific criterion or code of correspondence; Extensive / intensive (specific / general) – This duality is used to explain one of the basic characteristics of mathematical activity, namely generalization. This duality allows for the centre of attention to be the dialectics between the particular and the general, which is undoubtedly a key issue in the construction and application of mathematical knowledge; Unitary / systemic – In certain circumstances, mathematical objects participate as unitary entities, in others, they should be taken as the decomposition of others so that they can be studied.

A considerable number of researchers have been researching the nature of argumentation and types of proof (e.g. Harel and Sowder, 2007; Marrades and Gutiérrez 2000). In this theoretical frame, the following question is of particular interest: How can other models of Plane Geometry, other than the Euclidean one, help Secondary School students to develop deductive reasoning?

Two levels of accomplishment were set up for this work. The first one, being in a classroom environment with a class of 20 students (15-16 years of age) in 10th grade – Secondary School, from the social economics field in the 2004/2005 school year. The second level, situated on the study of the individual cognitive trajectories of two students (both girls 16 years of age) from the mentioned class, during their 11th grade (2005/2006 school year) which, even though it focused on the same questions as those defined for the class, allowed for a more detailed level of analysis. The empirical study in the second stage of the study was developed in an extra-classroom scenario in sessions of small work groups that ran in parallel to the mathematics class. In particular, we are concerned with the student's ability in argumentation and proof. The mediation offered by other model of plane geometry, other than Euclidean one, in the conceptualisation of meaning for parallelism concept and the methods of proof (e.g., method of proof by contraction) is very important for the cognitive aspects of proof. Following is the epistemic configuration and the cognitive configuration and trajectory of one of the problems proposed in the study process. The problem is written below:

The following diagram represents various hyperbolic lines (l, m, n and k) on the Poincaré half-plane, defined respectively by the conditions:

l: $(x - 7)^2 + y^2 = 16 \wedge y > 0$
m: $(x – 6,5)^2 + y^2 = 6,25 \wedge y > 0$
n: $(x - 3)^2 + y^2 = 1 \wedge y > 0$
k: $x = 11 \wedge y > 0$



Indicate if there are two parallel lines and two non-parallel lines. Justify.

Episode 1: After reading and analyzing the drawing supplied, the following dialogue took place:

> X. Teach', is the definition of parallels the same? ; Teacher: Yes, the definition is the same;
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> X. Then, two lines, no matter how far they are prolonged, never intercept; Y. These are not parallel (referring to l and n); X. But these two are (referring to l and m); Y. But they're not parallel…; X. How do you know?; Y. Oh, you can see… the distance from here to here and from here to here … (referring to the Euclidean distance between the two semi-circumferences, representative of the hyperbolic lines in question);X. But the distance doesn't have to be the same. It does, when they are parallel, this distance from here to here is always the same as from here to here and from here to here… (pointing at lines l and m). Isn't it?

Students are silent while observing the drawing; Student Y identified the value of the radius in hyperbolic lines l, m and n and noted it down next to the drawing.

> Y. Oh Teach', I have a question. Is it two lines or two straight lines? ; Teacher: Two lines. We had already decided that in hyperbolic geometry we speak of lines; Y. It's just that these don't intercept but they're not parallel either… (referring to l and m) the distance from here to here is not the same as from here to here; Teacher: Why do you say they are not parallel?; Y. Because the distance from here to here is not the same as from here to here; Teacher: Are you thinking of Euclidean geometry?; Y. Aha! Then they can be parallel …;X. Two parallels are l and m…aren't they?; Y. This here asks for two…

Episode 2: Setting up the justification. The following dialogue took place:

> X. You're only going to give one example…;Y. Yes…;X. I think we should first supply the more obvious ones let's try other interpretations… (the coordinates of the centers) The centers are seven, zero and six and a half, zero…and if you check, it's correct.

Next, and after the teacher's request, each student explained the reasoning set up by reading the respective solution.

> X. In Poincaré geometry, the definition of parallelism being the same as in Euclidean geometry, we can verify that m and l are parallel, since these lines never intercept and l and n are non-parallel since they intercept at one point; Y. Two lines are said to be parallel in any geometry when their interception is an empty set. So m is parallel to l and l is not parallel to n.; Teacher: It seems you all consider m and l to be parallel and that m, n and l, k and l, n are not parallel. Why?; X. Since the image …;Teacher: And couldn't you present a more convincing argument?; X. We can…we just need to know how (laughed); Teacher: In analytical geometry, when you wanted to determine the intersection of, for example, the straight lines of equation y equals two x plus four and y equals minus x plus two, how did you do it?; X. We would do the system and we'd have the point…This is Style Heading 3, if you need it.

The students then adopted an analytical approach to justify the answer put forward. Student Y resorted to the resolution of systems to verify the relationship of parallelism between lines k-l and l-n. When student X determined the point of interception of lines l and k, the following dialogue took place:

> X. Teach', this gives us a very weird point…I must have this wrong!; Teacher: And why is it weird?; X. Well, because it gives eleven, zero …;Teacher: And why is it weird?; X. Because the eleven should be farther up (student laughed);Y. It's not the eleven, it's the x;X. Oh, of course it is! Ok, I was seeing this backwards; Teacher: So, is it acceptable now?; Y. Yes, it is …;X. No: Y. Yes …eleven is: X. Alright…but y has to be greater than zero; it can't be zero; Y. But they intercept in one point…; X. That's right…but it's not valid because y has to be greater than zero; Teacher: So what do you conclude?; X and Y. So the only parallel ones here are l with m; Y. (Lines) m and n are also non-parallels because they intercept in point two, zero.

