On the Lyapunov and Stein Equations, II

Fernando C. Silva [*‡] Rita Simões [†‡]

January 8, 2007

Abstract

Let $L \in \mathbb{C}^{n\times n}$ and let $H, K \in \mathbb{C}^{n\times n}$ be Hermitian matrices.

Some already known results, including the general inertia theorem, give partial answers to the following problem: find a complete set of relations between the similarity class of L and the congruence classes of H and K, when the Lyapunov equation $LH + HL^* = K$ is satisfied.

In this paper, we solve this problem when L is nonderogatory, H is nonsingular and K has at least one eigenvalue with positive real part and one eigenvalue with negative real part. Our result generalizes a previous paper by L. M. DeAlba.

The corresponding problem with the Stein equation follows easily using a Cayley transform.

Mathematics Subject Classification: 93D05

Keywords: Lyapunov equation, Stein equation, Inertia of matrices

1 Introduction

Let $L \in \mathbb{C}^{n\times n}$ and let $H, K \in \mathbb{C}^{n\times n}$ be Hermitian matrices.

The main inertia theorem [3, 6, 8] can be viewed as giving a solution to the following problem, when K is positive definite.

[*]Departamento de Matemtica, Faculdade de Cincias, Universidade de Lisboa, 1749–016 Lisboa, Portugal. (fcsilva@fc.ul.pt)

[†]Departamento de Matemtica, Universidade de Aveiro, 3810–193 Aveiro, Portugal. (rsimoes@mat.ua.pt)

[‡]Research done within the activities of the *Centro de Estruturas Lineares e Combinatórias* and supported by *Fundação para a Ciência e a Tecnologia* (FCT). The work of the second author was also supported by FCT grant SFRH/BD/11133/2002.

Problem 1 *Find a complete set of relations between the similarity class of L and the congruence classes of H and K when the Lyapunov equation $LH + HL^* = K$ holds.*

Other partial answers can be found in [5, Theorems 1, 2, 3] and [7, Theorem 3]. See also [2, 4].

In this paper, we study this problem when L is nonderogatory, H is nonsingular and K is not semidefinite. (See Theorem 2 below.) Our result generalizes [5].

Let $A \in \mathbb{C}^{n\times n}$. The corresponding problem about the Stein equation consists of finding a complete set of relations between the similarity class of A and the congruence classes of H and K, when $H - AHA^* = K$ holds. A result analogous to Theorem 2 can be obtained as a corollary of Theorem 2, using a Cayley transform of the form

$$L_\theta = (\theta I_n + A)^{-1}(\theta I_n - A),$$

where θ is a complex number of modulus 1 chosen so that $\theta I_n + A$ is nonsingular. See [7], for more details.

2 Main Result

Let $L \in \mathbb{C}^{n\times n}$. The *inertia* of $L \in \mathbb{C}^{n\times n}$ is the triple $\mathrm{In}(L) = (\pi(L), \nu(L), \delta(L))$, where $\pi(L), \nu(L)$ and $\delta(L)$ denote, respectively, the number of eigenvalues of L with real positive part, with real negative part and with real part equal to zero. We shall say that $H, H' \in \mathbb{C}^{n\times n}$ are *congruent* if there exists a nonsingular matrix $S \in \mathbb{C}^{n\times n}$ such that $H' = SHS^*$. It is well-known that two Hermitian matrices are congruent if and only if they have the same inertia.

The following theorem is our main result and will be proved later.

Theorem 2 *Let $L \in \mathbb{C}^{n\times n}$. Let $\pi_h, \nu_h, \pi_k, \nu_k, \delta_k$ be nonnegative integers such that $\pi_h + \nu_h = \pi_k + \nu_k + \delta_k = n$.*

If L is nonderogatory and $\min\{\pi_k, \nu_k\} > 0$, then there exists a Hermitian matrix $H \in \mathbb{C}^{n\times n}$ such that $\mathrm{In}(H) = (\pi_h, \nu_h, 0)$ and $\mathrm{In}(LH + HL^*) = (\pi_k, \nu_k, \delta_k)$.

Note that, for every nonsingular matrix $S \in \mathbb{C}^{n\times n}$, $LH + HL^* = K$ is equivalent to

$$(SLS^{-1})(SHS^*) + (SHS^*)(SLS^{-1})^* = SKS^*.$$

Therefore, in order to prove Theorem 2, L can be replaced by any similar matrix.

