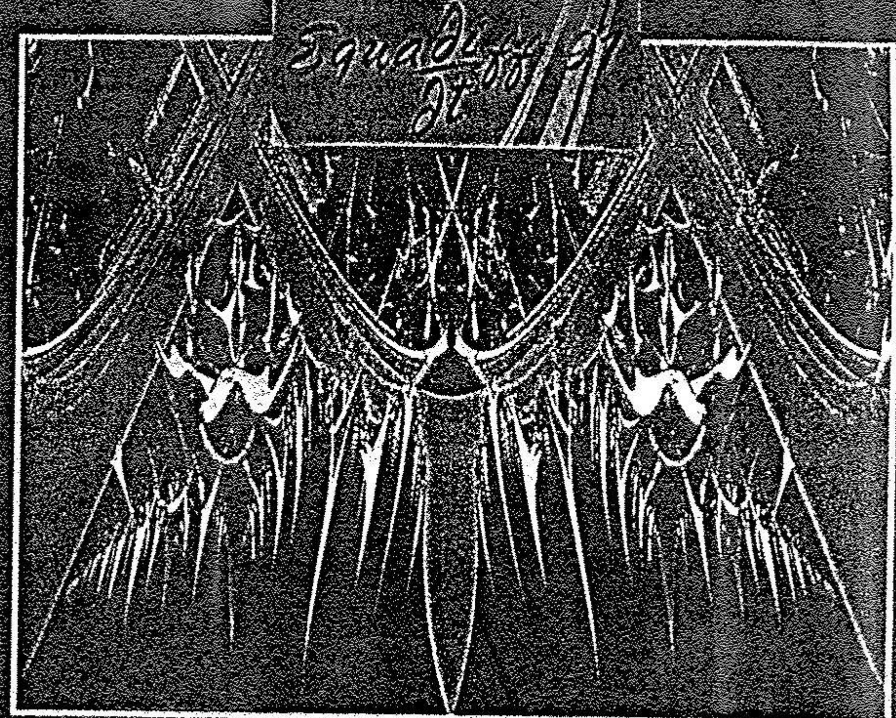

International Conference on

DIFFERENTIAL EQUATIONS

BARCELONA 1991

Edited by

C. Perelló, C. Simó and J. Solà-Morales

World Scientific

Perelló
Simó
Solà-Morales

International Conference on
DIFFERENTIAL EQUATIONS
BARCELONA 1991

Vol 2

ISBN 981-02-1992-X
ISBN 981-02-1023-X (Set)

QUALITATIVE PROPERTIES OF SOLUTION SETS TO LIPSCHITZEAN DIFFERENTIAL INCLUSIONS

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Abstract. We survey some results concerning existence and properties of continuous selections from solution sets to Lipschitzean differential inclusions.

Assume to have an ordinary differential equation in R^n such that the associated Cauchy problems admit solutions on a given interval [0, T]. Then the Peano Phenomenon holds : either the Cauchy problem admits a unique solution or it admits a continuum (i.e. a compact connected nonempty set) of solutions [17, p. 165]. If the Cauchy problem admits a unique solution depending continuously on the initial data and parameters then this problem is said to be *well posed.* If the solution is not unique then the Kneser's theorem say that the attainable set at time $t \le T$ is a continuum in R^n [11, p.15], hence it is connected. However the attainable set is not, in general, arcwise connected [17].

In what follows we shall present some results concerning a natural extension of the well posedness to Lipschitzean differential inclusions in terms of continuous selections from the solution map and some other properties of the solution sets to such differential inclusions. One of them says that any two continuous selections from the solution map are homotopic, hence in particular the sets of solutions are arcwise connected.

Let $I = [0,T]$, $T > 0$, X be a real, separable Banach space with the norm $\|\cdot\|$ and let 2^X be the family of all closed nonempty subsets of X endowed with the Hausdorff pseudodistance $d(.,.)$. Denote by AC(I,X) the Banach space of absolutely continuous functions $x: I \to X$ with the norm $\|x\|_{AC} = \|x(0)\| + \int_0^T \|\dot{x}(t)\|dt$, where $\dot{x}$ stands for the derivative of x.

