

# HIGHER-ORDER HAHN'S QUANTUM VARIATIONAL CALCULUS

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ABSTRACT. We prove a necessary optimality condition of Euler–Lagrange type for quantum variational problems involving Hahn's derivatives of higher-order.

## 1. INTRODUCTION

Many physical phenomena are described by equations involving nondifferentiable functions, e.g., generic trajectories of quantum mechanics [15]. Several different approaches to deal with nondifferentiable functions are followed in the literature of variational calculus, including the time scale approach, which typically deal with delta or nabla differentiable functions [14, 20, 23], the fractional approach, allowing to consider functions that have no first order derivative but have fractional derivatives of all orders less than one [3, 12, 16], and the quantum approach, which is particularly useful to model physical and economical systems [8, 10, 22].

Roughly speaking, a quantum calculus substitute the classical derivative by a difference operator, which allows to deal with sets of nondifferentiable functions. Several dialects of quantum calculus are available [13, 18]. For motivation to study a nondifferentiable quantum variational calculus we refer the reader to [4, 8, 10].

In 1949 Hahn introduced the difference operator  $D_{q,\omega}$  defined by

$$D_{q,\omega}[f](t) := \frac{f(qt + \omega) - f(t)}{(q-1)t + \omega},$$

where  $f$  is a real function, and  $q \in (0, 1)$  and  $\omega > 0$  are real fixed numbers [17]. The Hahn difference operator has been applied successfully in the construction of families of orthogonal polynomials as well as in approximation problems [6, 11, 25]. However, during 60 years, the construction of the proper inverse of Hahn's difference operator remained an open question. Eventually, the problem was solved in 2009 by Aldwoah [1] (see also [2, 7]). Here we introduce the higher-order Hahn's quantum variational calculus, proving the Hahn quantum analog of the higher-order Euler–Lagrange equation. As particular cases we obtain the  $q$ -calculus Euler–Lagrange equation [8] and the  $h$ -calculus Euler–Lagrange equation [9, 19].

Variational functionals that depend on higher derivatives arise in a natural way in applications of engineering, physics, and economics. Let us consider, for example, the equilibrium of an elastic bending beam. Let us denote by  $y(x)$  the deflection

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of the point  $x$  of the beam,  $E(x)$  the elastic stiffness of the material, that can vary with  $x$ , and  $\xi(x)$  the load that bends the beam. One may assume that, due to some constraints of physical nature, the dynamics does not depend on the usual derivative  $y'(x)$  but on some quantum derivative  $D_{q,\omega}[y](x)$ . In this condition, the equilibrium of the beam correspond to the solution of the following higher-order Hahn's quantum variational problem:

$$(1.1) \quad \int_0^L \left[ \frac{1}{2} (E(x) D_{q,\omega}^2[y](x))^2 - \xi(x)y(q^2x + q\omega + \omega) \right] dx \longrightarrow \min.$$

Note that we recover the classical problem of the equilibrium of the elastic bending beam when  $(\omega, q) \rightarrow (0, 1)$ . Problem (1.1) is a particular case of the problem (P) investigated in Section 3. Our higher-order Hahn's quantum Euler–Lagrange equation (Theorem 3.10) gives the main tool to solve such problems.

The paper is organized as follows. In Section 2 we summarize all the necessary definitions and properties of the Hahn difference operator and the associated  $q, \omega$ -integral. In Section 3 we formulate and prove our main results: in §3.1 we prove a higher-order fundamental Lemma of the calculus of variations with the Hahn operator (Lemma 3.8); in §3.2 we deduce a higher-order Euler–Lagrange equation for Hahn's variational calculus (Theorem 3.10); finally we provide in §3.3 a simple example of a quantum optimization problem where our Theorem 3.10 leads to the global minimizer, which is not a continuous function.

## 2. PRELIMINARIES

Let  $q \in (0, 1)$  and  $\omega > 0$ . We introduce the real number

$$\omega_0 := \frac{\omega}{1-q}.$$

Let  $I$  be a real interval containing  $\omega_0$ . For a function  $f$  defined on  $I$ , the *Hahn difference operator* of  $f$  is given by

$$D_{q,\omega}[f](t) := \frac{f(qt + \omega) - f(t)}{(q-1)t + \omega}, \text{ if } t \neq \omega_0,$$

and  $D_{q,\omega}[f](\omega_0) := f'(\omega_0)$ , provided  $f$  is differentiable at  $\omega_0$ . We sometimes call  $D_{q,\omega}[f]$  the  *$q, \omega$ -derivative of  $f$* , and  $f$  is said to be  *$q, \omega$ -differentiable on  $I$*  if  $D_{q,\omega}[f](\omega_0)$  exists.

*Remark 2.1.* The  $D_{q,\omega}$  operator generalizes (in the limit) the forward  $h$ -difference and the Jackson  $q$ -difference operators [13, 18]. Indeed, when  $q \rightarrow 1$  we obtain the forward  $h$ -difference

$$\Delta_h[f](t) := \frac{f(t+h) - f(t)}{h},$$

when  $\omega \rightarrow 0$  we obtain the Jackson  $q$ -difference operator

$$D_q[f](t) := \frac{f(qt) - f(t)}{t(q-1)}, \text{ if } t \neq 0,$$

and  $D_q[f](0) = f'(0)$  provided  $f'(0)$  exists. Notice also that, under appropriate conditions,

$$\lim_{\omega \rightarrow 0, q \rightarrow 1} D_{q,\omega}[f](t) = f'(t).$$

The Hahn difference operator has the following properties:

**Theorem 2.2** ([1, 2, 7]). *Let  $f$  and  $g$  be  $q, \omega$ -differentiable on  $I$  and  $t \in I$ . One has:*

- (1)  $D_{q,\omega}[f](t) \equiv 0$  on  $I$  if and only if  $f$  is constant;
- (2)  $D_{q,\omega}[f+g](t) = D_{q,\omega}[f](t) + D_{q,\omega}[g](t)$ ;
- (3)  $D_{q,\omega}[fg](t) = D_{q,\omega}[f](t)g(t) + f(qt+\omega)D_{q,\omega}[g](t)$ ;
- (4)  $D_{q,\omega}\left[\frac{f}{g}\right](t) = \frac{D_{q,\omega}[f](t)g(t) - f(t)D_{q,\omega}[g](t)}{g(t)g(qt+\omega)}$  if  $g(t)g(qt+\omega) \neq 0$ ;
- (5)  $f(qt+\omega) = f(t) + (t(q-1) + \omega)D_{q,\omega}[f](t)$ .

