MATRIX INTERPRETATION OF MULTIPLE ORTHOGONALITY

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ABSTRACT. In this work we give an interpretation of a $(s(d+1)+1)$ -term recurrence relation in terms of type II multiple orthogonal polynomials. We rewrite this recurrence relation in matrix form and we obtain a three-term recurrence relation for vector polynomials with matrix coefficients. We present a matrix interpretation of the type II multi-orthogonality conditions. We state a Favard type theorem and the expression for the resolvent function associated to the vector of linear functionals. Finally a reinterpretation of the type II Hermite-Padé approximation in matrix form is given.

1. INTRODUCTION

Multiple orthogonal polynomials are a generalization of orthogonal polynomials in the sense that they satisfy orthogonality conditions with respect to a number of measures. Such polynomials arise, in a natural way, in the study of simultaneous rational approximation, and in particular for the study of Hermite-Padé approximation for a system of $d \in \mathbb{Z}_+$ Markov functions (see [22]). In this way, multiple orthogonal polynomials are intimately related to Hermite-Padé approximation. In the literature we can find a lot of examples of multiple orthogonal polynomials (see [1, 2, 3, 13, 17, 19, 24, 25]).

Let $\vec{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$ which is called a *multi-index* with length $|\vec{n}| := n_1 + \dots + n_d$ and let $\{u^1, \dots, u^d\}$ be a set of linear functionals $u^j : \mathbb{P} \rightarrow \mathbb{C}$ with $j = 1, 2, \dots, d$.

Definition 1. Let $\{P_{\vec{n}}\}$ be a sequence of polynomials where the degree of $P_{\vec{n}}$ is at most $|\vec{n}|$. We say that $\{P_{\vec{n}}\}$ is a type II multiple orthogonal with respect to the set of linear functionals $\{u^1, \dots, u^d\}$ and multi-index

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$\vec{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$, if

$$(1) \quad u^j(x^m P_{\vec{n}}) = 0, \quad m = 0, 1, \dots, n_j - 1, \quad j = 1, \dots, d.$$

For the particular case in which the set of linear functionals is a system of integrals with respect to positive Borel measures, μ_j , on $I_j \subset \mathbb{R}$, $j = 1, \dots, d$, we have

$$u^j(x^k) = \int_{I_j} x^k d\mu_j, \quad k \in \mathbb{N}, \quad j = 1, \dots, d,$$

and the conditions of multi-orthogonality, (1), can be rewritten as

$$\int_{I_j} P_{\vec{n}}(x) x^k d\mu_j(x) = 0, \quad k = 0, 1, \dots, n_j - 1, \quad j = 1, \dots, d.$$

Definition 2. A multi-index $\vec{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$ is said to be normal for the set of linear functionals $\{u^1, \dots, u^d\}$, if for any non trivial solution $P_{\vec{n}}$ of (1), the degree of $P_{\vec{n}}$ is equal to $|\vec{n}|$. When all the multi-indices of a given family are normal, we say that the set of linear functionals $\{u^1, \dots, u^d\}$ is regular.

Definition 3. Let $u : \mathbb{P} \rightarrow \mathbb{C}$ be a linear functional, and $p \in \mathbb{P}$ a polynomial. The left product of u by p , is the linear functional $pu : \mathbb{P} \rightarrow \mathbb{C}$, defined by $pu(x^j) = u(p(x)x^j)$, $j \in \mathbb{N}$.

In the works of K. Douak and P. Maroni [14], P. Maroni [20, 21], V. Kaliaguine [18], J. Van Iseghem [27], and also in the work of V.N. Sorokin and J. Van Iseghem [23], it can be seen that a sequence of type II multiple orthogonal polynomials with respect to the set of linear functionals $\{u^1, \dots, u^d\}$ and multi-index $\vec{n} = (n_1, \dots, n_d) \in \mathcal{J}$, where

$$\mathcal{J} = \{(0, 0, \dots, 0), (1, 0, \dots, 0), \dots, (1, 1, \dots, 1), \\ (2, 1, \dots, 1), \dots, (2, 2, \dots, 2), \dots\},$$

satisfy a $(d+2)$ -term recurrence relation of type

$$xB_n = B_{n+1} + \sum_{k=0}^d a_{n-k}^n B_{n-k}, \quad a_{n-d}^n \neq 0, \quad \text{for } n = d, \dots$$

They call such polynomials d -orthogonal, where d corresponds to the number of functionals.

Now, if we multiply this recurrence equation $s-1$ times by x and using the recurrence relation property we arrive, for $n = sd, \dots$, to

$$(2) \quad x^s B_n = B_{n+s} + \sum_{k=0}^{s(d+1)-1} \tilde{a}_{n+s-1-k}^{n+s-1} B_{n+s-1-k}, \quad \tilde{a}_{n-sd}^{n+s+1} \neq 0,$$

which is our main object.

In this work we consider sequences of type II multiple orthogonal polynomials for more general sets of multi-indices, $\mathcal{J}$. We designate these multi-indices by quasi-diagonal of step s . In section 2 we build the sets of quasi-diagonal multi-indices $\mathcal{J}$. Next we give the type II multi-orthogonality conditions for a *sequence of monic polynomials*, $\{B_n\}$, i.e. $B_n = x^n + \dots$, $n = 0, 1, \dots$, with respect to the set of linear functionals $\{u^1, \dots, u^d\}$ and a family of quasi-diagonal multi-indices $\mathcal{J}$ of step s . We also prove that this sequence satisfies a $(s(d+1)+1)$ -term recurrence relation of the type (2). To finish this section, we rewrite the previous $(s(d+1)+1)$ -term recurrence relation in matrix form and we obtain a three-term recurrence relation for vector polynomials with matrix coefficients. We also give an example of multiple Hermite orthogonal polynomials satisfying a three term vector recurrence relation with matrix coefficients. In section 3 we present an algebraic theory which enables us to operate with the new presented objects. Here, our main goal is to present a matrix interpretation in terms of a vector of functionals, of the multi-orthogonality conditions presented in the section 2. We characterize the regularity (cf. [6, 12]) of a set of linear functional in terms of the regularity of a vector of linear functionals. Next we give a result of existence and uniqueness of a type II sequence of vector orthogonal polynomials with respect to a regular vector of linear functionals $\mathcal{U}$, and using a matrix three-term recurrence relations we establish a Favard type theorem. We remark that other characterization for sequences of orthogonal polynomials in terms of matrix three-term recurrence relations can be found in [15, 16]. In section 4 we express the resolvent function in terms of the matrix generating function associated to the vector of linear functionals. Note that in the recent paper [4], the authors applies the technique here exposed in the diagonal case, to describe the correspondence between dynamics of the coefficients of the operator defined by a Lax pair and its resolvent function. Finally, we give a reinterpretation of the type II multiple orthogonality, in terms of a Hermite-Padé approximation problem for the matrix generating function associated to the vector of linear functionals. We remark that Hermite-Padé approximation problems can be found for example in [22, 24], and in matrix form in [5, 7, 8, 9, 10, 11, 26].