For ξ in X we consider the Cauchy problem

(1) $$\dot{x} \in F(t, x) \ , \ x(0) = \xi,$$

where $F: I \times X \to 2^X$ is a multifunction satisfying the following assumptions:

(H_1) $t \to F(t,x)$ is (Lebesgue) measurable,

(H_2) there exists $k \in L^1(I,R)$ such that $d(F(t,x), F(t,y)) \le k(t)\|x-y\|$, for all $x, y \in X$, a.a. $t \in I$,

(H_3) there exists $\beta \in L^1(I, R)$ such that $d(0, F(t, 0)) \le \beta(t)$, for a.a. $t \in I$.

Let $\mathcal{S}(\xi) := \{x \in AC(I,X) : x(0)=\xi, \dot{x}(t) \in F(t,x(t)), \text{ a.e in } I\}$ be the set of all solutions of (1) and associate the attainable set $\mathcal{A}_T(\xi) := \{x(T) : x \in \mathcal{S}(\xi)\}$. Then by the Filippov's theorem in [10], improved by Himmelberg and Van Vleck in [12], it follows that under the assumptions (H_1) - (H_3) the solution set $\mathcal{S}(\xi)$ is nonempty for all $\xi \in X$.

We shall say that the Cauchy problem (1) is *well posed* if given $\xi_0 \in X$ and $x_0 \in \mathcal{S}(\xi_0)$ there exists a continuous selection $\varphi: X \to AC(I,X)$ from the multifunction $\xi \to \mathcal{S}(\xi)$ such that $\varphi(\xi_0) = x_0$. Recall that $\varphi(.)$ is a selection from $\xi \to \mathcal{S}(\xi)$ if $\varphi(\xi) \in \mathcal{S}(\xi)$ for all $\xi \in X$.

The first result concerning the existence of a continuous selection from the solution map $\xi \to \mathcal{S}(\xi)$ was obtained by Cellina in [4], under the assumptions that F is a Lipschitzean, that is satisfies (H_2), with compact values contained in a bounded set of R^n and ξ belongs to a compact subset of R^n. This result was improved to F with closed nonempty values in R^n by Cellina and Ornelas in [7].

The following result obtained jointly with Colombo, Fryszkowski and Rzezuchowski in [8] gives the natural extension of the well posedness property to Lipschitzean differential inclusions depending on parameter and contains the previous results as special cases.

Theorem 1. Let I and X be as before, S be a separable metric space and let $F: I \times X \times S \to 2^X$ satisfy the following assumptions :

(A_1) F is $\mathcal{L} \otimes \mathcal{B}(X \times S)$-measurable,

(A_2) there exists a continuous map $k : S \to L^1(I, R)$ such that $k(s)(t) > 0$ and for all $x, y \in X$ and $s \in S$: $d(F(t, x,s), F(t, y,s)) \le k(s)(t) \|x - y\|$ a.e. in I,

(A_3) for any $(t,x) \in I \times X$ the multivalued map $s \to F(t,x,s)$ is lower semicontinuous,

(A_4) there exists a continuous map $\beta: S \to L^1(I,R)$ such that for every s: $d(0,F(t,0,s)) \le \beta(s)(t)$ a.e. in I.

Let $\xi : S \to X$ be continuous and for $s \in S$ denote by $\mathcal{S}(s)$ the set of solutions of the Cauchy problem : $\dot{x} \in F(t, x,s)$, $x(0) = \xi(s)$. Then given $s_0 \in X$ and $x_0 \in \mathcal{S}(s_0)$ there exists $x: I \times S \to X$ with the following properties:

(i) $x(\cdot, s) \in \mathcal{S}(s)$ for each $s \in S$;

(ii) $s \to x(\cdot, s)$ is continuous from S into AC(I, X) ;

(iii) $x(\cdot, s_0) = x_0(\cdot)$.

A first consequence Theorem 1 and of Proposition 1 in [1, p. 80] is the following:

Corollary 2. Under the assumptions in Theorem 1 the multivalued map $s \to \mathcal{S}(s)$ is lower semicontinuous.