The following lemma is a simple generalization of [7, Lemma 6].

Lemma 3 *Let $a \in \mathbb{R}$, $\lambda, \mu \in \mathbb{C}$. Let π, ν, δ be nonnegative integers such that $\pi + \nu + \delta = 3$ and $\min\{\pi, \nu\} > 0$.*

Then, for every $z \in \mathbb{C} \setminus \{0\}$, there exists $y \in \mathbb{C}$ such that the matrix

$$T = \begin{bmatrix} \lambda & 1 & y \\ 0 & ia & z \\ 0 & 0 & \mu \end{bmatrix}$$

satisfies $\mathrm{In}(T + T^*) = (\pi, \nu, \delta)$.

Proof. Let

$$S = \begin{bmatrix} 1 & z^{-1}(2\Re(\mu)\bar{z}^{-1} - y) & -\bar{z}^{-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $T + T^*$ is congruent to

$$S(T + T^*)S^* = \left[2\Re(\lambda) + 2\Re(\mu)|z^{-1}|^2 - 2\Re(yz^{-1})\right] \oplus \begin{bmatrix} 0 & z \\ \bar{z} & 2\Re(\mu) \end{bmatrix}.$$

Clearly, y can be chosen so that $\mathrm{In}(T + T^*) = (\pi, \nu, \delta)$. ■

We shall say that an upper triangular matrix $T = [t_{i,j}] \in \mathbb{C}^{n \times n}$ is a *τ-matrix* if $t_{i,i+1} \neq 0$, for every $i \in \{1, \ldots, n-1\}$.

Lemma 4 *Let $L \in \mathbb{C}^{n \times n}$ be nonderogatory. Let π_k, ν_k, δ_k be nonnegative integers such that $\pi_k + \nu_k + \delta_k = n$ and $\min\{\pi_k, \nu_k\} > 0$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of L ordered so that*

(i_4) *if $\lambda_i = \lambda_j$, for some $i < j$, then $\lambda_i = \lambda_k$, for every $k \in \{i, \ldots, j\}$,*

(ii_4) *if $\Re(\lambda_i) < 0$, for some i, then $\Re(\lambda_k) < 0$, for every $k \in \{1, \ldots, i\}$,*

(iii_4) *if $\Re(\lambda_i) > 0$, for some i, then $\Re(\lambda_k) > 0$, for every $k \in \{i, \ldots, n\}$.*

Then there exists a τ-matrix $T = [t_{i,j}] \in \mathbb{C}^{n \times n}$ such that $t_{i,i} = \lambda_i$, $i \in \{1, \ldots, n\}$, and $\mathrm{In}(T + T^*) = (\pi_k, \nu_k, \delta_k)$.

Remark. In [7, Lemma 5], we have noticed that, if a τ-matrix $T = [t_{i,j}] \in \mathbb{C}^{n\times n}$ satisfies (i_4), then T is nonderogatory. Therefore, as the matrices L and T, referred to in Lemma 4, are nonderogatory and have the same eigenvalues with the same multiplicities, they are similar. This remark will be used in the proof of Theorem 2.

Proof of Lemma 4. If $n = 2$, the result is easy to prove. Suppose that $n \geq 3$.

If $\nu(L) = 0$, the result is a particular case of [7, Lemma 7]. If $\pi(L) = 0$, it can be proved with analogous arguments. Suppose that $\nu(L) > 0$ and $\pi(L) > 0$.

As $n \geq 3$, either $\pi(L) + \delta(L) \geq 2$ or $\nu(L) + \delta(L) \geq 2$. Without loss of generality, suppose that $\pi(L) + \delta(L) \geq 2$.

Let $n_- = \nu(L)$ and $n_+ = \pi(L) + \delta(L)$. Then L is similar to a matrix of the form $L_- \oplus L_+$, where $L_- \in \mathbb{C}^{n_-\times n_-}$ is nonderogatory and has eigenvalues $\lambda_1, \ldots, \lambda_{n_-}$, $L_+ \in \mathbb{C}^{n_+\times n_+}$ is nonderogatory and has eigenvalues $\lambda_{n_-+1}, \ldots, \lambda_n$.

