Closure Properties for the Class of Behavioral Models

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Abstract

Hidden k-logics can be considered as the underlying logics of program specification. They constitute natural generalizations of k-deductive systems and encompass deductive systems as well as hidden equational logics and inequational logics. In our abstract algebraic approach the data structures are sorted algebras endowed with a designated subset of their visible part, called a *filter*, which represents a set of truth values.

We present a hierarchy of hidden k-logics. The hidden k-logics in each class are characterized by three different kinds of conditions. Namely, properties of their Leibniz operator, closure properties of the class of their behavioral models, and by properties of equivalence systems. Using equivalence systems we obtain a new and more complete analysis of the axiomatization of the behavioral models. This is achieved by means of the Leibniz operator and its combinatorial properties.

Key words:
Behavioral Specification, Behavioral Equivalence, Hidden equational Logic, Leibniz Operator, Equivalential Logic.

1 Introduction

In the algebraic approach to program specification one intends to model programs by algebras which are considered as abstract machines in which the

⋆ This research was supported by *Fundação para a Ciência e a Tecnologia* (Portugal) through *Unidade de Investigação Matemática e Aplicações* of University of Aveiro.

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routines are to be run. Traditionally, equational logic has been used as the underlying logic; this is one way of giving a precise algebraic semantics for programs, against which the correctness of a program can be tested. However, there are properties inherent to object oriented (OO) programs which disable a straightforward application of equational methods. In this case, a more appropriate model for the abstract machine is a state transition system. A state of an OO program, like a state of a transition system, can be viewed as encapsulating all pertinent information about the abstract machine when it reaches that state during execution of the program. The standard equality predicate can then be replaced by *behavioral equivalence*; and in this way, the intrinsic properties of state transition systems can be translated into equational logic. Intuitively, two elements are considered *behaviorally equivalent* in a given implementation $\mathcal{A}$ if they cannot be "distinguished" in $\mathcal{A}$ by any visible program taking them as input. This analogy with state-transition systems suggests that the methods of coalgebra, in particular coinduction, might be useful in verifying behavioral validity. In fact, a considerable amount of research has been done on developing various forms of coinduction, usually in combination with the methods of standard equational logic, to verify behavioral validity for wide classes of hidden equational logics (see [18]). Our approach to behavioral specification differs substantially from that taken in most of the previous work in this area in the sense that it is greatly influenced by the theory of *abstract algebraic logic* (AAL) (see [6]). This is, in fact, the new feature of our work (see [29,30] and [31]).

AAL is an area of algebraic logic that focuses on the study of the relationship between logical equivalence and logical truth. More precisely, AAL is centered on the process of associating a class of algebras to a logical system (see [6]). A logical system, a *deductive system* as it has been called in the AAL field, is a pair formed by a signature Σ and a substitution-invariant closure relation on the set of terms over Σ. Using deductive systems (more precisely, k-deductive systems) we can deal with sentential logics, first-order logic (see [6]), equational logic and the logic of partially ordered algebras, as parts of a single unified theory.

The main paradigm in AAL is the representation of the classical propositional calculus in the equational theory of Boolean algebras by means of the so called *Lindenbaum-Tarski process*. In its traditional form, the Lindenbaum-Tarski process relies on the fact that the classical propositional calculus has a biconditional "$\leftrightarrow$" that defines logical equivalence. The set of all formulas is partitioned into logical equivalence classes and then the familiar algebraic process of forming the quotient algebra, called the *Lindenbaum-Tarski algebra*, is applied. There are many deductive systems that do not have a biconditional, and hence the Lindenbaum-Tarski process cannot be applied directly. However, there is an abstract notion of logical equivalence in every deductive system called the *Leibniz congruence* and in this way the Lindenbaum-Tarski process

can be generalized so as to apply to several deductive systems.

The Leibniz congruence $\Omega(T)$ on the term algebra over a theory T is characterized in the following way: for any pair α, β of terms, $\alpha \equiv \beta$ $(\Omega(T))$ if for every term φ and any variable p occurring in φ, $\varphi(p/\alpha) \in T$ if and only if $\varphi(p/\beta) \in T$. The Leibniz congruence is extended in a natural way to the power set of an arbitrary algebra. Given a Σ-algebra $\mathbf{A}$ and a designated subset F of A, the pair $\langle \mathbf{A}, F \rangle$ is called a *matrix*. The relation $\Omega(F)$ identifies any two elements which cannot be distinguished by any property defined by a formula. More precisely, for any pair of elements a, b of A, $a \equiv b$ $(\Omega(F))$ if for each formula $\varphi(x, u_0, \ldots, u_{k-1})$, and all parameters $\bar{c} \in A^k$, $\varphi^{\mathbf{A}}(a, \bar{c}) \in F$ if and only if $\varphi^{\mathbf{A}}(b, \bar{c}) \in F$. Moreover, $\Omega(F)$ is a congruence on $\mathbf{A}$.

In order to apply results and tools of AAL to the theory of specification of abstract data types, we have to look at the specification logic as a deductive system (i.e., as a substitution-invariant closure relation on an appropriate set of formulas) and behavioral equivalence as some generalized notion of Leibniz congruence. The class of deductive systems has to be expanded in order to include multisorted as well as one-sorted systems. The notion of k-deductive systems is generalized by considering the data to be heterogeneous in the sense that the data elements may be of different kinds. Specifically, there are the basic data, like integers, reals and Booleans, whose properties are well-known and for which well-defined and easily manipulated representations are available; and there are the auxiliary data such as arrays, lists, stacks, whose properties are specified by their behavior under the programs with visible output. Thus, we use distinct representations for each kind of data elements. We also distinguish the basic data and the auxiliary data by splitting the data in two kinds: the ones to which the user has direct access to (*visible data*) and those that the user only has access to by the meaning (output) of programs with visible output (*hidden data*). This advantage is central in the specification of OO systems. For some programs it is worth considering those kinds of encapsulated data representation either for security reasons or to simplify the process of updating and improving program implementations. The Leibniz congruence has to be considered in the context of the dichotomy of visible vs. hidden; namely, the formulas we used in the characterization of the Leibniz congruence also have to be restricted to an appropriate proper subset of all formulas, namely the visible formulas (called *contexts*).