A result concerning the lower semicontinuous dependence on parameters of the solution map to Lipschitzean differential inclusions has been proved by Naselli Ricceri and Ricceri in [15]. Since the solution sets are non-convex, in general, the existence of continuous selections from the solution map is stronger than the lower semicontinuity, in view of the well known

Michael's theorem [14]. For properties of the solution map to differential inclusions with upper semicontinuous convex valued right-hand side we refer to [1, 3, 8, 14].

Another consequence of Theorem 1 which generalizes the results by Cellina and by Cellina and Ornelas quoted before is the following:

Corollary 3. Assume that $F: I\times X \to 2^X$ satisfies (H_1) - (H_3) and let $\xi_0 \in X$ and $x_0 \in S(\xi_0)$. Then there exists a function $x(.,.) : I\times X \to X$ with the following properties:

(i) $x(.,\xi) \in S(\xi)$ for every $\xi \in X$;

(ii) $\xi \to x(.,\xi)$ is continuous from X into AC(I,X);

(iii) $x(.,\xi_0) = x_0(.)$.

Remark that a similar result can be formulated for the multifunction $\xi \to \mathcal{A}_T(\xi)$ and, by the same argument as for Corollary 2, we have that the multifunctions $\xi \to S(\xi)$ and $\mathcal{A}_T(\xi)$ are lower semicontinuous. An analog of the result in Corollary 3 to differential inclusions on closed sets has been obtained jointly with Cellina in [6]. By Corollary 3 and the Schauder fixed point theorem it follows the following result pointed out in [18]:

Corollary 4. Let $F(.,.)$ satisfy (H_1) - (H_3), K be a nonempty compact convex subset of X and assume that $\mathcal{A}_T(K) \subset K$, where $\mathcal{A}_T(K) = \{z: z \in \mathcal{A}_T(\xi), \xi \in K\}$. Then the boundary value problem $\dot{x} \in F(t, x)$, $x(0) = x(T) \in K$ admits a solution.

The following result, that we have proved in [18], is an analog for the solution map $\xi \to S(\xi)$ of a Michael's result in [14], concerning the continuous extension of selections from lower semicontinuous multifunctions with closed convex values.

Theorem 5. If F satisfies (H_1)-(H_3), C is a closed nonempty subset of X and $\varphi: C \to AC(I,X)$ is a continuous map such that $\varphi(\xi) \in S(\xi)$ for all $\xi \in C$ then there exists $\varphi^*: X \to AC(I,X)$, a continuous extension of φ, such that $\varphi^*(\xi) \in S(\xi)$ for all $\xi \in X$.

As an easy consequence of this result we obtain that any two solutions of (1) starting from two different points can be connected by a continuous arc of solutions which start from points on the segment defined by that two points. More precisely we have the following:

Corollary 6. Given $\xi_0, \xi_1 \in X$, $\xi_0 \neq \xi_1$, $x_0 \in S(\xi_0)$ and $x_1 \in S(\xi_1)$ there exists a continuous map $\psi: [0,1] \to AC(I,X)$ such that: $\psi(0) = x_0$, $\psi(1) = x_1$ and, for $\alpha \in [0,1]$, $\psi(\alpha) \in S(\xi_\alpha)$, where $\xi_\alpha = (1-\alpha)\xi_0 + \alpha\xi_1$.

In the joint paper with H. Wu [19] we have used Corollary 3 and an idea in [13] to prove the arcwise connectedness of the solution sets $S(\xi)$, for the case when F is time independent.

An analog result for differential inclusion on closed sets was obtained in [6]. Similar results have been obtained by De Blasi and Pianigiani in [9], by using the Baire category method.

In order to extend the simple construction in [19] for the case when F depends also on t we need the following improvement of Corollary 2, proved in [20]:

Lemma 7. Under the assumptions (H_1)-(H_3) there exists a continuous map $\varphi : I\times X \to AC(I,X)$ such that, for every $(\tau,\xi) \in I\times X$, $\varphi(\tau,\xi)$ is a solution of the Cauchy problem $\dot{x} \in F(t, x)$, $x(\tau) = \xi$.

Using this result we can prove the following:

Theorem 8. Let $F: I\times X \to 2^X$ satisfy (H_1) - (H_3). Then any two continuous selections from the multifunction $\xi \to S(\xi)$ are homotopic. In particular the solution set $S(\xi)$ is arcwise connected, for all $\xi \in X$. Similar results hold for the multivalued map $\xi \to \mathcal{A}_T(\xi)$.