For  $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$  define  $[k]_q := \frac{1-q^k}{1-q}$  and let  $\sigma(t) = qt + \omega$ ,  $t \in I$ . Note that  $\sigma$  is a contraction,  $\sigma(I) \subseteq I$ ,  $\sigma(t) < t$  for  $t > \omega_0$ ,  $\sigma(t) > t$  for  $t < \omega_0$ , and  $\sigma(\omega_0) = \omega_0$ . The following technical result is used several times in our paper:

**Lemma 2.3** ([1, 7]). *Let  $k \in \mathbb{N}$  and  $t \in I$ . Then,*

- (1)  $\sigma^k(t) = \underbrace{\sigma \circ \sigma \circ \dots \circ \sigma}_{k\text{-times}}(t) = q^k t + \omega[k]_q$ ;
- (2)  $(\sigma^k(t))^{-1} = \sigma^{-k}(t) = \frac{t - \omega[k]_q}{q^k}$ .

From now on  $I$  denotes an interval of  $\mathbb{R}$  containing  $\omega_0$ . Following [1, 2, 7] we define the notion of  $q, \omega$ -integral (also known as the *Jackson-Nörlund integral*) as follows:

**Definition 2.4.** Let  $a, b \in I$  and  $a < b$ . For  $f : I \rightarrow \mathbb{R}$  the  $q, \omega$ -integral of  $f$  from  $a$  to  $b$  is given by

$$\int_a^b f(t) d_{q,\omega} t := \int_{\omega_0}^b f(t) d_{q,\omega} t - \int_{\omega_0}^a f(t) d_{q,\omega} t,$$

where

$$\int_{\omega_0}^x f(t) d_{q,\omega} t := (x(1-q) - \omega) \sum_{k=0}^{+\infty} q^k f(xq^k + \omega[k]_q), \quad x \in I,$$

provided that the series converges at  $x = a$  and  $x = b$ . In that case,  $f$  is called  $q, \omega$ -integrable on  $[a, b]$ . We say that  $f$  is  $q, \omega$ -integrable over  $I$  if it is  $q, \omega$ -integrable over  $[a, b]$  for all  $a, b \in I$ .

*Remark 2.5.* The  $q, \omega$ -integral generalizes (in the limit) the Jackson  $q$ -integral and the Nörlund's sum [18]. When  $\omega \rightarrow 0$ , we obtain the Jackson  $q$ -integral

$$\int_a^b f(t) d_q t := \int_0^b f(t) d_q t - \int_0^a f(t) d_q t,$$

where

$$\int_0^x f(t) d_q t := x(1-q) \sum_{k=0}^{+\infty} q^k f(xq^k).$$

When  $q \rightarrow 1$ , we obtain the Nörlund's sum

$$\int_a^b f(t) \Delta_\omega t := \int_{+\infty}^b f(t) \Delta_\omega t - \int_{+\infty}^a f(t) \Delta_\omega t,$$

where

$$\int_{+\infty}^x f(t) \Delta_{\omega} t := -\omega \sum_{k=0}^{+\infty} f(x + k\omega).$$

It can be shown that if  $f : I \rightarrow \mathbb{R}$  is continuous at  $\omega_0$ , then  $f$  is  $q, \omega$ -integrable over  $I$  [1, 2, 7].

**Theorem 2.6** (Fundamental Theorem of Hahn's Calculus [1, 7]). *Assume that  $f : I \rightarrow \mathbb{R}$  is continuous at  $\omega_0$  and, for each  $x \in I$ , define*

$$F(x) := \int_{\omega_0}^x f(t) d_{q,\omega} t.$$

*Then  $F$  is continuous at  $\omega_0$ . Furthermore,  $D_{q,\omega}[F](x)$  exists for every  $x \in I$  with  $D_{q,\omega}[F](x) = f(x)$ . Conversely,  $\int_a^b D_{q,\omega}[f](t) d_{q,\omega} t = f(b) - f(a)$  for all  $a, b \in I$ .*

The  $q, \omega$ -integral has the following properties:

**Theorem 2.7** ([1, 2, 7]). *Let  $f, g : I \rightarrow \mathbb{R}$  be  $q, \omega$ -integrable on  $I$ ,  $a, b, c \in I$  and  $k \in \mathbb{R}$ . Then,*

- (1)  $\int_a^a f(t) d_{q,\omega} t = 0$ ;
- (2)  $\int_a^b k f(t) d_{q,\omega} t = k \int_a^b f(t) d_{q,\omega} t$ ;
- (3)  $\int_a^b f(t) d_{q,\omega} t = -\int_b^a f(t) d_{q,\omega} t$ ;
- (4)  $\int_a^b f(t) d_{q,\omega} t = \int_a^c f(t) d_{q,\omega} t + \int_c^b f(t) d_{q,\omega} t$ ;
- (5)  $\int_a^b (f(t) + g(t)) d_{q,\omega} t = \int_a^b f(t) d_{q,\omega} t + \int_a^b g(t) d_{q,\omega} t$ ;
- (6) *Every Riemann integrable function  $f$  on  $I$  is  $q, \omega$ -integrable on  $I$ ;*
- (7) *If  $f, g : I \rightarrow \mathbb{R}$  are  $q, \omega$ -differentiable and  $a, b \in I$ , then*

$$\int_a^b f(t) D_{q,\omega}[g](t) d_{q,\omega} t = f(t) g(t) \Big|_a^b - \int_a^b D_{q,\omega}[f](t) g(t) d_{q,\omega} t.$$