2. QUASI-DIAGONAL MULTI-INDICES

We call $\mathcal{J}$ a set of *quasi-diagonal multi-indices of step s* if

$$\mathcal{J} = \bigcup_{n \in \mathbb{N} \cup \{0\}} \mathcal{J}_n, \quad \mathcal{J}_n = \mathcal{J}_0 + \{n(s, s, \dots, s)\}, \quad n \in \mathbb{N},$$

and $\mathcal{J}_0$ is called *the pattern block* and is explicitly given by

$$\mathcal{J}_0 = \{(0, \dots, 0), (1, 0, \dots, 0), \dots, (k_i^1, \dots, k_i^d), \dots, (k_{sd-1}^1, \dots, k_{sd-1}^d)\},$$

where $(k_i^1, \dots, k_i^d)$, for $i = 0, \dots, sd-1$, verifies the following conditions:

- a) $|(k_i^1, \dots, k_i^d)| = i$,
- b) $(k_1^1, \dots, k_1^d) = (1, 0, \dots, 0)$,
- c) $(k_i^1, \dots, k_i^d) \leq (k_{i+1}^1, \dots, k_{i+1}^d)$, i.e. increasing structure in each component,
- d) $k_i^j \leq s$.

Notice that a) and c) implies that there exist a unique j' such that

$$k_{i+1}^{j'} = k_i^{j'} + 1, \text{ and } k_{i+1}^j = k_i^j, \ j \neq j'.$$

and d) implies that $(k_{sd-1}^1, \dots, k_{sd-1}^{d-1}, k_{sd-1}^d)$ is equal to $(s, \dots, s, s-1)$ up to a permutation. Remark that condition b) is not restrictive because we can always reorder the initial set of functionals.

For $s = 1$ we have that $\mathcal{J}_0$ can be given by,

$$\mathcal{J}_0 = \{(0, \dots, 0), (1, 0, \dots, 0), (1, 1, \dots, 0), \dots, (1, \dots, 1, 0)\},$$

whose *multi-indices* we designate by *diagonal*.

There is an one-to-one correspondence, $\mathbf{i}$, between our set of quasi-diagonal multi-indices $\mathcal{J} \subset \mathbb{N}^d$ and $\mathbb{N}$ given by, $\mathbf{i}(\vec{n}) = |\vec{n}| = n$.

Let us consider, $B_{\vec{n}}$, a sequence of type II multiple orthogonal polynomial with respect to the set of linear functionals $\{u^1, \dots, u^d\}$ and the set of quasi-diagonal multi-indices, $\mathcal{J}$. We identify $B_{\vec{n}} \equiv B_{|\vec{n}|} = B_n$, where $\vec{n} = (k_n^1, \dots, k_n^d)$.

Algorithm (Construction of linear functionals). *Let us consider a set of linear functionals $\{u^1, \dots, u^d\}$ and a set of quasi-diagonal multi-indices, $\mathcal{J}$, of step s .*

Let $v^1 = u^1$, $v^i = x^{k_{i-1}^j} u^j$, $i = 2, \dots, sd$ where j , for each i , is uniquely defined by the condition $k_i^j = k_{i-1}^j + 1$. Hence, we have

$$v^i \in \{x^k u^j : k = 0, 1, \dots, s-1, \ j = 1, 2, \dots, d\}, \ i = 1, 2, \dots, sd.$$

Example 1. *For the pattern block $\mathcal{J}_0 = \{(0, 0), (1, 0), (2, 0), (2, 1), (2, 2), (3, 2)\}$, we can obtain a new set of linear functionals, $\{v^1 = u^1, v^2 = xu^1, v^3 = u^2, v^4 = xu^2, v^5 = x^2u^1, v^6 = x^2u^2\}$. Notice that we have used that $(k_6^1, k_6^2) = (3, 3)$.*

Theorem 1. *The sequence of monic polynomials, $\{B_n\}$, is type II multiple orthogonal with respect to the regular set of linear functionals $\{u^1, \dots, u^d\}$ and the set of quasi-diagonal multi-indices $\mathcal{J}$ of step s if,*

and only if,

$$(3) \quad \begin{cases} v^j((x^s)^m B_{sdr+i}) = 0, & m = 0, 1, \dots, r-1, \quad j = 1, \dots, sd \\ v^\alpha((x^s)^r B_{sdr+i}) = 0, & \alpha = 1, \dots, i \\ v^{i+1}((x^s)^r B_{sdr+i}) \neq 0, \end{cases}$$

where $i = 0, 1, \dots, sd-1, r = 0, 1, \dots$, and the linear functionals v^j , $j = 1, \dots, sd$ are defined by the algorithm .

Proof. Let us consider the set of multi-indices

$$\mathcal{J}_0 = \{(0, \dots, 0), (1, 0, \dots, 0), \dots, (k_i^1, \dots, k_i^d), \dots, (k_{sd-1}^1, \dots, k_{sd-1}^d)\}.$$

The linear functionals $v^1, \dots, v^{sd}$ are defined by the algorithm . We obtain the multi-orthogonality conditions for the polynomials B_i , $i = 1, \dots, sd-1$. Let us consider the multi-index $(k_i^1, \dots, k_i^d)$ and let $j \in \{1, \dots, d\}$ be uniquely defined by the condition $k_i^j = k_{i-1}^j + 1$. We have

$$u^j(x^{k_{i-1}^j} B_i) = 0 \Leftrightarrow x^{k_{i-1}^j} u^j(B_i) = 0 \Leftrightarrow v^i(B_i) = 0.$$

By the increasing structure of the multi-indices, B_i complies with the multi-orthogonality conditions of $B_1, \dots, B_{i-1}$, so it holds,

$$v^j(B_i) = 0, \quad j = 1, \dots, i.$$

We obtain the multi-orthogonality conditions for the polynomials B_{rsd+i} , $i = 0, 1, \dots, sd-1, r = 1, \dots$.

Let us consider the multi-index $(k_i^1, \dots, k_i^d) + r(s, \dots, s)$ and let $j \in \{1, \dots, d\}$ be uniquely defined by the condition $k_i^j = k_{i-1}^j + 1$. We have

$$u^j(x^{k_{i-1}^j + rs} B_{rsd+i}) = 0 \Leftrightarrow x^{k_{i-1}^j} u^j((x^s)^r B_{rsd+i}) = 0 \Leftrightarrow v^i((x^s)^r B_{i+sd}) = 0.$$

By the increasing structure of the multi-indices, B_{rsd+i} complies with the multi-orthogonality conditions of $B_1, \dots, B_{rsd+i-1}$, so it holds that,

$$\begin{cases} v^j((x^s)^m B_{sdr+i}) = 0, & m = 0, 1, \dots, r-1, \quad j = 1, \dots, sd \\ v^\alpha((x^s)^r B_{sdr+i}) = 0, & \alpha = 1, \dots, i. \end{cases}$$

Finally, we show that $v^{i+1}((x^s)^r B_{sdr+i}) \neq 0$, for $r = 0, \dots, sd-1$ and $i = 0, 1, \dots$. In fact, if we suppose that $v^1(B_0) = 0$, we get that B_1 is of degree 0, which contradicts the normality of the multi-index $(1, 0, \dots, 0)$, and the third condition in (3) for $i = r = 0$ is achieved. Now, let us suppose that,

$$\begin{cases} v^j((x^s)^m B_{sdr+i}) = 0, & m = 0, 1, \dots, r-1, \quad j = 1, \dots, sd \\ v^\alpha((x^s)^r B_{sdr+i}) = 0, & \alpha = 1, \dots, i \\ v^{i+1}((x^s)^r B_{sdr+i}) = 0. \end{cases}$$

Then the polynomial B_{sdr+i} satisfy the multi-orthogonality conditions of the polynomial $B_{sdr+i+1}$ which contradicts the normality of the multi-indices. Hence, $v^{i+1}((x^s)^r B_{sdr+i}) \neq 0$.

