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Arcwise Connectedness of Solution Sets to Lipschitzean Differential Inclusions.

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Sunto. - *Si dimostra che è connesso per archi l'insieme delle soluzioni del problema di Cauchy $\dot{x} \in F(x)$, $x(0) = \xi$, dove $F: X \to 2^X$ è una multifunzione Lipschitziana a valori chiusi, ed X è uno spazio di Banach separabile.*

We consider for ξ in R^n the Cauchy problem

$$(P_\xi) \qquad \dot{x} \in F(x), \qquad x(0) = \xi,$$

and we denote by $\mathcal{T}(\xi)$ the set of solutions of (P_ξ) defined on a given interval I, i.e., the set of absolutely continuous functions $x: I \to R^n$ such that $x(0) = \xi$ and $\dot{x}(t) \in F(x(t))$ a.e. in I.

It is known that $\mathcal{T}(\xi)$ is a compact connected subset of $C(I, R^n)$ when F is an upper semicontinuous multifunction from R^n into the compact convex subsets of R^n (hence also when F is a continuous single valued map). However $\mathcal{T}(\xi)$ is not arcwise connected as an example in ([1], p. 203) shows.

The aim of this note is to prove that $\mathcal{T}(\xi)$ is arcwise connected if F is a Lipschitzean multifunction from a real separable Banach space X into the closed nonempty subsets of X. We obtain this result by using the existence of a continuous selection from $\xi \to \mathcal{T}(\xi)$.

Let $T > 0$, $I = [0, T]$ and let X be a real separable Banach space with norm $|\cdot|$. Denote by $C(I, X)$ the Banach space of continuous functions $x: I \to X$ with the norm $\|x\|_\infty = \sup\{|x(t)|: t \in I\}$ and by $AC(I, X)$ the Banach space of absolutely continuous functions $x: I \to X$ with the norm $\|x\|_{AC} = |x(0)| + \int_0^T |\dot{x}(t)|\, dt$.

Let $F: X \to 2^X$ be a multivalued map satisfying the following assumptions:

(H_1) the values of F are closed nonempty subsets of X.

(H_2) there exists $L > 0$ such that $d(F(x), F(y)) \leq L|x - y|$, for all $x, y \in X$, (here $d(\cdot, \cdot)$ stands for the Hausdorff distance).

The main result of this note is the following:

THEOREM. – *For every $\xi \in X$ the set $\mathcal{T}(\xi)$ is arcwise connected in $C(I, X)$.*

To prove it we shall use the following:

LEMMA. – ([2], [3]) *Let F satisfy (H_1)-(H_2), let $\xi_0 \in X$ and let $x_0 \in \mathcal{T}(\xi_0)$. Then there exists a continuous map $\varphi: X \to AC(I, X)$ such that $\varphi(\xi_0) = x_0$ and $\varphi(\xi) \in \mathcal{T}(\xi)$ for every $\xi \in X$.*

PROOF OF THEOREM. – Fix $\xi_0 \in X$ and let x, y be in $\mathcal{T}(\xi_0)$. By the Lemma, there exists a continuous map $\varphi: X \to AC(I, X)$ such that $\varphi(\xi_0) = x$ and $\varphi(\xi) \in \mathcal{T}(\xi)$ for every $\xi \in X$. Since $y(\cdot)$ is continuous on I, we have that the map $\lambda \to \varphi(y(\lambda T))$ is continuous from $[0, 1]$ to $AC(I, X)$. Moreover $\varphi(y(\lambda T)) \in \mathcal{T}(y(\lambda T))$, for each $\lambda \in [0, 1]$. Following an idea in [5] we define for $\lambda \in [0, 1]$

$$x_\lambda(t) = \begin{cases} y(t) & \text{if } 0 \leq t \leq \lambda T, \\ \varphi(y(\lambda T))(t - \lambda T) & \text{if } \lambda T \leq t \leq T, \end{cases}$$

and remark that $x_0(\cdot) = x(\cdot)$ and $x_1(\cdot) = y(\cdot)$. It is easy to see that $\dot{x}_\lambda(t) \in F(x_\lambda(t))$ for a.e. $t \in I$ and $x_\lambda(0) = \xi_0$, so that, $x_\lambda \in \mathcal{T}(\xi_0)$. To complete the proof it remains to be proved that $\lambda \to x_\lambda$ is continuous from $[0, 1]$ to $C(I, X)$. Let $\varepsilon > 0$ and $\lambda_0 \in [0, 1]$ be fixed. We show that there exists $\delta > 0$ such that for any λ in $[0, 1]$ with $|\lambda - \lambda_0| < \delta$ we have $\|x_\lambda - x_{\lambda_0}\|_\infty < \varepsilon$. For $t \in I$ we distinguish three situations:

(i) $0 \leq t \leq \lambda_0 T \leq \lambda T$, (ii) $\lambda_0 T \leq t \leq \lambda T$ and (iii) $\lambda_0 T \leq \lambda T \leq \leq t \leq T$.

In the case (i) we have $|x_\lambda(t) - x_{\lambda_0}(t)| = |y(t) - y(t)| = 0$.