Property 7 of Theorem 2.7 is known as  $q, \omega$ -integration by parts. Note that

$$\int_{\sigma(t)}^t f(\tau) d_{q,\omega} \tau = (t(1-q) - \omega) f(t).$$

**Lemma 2.8** (cf. [1, 7]). *Let  $b \in I$  and  $f$  be  $q, \omega$ -integrable over  $I$ . Suppose that*

$$f(t) \geq 0 \quad \forall t \in \left\{ q^n b + \omega[n]_q : n \in \mathbb{N}_0 \right\}.$$

- (1) *If  $\omega_0 \leq b$ , then*

$$\int_{\omega_0}^b f(t) d_{q,\omega} t \geq 0.$$

- (2) *If  $\omega_0 > b$ , then*

$$\int_b^{\omega_0} f(t) d_{q,\omega} t \geq 0.$$

**Remark 2.9.** There is an inconsistency in [1, 7]. Indeed, Lemma 6.2.7 of [1] is only valid if  $b \geq \omega_0$  and  $a \leq b$ . Similarly with respect to Lemma 3.7 of [7].

**Remark 2.10.** In general it is not true that

$$\left| \int_a^b f(t) d_{q,\omega} t \right| \leq \int_a^b |f(t)| d_{q,\omega} t, \quad a, b \in I.$$

For a counterexample see [1, 7]. This illustrates well the difference with other non-quantum integrals, e.g., the time scale integrals [21, 24].

For  $s \in I$  we define

$$(2.1) \quad [s]_{q,\omega} := \left\{ q^n s + \omega [n]_q : n \in \mathbb{N}_0 \right\} \cup \{\omega_0\}.$$

The following definition and lemma are important for our purposes.

**Definition 2.11.** Let  $s \in I$  and  $g : I \times (-\bar{\theta}, \bar{\theta}) \rightarrow \mathbb{R}$ . We say that  $g(t, \cdot)$  is differentiable at  $\theta_0$  uniformly in  $[s]_{q,\omega}$  if for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$0 < |\theta - \theta_0| < \delta \Rightarrow \left| \frac{g(t, \theta) - g(t, \theta_0)}{\theta - \theta_0} - \partial_2 g(t, \theta_0) \right| < \varepsilon$$

for all  $t \in [s]_{q,\omega}$ , where  $\partial_2 g = \frac{\partial g}{\partial \theta}$ .

**Lemma 2.12** (cf. [22]). Let  $s \in I$ . Assume that  $g : I \times (-\bar{\theta}, \bar{\theta}) \rightarrow \mathbb{R}$  is differentiable at  $\theta_0$  uniformly in  $[s]_{q,\omega}$ , and  $\int_{\omega_0}^s \partial_2 g(t, \theta_0) d_{q,\omega} t$  exist. Then,

$$G(\theta) := \int_{\omega_0}^s g(t, \theta) d_{q,\omega} t,$$

for  $\theta$  near  $\theta_0$ , is differentiable at  $\theta_0$  with  $G'(\theta_0) = \int_{\omega_0}^s \partial_2 g(t, \theta_0) d_{q,\omega} t$ .

### 3. MAIN RESULTS

We define the  $q, \omega$ -derivatives of higher-order in the usual way: the  $r$ th  $q, \omega$ -derivative ( $r \in \mathbb{N}$ ) of  $f : I \rightarrow \mathbb{R}$  is the function  $D_{q,\omega}^r[f] : I \rightarrow \mathbb{R}$  given by  $D_{q,\omega}^r[f] := D_{q,\omega}[D_{q,\omega}^{r-1}[f]]$ , provided  $D_{q,\omega}^{r-1}[f]$  is  $q, \omega$ -differentiable on  $I$  and where  $D_{q,\omega}^0[f] := f$ .

Let  $a, b \in I$  and  $a < b$ . We introduce the linear space  $\mathcal{Y}^r = \mathcal{Y}^r([a, b], \mathbb{R})$  by  $\mathcal{Y}^r := \{y : I \rightarrow \mathbb{R} \mid D_{q,\omega}^i[y], i = 0, \dots, r, \text{ are bounded on } [a, b] \text{ and continuous at } \omega_0\}$  endowed with the norm  $\|y\|_{r,\infty} := \sum_{i=0}^r \|D_{q,\omega}^i[y]\|_\infty$ , where  $\|y\|_\infty := \sup_{t \in [a,b]} |y(t)|$ . The following notations are in order:  $\sigma(t) = qt + \omega$ ,  $y^\sigma(t) = y^{\sigma^1}(t) = (y \circ \sigma)(t) = y(qt + \omega)$ , and  $y^{\sigma^k} = y \circ y^{\sigma^{k-1}}$ ,  $k = 2, 3, \dots$ . Our main goal is to establish necessary optimality conditions for the higher-order  $q, \omega$ -variational problem<sup>1</sup>

$$\begin{aligned} \mathcal{L}[y] &= \int_a^b L\left(t, y^{\sigma^r}(t), D_{q,\omega}\left[y^{\sigma^{r-1}}\right](t), \dots, D_{q,\omega}^r[y](t)\right) d_{q,\omega} t \longrightarrow \text{extr} \\ & y \in \mathcal{Y}^r([a, b], \mathbb{R}) \\ (P) \quad & y(a) = \alpha_0, \quad y(b) = \beta_0, \\ & \vdots \end{aligned}$$

$$D_{q,\omega}^{r-1}[y](a) = \alpha_{r-1}, \quad D_{q,\omega}^{r-1}[y](b) = \beta_{r-1},$$

where  $r \in \mathbb{N}$  and  $\alpha_i, \beta_i \in \mathbb{R}$ ,  $i = 0, \dots, r-1$ , are given.

**Definition 3.1.** We say that  $y$  is an admissible function for (P) if  $y \in \mathcal{Y}^r([a, b], \mathbb{R})$  and  $y$  satisfies the boundary conditions  $D_{q,\omega}^i[y](a) = \alpha_i$  and  $D_{q,\omega}^i[y](b) = \beta_i$  of problem (P),  $i = 0, \dots, r-1$ .