We call these generalized deductive systems *hidden k-logics*. We have been using them as underlying logics of program specification within the dichotomy visible vs hidden. They encompass deductive systems as well as the hidden versions of equational and inequational logics and Boolean logics, which are 1-dimensional multisorted logics with Boolean as the only visible sort and with equality-test operations for some of the hidden sorts in place of equality predicates (see [34]). Using hidden k-logics we obtain a unified treatment of

all these kinds of logics and, we can import tools and results from AAL to the specification and verification theory of OO systems.

Hidden k-logics were firstly introduced in [31], where the authors dedicated a special attention to the equational case to derive properties of the behavioral logic of hidden equational logics; the main result is the characterization of the behaviorally specifiable logics as the finitely equivalential ones. An extensive study concerning hidden k-logics was presented in [29]. The author has shown that hidden k-logics are a natural generalization of deductive systems. Moreover, he has used some theorems and arguments of AAL in order to establish results in the specification and verification theory of OO programs.

In this paper, an abstract algebraic approach is followed by considering data structures as sorted algebras endowed with a designated subset of the visible part of the algebra, called a *filter*, which represents the set of truth values. The properties of the specification are formalized by visible conditional equations. The restriction to visible axioms is natural since only visible programs are used in defining behavioral equivalence. In the equational case, allowing hidden equations as axioms may produce unexpected consequences in the behavior of the system. A straightforward consequence of allowing a hidden axiom is that it is trivially a behavioral theorem and we do not know *a priori* if it is actually a behavioral theorem of the original system we intend to specify. The correct procedure should be the following: given a hidden equation e and a hidden equational logic $\mathcal{L}$, if we are able to show that e is behaviorally valid in $\mathcal{L}$, then we may add it as a new axiom without altering the behavioral consequence relation. This can be done with any set of conditional equations. And we may be able to show that now some of the original axioms are redundant (i.e., consequences of the remaining ones together with the newly adjoined hidden axioms). Thus we can discard them and in this way we obtain a simpler specification (in [31] we illustrate this procedure by using the specification of stacks).

The present work is a significant contribution for the development of the generalized theory of AAL described above. The main aim of this paper is to establish a hierarchy of general hidden k-logics. The classes in the hierarchy are characterized by closure properties of the class of behavioral models. For instance, the protoalgebraic logics are characterized by the closure of their class of behavioral models under subdirect products.

This classification of hidden k-logics in terms of their behavior, as well as the characterizations by closure properties of the class of behavioral models, and the corresponding hierarchy of such classes of logics has never been considered in the context of hidden equational logics and observational logics. A similar axiomatization of behavioral equivalence has been considered by Bidoit and Hennicker (see for example [2]). They defined an axiomatization to be a set

of first order formulas which defines the Leibniz congruence over any filter. Here we go further, by presenting an extensive analysis of the various kinds of formulas that can be used in the axiomatization.

1.1 Outline of the paper

We start by presenting some basic notions and results on multisorted universal algebra which will be needed in the sequel. Then, we recall the notion of a hidden k-logic and we review some elementary aspects of its semantics.

As we said above, hidden k-logics are very important since they comprehend, not only the hidden and standard equational and inequational logics, but also Boolean logics and all sentential logics in the sense of AAL. The Leibniz congruence is introduced in this general context and its most relevant properties are formulated (for details see [31] and [29]).

The notion of protoalgebraic logic is generalized to the hidden case (Definition 18). The class of protoalgebraic logics seems to be the widest class whose behavior can be reasonably managed. This class is characterized syntactically by the existence of a special double sorted set of visible k-formulas. Closure properties of the set $\Omega(\mathrm{Th}(\mathcal{L})) := \{\Omega(T) : T \in \mathrm{Th}(\mathcal{L})\}$ are also discussed.

In Subsection 4.2, we introduce the notion of equivalence systems in the context of hidden k-logics and we develop their fundamental theory. Informally, an equivalence system of a hidden k-logic $\mathcal{L}$ is a sorted set of visible k-formulas which defines, for each theory T of $\mathcal{L}$, the Leibniz congruences on the term algebra over T (Theorem 28). Hence, an equivalence system of a logic $\mathcal{L}$ provides an axiomatization for its behavioral logic. Equivalence systems are the natural generalization of the well known phenomenon in the classical propositional calculus where the equivalence of formulas can be expressed by the equivalence symbol "$\leftrightarrow$" (see [35]), i.e., for each theory T, $\varphi \equiv \psi$ $(\Omega(T))$ iff $T \vdash \varphi \leftrightarrow \psi$. The formal definition of equivalence system, given here, is syntactic, but we present model theoretical characterizations for a sorted set of formulas to be an equivalence system. Proposition 31 is a very useful tool; we show, using this proposition, that the specification of stacks is not finitely equivalential (see Subsection 4.4.3).
Using the theory of equivalence systems, in subsection 4.3, we establish a hierarchy in the class of hidden k-logics by properties of the respective equivalence systems. The classes considered in that hierarchy are the protoalgebraic, the parameterized finitely equivalential, the equivalential and the finitely equivalential logics.
Such classes are also characterized by properties of the Leibniz operator, as well as by closure properties of the class of behavioral models. The character-

izations are established using Theorem 35 which provides an axiomatization of the class of behavioral models using equivalence systems. This theorem is essentially a consequence of the fact that the equivalence systems of a hidden k-logic define the Leibniz congruence on $\mathbf{A}$ over F for each model $\langle \mathbf{A}, F \rangle$ of $\mathcal{L}$ (Theorem 28).

To clarify the several abstract notions of equivalence systems we present some examples in Subsection 4.4; in particular, we show an example of a hidden equational logic which is not equivalential.

We finish Section 4 by presenting a further topic of research: the definability of the set of behavioral theorems. We just develop the introductory theory, but the analogy with the theory of equivalence systems suggests that the same tools will work. At the end of the paper we establish connections with related work.