Reciprocally, for $n = sdr + i$, $i = 1, \dots, sd$

$$\begin{cases} v^j((x^s)^m B_{sdr+i}) = 0, & m = 0, 1, \dots, r-1, \quad j = 1, \dots, sd \\ v^\alpha((x^s)^r B_{sdr+i}) = 0, & \alpha = 1, \dots, i, \end{cases}$$

and considering that the degree of B_n is equal to n by the normality of each of the multi-indices which implies the uniqueness of the monic type II multiple orthogonal polynomial sequence, B_n , with respect to the set of linear functionals $\{u^1, \dots, u^d\}$ and quasi-diagonal multi-index $\vec{n}$ such that $|\vec{n}| = n$. $\square$

Here, we show that the sequence of monic type II multiple orthogonal polynomials, $\{B_n\}$, with respect to the regular set of linear functionals $\{u^1, \dots, u^d\}$ and a set of quasi-diagonal multi-indices $\mathcal{J}$ of step s , satisfies a $(s(d+1) + 1)$ -term recurrence relation.

Theorem 2. *Let $\{B_n\}$ be a monic type II multiple orthogonal polynomials sequence, with respect to a regular set of linear functionals $\{u^1, \dots, u^d\}$ and a set of quasi-diagonal multi-indices $\mathcal{J}$ of step s . Then, there are sequences $(a_{n+s-1-k}^{n+s-1}) \subset \mathbb{C}$, $k = 0, 1, \dots, s(d+1)-1$, such that,*

$$(4) \quad x^s B_n(x) = B_{n+s}(x) + \sum_{k=0}^{s(d+1)-1} a_{n+s-1-k}^{n+s-1} B_{n+s-1-k}(x),$$

for $n = sd, sd+1, \dots$, where $a_{n-sd}^{n+s-1} \neq 0$ and $B_0, B_1, \dots, B_{sd-1}$ are given.

Proof. As the degree of the monic polynomials B_n is equal to n , there is an unique sequence $(a_j^{n+s-1}) \subset \mathbb{C}$, such that:

$$x^s B_n = B_{n+s} + \sum_{j=0}^{n+s-1} a_j^{n+s-1} B_j.$$

Substituting n by $sdr + i$ where $i = 0, 1, \dots, sd-1$ and $r = 0, 1, \dots$, in the above identity, we have

$$x^s B_{sdr+i} - B_{sdr+i+s} = \sum_{j=0}^{sdr+i+s-1} a_j^{sdr+i+s-1} B_j.$$

Now, considering the orthogonality conditions (3), and applying successively to both members of the above equation the functionals

$$v^1, \dots, v^{sd}, \dots, (x^s)^{r-2} v^1, \dots, (x^s)^{r-2} v^{sd}, (x^s)^{r-1} v^1, \dots, (x^s)^{r-1} v^i$$

we obtain successively that $a_j^{sdr+i+s-1} = 0$, for $j = 0, 1, \dots, (r-1)sd+i$, and the theorem is proved. $\square$

Now we give a matrix interpretation for the recurrence relation(4).

Lemma 1. *Let $\{B_n\}$ be a monic sequence of polynomials. Then, the following conditions are equivalent:*

a) *The sequence of polynomials $\{B_n\}$ satisfies a $(s(d+1)+1)$ -term relation,*

$$x^s B_n(x) = B_{n+s}(x) + \sum_{k=0}^{s(d+1)-1} a_{n+s-1-k}^{n+s-1} B_{n+s-1-k}(x), \quad n = sd, sd+1, \dots,$$

where $a_{n-sd}^{n+s-1} \neq 0$ and $B_0, B_1, \dots, B_{sd-1}$ are given.

b) *The vector sequence of polynomials $\{\mathcal{B}_m\}$, where*

$$\mathcal{B}_m = [B_{msd} \quad \dots \quad B_{(m+1)sd-1}]^T, \quad m \in \mathbb{N}$$

satisfies a three-term recurrence relation with $sd \times sd$ matrix coefficients, for all $m = 0, 1, \dots$,

$$(5) \quad x^s \mathcal{B}_m(x) = \alpha_m^{s,d} \mathcal{B}_{m+1}(x) + \beta_m^{s,d} \mathcal{B}_m(x) + \gamma_m^{s,d} \mathcal{B}_{m-1}(x),$$

with $\mathcal{B}_{-1} = 0_{sd \times 1}$ and $\mathcal{B}_0$ given, where $\alpha_m^{s,d}$, $\beta_m^{s,d}$ and $\gamma_m^{s,d}$ are respectively given by

$$\begin{bmatrix} 1 \\ a_{(m+1)sd}^{(m+1)sd} & \ddots \\ \vdots & \ddots & 1 \\ a_{(m+1)sd}^{(m+1)sd+s-2} & \dots & a_{(m+1)sd+s-2}^{(m+1)sd+s-2} & 1 \end{bmatrix};$$

$$\begin{bmatrix} a_{msd}^{msd+s-1} & \dots & a_{msd+s-1}^{msd+s-1} & 1 \\ \vdots & & \vdots & \ddots & \ddots \\ a_{msd}^{(m+1)sd-1} & \dots & a_{msd+s-1}^{(m+1)sd-1} & \dots & a_{(m+1)sd-2}^{(m+1)sd-2} & 1 \\ \vdots & & \vdots & & \vdots & \vdots \\ a_{msd}^{(m+1)sd+s-2} & \dots & a_{msd+s-1}^{(m+1)sd+s-2} & \dots & a_{(m+1)sd-2}^{(m+1)sd+s-2} & a_{(m+1)sd-1}^{(m+1)sd+s-2} \end{bmatrix};$$

$$\begin{bmatrix} a_{(m-1)sd}^{msd+s-1} & \dots & a_{msd-1}^{msd+s-1} \\ & \ddots & \vdots \\ & & a_{msd-1}^{(m+1)sd+s-2} \end{bmatrix}.$$

Example 2. The Hermite multiple orthogonal polynomials for any multi-index (n_1, n_2) are given by the Rodrigues formula (see [2])

$$H_{(n_1, n_2)}(x) = \left(-\frac{1}{2}\right)^{n_1+n_2} e^{-b_2x+x^2} \frac{\partial^{n_2}}{\partial x^{n_2}} \left(e^{(-b_1+b_2)x} \frac{\partial^{n_1}}{\partial x^{n_1}} \left(e^{-b_1x+x^2} \right) \right)$$

For the pattern block $\mathcal{J}_0 = \{(0, 0), (1, 0), (2, 0), (2, 1)\}$ correspond to the case $s = d = 2$ and the associated set of weight functions with support in $\mathbb{R}$ are given by $\{e^{-x^2+b_1x}, xe^{-x^2+b_1x}, e^{-x^2+b_2x}, xe^{-x^2+b_2x}\}$. Then, the three term recurrence relations with matrix coefficients $\alpha_m^{2,2}$, $\beta_m^{2,2}$ and $\gamma_m^{2,2}$

$$x^2 \mathcal{B}_m(x) = \alpha_m^{2,2} \mathcal{B}_{m+1}(x) + \beta_m^{2,2} \mathcal{B}_m(x) + \gamma_m^{2,2} \mathcal{B}_{m-1}(x), \quad m = 0, 1, \dots,$$

where

$$\mathcal{B}_m(x) = [H_{(2n, 2n)}(x), H_{(2n+1, 2n)}(x), H_{(2n+2, 2n)}(x), H_{(2n+2, 2n+1)}(x)]^T$$