<sup>1</sup>In problem (P) “extr” denotes “extremize” (i.e., minimize or maximize).

The Lagrangian  $L$  is assumed to satisfy the following hypotheses:

- (H1)  $(u_0, \dots, u_r) \rightarrow L(t, u_0, \dots, u_r)$  is a  $C^1(\mathbb{R}^{r+1}, \mathbb{R})$  function for any  $t \in [a, b]$ ;
- (H2)  $t \rightarrow L(t, y(t), D_{q,\omega}[y](t), \dots, D_{q,\omega}^r[y](t))$  is continuous at  $\omega_0$  for any admissible  $y$ ;
- (H3) functions  $t \rightarrow \partial_{i+2} L(t, y(t), D_{q,\omega}[y](t), \dots, D_{q,\omega}^r[y](t))$ ,  $i = 0, 1, \dots, r$ , belong to  $\mathcal{Y}^1([a, b], \mathbb{R})$  for all admissible  $y$ .

**Definition 3.2.** We say that  $y_*$  is a local minimizer (resp. local maximizer) for problem (P) if  $y_*$  is an admissible function and there exists  $\delta > 0$  such that

$$\mathcal{L}[y_*] \leq \mathcal{L}[y] \quad (\text{resp. } \mathcal{L}[y_*] \geq \mathcal{L}[y])$$

for all admissible  $y$  with  $\|y_* - y\|_{r,\infty} < \delta$ .

**Definition 3.3.** We say that  $\eta \in \mathcal{Y}^r([a, b], \mathbb{R})$  is a *variation* if  $\eta(a) = \eta(b) = 0$ ,  $\dots$ ,  $D_{q,\omega}^{r-1}[\eta](a) = D_{q,\omega}^{r-1}[\eta](b) = 0$ .

We define the  $q, \omega$ -interval from  $a$  to  $b$  by

$$[a, b]_{q,\omega} := \left\{ q^n a + \omega[n]_q : n \in \mathbb{N}_0 \right\} \cup \left\{ q^n b + \omega[n]_q : n \in \mathbb{N}_0 \right\} \cup \{\omega_0\},$$

i.e.,  $[a, b]_{q,\omega} = [a]_{q,\omega} \cup [b]_{q,\omega}$ , where  $[a]_{q,\omega}$  and  $[b]_{q,\omega}$  are given by (2.1).

**3.1. Higher-order fundamental lemma of Hahn's variational calculus.** The chain rule, as known from classical calculus, does not hold in Hahn's quantum context (see a counterexample in [1, 7]). However, we can prove the following.

**Lemma 3.4.** *If  $f$  is  $q, \omega$ -differentiable on  $I$ , then the following equality holds:*

$$D_{q,\omega}[f^\sigma](t) = q(D_{q,\omega}[f])^\sigma(t), \quad t \in I.$$

*Proof.* For  $t \neq \omega_0$  we have

$$(D_{q,\omega}[f])^\sigma(t) = \frac{f(q(qt + \omega) + \omega) - f(qt + \omega)}{(q-1)(qt + \omega) + \omega} = \frac{f(q(qt + \omega) + \omega) - f(qt + \omega)}{q((q-1)t + \omega)}$$

and

$$D_{q,\omega}[f^\sigma](t) = \frac{f^\sigma(qt + \omega) - f^\sigma(t)}{(q-1)t + \omega} = \frac{f(q(qt + \omega) + \omega) - f(qt + \omega)}{(q-1)t + \omega}.$$

Therefore,  $D_{q,\omega}[f^\sigma](t) = q(D_{q,\omega}[f])^\sigma(t)$ . If  $t = \omega_0$ , then  $\sigma(\omega_0) = \omega_0$ . Thus,

$$(D_{q,\omega}[f])^\sigma(\omega_0) = (D_{q,\omega}[f])(\sigma(\omega_0)) = (D_{q,\omega}[f])(\omega_0) = f'(\omega_0)$$

and  $D_{q,\omega}[f^\sigma](\omega_0) = [f^\sigma]'(\omega_0) = f'(\sigma(\omega_0))\sigma'(\omega_0) = qf'(\omega_0)$ .  $\square$

**Lemma 3.5.** *If  $\eta \in \mathcal{Y}^r([a, b], \mathbb{R})$  is such that  $D_{q,\omega}^i[\eta](a) = 0$  (resp.  $D_{q,\omega}^i[\eta](b) = 0$ ) for all  $i \in \{0, 1, \dots, r\}$ , then  $D_{q,\omega}^{i-1}[\eta^\sigma](a) = 0$  (resp.  $D_{q,\omega}^{i-1}[\eta^\sigma](b) = 0$ ) for all  $i \in \{1, \dots, r\}$ .*

*Proof.* If  $a = \omega_0$  the result is trivial (because  $\sigma(\omega_0) = \omega_0$ ). Suppose now that  $a \neq \omega_0$  and fix  $i \in \{1, \dots, r\}$ . Note that

$$D_{q,\omega}^i[\eta](a) = \frac{(D_{q,\omega}^{i-1}[\eta])^\sigma(a) - D_{q,\omega}^{i-1}[\eta](a)}{(q-1)a + \omega}.$$

Since, by hypothesis,  $D_{q,\omega}^i[\eta](a) = 0$  and  $D_{q,\omega}^{i-1}[\eta](a) = 0$ , then  $(D_{q,\omega}^{i-1}[\eta])^\sigma(a) = 0$ . Lemma 3.4 shows that

$$(D_{q,\omega}^{i-1}[\eta])^\sigma(a) = \left(\frac{1}{q}\right)^{i-1} D_{q,\omega}^{i-1}[\eta^\sigma](a).$$

We conclude that  $D_{q,\omega}^{i-1}[\eta^\sigma](a) = 0$ . The case  $t = b$  is proved in the same way.  $\square$

**Lemma 3.6.** *Suppose that  $f \in \mathcal{Y}^1([a, b], \mathbb{R})$ . One has*

$$\int_a^b f(t) D_{q,\omega}[\eta](t) d_{q,\omega}t = 0$$

for all functions  $\eta \in \mathcal{Y}^1([a, b], \mathbb{R})$  such that  $\eta(a) = \eta(b) = 0$  if and only if  $f(t) = c$ ,  $c \in \mathbb{R}$ , for all  $t \in [a, b]_{q,\omega}$ .