Sketch of the proof. Let φ_1, φ_2 be two continuous selections from $\xi \to S(\xi)$ and let $\varphi: I\times X \to AC(I,X)$ be given by Lemma 7. Then define

$$H_1(\lambda,\xi)(t) = \begin{cases} \varphi_1(\xi)(t) & \text{if } 0 \le t \le \lambda T \\ \varphi(\lambda T, \varphi_1(\xi)(\lambda T))(t) & \text{if } \lambda T \le t \le T \end{cases},$$

$$H_2(\lambda,\xi)(t) = \begin{cases} \varphi_2(\xi)(t) & \text{if } 0 \le t \le \lambda T \\ \varphi(\lambda T, \varphi_2(\xi)(\lambda T))(t) & \text{if } \lambda T \le t \le T \end{cases},$$

and remark that $H(.,.)$ defined by

$$H(\lambda,\xi) = \begin{cases} H_1(1-2\lambda,\xi) & \text{if } 0 \le \lambda \le \frac{1}{2} \\ H_2(2\lambda-1,\xi) & \text{if } \frac{1}{2} \le \lambda \le 1 \end{cases}$$

is the desired homotopy from φ_1 to φ_2, i.e. : $H(.,.)$ is continuous, $H(0,\xi) = \varphi_1(\xi)$, $H(1,\xi) = \varphi_2(\xi)$ and $H(\lambda,\xi) \in S(\xi)$ for all $\xi \in X$.

We refer to [21] for the complete proof of this result and for a notion of topological degree.

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On Uniform Regularity Estimates and Robust Approximations for Parameter Dependent Problems

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Abstract

We consider as examples two parameter-dependent problems, the Timoshenko beam (parameter = thickness t) and 2-d elasticity (parameter = Poisson ratio ν). As the parameter tends to a limiting value ($t \to 0$ or $\nu \to 0.5$), the Galerkin approximation is known to deteriorate for several numerical schemes, a phenomenon called *locking*. We show how a uniform regularity estimate plays a crucial role in determining the robustness of numerical schemes with respect to the parameter.

In this paper, we will be interested in the regularity properties and numerical approximation of certain *parameter-dependent* problems. Our theory is applicable to a broad class of problems; we illustrate it here by means of two examples. The first is a 1-d problem $(P1)$, of a Timoshenko beam, subject to a vertical body force of $-tg(x)$,

$$-\phi_t'' + \frac{1}{t^2}(\phi_t - w_t') = f \quad \text{on} \quad \Omega = (0,1) \tag{1}$$

$$\frac{1}{t^2}(\phi_t - w_t')' = g \quad \text{on} \quad \Omega \tag{2}$$

$$\phi_t(0) = \phi_t(1) = w_t(0) = w_t(1) = 0 \tag{3}$$

where w_t and ϕ_t represent the vertical displacement and rotation, respectively, with the parameter t (the thickness) lying in $(0,1]$ (f can be related to moments in applications).

Our second problem $(P2)$ is that of 2-d elasticity over a domain Ω with boundary Γ,

$$-\frac{E}{2(1+\nu)}\Delta \mathbf{u}_\nu - \frac{E}{2(1+\nu)(1-2\nu)}\,\text{grad div}\,\mathbf{u}_\nu = \mathbf{f} \quad \text{in} \quad \Omega \tag{4}$$

$$\frac{E}{(1+\nu)}\sum_{j=1}^{2}\left[\epsilon_{ij}(\mathbf{u}_\nu) + \delta_{ij}\frac{\nu}{1-2\nu}\,\text{div}\,\mathbf{u}_\nu\right] n_j = g_i \quad \text{on } \Gamma,\ i = 1,2 \tag{5}$$

where $\mathbf{u}_\nu = (u_1, u_2)$, $\{\epsilon_{ij}\}$ is the strain tensor and E the modulus of elasticity. Here the parameter is ν, the Poisson ratio, which is in $[0, 0.5)$. (We assume a compatibility condition on $\mathbf{f}$ and $\mathbf{g}$ to ensure a solution unique up to rigid body motions.)

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[1] Research partially supported by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under Grant #AFOSR 89-0252