Case 1. Suppose that $\Re(\lambda_{n_-+1}) > 0$. Let

$$\begin{aligned}
\nu_- &= \min\{n_-, \nu_k\}, \\
\nu_+ &= \nu_k - \nu_-, && (1)\\
\pi_+ &= \min\{\pi_k, n_+ - \nu_+\}, && (2)\\
\pi_- &= \pi_k - \pi_+, && (3)\\
\delta_- &= n_- - \nu_- - \pi_-, && (4)\\
\delta_+ &= n_+ - \nu_+ - \pi_+. && (5)
\end{aligned}$$

It is not hard to see that ν_-, π_+ are positive integers and $\nu_+, \pi_-, \delta_-, \delta_+$ are nonnegative integers.

It follows from [7, Lemma 7] that there exists a τ-matrix $T_- \in \mathbb{C}^{n_-\times n_-}$ with main diagonal $(\lambda_1, \ldots, \lambda_{n_-})$ such that $\mathrm{In}(T_- + T_-^*) = (\pi_-, \nu_-, \delta_-)$; and there exists a τ-matrix $T_+ \in \mathbb{C}^{n_+\times n_+}$ with main diagonal $(\lambda_{n_-+1}, \ldots, \lambda_n)$ such that $\mathrm{In}(T_+ + T_+^*) = (\pi_+, \nu_+, \delta_+)$.

Suppose that

$$T_- = \begin{bmatrix} * & t_- \\ 0 & \lambda_{n_-} \end{bmatrix} \in \mathbb{C}^{n_-\times n_-}, \quad \text{where} \quad t_- \in \mathbb{C}^{(n_--1)\times 1}, \tag{6}$$

$$T_+ = \begin{bmatrix} \lambda_{n_-+1} & t_+ \\ 0 & * \end{bmatrix} \in \mathbb{C}^{n_+\times n_+}, \quad \text{where} \quad t_+ \in \mathbb{C}^{1\times(n_+-1)}.$$

Let

$$X_- = \begin{bmatrix} I_{n_--1} & -(2\Re(\lambda_{n_-}))^{-1}t_- \\ 0 & 1 \end{bmatrix} \in \mathbb{C}^{n_-\times n_-}, \tag{7}$$

$$X_+ = \begin{bmatrix} 1 & 0 \\ -(2\Re(\lambda_{n_-+1}))^{-1}t_+^* & I_{n_+-1} \end{bmatrix} \in \mathbb{C}^{n_+\times n_+}.$$

Note that, by $(i_4)-(iii_4)$ and by the definition of n_-, $\Re(\lambda_{n_-}) \neq 0$.

Then $T_- + T_-^*$ is congruent to

$$X_-(T_- + T_-^*)X_-^* = S_- \oplus [2\Re(\lambda_{n_-})], \quad \text{for some } S_- \in \mathbb{C}^{(n_--1)\times(n_--1)}, \tag{8}$$

and $T_+ + T_+^*$ is congruent to

$$X_+(T_+ + T_+^*)X_+^* = [2\Re(\lambda_{n_-+1})] \oplus S_+, \quad \text{for some } S_+ \in \mathbb{C}^{(n_+-1)\times(n_+-1)}.$$

Note that $\mathrm{In}(S_-) = (\pi_-, \nu_- - 1, \delta_-)$ and $\mathrm{In}(S_+) = (\pi_+ - 1, \nu_+, \delta_+)$.

Let

$$T = \left[\begin{array}{c|cc} T_- & (2\Re(\lambda_{n_-}))^{-1}t_- & (4\Re(\lambda_{n_-})\Re(\lambda_{n_-+1}))^{-1}t_-t_+ \\ & 1 & (2\Re(\lambda_{n_-+1}))^{-1}t_+ \\ \hline 0 & \multicolumn{2}{c}{T_+} \end{array}\right].$$

Let $X = X_- \oplus X_+$. Then $T + T^*$ is congruent to

$$X(T+T^*)X^* = S_- \oplus \begin{bmatrix} 2\Re(\lambda_{n_-}) & 1 \\ 1 & 2\Re(\lambda_{n_-+1}) \end{bmatrix} \oplus S_+.$$

Hence

$$\mathrm{In}(T+T^*) = \mathrm{In}(S_-) + (1,1,0) + \mathrm{In}(S_+) = (\pi_k, \nu_k, \delta_k).$$

Case 2. Suppose that $\Re(\lambda_{n_-+1}) = 0$.