2 Hidden algebra

Let SORT be a nonempty set whose elements are called *sorts*. A *type over* SORT is a nonempty finite sequence $S_0, \ldots, S_n$ of sorts in SORT. We will write a type as $S_0, \ldots, S_{n-1} \to S_n$. The set of all types is denoted by TYPE. The dichotomy visible vs hidden is taken in account from the beginning: the set of sorts is split in two parts, visible and hidden, in the definition of signature. A *hidden* (*sorted*) *signature* is a triple $\Sigma = \langle \mathrm{SORT}, \mathrm{VIS}, \langle \mathrm{OP}_\tau : \tau \in \mathrm{TYPE} \rangle \rangle$, where SORT is a countable nonempty set of sorts, VIS is a subset of SORT, which we call the set of *visible sorts*, and OP_τ is a countable set of operation symbols of type τ. The sorts in $\mathrm{HID} := \mathrm{SORT} \setminus \mathrm{VIS}$ are called *hidden sorts*. Σ is said to be *standard* if there is a ground term of every sort.

Example 1 (FLAGS) The hidden signature of Flags, Σ_{flags}, is the signature used to specify the semaphore systems. These systems are used to schedule resources in the following way. We associate a flag to each resource. When a resource is being used by some process, its flag is put "up" to indicate forbidden access. After being used, its flag is put "down", which means that the resource is available to be used by another process. The user does not have access to the flag itself (i.e., flag is the hidden sort). The only access is through the operation *up*?, which is used to test the state of the semaphore and returns a Boolean value. In case of the implementation which has as its Boolean part the 2-element Boolean algebra, the result of *up*? is true or false with the meaning that the resource is available or not, respectively. Σ_{flags} is the hidden signature $\langle \mathrm{SORT}, \mathrm{VIS}, \mathrm{OP} \rangle$ presented in Fig. 1.

$\diamondsuit$

SORT : $bool,\ flag$
VIS : $bool$

Operation symbols:

$up : flag \to flag;$	$false : \ \to bool;$
$rev : flag \to flag;$	$\neg : bool \to bool;$
$dn : flag \to flag;$	$\wedge : bool, bool \to bool;$
$up? : flag \to bool;$	$\vee : bool, bool \to bool.$
$true : \ \to bool;$	

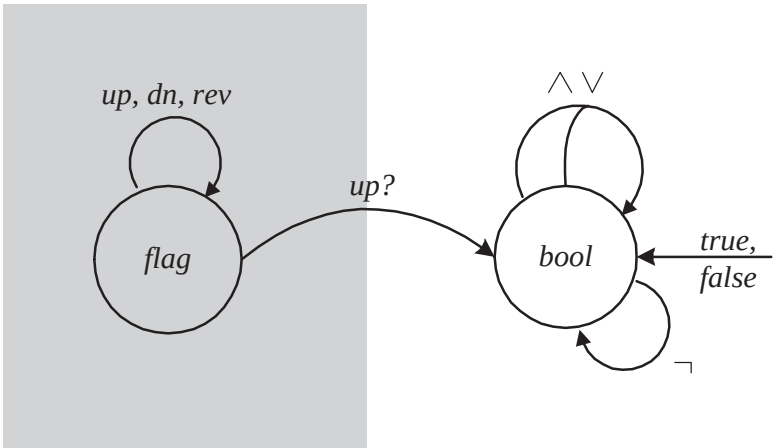

Fig. 1. Signature of Flags, Σ_{flags}.

By a Σ-*algebra*[1] (we simply say an algebra, if Σ is clear from the context) we mean a pair $\mathbf{A} = \Big\langle \langle A_S : S \in \mathrm{SORT}\rangle, \langle \mathrm{OP}^{\mathbf{A}}_{\tau} : \tau \in \mathrm{TYPE}\rangle \Big\rangle$, where $A = \langle A_S : S \in \mathrm{SORT}\rangle$ is a nonempty sorted set and for each $\tau \in \mathrm{TYPE}$ ($\tau = S_0, \ldots, S_{n-1} \to S_n$), $\mathrm{OP}^{\mathbf{A}}_{\tau} = \langle O^{\mathbf{A}} : O \in \mathrm{OP}_{\tau}\rangle$, where $O^{\mathbf{A}}$ is an operation on A of type τ, that is $O^{\mathbf{A}} : A_{S_0} \times \cdots \times A_{S_{n-1}} \to A_{S_n}$. We call the sorted set $A = \langle A_S : S \in \mathrm{SORT}\rangle$ the *universe* (or *carrier*) *of* $\mathbf{A}$ and the sets A_S, for $S \in \mathrm{SORT}$, are called the *domains of* $\mathbf{A}$.

We say that a sorted set A is *locally countable* (*finite*), if for every sort S, A_S is a countable (finite) set; and A is said to be *globally finite*, if A is locally finite and A_S is empty except for a finite number of sorts. Note that, if SORT is finite then global and local finiteness are equivalent. We write $B \subseteq_{\mathcal{GF}} A$ if B is a sorted subset of A and B is globally finite. The set of all globally finite sorted subsets of A is denoted by $\mathcal{P}_{\mathcal{GF}}(A)$, i.e., $\mathcal{P}_{\mathcal{GF}}(A) = \{B : B \subseteq_{\mathcal{GF}} A\}$.

[1] Throughout this paper we assume that $A_S \neq \emptyset$, for all $S \in \mathrm{SORT}$. With this assumption we exclude some data structures of practical interest. However, the metamathematics is simpler in this case and most results of universal algebra hold in their usual form. More generally, the assumption holds automatically if Σ is standard.