$\mathcal{B}_{-1}(x) = 0_{4 \times 1}$, are given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ (b_1 + b_2)2 & 1 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} \frac{2+b_1^2+16n}{4} & b_1 & 1 & 0 \\ \frac{b_1+2b_1n+2b_2n}{2} & \frac{6+b_1^2+16n}{4} & \frac{b_1+b_2}{2} & 1 \\ \frac{1+6n-b_1b_2n+b_2^2n+8n^2}{2} & b_1 + b_1n + b_2n & \frac{10+b_2^2+16n}{4} & b_2 \\ \frac{(b_1-b_2)(1+2n)}{4} & \frac{3+b_1^2-b_1b_2+10n+b_1^2n-b_1b_2n+8n^2}{2} & \frac{2b_1+b_2+2b_1n+2b_2n}{2} & \frac{14+b_2^2+16n}{4} \end{bmatrix},$$

$$\begin{bmatrix} \frac{(b_1-b_2)^2n(-1+2n)}{8} & \frac{(b_1-b_2)n(-2+b_1^2-b_1b_2+8n)}{4} & \frac{n(-2+b_1^2-b_1b_2+8n)}{2} & (b_1 + b_2)n \\ 0 & \frac{(b_1-b_2)^2n(1+2n)}{8} & \frac{(b_1-b_2)n}{2} & \frac{n(2-b_1b_2+b_2^2+8n)}{2} \\ 0 & 0 & \frac{(b_1-b_2)^2(-1+2n)}{8} & -\frac{((b_1-b_2)n(2-b_1b_2+b_2^2+8n))}{2} \\ 0 & 0 & 0 & \frac{(b_1-b_2)^2n(1+2n)}{8} \end{bmatrix}.$$

3. MATRIX INTERPRETATION OF TYPE II MULTI-ORTHOGONALITY

Let us consider

$$\mathbb{P}^{sd} = \{[P_1 \ \dots \ P_{sd}]^T : P_j \in \mathbb{P}\}.$$

We denote by $\mathcal{M}_{sd \times sd}$ the set of $sd \times sd$ matrices with entries in $\mathbb{C}$.

Let $\{\mathcal{P}_j\}$ be a sequence of vectors of polynomials given by

$$(6) \quad \mathcal{P}_j = [x^{j_{sd}} \ \dots \ x^{(j+1)_{sd}-1}]^T, \quad j \in \mathbb{N}.$$

Definition 4. Let $v^j : \mathbb{P} \rightarrow \mathbb{C}$ with $j = 1, \dots, sd$ be linear functionals. We define the vector of functionals $\mathcal{U} = [v^1 \ \dots \ v^{sd}]^T$ acting in $\mathbb{P}^{sd}$ over $\mathcal{M}_{sd \times sd}$, by

$$\mathcal{U}(\mathcal{P}) := (\mathcal{U} \cdot \mathcal{P}^T)^T = \begin{bmatrix} v^1(P_1) & \dots & v^{sd}(P_1) \\ \vdots & \ddots & \vdots \\ v^1(P_{sd}) & \dots & v^{sd}(P_{sd}) \end{bmatrix},$$

where “ $\cdot$ ” means the symbolic product of the vectors $\mathcal{U}$ and $\mathcal{P}^T$.

Now we define an operation called *left multiplication of a vector of functionals by a polynomial*.

Definition 5. Let $\hat{A} = \sum_{k=0}^l A_k x^k$ be a matrix polynomial of degree l where $A_k \in \mathcal{M}_{sd \times sd}$ and $\mathcal{U}$ a vector of linear functionals. We define the vector of linear functionals, left multiplication of $\mathcal{U}$ by a matrix polynomial $\hat{A}$, and denote it by $\hat{A}\mathcal{U}$, the map of $\mathbb{P}^{sd}$ to $\mathcal{M}_{sd \times sd}$, defined by:

$$(\hat{A}\mathcal{U})(\mathcal{P}) := (\hat{A}\mathcal{U} \cdot \mathcal{P}^T)^T = \sum_{k=0}^l (x^k \mathcal{U})(\mathcal{P})(A_k)^T.$$

Theorem 3. A sequence of monic polynomials $\{B_m\}$, is type II multiple orthogonal with respect to the regular set of linear functionals $\{u^1, \dots, u^d\}$ and the set of quasi-diagonal multi-indices $\mathcal{J}$ of step s if, and only if, the vector sequence of polynomials $\{\mathcal{B}_m\}$, $\mathcal{B}_m = [B_{msd} \ \dots \ B_{(m+1)sd-1}]^T$, $m \in \mathbb{N}$, satisfies:

$$(7) \quad \left. \begin{array}{l} i) ((x^s)^k \mathcal{U})(\mathcal{B}_m) = 0_{sd \times sd}, \ k = 0, 1, \dots, m-1 \\ ii) ((x^s)^m \mathcal{U})(\mathcal{B}_m) = \Delta_m, \end{array} \right\}$$

where $\mathcal{U} = [v^1 \ \dots \ v^{sd}]^T$, v^j , $j = 1, \dots, sd$ are defined by the algorithm, and Δ_m is a regular upper triangular $sd \times sd$ matrix.

Proof. By Definition 4, we have

$$((x^s)^k \mathcal{U})(\mathcal{B}_m) = \begin{bmatrix} v^1((x^s)^k B_{msd}) & \dots & v^{sd}((x^s)^k B_{msd}) \\ \vdots & \ddots & \vdots \\ v^1((x^s)^k B_{(m+1)sd-1}) & \dots & v^{sd}((x^s)^k B_{(m+1)sd-1}) \end{bmatrix}.$$

Using the orthogonality conditions of type II in Theorem 1 we have the conditions (7), and reciprocally. $\square$

Now we introduce the notions of *moments* and *Hankel matrices* by blocks associated to the vector of linear functionals $\mathcal{U} = [v^1 \ \dots \ v^{sd}]^T$.

Definition 6. We define the moments of order $j \in \mathbb{N}$ associated to the vector of linear functionals $(x^s)^k \mathcal{U}$, by

$$(8) \quad \mathcal{U}_j^k = ((x^s)^k \mathcal{U})(\mathcal{P}_j) = \begin{bmatrix} v^1(x^{j sd + ks}) & \dots & v^{sd}(x^{j sd + ks}) \\ \vdots & \ddots & \vdots \\ v^1(x^{(j+1)sd + ks - 1}) & \dots & v^{sd}(x^{(j+1)sd + ks - 1}) \end{bmatrix}$$

we define Hankel matrices by

$$(9) \quad \mathcal{H}_m = \begin{bmatrix} \mathcal{U}_0^0 & \dots & \mathcal{U}_0^m \\ \vdots & \ddots & \vdots \\ \mathcal{U}_m^0 & \dots & \mathcal{U}_m^m \end{bmatrix}, \quad m \in \mathbb{N},$$

and we say that the vector of linear functionals $\mathcal{U}$ is regular if the principal minors of the matrix $\mathcal{H}_m$, are regular for $m \in \mathbb{N}$.