*Proof.* The implication “ $\Leftarrow$ ” is obvious. We prove “ $\Rightarrow$ ”. We begin noting that

$$\underbrace{\int_a^b f(t) D_{q,\omega}[\eta](t) d_{q,\omega}t}_{=0} = \underbrace{f(t) \eta(t)}_{=0} \Big|_a^b - \int_a^b D_{q,\omega}[f](t) \eta^\sigma(t) d_{q,\omega}t.$$

Hence,

$$\int_a^b D_{q,\omega}[f](t) \eta(qt + \omega) d_{q,\omega}t = 0$$

for any  $\eta \in \mathcal{Y}^1([a, b], \mathbb{R})$  such that  $\eta(a) = \eta(b) = 0$ . We need to prove that, for some  $c \in \mathbb{R}$ ,  $f(t) = c$  for all  $t \in [a, b]_{q,\omega}$ , that is,  $D_{q,\omega}[f](t) = 0$  for all  $t \in [a, b]_{q,\omega}$ . Suppose, by contradiction, that there exists  $p \in [a, b]_{q,\omega}$  such that  $D_{q,\omega}[f](p) \neq 0$ .

(1) If  $p \neq \omega_0$ , then  $p = q^k a + \omega[k]_q$  or  $p = q^k b + \omega[k]_q$  for some  $k \in \mathbb{N}_0$ . Observe that  $a(1-q) - \omega$  and  $b(1-q) - \omega$  cannot vanish simultaneously.

(a) Suppose that  $a(1-q) - \omega \neq 0$  and  $b(1-q) - \omega \neq 0$ . In this case we can assume, without loss of generality, that  $p = q^k a + \omega[k]_q$  and we can define

$$\eta(t) = \begin{cases} D_{q,\omega}[f](q^k a + \omega[k]_q) & \text{if } t = q^{k+1} a + \omega[k+1]_q \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned} & \int_a^b D_{q,\omega}[f](t) \cdot \eta(qt + \omega) d_{q,\omega}t \\ &= -(a(1-q) - \omega) q^k D_{q,\omega}[f](q^k a + \omega[k]_q) \cdot D_{q,\omega}[f](q^k a + \omega[k]_q) \neq 0, \end{aligned}$$

which is a contradiction.

(b) If  $a(1-q) - \omega \neq 0$  and  $b(1-q) - \omega = 0$ , then  $b = \omega_0$ . Since  $q^k \omega_0 + \omega[k]_q = \omega_0$  for all  $k \in \mathbb{N}_0$ , then  $p \neq q^k b + \omega[k]_q \forall k \in \mathbb{N}_0$  and, therefore,

$$p = q^k a + \omega[k]_{q,\omega} \text{ for some } k \in \mathbb{N}_0.$$

Repeating the proof of (a) we obtain again a contradiction.

(c) If  $a(1-q) - \omega = 0$  and  $b(1-q) - \omega \neq 0$  then the proof is similar to (b).

(2) If  $p = \omega_0$  then, without loss of generality, we can assume  $D_{q,\omega}[f](\omega_0) > 0$ . Since

$$\lim_{n \rightarrow +\infty} (q^n a + \omega[k]_q) = \lim_{n \rightarrow +\infty} (q^n b + \omega[k]_q) = \omega_0$$

(see [1]) and  $D_{q,\omega}[f]$  is continuous at  $\omega_0$ , then

$$\lim_{n \rightarrow +\infty} D_{q,\omega}[f](q^n a + \omega[k]_q) = \lim_{n \rightarrow +\infty} D_{q,\omega}[f](q^n b + \omega[k]_q) = D_{q,\omega}[f](\omega_0) > 0.$$

Thus, there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$  one has  $D_{q,\omega}[f](q^n a + \omega[k]_q) > 0$

and  $D_{q,\omega}[f](q^n b + \omega[k]_q) > 0$ .

(a) If  $\omega_0 \neq a$  and  $\omega_0 \neq b$ , then we can define

$$\eta(t) = \begin{cases} D_{q,\omega}[f](q^N b + \omega[N]_q) & \text{if } t = q^{N+1}a + \omega[N+1]_q \\ D_{q,\omega}[f](q^N a + \omega[N]_q) & \text{if } t = q^{N+1}b + \omega[N+1]_q \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\begin{aligned} & \int_a^b D_{q,\omega}[f](t) \eta(qt + \omega) d_{q,\omega} t \\ &= (b-a)(1-q)q^N D_{q,\omega}[f](q^N b + \omega[N]_q) \cdot D_{q,\omega}[f](q^N a + \omega[N]_q) \neq 0, \end{aligned}$$

which is a contradiction.

(b) If  $\omega_0 = b$ , then we define

$$\eta(t) = \begin{cases} D_{q,\omega}[f](\omega_0) & \text{if } t = q^{N+1}a + \omega[N+1]_q \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} & \int_a^b D_{q,\omega}[f](t) \eta(qt + \omega) d_{q,\omega} t \\ &= - \int_{\omega_0}^a D_{q,\omega}[f](t) \eta(qt + \omega) d_{q,\omega} t \\ &= -(a(1-q) - \omega) q^N D_{q,\omega}[f](q^N a + \omega[k]_q) \cdot D_{q,\omega}[f](\omega_0) \neq 0, \end{aligned}$$

which is a contradiction.