2.1 Homomorphisms and congruences

A mapping $f : A \to B$ between the universes of two Σ-algebras $\mathbf{A}$ and $\mathbf{B}$ is a (Σ-*algebra*) *homomorphism* from $\mathbf{A}$ to $\mathbf{B}$, denoted by $f : \mathbf{A} \to \mathbf{B}$, if for each operation symbol O of type $S_0, \ldots, S_{n-1} \to S_n$ and all $a_0 \in A_{S_0}, \ldots, a_{n-1} \in A_{S_{n-1}}$, $f_{S_n}(O^{\mathbf{A}}(a_0, \ldots, a_{n-1})) = O^{\mathbf{B}}(f_{S_0}(a_0), \ldots, f_{S_{n-1}}(a_{n-1}))$. An injective homomorphism f is called a *monomorphism*. If f is surjective, it is called an *epimorphism*. We say that h is an *isomorphism* if it is both an injective and a surjective homomorphism. By a sorted congruence on a Σ-algebra $\mathbf{A}$ we mean a sorted binary relation $\theta \subseteq A^2$ such that, for each $S \in \mathrm{SORT}$, θ_S is an equivalence relation on A_S and for every operation symbol O, say of type $S_0, \ldots, S_{n-1} \to S_n$, and all $a_0, a'_0 \in A_{S_0}, \ldots, a_{n-1}, a'_{n-1} \in A_{S_{n-1}}$, θ satisfies the *congruence condition* (sometimes called *substitutivity condition*): $O^{\mathbf{A}}(a_0, \ldots, a_{n-1}) \equiv O^{\mathbf{A}}(a'_0, \ldots, a'_{n-1})$ (θ_{S_n}), whenever $a_i \equiv a'_i$ (θ_{S_i}) for each $i < n$. We will represent congruence relations by the symbol $\equiv$ or by the Greek letter θ . The set of all congruences on $\mathbf{A}$ is denoted by $\mathrm{Con}(\mathbf{A})$. If $h : A \to B$ is a homomorphism from $\mathbf{A}$ to $\mathbf{B}$ then $ker(h)$ is a sorted congruence on $\mathbf{A}$. Let F be a sorted set of pairs of an algebra $\mathbf{A}$; the *congruence generated by F on* $\mathbf{A}$, denoted by $\Theta(F)$, is the intersection of all congruences of $\mathbf{A}$ that contain F, i.e., the smallest congruence on $\mathbf{A}$ which contains F. The *quotient* of $\mathbf{A}$ by the congruence θ is the algebra $\mathbf{A}/\theta = \langle A/\theta, \mathrm{OP}^{\mathbf{A}/\theta}\rangle$, where for each operation symbol O of type $S_0, \ldots, S_{n-1} \to S_n$ and all $a_i/\theta_{S_i} \in A_{S_i}/\theta_{S_i}$, $i < n$, $O^{\mathbf{A}/\theta}(a_0/\theta_{S_0}, \ldots, a_{n-1}/\theta_{S_{n-1}}) = O^{\mathbf{A}}(a_0, \ldots, a_{n-1})/\theta_{S_n}$.

The homomorphism theorems of (unsorted) universal algebra all extend naturally to sorted universal algebra (see [32]). In particular, the first homomorphism theorem says that a surjective homomorphism $h : \mathbf{A} \to \mathbf{B}$ of sorted algebras can be factored uniquely by the natural mapping $nat : \mathbf{A} \to \mathbf{A}/ker(h)$.

Theorem 2 (Homomorphism Theorem) *Let $h : \mathbf{A} \to \mathbf{B}$ be a homomorphism from $\mathbf{A}$ to $\mathbf{B}$. Then, there is a unique monomorphism $g : \mathbf{A}/ker(h) \to \mathbf{B}$ such that $g(a/ker(h)) = h(a)$. Moreover, if h is surjective then g is an isomorphism.*

2.2 Products and Filtered Products

The *direct product* of a family of Σ-algebras $\mathbf{A}_i$, $i \in I$, is denoted by $\prod_{i\in I} \mathbf{A}_i$. Its universe is $\prod_{i\in I} A_i = \langle \prod_{i\in I}(A_i)_S : S \in \mathrm{SORT}\rangle$, and its operations are defined componentwise as usual. If I is the empty set, then $\prod_{i\in I} \mathbf{A}_i$ is by definition a trivial algebra. For each $i \in I$, the *projection* $\pi_i : \prod_{i\in I} A_i \to A_i$ is a surjective homomorphism from $\prod_{i\in I} \mathbf{A}_i$ onto $\mathbf{A}_i$. $\mathbf{B}$ is a subdirect product of a family of Σ-algebras $\mathbf{A}_i$ if $\mathbf{B} \subseteq \prod_{i\in I} \mathbf{A}_i$ and $\pi_i(B) = A_i$, for each $i \in I$.

We denote a subdirect product by $\mathbf{B} \subseteq_{SD} \prod_{i \in I} \mathbf{A}_i$.

Let I be a nonempty set. A *filter on* I is a set $\mathcal{F}$ of subsets of I which satisfies the following conditions:

- $I \in \mathcal{F}$;
- if $J \in \mathcal{F}$ and $J \subseteq K \subseteq I$, then $K \in \mathcal{F}$;
- if $J,\ K \in \mathcal{F}$ then $J \cap K \in \mathcal{F}$.

A filter $\mathcal{F}$ that does not contain the empty set ($\emptyset \notin \mathcal{F}$) is called a *proper* filter. Using Zorn's Lemma we can prove that every proper filter can be extended to a proper filter $\mathcal{F}$, which is maximal with respect to set inclusion (i.e., there is no proper filter on I strictly including $\mathcal{F}$). A proper filter $\mathcal{F}$ which is maximal with respect to set inclusion is called an *ultrafilter*. Sometimes it is convenient to consider the equivalent condition: either $X \in \mathcal{F}$ or $\overline{X} \in \mathcal{F}$ but not both, for every $X \subseteq I$.

Let $\mathbf{A}_i$, $i \in I$, be a family of Σ-algebras and $\mathcal{F}$ a filter on I. We define a sorted binary relation $\theta(\mathcal{F}) = \Big\langle (\theta(\mathcal{F}))_S : S \in \mathrm{SORT}) \Big\rangle$ on $\prod_{i \in I} A_i$ by,

$$\langle a_i : i \in I \rangle \equiv \langle b_i : i \in I \rangle ((\theta(\mathcal{F}))_S) \text{ iff } \{i \in I : a_i = b_i\} \in \mathcal{F},$$

for all $\langle a_i : i \in I \rangle, \langle b_i : i \in I \rangle \in \prod_{i \in I} (A_i)_S$. In fact, $\theta(\mathcal{F})$ is a congruence on $\prod_{i \in I} \mathbf{A}_i$. Thus, we can form the quotient $(\prod_{i \in I} \mathbf{A}_i)/\theta(\mathcal{F})$, which is called the *reduced product of* the family $\mathbf{A}_i$, $i \in I$, by the filter $\mathcal{F}$. When $\mathcal{F}$ is an ultrafilter, $(\prod_{i \in I} \mathbf{A}_i)/\theta(\mathcal{F})$ is called an *ultraproduct*.