If $\nu_k \geq 2$, let

$$\nu_- = \min\{n_-, \nu_k - 1\},$$

and define $\nu_+, \pi_+, \pi_-, \delta_-, \delta_+$ as in (1)–(5).

If $\nu_k = 1$, let $\nu_- = 1$, $\nu_+ = 1$, define π_+ as in (2), let

$$\pi_- = \max\{0, \pi_k - \pi_+ - 1\},$$

and define δ_-, δ_+ as in (4), (5).

It is not hard to see that, in any case, ν_-, ν_+, π_+ are positive integers and $\pi_-, \delta_-, \delta_+$ are nonnegative integers.

It follows from [7, Lemma 7] that there exists a τ-matrix $T_- \in \mathbb{C}^{n_- \times n_-}$ with main diagonal $(\lambda_1, \ldots, \lambda_{n_-})$ such that $\mathrm{In}(T_- + T_-^*) = (\pi_-, \nu_-, \delta_-)$; and there exists a τ-matrix $T_+ \in \mathbb{C}^{n_+ \times n_+}$ with main diagonal $(\lambda_{n_-+1}, \ldots, \lambda_n)$ such that $\mathrm{In}(T_+ + T_+^*) = (\pi_+, \nu_+, \delta_+)$.

Suppose that T_- is partitioned as in (6). Suppose that

$$T_+ = \begin{bmatrix} T_{1,1} & T_{1,2} \\ 0 & T_{2,2} \end{bmatrix} \in \mathbb{C}^{n_+ \times n_+}, \quad \text{where} \quad T_{1,1} \in \mathbb{C}^{2\times 2}.$$

As $\Re(\lambda_{n_-+1}) = 0$ and the entry $(1,2)$ of $T_{1,1}$ is nonzero,

$$\mathrm{In}(T_{1,1} + T_{1,1}^*) = (1, 1, 0).$$

Define X_- as in (7) and let

$$X_+ = \begin{bmatrix} I_2 & 0 \\ -T_{1,2}^*(T_{1,1} + T_{1,1}^*)^{-1} & I_{n_+-2} \end{bmatrix} \in \mathbb{C}^{n_+ \times n_+}.$$

Then $T_- + T_-^*$ is congruent to (8) and $T_+ + T_+^*$ is congruent to

$$X_+(T_+ + T_+^*)X_+^* = (T_{1,1} + T_{1,1}^*) \oplus S_+, \quad \text{for some } S_+ \in \mathbb{C}^{(n_+-2)\times(n_+-2)}.$$

Note that $\mathrm{In}(S_-) = (\pi_-, \nu_- - 1, \delta_-)$ and $\mathrm{In}(S_+) = (\pi_+ - 1, \nu_+ - 1, \delta_+)$.

According to Lemma 3, there exists $y \in \mathbb{C}$ such that

$$R = \left[\begin{array}{c|cc} \lambda_{n_-} & 1 & y \\ \hline 0 & \multicolumn{2}{c}{T_{1,1}} \end{array}\right]$$

satisfies

$$\mathrm{In}(R + R^*) = (\pi_k, \nu_k, \delta_k) - \mathrm{In}(S_-) - \mathrm{In}(S_+) \geq (1, 1, 0).$$

Let $M = \begin{bmatrix} 1 & y \end{bmatrix}$. Let

$$T = \left[\begin{array}{c|cc} T_- & (2\Re(\lambda_{n_-}))^{-1} t_- M & (2\Re(\lambda_{n_-}))^{-1} t_- M (T_{1,1} + T_{1,1}^*)^{-1} T_{1,2} \\ & M & M(T_{1,1} + T_{1,1}^*)^{-1} T_{1,2} \\ \hline 0 & \multicolumn{2}{c}{T_+} \end{array}\right].$$