If we consider the matrix $A = (a_{i,j})$, we denote by $A_{r,s}$, $r, s \in \mathbb{N}$ the principal submatrix obtained from A of size $r \times s$, that is $A_{r,s} = (a_{i,j}), i = 1, \dots, r, j = 1, \dots, s$.

Theorem 4. Let us consider a set of linear functionals $\{u^1, \dots, u^d\}$ and the set of quasi-diagonal multi-indices $\mathcal{J}$ of step s and let $\mathcal{U} = [v^1 \dots v^{sd}]^T$ be the vector of linear functionals where v^j , $j = 1, \dots, sd$ are defined by the algorithm. Then $\mathcal{U}$ is regular if, and only if, $\{u^1, \dots, u^d\}$ is regular.

Proof. Let us suppose that the set of linear functionals $\{u^1, \dots, u^d\}$ is regular. Let $\{B_n\}$ be the sequence of monic polynomials multiple orthogonal with respect to this set of linear functionals and the set of quasi-diagonal multi-indices $\mathcal{J}$, where $\deg B_n = n$. Let us consider the vector sequence of polynomials associated $\{\mathcal{B}_m\}$, given by $\mathcal{B}_m = [B_{msd} \dots B_{(m+1)sd-1}]^T$, $m \in \mathbb{N}$. We can write $\mathcal{B}_m = \sum_{j=0}^m B_j^m \mathcal{P}_j$ where $B_j^m \in \mathcal{M}_{sd \times sd}$ and B_m^m a regular lower triangular matrix. By the multi-orthogonality conditions (7) the vector sequence of polynomials $\{\mathcal{B}_m\}$ satisfies

$$((x^s)^k \mathcal{U})(\mathcal{B}_m) = ((x^s)^k \mathcal{U}) \left(\sum_{j=0}^m B_j^m \mathcal{P}_j \right) = \sum_{j=0}^m B_j^m ((x^s)^k \mathcal{U})(\mathcal{P}_j) = 0_{sd \times sd},$$

and for all $m \in \mathbb{N}$,

$$(10) \quad ((x^s)^m \mathcal{U})(\mathcal{B}_m) = \sum_{j=0}^m B_j^m ((x^s)^m \mathcal{U})(\mathcal{P}_j) = \Delta_m.$$

where Δ_m are regular upper triangular matrices of order $sd \times sd$. In matrix form we have,

$$\begin{bmatrix} B_0^m & \cdots & B_m^m \end{bmatrix} \begin{bmatrix} \mathcal{U}_0^0 & \cdots & \mathcal{U}_0^m \\ \vdots & \ddots & \vdots \\ \mathcal{U}_m^0 & \cdots & \mathcal{U}_m^m \end{bmatrix} = \begin{bmatrix} 0_{sd \times sd} & \cdots & 0_{sd \times sd} & \Delta_m \end{bmatrix}.$$

For $m = 0$, we have $B_0^0 \mathcal{U}_0^0 = \Delta_0$. Using the regularity of the principal minors of the matrices $\mathcal{B}_0^0$ and Δ_0 we have that the principal minors of $\mathcal{U}_0^0$ are regular matrices.

For $m = 1$ we have

$$\begin{cases} B_0^1 \mathcal{U}_0^0 + B_1^1 \mathcal{U}_1^0 = 0_{sd \times sd} \\ B_0^1 \mathcal{U}_0^1 + B_1^1 \mathcal{U}_1^1 = \Delta_1, \end{cases} \quad \text{i.e.} \quad B_1^1(\mathcal{U}_1^1 - \mathcal{U}_1^0(\mathcal{U}_0^0)^{-1}\mathcal{U}_0^1) = \Delta_1.$$

Using the regularity of the principal minors of the matrices $\mathcal{B}_1^1$ and Δ_1 we have that the principal minors of $(\mathcal{U}_1^1 - \mathcal{U}_1^0(\mathcal{U}_0^0)^{-1}\mathcal{U}_0^1)$ are regular matrices, and using the triangular structure by blocks we have that the principal minors of $\mathcal{H}_1$ are regular matrices. This argument can be inductively repeated and we obtain the regularity of $\mathcal{U}$.

Reciprocally, supposing the regularity of the vector of linear functionals $\mathcal{U}$, let us consider $B_n(x) = a_{n,n}x^n + a_{n,n-1}x^{n-1} + \cdots + a_{n,0}$. The multi-orthogonal conditions for the multi-index $\vec{n}$, such that $|\vec{n}| = msd + i$, for $m \in \mathbb{N}$, $0 \leq i \leq sd - 1$, are:

$$\begin{bmatrix} a_{msd+i,0} & \cdots & a_{msd+i,msd+i} \end{bmatrix} \begin{bmatrix} \mathcal{U}_0^0 & \cdots & \mathcal{U}_0^m \\ \vdots & \ddots & \vdots \\ \mathcal{U}_m^0 & \cdots & \mathcal{U}_m^m \end{bmatrix}_{msd+i+1,msd+i} = 0_{1 \times (msd+i)}.$$

Let us suppose that there exist a solution for this linear system of equation of degree less than $msd + i$, that is $a_{msd+i,msd+i} = 0$. In this case we would have a linear homogeneous system of $msd + i$ equations and unknowns and the matrix of coefficients being a regular matrix. This would imply that the unique solution is the trivial solution and so the multi-index $\vec{n}$ such that $|\vec{n}| = msd + i$ is a normal index. $\square$

Theorem 5. *Let $\mathcal{U} = [v^1 \ \cdots \ v^{sd}]^T$ be a vector of linear functionals. Then $\mathcal{U} = [v^1 \ \cdots \ v^{sd}]^T$ is regular if, and only if, there exist a unique vector sequence of polynomials $\{\mathcal{B}_m\}$, $\mathcal{B}_m = [B_{msd} \ \cdots \ B_{(m+1)sd-1}]$, where B_n is a monic polynomial of degree n and a unique sequence $\Delta_m, m \in \mathbb{N}$, of regular upper triangular $sd \times sd$ matrices such that:*

$$((x^s)^k \mathcal{U})(\mathcal{B}_m) = \Delta_m \delta_{k,m}, \quad k = 0, 1, \dots, m, \quad m \in \mathbb{N},$$

Proof. Let us suppose that $\mathcal{U}$ is regular. To find the vector polynomial sequence $\{\mathcal{B}_m\}$, where $\mathcal{B}_m = [B_{msd} \cdots B_{(m+1)sd-1}]^T$, $m \in \mathbb{N}$, and B_n a monic polynomial of degree n , that satisfies

$$((x^s)^k \mathcal{U})(\mathcal{B}_m) = \Delta_m \delta_{k,m}, \quad k = 0, 1, \dots, m, \quad m \in \mathbb{N},$$

is equivalent to solve

$$\begin{bmatrix} B_0^m & \cdots & B_m^m \end{bmatrix} \begin{bmatrix} \mathcal{U}_0^0 & \cdots & \mathcal{U}_0^m \\ \vdots & \ddots & \vdots \\ \mathcal{U}_m^0 & \cdots & \mathcal{U}_m^m \end{bmatrix} = \begin{bmatrix} 0_{sd \times sd} & \cdots & 0_{sd \times sd} & \Delta_m \end{bmatrix},$$

where we write $\mathcal{B}_m = \sum_{j=0}^m B_j^m \mathcal{P}_j$, with $B_j^m \in \mathcal{M}_{sd \times sd}$ and B_m^m a regular lower triangular matrix.