2.3 Data structures

The *visible part* of a sorted set A is the sorted set $\langle A_V : V \in \mathrm{VIS} \rangle$, which we denote by A_{VIS}. A *visible k-data structure* (or simply a *k-data structure*) over Σ is a pair $\mathcal{A} = \langle \mathbf{A}, F \rangle$, where $\mathbf{A}$ is a Σ-algebra and $F \subseteq A^k_{\mathrm{VIS}}$; $\mathbf{A}$ is called the *underlying algebra* of $\mathcal{A}$ and F is called the *filter* of $\mathcal{A}$. An example of a 2-data structure is any model of the free hidden equational logic over Σ (HEL_Σ) considered below (Definition 9). The standard model of HEL_Σ is of the form $\langle \mathbf{A}, id_{A_{\mathrm{VIS}}} \rangle$, where $\mathbf{A}$ is a Σ-algebra and $id_{A_{\mathrm{VIS}}}$ is the identity relation on the visible part of A, but one gets more general 2-data structures as models by taking any congruence relation on the visible part of A in place of $id_{A_{\mathrm{VIS}}}$. We can also consider the free Boolean logic over Σ, if Σ has a Boolean sort. Here the standard models are the 1-data structures $\langle \mathbf{A}, \{true\} \rangle$, where $\mathbf{A}$ is a Σ-algebra such that A_{VIS} is the two-element Boolean algebra. In a general model, A_{VIS} is an arbitrary Boolean algebra and $\{true\}$ is replaced by an arbitrary filter on A_{VIS}.

2.3.1 Data structure homomorphisms, products and filtered products

Let Σ be a hidden signature and $\mathcal{A} = \langle \mathbf{A}, F\rangle$, $\mathcal{B} = \langle \mathbf{B}, G\rangle$ be k-data structures over Σ. We say that $\mathcal{A} = \langle \mathbf{A}, F\rangle$ is a *k-data substructure* of $\mathcal{B} = \langle \mathbf{B}, G\rangle$, in symbols $\mathcal{A} \subseteq \mathcal{B}$, if $\mathbf{A} \subseteq \mathbf{B}$ and $G \cap A^k_{\mathrm{VIS}} = F$. Let $h : \mathbf{A} \to \mathbf{B}$ be a sorted homomorphism between the underlying algebras. We say that h is a *data structure homomorphism* from $\mathcal{A}$ to $\mathcal{B}$ if $h(F) \subseteq G$, or equivalently, if $F \subseteq h^{-1}(G)$, and we denote it by $h : \mathcal{A} \to \mathcal{B}$. If $F = h^{-1}(G)$, h is said to be *strict*. We say that h is *surjective* if it is surjective as a sorted algebra homomorphism and it is denoted by $h : \mathcal{A} \twoheadrightarrow \mathcal{B}$. Moreover, h is said to be *injective* if it is injective as an algebra homomorphism and $h(F) = G \cap h(A^k)$, equivalently $h^{-1}(G) = F$ and we denote it by $h : \mathcal{A} \rightarrowtail \mathcal{B}$. We say that h is a data structure *isomorphism* if it is injective and surjective, i.e., h is an algebra isomorphism and $h(F) = G$. In this case we write $\mathcal{A} \cong \mathcal{B}$. We also use the notation $\mathcal{A} \cong; \subseteq \mathcal{B}$ to say that *there is a k-data structure $\mathcal{C}$ such that $\mathcal{A} \cong \mathcal{C} \subseteq \mathcal{B}$.*

Let $\mathcal{A}_i = \langle \mathbf{A}_i, F_i\rangle$, $i \in I$, be a family of k-data structures. We define the *direct product* of $\mathcal{A}_i$, $i \in I$ to be the k-data structure

$$\textstyle\prod_{i\in I} \mathcal{A}_i := \Big\langle \prod_{i\in I} \mathbf{A}_i, \prod_{i\in I} F_i \Big\rangle,$$

where $\prod_{i\in I} \mathbf{A}_i$ is the direct product of the family of algebras $\mathbf{A}_i$, $i \in I$, and, $\prod_{i\in I} F_i = \Big\langle \prod_{i\in I}(F_i)_V : V \in \mathrm{VIS}\Big\rangle$. The index set may be empty. In this case $\prod_{i\in I} \mathbf{A}_i$ is the trivial one-element algebra.

A k-data structure $\mathcal{B} \subseteq \prod_{i\in I} \mathcal{A}_i$ is called a *subdirect product* of the family $\mathcal{A}_i$, $i \in I$, in symbols $\mathcal{B} \subseteq_{\mathrm{SD}} \prod_{i\in I} \mathcal{A}_i$, if the projections $\pi_i : B \to A_i$ are surjective for all $i \in I$.

Let $\mathcal{F}$ be a lattice filter of $\langle \mathcal{P}(I), \cap, \cup\rangle$. For each visible sort V we define:

$$(D^{\mathcal{F}}_{\prod_{i\in I} A_i})_V := \Big\{\bar{f} \in (\textstyle\prod_{i\in I}(A_i)_V)^k : \{i \in I : \langle f_0(i), \ldots, f_{k-1}(i)\rangle \in (F_i)_V\} \in \mathcal{F}\Big\}$$

$$\textstyle\prod^{\mathcal{F}}_{i\in I} A_i := \langle \prod_{i\in I} A_i, D^{\mathcal{F}}_{\prod_{i\in I} A_i}\rangle.$$

We now define a relation $\theta(\mathcal{F})$ on $\prod_{i\in I} A_i$ by

$$\theta(\mathcal{F})_S := \{\langle f, g\rangle \in (\textstyle\prod_{i\in I} A_i)^2_S : \{i : f(i) = g(i)\} \in \mathcal{F}\}.$$

Let $(\prod_{i\in I} A_i)/\mathcal{F} := (\prod_{i\in I} A_i)/\theta(\mathcal{F})$ and $D_{(\prod_{i\in I} A_i)/\mathcal{F}} := D^{\mathcal{F}}_{(\prod_{i\in I} A_i)}/\theta(\mathcal{F})$. $(\prod_{i\in I} A_i)/\mathcal{F}$ is the usual reduced product of algebras. The *data structure filtered product* of the family $\mathcal{A}_i$, $i \in I$, by $\mathcal{F}$ is

$$(\textstyle\prod_{i\in I} \mathcal{A}_i)/\mathcal{F} := \Big\langle (\prod_{i\in I} A_i)/\mathcal{F}, D_{(\prod_{i\in I} A_i)/\mathcal{F}}\Big\rangle.$$