Let $X = X_- \oplus X_+$. Then $T + T^*$ is congruent to

$$X(T + T^*)X^* = S_- \oplus (R + R^*) \oplus S_+.$$

Hence $\mathrm{In}(T + T^*) = (\pi_k, \nu_k, \delta_k)$. ■

Proof of Theorem 2, when $\delta(L) = n$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of L, ordered so that (i_4) is satisfied. Choose a negative real number h such that, for every $i \in \{1, \ldots, n\}$, if $\lambda_i \neq 0$, then $h\lambda_i$ is not an eigenvalue of L. According to Lemma 4, there exists a τ-matrix $T \in \mathbb{C}^{n\times n}$, with main diagonal $(h\lambda_1, \ldots, h\lambda_{\nu_h}, \lambda_{\nu_h+1}, \ldots, \lambda_n)$, such that $\mathrm{In}(T + T^*) = (\pi_k, \nu_k, \delta_k)$. Let $H = (hI_{\nu_h}) \oplus I_{\pi_h}$. Then $\mathrm{In}(H) = (\pi_h, \nu_h, 0)$, TH^{-1} is similar to L and

$$\mathrm{In}((TH^{-1})H + H(TH^{-1})^*) = (\pi_k, \nu_k, \delta_k). \tag{9}$$

■

Proof of Theorem 2, when $\delta(L) = 0$. *Case* 1. Suppose that $\nu_h \leq \pi(L)$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of L, ordered so that (i_4), (ii_4) and (iii_4) are satisfied. Choose a negative real number h such that, for every $i \in \{1, \ldots, n\}$, $h\lambda_i$ is not an eigenvalue of L. According to Lemma 4, there exists a τ-matrix $T \in \mathbb{C}^{n\times n}$, with main diagonal

$$(\lambda_1, \ldots, \lambda_{\nu(L)}, h\lambda_{\nu(L)+1}, \ldots, h\lambda_{\nu(L)+\nu_h}, \lambda_{\nu(L)+\nu_h+1}, \ldots, \lambda_n),$$

such that $\mathrm{In}(T + T^*) = (\pi_k, \nu_k, \delta_k)$. Let

$$H = I_{\nu(L)} \oplus (hI_{\nu_h}) \oplus I_{\pi(L)-\nu_h}.$$

Then $\mathrm{In}(H) = (\pi_h, \nu_h, 0)$, TH^{-1} is similar to L and (9) holds.

Case 2. Suppose that $\nu_h > \pi(L)$. Then $\pi_h = n - \nu_h < n - \pi(L) = \nu(L) = \pi(-L)$. According to Case 1, there exists a matrix $T \in \mathbb{C}^{n\times n}$ similar to $-L$ and there exists a Hermitian matrix $H \in \mathbb{C}^{n\times n}$ such that $\mathrm{In}(H) = (\nu_h, \pi_h, 0)$ and $\mathrm{In}(TH + HT^*) = (\pi_k, \nu_k, \delta_k)$. Then $-T$ is similar to L, $\mathrm{In}(-H) = (\pi_h, \nu_h, 0)$ and

$$\mathrm{In}((-T)(-H) + (-H)(-T^*)) = (\pi_k, \nu_k, \delta_k).$$

■

Proof of Theorem 2, when $0 < \delta(L) < n$. The matrix L is similar to a matrix of the form $L_0 \oplus L_1$, where $L_0 \in \mathbb{C}^{n_0\times n_0}$ is nonderogatory, $\delta(L_0) = n_0$, $L_1 \in \mathbb{C}^{n_1\times n_1}$ is nonderogatory, $\delta(L_1) = 0$.

Without loss of generality, suppose that $\pi_k \geq \nu_k$. (The case $\pi_k \leq \nu_k$ is analogous.)

Choose nonnegative integers $\mathfrak{p}_0, \mathfrak{n}_0, \mathfrak{p}_1, \mathfrak{n}_1$ such that $\mathfrak{p}_0 + \mathfrak{n}_0 = n_0$, $\mathfrak{p}_1 + \mathfrak{n}_1 = n_1$, $\mathfrak{p}_0 + \mathfrak{p}_1 = \pi_h$ and $\mathfrak{n}_0 + \mathfrak{n}_1 = \nu_h$.

We shall define nonnegative integers $\pi_0, \nu_0, \delta_0, \pi_1, \nu_1, \delta_1$ and Hermitian matrices $H_0 \in \mathbb{C}^{n_0\times n_0}$, $H_1 \in \mathbb{C}^{n_1\times n_1}$ as follows.