(c) When  $\omega_0 = a$ , the proof is similar to (b).  $\square$

**Lemma 3.7** (Fundamental lemma of Hahn's variational calculus). *Let  $f, g \in \mathcal{Y}^1([a, b], \mathbb{R})$ . If*

$$\int_a^b (f(t) \eta^\sigma(t) + g(t) D_{q,\omega}[\eta](t)) d_{q,\omega} t = 0$$

*for all  $\eta \in \mathcal{Y}^1([a, b], \mathbb{R})$  such that  $\eta(a) = \eta(b) = 0$ , then*

$$D_{q,\omega}[g](t) = f(t) \quad \forall t \in [a, b]_{q,\omega}.$$



*Proof.* Define the function  $A$  by  $A(t) := \int_{\omega_0}^t f(\tau) d_{q,\omega}\tau$ . Then  $D_{q,\omega}[A](t) = f(t)$  for all  $t \in [a, b]$  and

$$\begin{aligned} \int_a^b A(t) D_{q,\omega}[\eta](t) d_{q,\omega}t &= A(t) \eta(t) \Big|_a^b - \int_a^b D_{q,\omega}[A](t) \eta^\sigma(t) d_{q,\omega}t \\ &= - \int_a^b D_{q,\omega}[A](t) \eta^\sigma(t) d_{q,\omega}t \\ &= - \int_a^b f(t) \eta^\sigma(t) d_{q,\omega}t. \end{aligned}$$

Hence,

$$\begin{aligned} \int_a^b (f(t) \eta^\sigma(t) + g(t) D_{q,\omega}[\eta](t)) d_{q,\omega}t &= 0 \\ \Leftrightarrow \int_a^b (-A(t) + g(t)) D_{q,\omega}[\eta](t) d_{q,\omega}t &= 0. \end{aligned}$$

By Lemma 3.6 there is a  $c \in \mathbb{R}$  such that  $-A(t) + g(t) = c$  for all  $t \in [a, b]_{q,\omega}$ . Hence  $D_{q,\omega}[A](t) = D_{q,\omega}[g](t)$  for  $t \in [a, b]_{q,\omega}$ , which provides the desired result:  $D_{q,\omega}[g](t) = f(t) \quad \forall t \in [a, b]_{q,\omega}$ .  $\square$

We are now in conditions to deduce the higher-order fundamental Lemma of Hahn's quantum variational calculus.

**Lemma 3.8** (Higher-order fundamental lemma of Hahn's variational calculus). *Let  $f_0, f_1, \dots, f_r \in \mathcal{Y}^1([a, b], \mathbb{R})$ . If*

$$\int_a^b \left( \sum_{i=0}^r f_i(t) D_{q,\omega}^i [\eta^{\sigma^{r-i}}](t) \right) d_{q,\omega}t = 0$$

*for any variation  $\eta$ , then*

$$\sum_{i=0}^r (-1)^i \left( \frac{1}{q} \right)^{\frac{(i-1)i}{2}} D_{q,\omega}^i [f_i](t) = 0$$

*for all  $t \in [a, b]_{q,\omega}$ .*

*Proof.* We proceed by mathematical induction. If  $r = 1$  the result is true by Lemma 3.7. Assume that

$$\int_a^b \left( \sum_{i=0}^{r+1} f_i(t) D_{q,\omega}^i [\eta^{\sigma^{r+1-i}}](t) \right) d_{q,\omega}t = 0$$

for all functions  $\eta$  such that  $\eta(a) = \eta(b) = 0, \dots, D_{q,\omega}^r[\eta](a) = D_{q,\omega}^r[\eta](b) = 0$ . Note that

$$\begin{aligned} \int_a^b f_{r+1}(t) D_{q,\omega}^{r+1}[\eta](t) d_{q,\omega}t \\ &= f_{r+1}(t) D_{q,\omega}^r[\eta](t) \Big|_a^b - \int_a^b D_{q,\omega}[f_{r+1}](t) (D_{q,\omega}^r[\eta])^\sigma(t) d_{q,\omega}t \\ &= - \int_a^b D_{q,\omega}[f_{r+1}](t) (D_{q,\omega}^r[\eta])^\sigma(t) d_{q,\omega}t \end{aligned}$$

and, by Lemma 3.4,

$$\int_a^b f_{r+1}(t) D_{q,\omega}^{r+1}[\eta](t) d_{q,\omega}t = - \int_a^b D_{q,\omega}[f_{r+1}](t) \left(\frac{1}{q}\right)^r D_{q,\omega}^r[\eta^\sigma](t) d_{q,\omega}t.$$

Therefore,

$$\begin{aligned} & \int_a^b \left( \sum_{i=0}^{r+1} f_i(t) D_{q,\omega}^i[\eta^{\sigma^{r+1-i}}](t) \right) d_{q,\omega}t \\ &= \int_a^b \left( \sum_{i=0}^r f_i(t) D_{q,\omega}^i[\eta^{\sigma^{r+1-i}}](t) \right) d_{q,\omega}t \\ & \quad - \int_a^b D_{q,\omega}[f_{r+1}](t) \left(\frac{1}{q}\right)^r D_{q,\omega}^r[\eta^\sigma](t) d_{q,\omega}t \\ &= \int_a^b \left[ \sum_{i=0}^{r-1} f_i(t) D_{q,\omega}^i[(\eta^\sigma)^{\sigma^{r-i}}](t) d_{q,\omega}t \right. \\ & \quad \left. + \left( f_r - \left(\frac{1}{q}\right)^r D_{q,\omega}[f_{r+1}] \right)(t) D_{q,\omega}^r[\eta^\sigma](t) \right] d_{q,\omega}t. \end{aligned}$$