If $\mathcal{F}$ is an ultrafilter then the data structure filtered product is called a *data structure ultrafiltered product.*

Let K be a class of k-data structures. We define

$\mathbf{H}(K) := \{\mathcal{B} :$ there exists a surjective data-structure homomorphism $h : \mathcal{A} \to \mathcal{B},$ for some $\mathcal{A} \in K\}$

$\mathbf{S}(K) := \{\mathcal{A} : \mathcal{A} \cong; \subseteq \mathcal{B},$ for some $\mathcal{B} \in K\}$

$\mathbf{P}(K) := \{\mathcal{A} : \mathcal{A} \cong \prod_{i \in I} \mathcal{B}_i,$ for some $\mathcal{B}_i \in K, i \in I\}$

$\mathbf{P}_{\mathrm{SD}}(K) := \{\mathcal{A} : \mathcal{A} \cong; \subseteq_{\mathrm{SD}} (\prod_{i \in I} \mathcal{B}_i),$ for some $\mathcal{B}_i \in K, i \in I\}$

$\mathbf{P}_u(K) := \{\mathcal{A} : \mathcal{A} \cong (\prod_{i \in I} \mathcal{B}_i)/\mathcal{F},$ for some $\mathcal{B}_i \in K, i \in I,$ and some ultrafilter $\mathcal{F}$ of $\mathcal{P}(I)\}$.

We say that a class K is *closed under data substructures, products, ultraproducts and subdirect products* if $\mathbf{S}(K) = K$, $\mathbf{P}(K) = K$, $\mathbf{P}_u(K) = K$ and $\mathbf{P}_{\mathrm{SD}}(K) = K$, respectively.

The following theorem states some basic properties concerning these classes of structures. We will omit the proof since it can be easily obtained from its one-sorted version.

Theorem 3 *Let K be a class of k-data structures. Then,*

(i) $\mathbf{SH}(K) \subseteq \mathbf{HS}(K)$*;*
(ii) $\mathbf{PS}(K) \subseteq \mathbf{SP}(K)$*;*
(iii) $\mathbf{PH}(K) \subseteq \mathbf{HP}(K)$*;*
(iv) $\mathbf{S}, \mathbf{H}, \mathbf{P}, \mathbf{SP}, \mathbf{SH}, \mathbf{HP}, \mathbf{HSP}$ *are closure operators on the class of all k-data structures.*

2.3.2 Leibniz Congruence

Let $\langle \mathbf{A}, F \rangle$ be a k-data structure. A congruence relation θ on $\mathbf{A}$ is VIS-*compatible* (*or simply compatible*) *with* F if for all $V \in$ VIS and for all $\bar{a}, \bar{a}' \in A_V^k$, $a_i \equiv a_i'(\theta_V)$ for all $i < k$, implies that ($\bar{a} \in F_V$ iff $\bar{a}' \in F_V$). Equivalently, we have that θ is compatible with F if and only if for every $V \in$ VIS, F_V is the union of the cartesian product of θ_V-classes i.e.,

$$F_V = \bigcup_{\bar{a} \in F_V} (a_0/\theta_V) \times (a_1/\theta_V) \times \cdots \times (a_{k-1}/\theta_V).$$

We now point out some properties of the relation of compatibility that will be used below without further reference. If a congruence θ is compatible with F, then any congruence that is contained in θ is also compatible with F. But θ being compatible with F does not imply that θ is compatible with any G contained in F. However, if θ is compatible with each member of a set U of filters ($U \subseteq \mathcal{P}(A^k)$) then it is compatible with its intersection $\bigcap U$. The fact that a congruence θ is compatible with F also does not imply that it is compatible with any G that contains F.

It is not difficult to see that the largest congruence compatible with a filter F always exists. In fact, given two congruences θ_0 and θ_1 of $\mathbf{A}$ compatible with F, the relative product $\theta_0 \circ \theta_1$ is also compatible with F. This implies that the join $\theta_0 \vee \theta_1$, is also compatible with F. Hence, the set $\{\theta \in \mathrm{Con}(A) : \theta$ is compatible with $F\}$ is directed under set theoretical inclusion and so, the union is also a congruence compatible with F (for the details see [31]).

Definition 4 *Let $\langle \mathbf{A}, F\rangle$ be a k-data structure. Then the* Leibniz congruence *of F on $\mathbf{A}$ is the largest congruence relation on $\mathbf{A}$ compatible with F. We denote it by $\Omega_{\mathbf{A}}(F)$; we simply write $\Omega(F)$, when $\mathbf{A}$ is clear from the context.*

One of the main properties of the Leibniz congruence is its preservation under inverse images of surjective homomorphisms. This is given in the following lemma. Let $h : B \to A$ be a mapping between sets. If $\bar{b} = \langle b_0, \ldots, b_{k-1}\rangle \in B^k$, then $h(\bar{b}) := \langle h(b_0), \ldots, h(b_{k-1})\rangle \in A^k$, and if $\bar{a} = \langle a_0, \ldots, a_{k-1}\rangle \in A^k$, $h^{-1}(\bar{a}) := \{\, \bar{b} \in B^k : h(\bar{b}) = \bar{a} \,\}$.