If $n_0 \geq 2$, let

$$\begin{aligned}\pi_0 &= \min\{\pi_k, n_0 - 1\},\\ \nu_0 &= \min\{\nu_k, n_0 - \pi_0\},\\ \delta_0 &= n_0 - \pi_0 - \nu_0;\end{aligned}$$

then $\min\{\pi_0, \nu_0\} > 0$ and $\delta_0 \geq 0$; according to a proof already done, there exists a Hermitian matrix $H_0 \in \mathbb{C}^{n_0 \times n_0}$ such that

$$\mathrm{In}(H_0) = (\mathfrak{p}_0, \mathfrak{n}_0, 0), \tag{10}$$

$$\mathrm{In}(L_0 H_0 + H_0 L_0^*) = (\pi_0, \nu_0, \delta_0). \tag{11}$$

If $n_0 = 1$, choose $h_0 \in \{1, -1\}$ so that $H_0 = [h_0]$ satisfies (10) and let π_0, ν_0, δ_0 be the integers that satisfy (11).

If $n_1 \geq 2$, let

$$\begin{aligned}\pi_1 &= \min\{\pi_k, n_1 - 1\},\\ \nu_1 &= \min\{\nu_k, n_1 - \pi_1\},\\ \delta_1 &= n_1 - \pi_1 - \nu_1;\end{aligned}$$

then $\min\{\pi_1, \nu_1\} > 0$ and $\delta_1 \geq 0$; according to a proof already done, there exists a Hermitian matrix $H_1 \in \mathbb{C}^{n_1 \times n_1}$ such that

$$\mathrm{In}(H_1) = (\mathfrak{p}_1, \mathfrak{n}_1, 0), \tag{12}$$

$$\mathrm{In}(L_1 H_1 + H_1 L_1^*) = (\pi_1, \nu_1, \delta_1). \tag{13}$$

If $n_1 = 1$, choose $h_1 \in \{1, -1\}$ so that $H_1 = [h_1]$ satisfies (12) and let π_1, ν_1, δ_1 be the integers that satisfy (13).

In any case, it follows from [1] that there exists $X \in \mathbb{C}^{n_0 \times n_1}$ such that

$$\mathrm{In}\begin{bmatrix} L_0 H_0 + H_0 L_0^* & X \\ X^* & L_1 H_1 + H_1 L_1^* \end{bmatrix} = (\pi_k, \nu_k, \delta_k).$$

As L_0 and L_1 do not have common eigenvalues, L is similar to

$$\begin{bmatrix} L_0 & X H_1^{-1} \\ 0 & L_1 \end{bmatrix}.$$

Without loss of generality, suppose that L has this form. Let $H = H_0 \oplus H_1$. Then $\mathrm{In}(H) = (\pi_h, \nu_h, 0)$ and $\mathrm{In}(LH + HL^*) = (\pi_k, \nu_k, \delta_k)$. ■

References

[1] B. E. Cain and E. M. Sá, The inertia of a Hermitian matrix having prescribed complementary principal submatrices, *Linear Algebra Appl.* 37 (1981), 161–171.

[2] D. Carlson and H. Schneider, Inertia theorems for matrices: the semidefinite case, *J. Math. Anal. Appl.* 6 (1963), 430–446.

[3] J. L. Daleckii and M. G. Krein, *Stability of Solutions of Differential Equations in Banach Space*, Amer. Math. Soc., Providence, 1974.

[4] L. M. DeAlba and C. R. Johnson, Possible inertia combinations in the Stein and Lyapunov equations, *Linear Algebra Appl.* 222 (1995), 227–240.

[5] L. M. DeAlba, Inertia of the Stein transformation with respect to some nonderogatory matrices, *Linear Algebra Appl.* 241/243 (1996), 191–201.

[6] A. Ostrowski and H. Schneider, Some theorems on the inertia of general matrices, *J. Math. Anal. Appl.* 4 (1962), 72–84.

[7] F. C. Silva and R. Simões, On the Lyapunov and Stein equations, *Linear Algebra Appl.* 420 (2007), 329-338.

[8] O. Taussky, A generalization of a theorem of Lyapunov, *SIAM J. Appl. Math.* 9 (1961), 640–643.