If $\lambda_0 T \leq t \leq \lambda T$ then $|x_{\lambda_0}(t) - x_\lambda(t)| = |\varphi(y(\lambda_0 T))(t - \lambda_0 T) - y(t)|$. Since $\varphi(y(\lambda_0 T))(\cdot)$ and $y(\cdot)$ are uniformly continuous on I, there exists $\delta_1 > 0$ such that for any t' and t'' in I with $|t' - t''| < \delta_1$ we have

$$|\varphi(y(\lambda_0 T))(t') - \varphi(y(\lambda_0 T))(t'')| < \varepsilon/2 \qquad \text{and} \qquad |y(t') - y(t'')| < \varepsilon/2 .$$

Let $|\lambda_0 - \lambda| < \delta_1/T$. Then $|t - \lambda_0 T| \leqslant |\lambda - \lambda_0| T \leqslant \delta_1$ and

$$|x_{\lambda_0}(t) - x_\lambda(t)| \leqslant |\varphi(y(\lambda_0 T))(t - \lambda_0 T) -$$

$$- \varphi(y(\lambda_0 T))(0)| + |\varphi(y(\lambda_0 T))(0) - y(t)| < \varepsilon/2 + \varepsilon/2 = \varepsilon .$$

Finally, if (iii) holds then

$$|x_{\lambda_0}(t) - x_\lambda(t)| = |\varphi(y(\lambda_0 T))(t - \lambda_0 T) - \varphi(y(\lambda T))(t - \lambda T)| \leqslant \tag{1}$$

$$\leqslant |\varphi(y(\lambda_0 T))(t - \lambda_0 T) - \varphi(y(\lambda_0 T))(t - \lambda T)| +$$

$$+ |\varphi(y(\lambda_0 T))(t - \lambda T) - \varphi(y(\lambda T))(t - \lambda T)| .$$

Since $\lambda \to \varphi(y(\lambda T))$ is continuous from $[0, 1]$ to $C(I, X)$ there exists $\delta_2 > 0$ such that

$$|\lambda - \lambda_0| < \delta_2 \text{ implies } \|\varphi(y(\lambda T))(\cdot) - \varphi(y(\lambda_0 T))(\cdot)\|_\infty < \varepsilon/2$$

so that for $|\lambda - \lambda_0| < \delta_2$ we have

$$|\varphi(y(\lambda T))(t - \lambda T) - \varphi(y(\lambda_0 T))(t - \lambda T)| < \varepsilon/2 . \tag{2}$$

Moreover since $t \to \varphi(y(\lambda_0 T))(t)$ is uniformly continuous on I, there exists $\delta_3 > 0$ such that

$$|t' - t''| < \delta_3 \text{ implies } |\varphi(y(\lambda_0 T))(t') - \varphi(y(\lambda_0 T))(t'')| < \varepsilon/2 .$$

Then if $|\lambda - \lambda_0| < \delta_3/T$ we have that $|t - \lambda T - t + \lambda_0 T| \leqslant |\lambda - \lambda_0| T \leqslant \delta_3$ and

$$|\varphi(y(\lambda_0 T))(t - \lambda T) - \varphi(y(\lambda_0 T))(t - \lambda_0 T)| < \varepsilon/2 . \tag{3}$$

By (1), (2) and (3), we have that if $|\lambda - \lambda_0| < \min\{\delta_2, \delta_3/T\}$ then $|x_\lambda(t) - x_0(t)| < \varepsilon$.

Let $\delta := \min\{\delta_1/T, \delta_2, \delta_3/T\}$. We have shown that if $\lambda_0 \leqslant \lambda$ and $|\lambda - \lambda_0| < \delta$ then, for any t in I, $|x_{\lambda_0}(t) - x_\lambda(t)| < \varepsilon$, that is $\|x_{\lambda_0} - x_\lambda\|_\infty < \varepsilon$. For $\lambda \leqslant \lambda_0$ the proof is similar. ■

PROPOSITION. – *$\mathcal{T} := \cup\{\mathcal{T}(\xi) \colon \xi \in X\}$ is arcwise connected in $C(I, X)$.*

PROOF. – Let x, y in $\mathcal{T}$ and let ξ and ξ_0 in X be such that $x \in \mathcal{T}(\xi_0)$ and $y \in \mathcal{T}(\xi)$. If $\xi \neq \xi_0$ then, by Corollary 4.3 in [6], there exists a continuous map h: $[0, 1] \to AC(I, X)$ such that $h(0) = x$, $h(1) = y$ and, for $\lambda \in [0, 1]$ $h(\lambda) \in \mathcal{T}(\xi_\lambda)$, where $\xi_\lambda = (1 - \lambda)\xi_0 + \lambda\xi$.

If $\xi = \xi_0$, then the existence of a continuous map h: $[0, 1] \to C(I, X)$ such that $h(0) = x$, $h(1) = y$ follows from the Theorem. ∎

REMARK. – Similar properties of solution sets to differential inclusions have been obtained by F. S. De Blasi and G. Pianigiani in [4], by using the Baire cathegory method.

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