Lemma 5 ([31]) *Let $\mathcal{A} = \langle \mathbf{A}, F\rangle$ be a k-data structure over Σ, and let $\mathbf{B}$ be a Σ algebra and $h : \mathbf{B} \to \mathbf{A}$ a surjective homomorphism. Then $h^{-1}(\Omega_{\mathbf{A}}(F)) = \Omega_{\mathbf{B}}(h^{-1}(F))$.*

If $\mathcal{A} = \langle \mathbf{A}, F\rangle$ is a k-data structure over Σ, we can form the quotient structure $\mathcal{A}/\Omega(F) = \langle \mathbf{A}, F\rangle/\Omega(F) = \langle \mathbf{A}/\Omega(F),\, F/\Omega(F)\rangle$, where $A/\Omega(F)$ is the quotient of A by $\Omega(F)$ and $F/\Omega(F) = \{\, (a_0/\Omega(F), \ldots, a_{k-1}/\Omega(F)) : \langle a_0, \ldots, a_{k-1}\rangle \in F \,\}$. $\mathcal{A}$ is said to be *reduced* if $\Omega(F)$ is the identity congruence on $\mathbf{A}$. In [31] it is shown that the quotient of any k-data structure by the Leibniz congruence is always reduced. In 2-data structures the filters are themselves sets of pairs. Hence, a strict relationship between them and their Leibniz congruence can be established (see [29]). For example, given a 2-data structure $\mathcal{A} = \langle \mathbf{A}, F\rangle$ and a congruence θ on $\mathbf{A}$ we have: (a) θ is compatible with F if and only if $\theta_{\mathrm{VIS}} \circ F \circ \theta_{\mathrm{VIS}} \subseteq F$ and (b) if F is reflexive (on A_{VIS}) and transitive then θ is compatible with F if and only if $\theta_{\mathrm{VIS}} \subseteq F$.

From (b), if $\Theta(F \cup \{\langle a, a'\rangle\})_{\mathrm{VIS}} = F$ and, F is reflexive and transitive (on A_{VIS}), then $\Theta(F \cup \{\langle a, a'\rangle\})$ is compatible with F. Hence, $\Theta(F \cup \{\langle a, a'\rangle\}) \subseteq \Omega(F)$. Therefore we have that $\Theta(F \cup \{\langle a, a'\rangle\})_{\mathrm{VIS}} = F$ implies $a \equiv a'$ $(\Omega(F))$.

3 Hidden logic

Let $X = \langle X_S : S \in \mathrm{SORT}\rangle$ be a fixed locally countable sorted set of variables. We define the sorted set $\mathrm{Te}_\Sigma(X)$ of terms in the signature Σ as usual, and by defining, in the natural way, operations in $\mathrm{Te}_\Sigma(X)$ we get the *term algebra* over the signature Σ. It is well known that $\mathrm{Te}_\Sigma(X)$ has the universal mapping

property over X in the sense that, for every Σ-algebra A and every sorted map $h : X \to A$, called an *assignment*, there is a unique sorted homomorphism $h^* : \mathrm{Te}_\Sigma(X) \to A$, which extends h. In particular a map from X to the set of terms, and its unique extension to an endomorphism of $\mathrm{Te}_\Sigma(X)$, is called a *substitution*. Substitutions are represented by the Greek letters $\sigma, \tau, \ldots$ Since X is assumed fixed throughout the paper, we normally write Te_Σ in place of $\mathrm{Te}_\Sigma(X)$; moreover, we may write simply Te when Σ is clear from context. We also define a mapping $\bar{h}^* : \mathrm{Te}^k_\Sigma \to A^k$ by $\bar{h}^*(\langle \varphi_0, \ldots, \varphi_{k-1} \rangle) := \langle h^*(\varphi_0), \ldots, h^*(\varphi_{k-1}) \rangle$. In the sequel, if it is clear from the context, we will denote all these mappings by the same symbol h.

Let $t(x_0{:}S_0, \ldots, x_{n-1}{:}S_{n-1}) \in \mathrm{Te}_\Sigma(X)_S$. For each Σ-algebra $\mathbf{A}$, the *interpretation of* t in $\mathbf{A}$ is a mapping, called a *derived operation* (*or term operation*) *on* $\mathbf{A}$, $t^{\mathbf{A}} : A_{S_0} \times \cdots \times A_{S_{n-1}} \to A_S$ such that $t^{\mathbf{A}}(a_0, \ldots, a_{n-1}) = h(t)$, where h is any assignment such that $h(x_i) = a_i$, for $i = 0, \ldots, n-1$.

Assume a fixed order in all variables of all types with the property that for any finite set of variables X of mixed type, and any sort S, there exists a variable of type S which is larger than all the variables in X. Given any ordinary sorted signature $\Sigma = \langle \mathrm{SORT}, \mathrm{OP} \rangle$, $\mathrm{DER}(\Sigma) = \langle \mathrm{SORT}, \mathrm{OP}_{\mathrm{DER}(\Sigma)} \rangle$ is the signature whose set of sorts is the same as that of Σ, and $\mathrm{OP}_{\mathrm{DER}(\Sigma)}$ is the set of special terms of the form $t(x_0, x_1, \ldots, x_{n-1})$ where $\langle x_0, x_1, \ldots, x_{n-1} \rangle$ is any n-tuple of variables, in the same order as they occur in the fixed order, such that every variable occurring in t appears exactly once in the list. Each term of this form is considered an operation symbol of $\mathrm{DER}(\Sigma)$ of type $S_0, \ldots, S_{n-1} \to S$. Thus $\mathrm{OP}_{\mathrm{DER}(\Sigma)} = \mathrm{Te}_\Sigma(X)$. We can consider $\mathrm{DER}(\Sigma)$ as an enrichment of Σ by identifying each operation symbol $O \in \mathrm{OP}$ of type $S_0, \ldots, S_{n-1} \to S$ with the unique term $O(x_0{:}S_0, \ldots, x_{n-1}{:}S_{n-1}){:}S$, where $\langle x_0, \ldots, x_{n-1} \rangle$ is the first subsequence, without repetitions, of the fixed ordering of variables whose sorts are respectively $S_0, \ldots, S_{n-1}$. We should note that if $\mathbf{A}$ is a $\mathrm{DER}(\Sigma)$-algebra then a binary relation on A is a congruence on $\mathbf{A}$ only if it is a Σ-congruence on the reduct $\mathbf{A}\restriction_\Sigma$.

To provide a context that allows us to deal simultaneously with specification logics that are sentential (for example logics with a Boolean sort) and equational, we introduce the notion of a k-term for any nonzero natural number k. A *k-term* of sort S over Σ is just a sequence of k Σ-terms, all of the same sort S. We indicate k-terms by overlining, so $\bar{\varphi}{:}S = \langle \varphi_0{:}S, \ldots, \varphi_{k-1}{:}S \rangle$. When we do not want to make the common sort of each term of $\bar{\varphi}$ explicit, we simply write it as $\bar{\varphi}$. Te^k_Σ denotes the sorted set of all k-terms over Σ, i.e., $\mathrm{Te}^k_\Sigma = \langle (\mathrm{Te}_\Sigma)^k_S : S \in \mathrm{SORT} \rangle$. The set of all *visible k-terms* $(\mathrm{Te}^k_\Sigma)_{\mathrm{VIS}}$ is the set $\langle (\mathrm{Te}^k_\Sigma)_V : V \in \mathrm{VIS} \rangle$. A *$k$-variable* is the special k-term $\bar{x}$ consisting of k distinct variables $\langle x_0{:}S, \ldots, x_{k-1}{:}S \rangle$ all of them of the same sort.