By Lemma 3.5,  $\eta^\sigma$  is a variation. Hence, using the induction hypothesis,

$$\begin{aligned} & \sum_{i=0}^{r-1} (-1)^i \left(\frac{1}{q}\right)^{\frac{(i-1)i}{2}} D_{q,\omega}^i[f_i](t) \\ & \quad + (-1)^r \left(\frac{1}{q}\right)^{\frac{(r-1)r}{2}} D_{q,\omega}^r \left[ \left( f_r - \frac{1}{q^r} D_{q,\omega}[f_{r+1}] \right) \right](t) \\ &= \sum_{i=0}^{r-1} (-1)^i \left(\frac{1}{q}\right)^{\frac{(i-1)i}{2}} D_{q,\omega}^i[f_i](t) + (-1)^r \left(\frac{1}{q}\right)^{\frac{(r-1)r}{2}} D_{q,\omega}^r[f_r](t) \\ & \quad + (-1)^{r+1} \left(\frac{1}{q}\right)^{\frac{(r-1)r}{2}} \frac{1}{q^r} D_{q,\omega}^r[D_{q,\omega}[f_{r+1}]](t) \\ &= 0 \end{aligned}$$

for all  $t \in [a, b]_{q,\omega}$ , which leads to

$$\sum_{i=0}^{r+1} (-1)^i \left(\frac{1}{q}\right)^{\frac{(i-1)i}{2}} D_{q,\omega}^i[f_i](t) = 0, \quad t \in [a, b]_{q,\omega}.$$

□

**3.2. Higher-order Hahn's quantum Euler–Lagrange equation.** For a variation  $\eta$  and an admissible function  $y$ , we define the function  $\phi : (-\bar{\epsilon}, \bar{\epsilon}) \rightarrow \mathbb{R}$  by

$$\phi(\epsilon) = \phi(\epsilon, y, \eta) := \mathcal{L}[y + \epsilon\eta].$$

The first variation of the variational problem (P) is defined by

$$\delta\mathcal{L}[y, \eta] := \phi'(0).$$

Observe that

$$\begin{aligned}\mathcal{L}[y + \epsilon\eta] &= \int_a^b L\left(t, y^{\sigma^r}(t) + \epsilon\eta^{\sigma^r}(t), D_{q,\omega}\left[y^{\sigma^{r-1}}\right](t) + \epsilon D_{q,\omega}\left[\eta^{\sigma^{r-1}}\right](t), \right. \\ &\quad \left. \dots, D_{q,\omega}^r[y](t) + \epsilon D_{q,\omega}^r[\eta](t)\right) d_{q,\omega}t \\ &= \mathcal{L}_b[y + \epsilon\eta] - \mathcal{L}_a[y + \epsilon\eta]\end{aligned}$$

with

$$\begin{aligned}\mathcal{L}_\xi[y + \epsilon\eta] &= \int_{\omega_0}^\xi L\left(t, y^{\sigma^r}(t) + \epsilon\eta^{\sigma^r}(t), D_{q,\omega}\left[y^{\sigma^{r-1}}\right](t) + \epsilon D_{q,\omega}\left[\eta^{\sigma^{r-1}}\right](t), \right. \\ &\quad \left. \dots, D_{q,\omega}^r[y](t) + \epsilon D_{q,\omega}^r[\eta](t)\right) d_{q,\omega}t,\end{aligned}$$

$\xi \in \{a, b\}$ . Therefore,

$$(3.1) \quad \delta\mathcal{L}[y, \eta] = \delta\mathcal{L}_b[y, \eta] - \delta\mathcal{L}_a[y, \eta].$$

Considering (3.1), the following lemma is a direct consequence of Lemma 2.12:

**Lemma 3.9.** *For a variation  $\eta$  and an admissible function  $y$ , let*

$$\begin{aligned}g(t, \epsilon) &:= L\left(t, y^{\sigma^r}(t) + \epsilon\eta^{\sigma^r}(t), D_{q,\omega}\left[y^{\sigma^{r-1}}\right](t) + \epsilon D_{q,\omega}\left[\eta^{\sigma^{r-1}}\right](t), \right. \\ &\quad \left. \dots, D_{q,\omega}^r[y](t) + \epsilon D_{q,\omega}^r[\eta](t)\right),\end{aligned}$$

$\epsilon \in (-\bar{\epsilon}, \bar{\epsilon})$ . Assume that:

- (1)  $g(t, \cdot)$  is differentiable at 0 uniformly in  $t \in [a, b]_{q,\omega}$ ;
- (2)  $\mathcal{L}_a[y + \epsilon\eta] = \int_{\omega_0}^a g(t, \epsilon) d_{q,\omega}t$  and  $\mathcal{L}_b[y + \epsilon\eta] = \int_{\omega_0}^b g(t, \epsilon) d_{q,\omega}t$  exist for  $\epsilon \approx 0$ ;
- (3)  $\int_{\omega_0}^a \partial_2 g(t, 0) d_{q,\omega}t$  and  $\int_{\omega_0}^b \partial_2 g(t, 0) d_{q,\omega}t$  exist.

Then

$$\begin{aligned}\phi'(0) = \delta\mathcal{L}[y, \eta] &= \int_a^b \left( \sum_{i=0}^r \partial_{i+2} L\left(t, y^{\sigma^r}(t), D_{q,\omega}\left[y^{\sigma^{r-1}}\right](t), \dots, D_{q,\omega}^r[y](t)\right) \right. \\ &\quad \left. \cdot D_{q,\omega}^i\left[\eta^{\sigma^{r-i}}\right](t) \right) d_{q,\omega}t,\end{aligned}$$

where  $\partial_i L$  denotes the partial derivative of  $L$  with respect to its  $i$ th argument.

The following result gives a necessary condition of Euler–Lagrange type for an admissible function to be a local extremizer for (P).

**Theorem 3.10** (Higher-order Hahn's quantum Euler–Lagrange equation). *Under hypotheses (H1)–(H3) and conditions (1)–(3) of Lemma 3.9 on the Lagrangian  $L$ ,*

if  $y_* \in \mathcal{Y}^r$  is a local extremizer for problem (P), then  $y_*$  satisfies the  $q, \omega$ -Euler-Lagrange equation

$$(3.2) \quad \sum_{i=0}^r (-1)^i \left( \frac{1}{q} \right)^{\frac{(i-1)i}{2}} D_{q,\omega}^i [\partial_{i+2} L] \left( t, y^{\sigma^r}(t), D_{q,\omega} [y^{\sigma^{r-1}}] (t), \dots, D_{q,\omega}^r [y] (t) \right) = 0$$

for all  $t \in [a, b]_{q,\omega}$ .