After solving the two compound systems of equations, the following dialogue took place:

> Y. That definition of parallelism, when we say no matter how far they are prolonged, is wrong for circumferences because take a look at these; X. I see what you mean…;Y. We don't have to say no matter how far they are prolonged. […]; X. (Lines) l and n are the only ones that do not intercept.

Note that student X uses the designation of straight lines and not lines, she follows the definition of parallelism associated to the existence of intersection and no longer associates parallelism to the initial expression "[…] no matter how far they are prolonged, they never meet.[…]"

As for the procedures adopted, student X's choice for the algebraic one is evident. In spite of this student visualizing point B, of interception of lines m and n, she resolves a system and indicates coordinates of that point, with figures rounded off to the hundredths. The algebraization of the problem helped clarify likely doubts on the parallelism of some lines. It seems that the visualization of the drawing did not induce wrong reasoning. The justification put forward is based on the previous procedures and had a deductive nature, where the specific examples were used to support the organization of the justifications – thought-out experimentation. Student Y used graph and algebra languages, as aids in identifying parallel and non-parallel lines. The drawing supplied in the exposition comprises an aid in identifying parallel and non-parallel lines. The situation put forward aimed at strengthening visualization and valuing the role of the Poincaré half-plane definition in justifying the indication of parallel and non-parallel lines. Algebraic language aids in clarifying likely doubts on the parallelism of some lines, such as lines l and k. The problem also gave rise to the approach of concepts, properties (e.g., definition of parallel lines in an abstract geometry). The justification was of the conceptual type, based on the definitions of the Poincaré half-plane and of parallel lines. The sequence of procedures adopted by the students was visualisation – reasoning. But could visualisation, in this case, have induced wrong reasoning? Visualization, in the ascending phase of problem resolution, gave rise to the intuition of some parallel lines (e.g., n and m) which in reality were not. In fact, the relationships of parallelism between the lines given in the problem statement was not intuitive, it was not obvious and they were accepted based on carrying out a more formal verification (resorting to the resolution of systems, resorting to the Poincaré half-plane definition…). Next we expose an interpretation centered on the student´s arguments applying the contextual attributes: Ostensive – non-ostensive- Student X, used points A and B to mark, respectively, the intersection of lines l,n and m,n. Nevertheless, it seems to us that she felt the need to determine the coordinates of the points to recognize the non-ostensive (non-parallel lines and parallel lines). Therefore, the ostensive objects brought forward in presenting the solution to the problem were the representation of points A and B in the drawing supplied in the exposition and the systems of the respective conditions which define the hyperbolic lines in question. Student Y used: the "//" notation (ostensive) to refer to the relationship of parallelism (non-ostensive) between lines; the algebraic language and the symbol (if…then…) when joining sentences; Extensive – Intensive- Student X used the condition given in the statement as support to identify the centers of the semi-circumferences. The definition given in the beginning "Parallelism – when two lines, no matter how far thy are prolonged, never intersect" is adopted by the student for hyperbolic geometry, which she designates as Poincaré geometry. However, in the solution of the problem, she only refers to the existence or not of intersection. Student Y started by writing: Two lines are said to be parallel (in any geometry) when their intersection is an empty set. In other words, she thought of the definition of parallel lines and only then she focus on the extensive objects represented in the problem statement; Institutional – personal: If, on the one hand, visualisation is revealed to be a means to provide a solution to the problem, on the other, the more recent experiences of these students in the scope of parallelism of lines, in Euclidean geometry, was carried out according to an analytical approach and by resorting to the resolution of equation systems. Therefore, at personal cognition level, the problem situation generated the following conflicts in terms of defining parallel lines: Student Y used the ostensive of parallel lines of Euclidean geometry, in the context of hyperbolic geometry (according to episode 1). Student X presented a definition of parallel lines right in the beginning of the written solution (drawing …) where she refers "…no matter how far they are prolonged, they never intersect" and confronted by the maladjustment of this definition – by student Y – she does not present any arguments; Unitary – systemic - The analysis carried out by both students' displays different aspects. Student X feels the need to break down the exposition, recording the coordinates of the centers of the semi-circumferences and the points of intersection of lines l, n and m, n. Student Y, upon breaking down the exposition, records the value of the radii of the mentioned semi-circumferences and focuses on the distance between them. In student X’s case, she refers to the only “straight lines” that are not parallel and then states: “All the others are // between themselves because they never intersect since y=0 does not belong to the half-plane”. In student Y’s case, the conclusion includes reference to the relationship of parallelism between the lines two by two; Expression – content - The problem situation induces the definition of parallel lines in a context of hyperbolic geometry. The students revealed a command of algebraic calculus but in terms of the language, student X seems not to be familiar with some issues of hyperbolic geometry language.

The justification they present is of a conceptual nature – based on the definition of parallel lines in an abstract geometry, formulation of properties (Properties of the relationship of parallelism) and on algebraic calculus (symbolic calculus). The justification is based on the resolution of systems of equations, on the use of formalized symbolic expressions. The evolution from an ascending phase, characterized by empirical activity, to a descending phase, in which the students produce deductive justification, was clear.

The problems proposed created conflicts between an intuitive interpretation and formal argumentation. The resolution of these conflicts allowed for an evolution of knowledge and argumentative skills (e.g., the role attributed to definitions).

The study suggests that a diversified geometric approach, through various models of plane geometry, promoted a different understanding of the processes leading to the deductive reasoning.

Mathematical argumentation can be better understood and assessed if we are aware that the arguments are interconnected with the primary and secondary objects defined in the onto-semiotic focus of mathematical cognition.

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