For $m = 0$, we have $B_0^0 \mathcal{U}_0^0 = \Delta_0$.

Using the regularity of the principal minors of the matrices $\mathcal{U}_0^0$, and the LU decomposition, we can find uniquely B_0^0 a regular lower triangular matrix with entries equal to 1 in the main diagonal, and Δ_0 a regular upper triangular matrix such that $B_0^0 \mathcal{U}_0^0 = \Delta_0$.

For $m = 1$ we have

$$\begin{cases} B_0^1 \mathcal{U}_0^0 + B_1^1 \mathcal{U}_1^0 = 0_{sd \times sd} \\ B_0^1 \mathcal{U}_0^1 + B_1^1 \mathcal{U}_1^1 = \Delta_1, \end{cases} \quad \text{i.e. } B_1^1 (\mathcal{U}_1^1 - \mathcal{U}_1^0 (\mathcal{U}_0^0)^{-1} \mathcal{U}_0^1) = \Delta_1.$$

Again using the regularity of the principal minors of the matrices $\mathcal{U}_1^1 - \mathcal{U}_1^0 (\mathcal{U}_0^0)^{-1} \mathcal{U}_0^1$, and the LU decomposition, we can find uniquely B_1^1 a regular lower triangular matrix with entries equal to 1 in the main diagonal, and Δ_1 a regular upper triangular matrix such that $B_1^1 (\mathcal{U}_1^1 - \mathcal{U}_1^0 (\mathcal{U}_0^0)^{-1} \mathcal{U}_0^1) = \Delta_1$. We also obtain from $B_0^1 \mathcal{U}_0^0 + B_1^1 \mathcal{U}_1^0 = 0_{sd \times sd}$, uniquely the matrix B_0^1 . This argument can be inductively repeated and we obtain the stated result.

The converse is true following the same reasoning as in Theorem 4. $\square$

In section 2 we have proven that a sequence of monic type II multiple orthogonal polynomials, $\{B_n\}$, with respect to the regular set of linear functionals $\{u^1, \dots, u^d\}$ and the set of quasi-diagonal multi-indices $\mathcal{J}$ of step s satisfy a $(s(d+1)+1)$ -term recurrence relation and we rewrote this recurrence relation in matrix form, obtaining a three-term recurrence relation for vector polynomials with matrix coefficients. Now we prove the converse of this result which is called the *Favard type theorem*. Note that in the given literature (see for instance [15, 16]), the coefficients of matrix three term recurrence relation, are regular Hermitian matrices, and in (5) this is not the case.

Theorem 6. *Let $\{B_n\}$ be a sequence of monic polynomials and let us consider the vector sequence of polynomials associated $\{\mathcal{B}_m\}$, $\mathcal{B}_m =$*

$[B_{msd} \ \cdots \ B_{(m+1)sd-1}]$ and let $\mathcal{U} = [v^1 \ \cdots \ v^{sd}]^T$ be a vector of linear functionals. Then, the following conditions are equivalent:

a) The vector sequence of polynomials $\{\mathcal{B}_m\}$ satisfies:

$$(11) \quad ((x^s)^k \mathcal{U})(\mathcal{B}_m) = \Delta_m \delta_{k,m}, \quad k = 0, 1, \dots, m, \quad m \in \mathbb{N},$$

where Δ_m is a regular upper triangular $sd \times sd$ matrix.

b) There exist sequences of $sd \times sd$ matrices $(\alpha_m^{s,d})$, $(\beta_m^{s,d})$ and $(\gamma_m^{s,d})$, $m \in \mathbb{N}$, with $\gamma_m^{s,d}$ regular upper triangular matrix such that $\mathcal{B}_m$ is defined by the three-term recurrence relation with matrix coefficients in $\mathcal{M}_{sd \times sd}$ given for all $m = 0, 1, \dots$ by

$$(12) \quad x^s \mathcal{B}_m(x) = \alpha_m^{s,d} \mathcal{B}_{m+1}(x) + \beta_m^{s,d} \mathcal{B}_m(x) + \gamma_m^{s,d} \mathcal{B}_{m-1}(x),$$

with $\mathcal{B}_{-1} = 0_{d \times 1}$ and $\mathcal{B}_0$ given.

Furthermore

$$\Delta_m = \gamma_m^{s,d} \cdots \gamma_1^{s,d} \Delta_0, \quad m = 1, 2, \dots$$

Proof. a) $\Rightarrow$ b). We can express:

$$\begin{aligned} x^s \mathcal{B}_m(x) &= \alpha_m^{s,d} \mathcal{B}_{m+1}(x) + \beta_m^{s,d} \mathcal{B}_m(x) + \gamma_m^{s,d} \mathcal{B}_{m-1}(x) \\ &\quad + \sum_{j=0}^{m-2} \delta_{m,j}^{s,d} \mathcal{B}_j(x), \quad m = 0, 1, \dots \end{aligned}$$

where $\alpha_m^{s,d}, \beta_m^{s,d}, \gamma_m^{s,d}, \delta_{m,j}^{s,d} \in \mathcal{M}_{sd \times sd}$ and they are uniquely determined. For $m \geq 2$, let us multiply both members of this equation by $(x^s)^k$,

$$\begin{aligned} (x^s)^{(k+1)} \mathcal{B}_m(x) &= \alpha_m^{s,d} (x^s)^k \mathcal{B}_{m+1}(x) + \beta_m^{s,d} (x^s)^k \mathcal{B}_m(x) \\ &\quad + \gamma_m^{s,d} (x^s)^k \mathcal{B}_{m-1}(x) + \sum_{j=0}^{m-2} \delta_{m,j}^{s,d} (x^s)^k \mathcal{B}_j(x). \end{aligned}$$

For $k = 0, \dots, m-2$ we apply successively the vector of functionals $\mathcal{U}$, use the linearity, the orthogonality condition (11), and we obtain

$$0_{sd \times sd} = \delta_{m,j}^{s,d} \Delta_j, \quad j = 0, \dots, m-2;$$

from the regularity of the matrix Δ_j , for $j = 0, \dots, m-2$ we get that

$$0_{sd \times sd} = \delta_{m,m-1}^{s,d}, \quad j = 0, \dots, m-2.$$

For $k = m-1$ we obtain $\Delta_m = \gamma_m^{s,d} \Delta_{m-1}$ so it holds that $\gamma_m^{s,d}$ is a regular upper triangular matrix.

b) $\Rightarrow$ a). We build a vector of linear functionals $\mathcal{U}$ that verifies (11) defined uniquely taking into account its moments $\mathcal{U}_m^k$. For each $m \in \mathbb{N}$, there is an unique sequence $(B_j^m) \subset \mathcal{M}_{sd \times sd}$, such that, $\mathcal{B}_m =$

$$\sum_{j=0}^m B_j^m \mathcal{P}_j.$$

• Let $k = 0$. We have

$$\mathcal{U}(\mathcal{B}_0) = B_0^0 \mathcal{U}(\mathcal{P}_0)$$

and so $\mathcal{U}_0^0 = (B_0^0)^{-1} \mathcal{U}(\mathcal{B}_0)$,

$$\mathcal{U}(\mathcal{B}_m) = \sum_{j=0}^m B_j^m \mathcal{U}(\mathcal{P}_j), \text{ i.e. } \mathcal{U}_m^0 = - \sum_{j=0}^{m-1} (B_m^m)^{-1} B_j^m \mathcal{U}_j^0, \quad m = 1, 2, \dots$$