*Proof.* Let  $y_*$  be a local extremizer for problem (P) and  $\eta$  a variation. Define  $\phi : (-\bar{\epsilon}, \bar{\epsilon}) \rightarrow \mathbb{R}$  by  $\phi(\epsilon) := \mathcal{L}[y_* + \epsilon\eta]$ . A necessary condition for  $y_*$  to be an extremizer is given by  $\phi'(0) = 0$ . By Lemma 3.9 we conclude that

$$\int_a^b \left( \sum_{i=0}^r \partial_{i+2} L \left( t, y^{\sigma^r}(t), D_{q,\omega} [y^{\sigma^{r-1}}] (t), \dots, D_{q,\omega}^r [y] (t) \right) \cdot D_{q,\omega}^i [\eta^{\sigma^{r-i}}] (t) \right) d_{q,\omega} t = 0$$

and (3.2) follows from Lemma 3.8.  $\square$

*Remark 3.11.* In practical terms the hypotheses of Theorem 3.10 are not so easy to verify *a priori*. One can, however, assume that all hypotheses are satisfied and apply the  $q, \omega$ -Euler-Lagrange equation (3.2) heuristically to obtain a *candidate*. If such a candidate is, or not, a solution to problem (P) is a different question that always requires further analysis (see an example in §3.3).

When  $\omega \rightarrow 0$  one obtains from (3.2) the higher-order  $q$ -Euler-Lagrange equation:

$$\sum_{i=0}^r (-1)^i \left( \frac{1}{q} \right)^{\frac{(i-1)i}{2}} D_q^i [\partial_{i+2} L] \left( t, y^{\sigma^r}(t), D_q [y^{\sigma^{r-1}}] (t), \dots, D_q^r [y] (t) \right) = 0$$

for all  $t \in \{aq^n : n \in \mathbb{N}_0\} \cup \{bq^n : n \in \mathbb{N}_0\} \cup \{0\}$ . The higher-order  $h$ -Euler-Lagrange equation is obtained from (3.2) taking the limit  $q \rightarrow 1$ :

$$\sum_{i=0}^r (-1)^i \Delta_h^i [\partial_{i+2} L] \left( t, y^{\sigma^r}(t), \Delta_h [y^{\sigma^{r-1}}] (t), \dots, \Delta_h^r [y] (t) \right) = 0$$

for all  $t \in \{a + nh : n \in \mathbb{N}_0\} \cup \{b + nh : n \in \mathbb{N}_0\}$ . The classical Euler-Lagrange equation [26] is recovered when  $(\omega, q) \rightarrow (0, 1)$ :

$$\sum_{i=0}^r (-1)^i \frac{d^i}{dt^i} \partial_{i+2} L \left( t, y(t), y'(t), \dots, y^{(r)}(t) \right) = 0$$

for all  $t \in [a, b]$ .

We now illustrate the usefulness of our Theorem 3.10 by means of an example that is not covered by previous available results in the literature.

**3.3. An Example.** Let  $q = \frac{1}{2}$  and  $\omega = \frac{1}{2}$ . Consider the following problem:

$$(3.3) \quad \mathcal{L}[y] = \int_{-1}^1 \left( y^\sigma(t) + \frac{1}{2} \right)^2 \left( (D_{q,\omega} [y] (t))^2 - 1 \right)^2 d_{q,\omega} t \longrightarrow \min$$

over all  $y \in \mathcal{Y}^1$  satisfying the boundary conditions

$$(3.4) \quad y(-1) = 0 \quad \text{and} \quad y(1) = -1.$$

This is an example of problem (P) with  $r = 1$ . Our  $q, \omega$ -Euler-Lagrange equation (3.2) takes the form

$$D_{q,\omega} [\partial_3 L] (t, y^\sigma(t), D_{q,\omega} [y](t)) = \partial_2 L (t, y^\sigma(t), D_{q,\omega} [y](t)).$$

Therefore, we look for an admissible function  $y_*$  of (3.3)-(3.4) satisfying

$$(3.5) \quad D_{q,\omega} \left[ 4 \left( y^\sigma + \frac{1}{2} \right)^2 \left( (D_{q,\omega} [y])^2 - 1 \right) D_{q,\omega} [y] \right] (t) \\ = 2 \left( y^\sigma(t) + \frac{1}{2} \right) \left( (D_{q,\omega} [y](t))^2 - 1 \right)$$

for all  $t \in [-1, 1]_{q,\omega}$ . It is easy to see that

$$y_*(t) = \begin{cases} -t & \text{if } t \in (-1, 0) \cup (0, 1] \\ 0 & \text{if } t = -1 \\ 1 & \text{if } t = 0 \end{cases}$$

is an admissible function for (3.3)-(3.4) with

$$D_{q,\omega} [y_*] (t) = \begin{cases} -1 & \text{if } t \in (-1, 0) \cup (0, 1] \\ 1 & \text{if } t = -1 \\ -3 & \text{if } t = 0, \end{cases}$$

satisfying the  $q, \omega$ -Euler-Lagrange equation (3.5). We now prove that the *candidate*  $y_*$  is indeed a minimizer for (3.3)-(3.4). Note that here  $\omega_0 = 1$  and, by Lemma 2.8 and item (3) of Theorem 2.7,

$$(3.6) \quad \mathcal{L} [y] = \int_{-1}^1 \left( y^\sigma(t) + \frac{1}{2} \right)^2 \left( (D_{q,\omega} [y](t))^2 - 1 \right)^2 d_{q,\omega} t \geq 0$$

for all admissible functions  $y \in \mathcal{Y}^1([-1, 1], \mathbb{R})$ . Since  $\mathcal{L} [y_*] = 0$ , we conclude that  $y_*$  is a minimizer for problem (3.3)-(3.4).

It is worth to mention that the minimizer  $y_*$  of (3.3)-(3.4) is not continuous while the classical calculus of variations [26], the calculus of variations on time scales [14, 20, 23], or the nondifferentiable scale variational calculus [4, 5, 10], deal with functions which are necessarily continuous. As an open question, we pose the problem of determining conditions on the data of problem (P) assuring, *a priori*, the minimizer to be regular.

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