• Let $k = 1, 2, \dots$. Using (12) we have

$$(x^s)^k \mathcal{B}_m = \alpha_m^{s,d} x^{s(k-1)} \mathcal{B}_{m+1} + \beta_m^{s,d} x^{s(k-1)} \mathcal{B}_m + \gamma_m^{s,d} x^{s(k-1)} \mathcal{B}_{m-1}.$$

For $m = 0$ we have

$$\mathcal{U}((x^s)^k \mathcal{B}_0) = \alpha_0^{s,d} \mathcal{U}(x^{s(k-1)} \mathcal{B}_1) + \beta_0^{s,d} \mathcal{U}(x^{s(k-1)} \mathcal{B}_0),$$

$$\text{i.e. } \mathcal{U}_0^k = (B_0^0)^{-1} \left[\alpha_0^{s,d} B_1^1 \mathcal{U}_1^{s(k-1)} + (\alpha_0^{s,d} B_0^1 + \beta_0^{s,d} B_0^0) \right] \mathcal{U}_0^{s(k-1)}.$$

For $m \leq k$, we have

$$\begin{aligned} \mathcal{U}((x^s)^k \mathcal{B}_m) &= \alpha_m^{s,d} \mathcal{U}(x^{s(k-1)} \mathcal{B}_{m+1}) + \beta_m^{s,d} \mathcal{U}(x^{s(k-1)} \mathcal{B}_m) \\ &\quad + \gamma_m^{s,d} \mathcal{U}(x^{s(k-1)} \mathcal{B}_{m-1}), \end{aligned}$$

$$\begin{aligned} \mathcal{U}((x^s)^k \mathcal{B}_m) &= \alpha_m^{s,d} \sum_{j=0}^{m+1} B_j^{m+1} \mathcal{U}_j^{k-1} + \beta_m^{s,d} \sum_{j=0}^m B_j^m \mathcal{U}_j^{k-1} \\ &\quad + \gamma_m^{s,d} \sum_{j=0}^{m-1} B_j^{m-1} \mathcal{U}_j^{k-1}, \end{aligned}$$

$$\begin{aligned} \mathcal{U}((x^s)^k \mathcal{B}_m) &= \sum_{j=0}^{m-1} (\alpha_m^{s,d} B_j^{m+1} + \beta_m^{s,d} B_j^m + \gamma_m^{s,d} B_j^{m-1}) \mathcal{U}_j^{k-1} \\ &\quad + (\alpha_m^{s,d} B_m^{m+1} + \beta_m^{s,d} B_m^m) \mathcal{U}_m^{k-1} + \alpha_m^{s,d} B_{m+1}^{m+1} \mathcal{U}_{m+1}^{k-1}. \end{aligned}$$

Taking into account that,

$$\mathcal{U}((x^s)^k \mathcal{B}_m) = \mathcal{U}((x^s)^k \sum_{j=0}^m B_j^m \mathcal{P}_j) = B_m^m \mathcal{U}_m^k + \sum_{j=0}^{m-1} B_j^m \mathcal{U}_j^k,$$

we have

$$\begin{aligned} \mathcal{U}_m^k &= (B_m^m)^{-1} \sum_{j=0}^{m-1} (\alpha_m^{s,d} B_j^{m+1} + \beta_m^{s,d} B_j^m + \gamma_m^{s,d} B_j^{m-1}) \mathcal{U}_j^{k-1} \\ &\quad + (B_m^m)^{-1} ((\alpha_m^{s,d} B_m^{m+1} + \beta_m^{s,d} B_m^m) \mathcal{U}_m^{k-1} + \alpha_m^{s,d} B_{m+1}^{m+1} \mathcal{U}_{m+1}^{k-1} - \sum_{j=0}^{m-1} B_j^m \mathcal{U}_j^k). \end{aligned}$$

For $m = k$ we have

$$\mathcal{U}((x^s)^k \mathcal{B}_k) = \gamma_k^{s,d} \gamma_{k-1}^{s,d} \cdots \gamma_1^{s,d} B_0^0 \mathcal{U}_0^0,$$

and so,

$$\mathcal{U}_k^k = (B_k^k)^{-1} (\gamma_k^{s,d} \gamma_{k-1}^{s,d} \cdots \gamma_1^{s,d} B_0^0 \mathcal{U}_0^0 - \sum_{j=0}^{k-1} B_j^k \mathcal{U}_j^k).$$

For $m > k$ we have $\mathcal{U}((x^s)^k \mathcal{B}_m) = 0_{sd \times sd}$, i.e.

$$\mathcal{U}_m^k = - \sum_{j=0}^{m-1} (B_m^m)^{-1} B_j^m \mathcal{U}_j^k.$$

Therefore, the moments associated to the vector of linear functionals $\mathcal{U}$ are uniquely determined from (11). Hence, the result is proved. $\square$

Notice that, in matrix notation the three-term recurrence relation of the previous Theorem, (12), is written by

$$(13) \quad J \begin{bmatrix} \mathcal{B}_0 \\ \vdots \\ \mathcal{B}_m \\ \vdots \end{bmatrix} = x^s \begin{bmatrix} \mathcal{B}_0 \\ \vdots \\ \mathcal{B}_m \\ \vdots \end{bmatrix},$$

where the tridiagonal matrix by blocks

$$(14) \quad J = \begin{bmatrix} \beta_0^{s,d} & \alpha_0^{s,d} & 0_{sd \times sd} & & & \\ \gamma_1^{s,d} & \beta_1^{s,d} & \alpha_1^{s,d} & 0_{sd \times sd} & & \\ 0_{sd \times sd} & \gamma_2^{s,d} & \beta_2^{s,d} & \alpha_2^{s,d} & 0_{sd \times sd} & \\ & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

is called *block Jacobi matrix* associated to the vector of functionals $\mathcal{U}$, and the uniquely determined vector polynomial sequence $\{\mathcal{B}_m\}$ associated to the vector of functionals $\mathcal{U}$, is called *type II multiple vector orthogonal sequence* with respect to the vector of functionals $\mathcal{U}$.

4. TYPE II HERMITE-PADÉ APPROXIMATION

Definition 7. Let $\mathcal{U} = [v^1 \ \cdots \ v^{sd}]$ be a vector of linear functionals. We define the matrix generating function associated to $\mathcal{U}$, $\mathcal{F}$, by

$$(15) \quad \mathcal{F}(z) := \mathcal{U}_x \left(\frac{\mathcal{P}_0(x)}{z - x^s} \right) = \begin{bmatrix} v_x^1 \left(\frac{1}{z - x^s} \right) & \cdots & v_x^{sd} \left(\frac{1}{z - x^s} \right) \\ \vdots & \ddots & \vdots \\ v_x^1 \left(\frac{x^{sd-1}}{z - x^s} \right) & \cdots & v_x^{sd} \left(\frac{x^{sd-1}}{z - x^s} \right) \end{bmatrix}.$$

Being,

$$(16) \quad \frac{1}{z - x^s} = \frac{1}{z} \sum_{k=0}^{\infty} \left(\frac{x^s}{z} \right)^k \quad \text{for } |x^s| < |z|,$$

$$\text{we have } \mathcal{F}(z) = \sum_{k=0}^{\infty} \frac{((x^s)^k \mathcal{U}_x)(\mathcal{P}_0(x))}{z^{k+1}}.$$

Theorem 7. Let $\mathcal{U} = [v^1 \ \dots \ v^{sd}]$ be a regular vector of linear functionals, $\{\mathcal{B}_m\}$ the vector type II multiple orthogonal polynomials sequence with respect to $\mathcal{U}$, J the block Jacobi matrix associated, given in (14) and $\mathcal{R}$ the resolvent function associated, i.e.

$$\mathcal{R}(z) = \sum_{n=0}^{\infty} \frac{e_0^t J^n e_0}{z^{n+1}}, \quad \text{where } e_0 = [I_{sd \times sd} \ 0_{sd \times sd} \ \dots]^T.$$

Then, $\mathcal{R}(z) = B_0^0 \mathcal{F}(z) (\mathcal{U}(\mathcal{P}_0))^{-1} (B_0^0)^{-1}$, where B_0^0 is the matrix coefficient in $\mathcal{B}_0 = B_0^0 \mathcal{P}_0$.

Proof. In order to determine the value of $e_0^t J^n e_0$, $n \in \mathbb{N}$, we consider the matrix identity (13), from which we can obtain,

$$(17) \quad J^n \begin{bmatrix} \mathcal{B}_0(x) \\ \vdots \\ \mathcal{B}_m(x) \\ \vdots \end{bmatrix} = (x^s)^n \begin{bmatrix} \mathcal{B}_0(x) \\ \vdots \\ \mathcal{B}_m(x) \\ \vdots \end{bmatrix}, \quad n \in \mathbb{N}.$$

$$\text{Let } (x^s)^n \mathcal{B}_0(x) = \sum_{j=0}^n \eta_{j,n}^0 \mathcal{B}_j(x).$$

By (17), $e_0^t J^n e_0$, $n \in \mathbb{N}$, is given by $\eta_{0,n}^0$. Applying the vector of linear functionals $\mathcal{U}$ to both members of the previous matrix identity, we have

$$\eta_{0,n}^0 = ((x^s)^n \mathcal{U})(\mathcal{B}_0)(\mathcal{U}(\mathcal{B}_0))^{-1}.$$

Using $\mathcal{B}_0 = B_0^0 \mathcal{P}_0$, we have

$$\eta_{0,n}^0 = B_0^0 ((x^s)^n \mathcal{U})(\mathcal{P}_0)(\mathcal{U}(\mathcal{P}_0))^{-1} (B_0^0)^{-1}.$$

Hence,

$$\mathcal{R}(z) = B_0^0 \left\{ \sum_{n=0}^{\infty} \frac{((x^s)^n \mathcal{U})(\mathcal{P}_0)(\mathcal{U}(\mathcal{P}_0))^{-1}}{z^{n+1}} \right\} (B_0^0)^{-1},$$

as we wanted to prove. □

The vector sequence of polynomials $\{\mathcal{B}_m\}$, where

$$\mathcal{B}_m = [B_{msd} \ \cdots \ B_{(m+1)sd-1}]^T, \ m \in \mathbb{N}$$

and B_n is a monic polynomial of degree n can be written as

$$\mathcal{B}_n = \sum_{j=0}^n B_j^n \mathcal{P}_j, \ B_j^n \in \mathcal{M}_{sd \times sd},$$

where the matrix coefficients B_j^n , $j = 0, 1, \dots, n$ are uniquely determined and B_n^n is a regular lower triangular matrix.

Taking into account (6) we have that $\mathcal{P}_j = (x^{sd})^j \mathcal{P}_0$, $j \in \mathbb{N}$. Therefore, $\mathcal{B}_n = V_n(x^{sd}) \mathcal{P}_0$, where V_n is a matrix polynomial of degree n and dimension sd , given by $V_n(x) = \sum_{j=0}^n B_j^n x^j$, $B_j^n \in \mathcal{M}_{sd \times sd}$. Now, we present a reinterpretation of type II Hermite-Padé approximation in terms of the matrix functions.

Definition 8. Let $\{\mathcal{B}_m\}$ be a vector sequence of polynomials and $\mathcal{U}$ a vector of linear functionals. To the sequence of polynomials $\{\mathcal{B}_{m-1}^{(1)}\}$ given by

$$\mathcal{B}_{m-1}^{(1)}(z) := \mathcal{U}_x \left(\frac{V_m(z^d) - V_m(x^{sd})}{z - x^s} \mathcal{P}_0(x) \right),$$

where $\mathcal{U}_x$ represents the action of $\mathcal{U}$ over the variable x , and $\mathcal{B}_n = V_n(x^{sd}) \mathcal{P}_0$, we designate sequence of polynomials associated to $\{\mathcal{B}_m\}$ and to $\mathcal{U}$.

Theorem 8. Let $\mathcal{U}$ be a regular vector of linear functionals, $\{\mathcal{B}_m\}$ a vector sequence of polynomials, $\mathcal{B}_n = V_n(x^{sd}) \mathcal{P}_0$, where $V_n(x) = \sum_{j=0}^n B_j^n x^j$, $B_j^n \in \mathcal{M}_{sd \times sd}$, B_n^n is a regular lower triangular matrix with entries equal to 1 in the diagonal. $\{\mathcal{B}_{m-1}^{(1)}\}$ the sequence of associated polynomials and $\mathcal{F}$ the matrix generating function defined in (15). Then $\{\mathcal{B}_m\}$ is the vector type II multiple orthogonal with respect to the vector of linear functionals $\mathcal{U}$ if, and only if,

$$V_m(z^d) \mathcal{F}(z) - \mathcal{B}_{m-1}^{(1)}(z) = \sum_{k=m}^{\infty} \frac{((x^s)^k \mathcal{U}_x)(\mathcal{B}_m(x))}{z^{k+1}}.$$

and $((x^s)^k \mathcal{U}_x)(\mathcal{B}_m(x)) = \Delta_m$, where Δ_m is a regular upper triangular matrix.

Proof. Taking into account the Definition 8, we have

$$\begin{aligned} \mathcal{B}_{m-1}^{(1)}(z) &= \mathcal{U}_x \left(\frac{V_m(z^d) - V_m(x^{sd})}{z - x^s} \mathcal{P}_0(x) \right) \\ &= V_m(z^d) \mathcal{F}(z) - \mathcal{U}_x \left(\frac{V_m(x^{sd})}{z - x^s} \mathcal{P}_0(x) \right), \end{aligned}$$

i.e. $V_m(z^d)\mathcal{F}(z) - \mathcal{B}_{m-1}^{(1)}(z) = \mathcal{U}_x\left(\frac{V_m(x^{sd})}{z-x^s}\mathcal{P}_0(x)\right)$.

Taking into account (16) we have

$$V_m(z^d)\mathcal{F}(z) - \mathcal{B}_{m-1}^{(1)}(z) = \sum_{k=0}^{\infty} \frac{((x^s)^k \mathcal{U}_x)(\mathcal{B}_m(x))}{z^{k+1}}.$$

Hence, we get the desired result. $\square$

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