For the purposes of this work it is convenient to define a hidden logic as an

abstract closure relation on the set of k-terms, independently of any specific choice of axioms and rules of inference. By a *closure relation* on a (sorted) subset Λ of Te^k_Σ we mean a binary relation $\vdash \subseteq \mathcal{P}(\Lambda) \times \Lambda$ between subsets of Λ of k-terms and individual k-terms in Λ satisfying for all $\Gamma, \Delta \subseteq \Lambda$ the following conditions: (1) $\Gamma \vdash \bar{\gamma}$ for each $\bar{\gamma} \in \Gamma$; (2) $\Gamma \vdash \bar{\varphi}$ and $\Delta \vdash \bar{\gamma}$ for each $\bar{\gamma} \in \Gamma$, imply $\Delta \vdash \bar{\varphi}$. The closure relation is *finitary* (or *compact*) if $\Gamma \vdash \bar{\varphi}$ implies $\Delta \vdash \bar{\varphi}$ for some globally finite subset Δ of Γ. It is *substitution-invariant* if $\Gamma \vdash \bar{\varphi}$ implies $\sigma(\Gamma) \vdash \sigma(\bar{\varphi})$ for every substitution $\sigma : X \to \mathrm{Te}_\Sigma$. Every closure relation $\vdash$ on Te^k_Σ has a natural extension to a relation, also denoted by $\vdash$, between subsets of Te^k_Σ. It is defined by $\Gamma \vdash \Delta$ if $\Gamma \vdash \bar{\varphi}$ for each $\bar{\varphi} \in \Delta$.

Definition 6 *A* hidden logical k-system (*or simply a* hidden k-logic)[2] *over a hidden signature Σ is a pair $\mathcal{L} = \langle \Sigma, \vdash_{\mathcal{L}} \rangle$, where Σ is a hidden signature and $\vdash_{\mathcal{L}}$ is a substitution-invariant closure relation on the set $(\mathrm{Te}^k_\Sigma)_{\mathrm{VIS}}$ of visible k-terms. A hidden k-logic is* specifiable *if $\vdash_{\mathcal{L}}$ is finitary. We call $\vdash_{\mathcal{L}}$ the consequence relation of $\mathcal{L}$ (if it is clear from the context, we simply write $\vdash$ for $\vdash_{\mathcal{L}}$).*

We say that a hidden k-logic over a hidden signature Σ is *standard* if Σ is standard. A hidden k-logic with VIS = SORT will be called a *visible k-logic*, or simply a *k-logic*. By a *sentential logic* we mean a homogeneous (one-sorted) specifiable visible 1-logic. As usual, in this framework terms (k-terms) will be called formulas (*k-formulas*, resp.) and the set Te_Σ (Te^k_Σ) will be represented by $\mathrm{Fm}(\mathcal{L})$ ($\mathrm{Fm}^k(\mathcal{L})$, resp.).

Given any set of visible k-formulas Γ, we define the *set of all consequences of* Γ, in symbols $\mathrm{Cn}_{\mathcal{L}}(\Gamma)$, as the set of k-formulas $\mathrm{Cn}_{\mathcal{L}}(\Gamma) = \{\, \bar{\varphi} \in (\mathrm{Fm}^k(\mathcal{L}))_{\mathrm{VIS}} : \Gamma \vdash_{\mathcal{L}} \bar{\varphi}\}$. By a *theorem* of $\mathcal{L}$ we mean a (necessarily visible) k-formula $\bar{\varphi}$ such that $\vdash_{\mathcal{L}} \bar{\varphi}$, i.e., $\emptyset \vdash_{\mathcal{L}} \bar{\varphi}$. The set of all theorems is denoted by $\mathrm{Thm}(\mathcal{L})$. A set of visible k-formulas T closed under the consequence relation, i.e., $T \vdash_{\mathcal{L}} \bar{\varphi}$ implies $\bar{\varphi} \in T$, is called a *theory* of $\mathcal{L}$. The set of all theories is denoted by $\mathrm{Th}(\mathcal{L})$. It can be shown that the set of all theories $\mathrm{Th}(\mathcal{L})$ constitutes a closed set system, i.e., it is closed under arbitrary intersections. If Γ is a set of visible k-formulas, the set of all consequences of Γ, $\mathrm{Cn}_{\mathcal{L}}(\Gamma)$, is the smallest $\mathcal{L}$-theory that contains Γ. Moreover, $\Gamma \vdash_{\mathcal{L}} \bar{\varphi}$ if and only if for each $T \in \mathrm{Th}(\mathcal{L})$, $\Gamma \subseteq T$ implies $\bar{\varphi} \in T$. Hence, T is a theory of $\mathcal{L}$ if and only if $\mathrm{Cn}_{\mathcal{L}}(T) = T$.

By the substitution-invariance of $\mathcal{L}$ we have that $\mathrm{Th}(\mathcal{L})$ is closed under inverse substitutions, that is, for any $T \in \mathrm{Th}(\mathcal{L})$ and any substitution $\sigma : \mathrm{Te} \to \mathrm{Te}$, $\sigma^{-1}(T) \in \mathrm{Th}(\mathcal{L})$. To see this let $T \in \mathrm{Th}(\mathcal{L})$ and $\bar{\varphi} \in \mathrm{Fm}^k(\mathcal{L})$. Suppose that $\sigma^{-1}(T) \vdash_{\mathcal{L}} \bar{\varphi}$. Hence, by substitution invariance of $\mathcal{L}$, $\sigma(\sigma^{-1}(T)) \vdash_{\mathcal{L}} \sigma(\bar{\varphi})$. Since

[2] A similar notion of a general logic, also defined as a closure relation, is due to Meseguer [28]. Meseguer's system is called *entailment system* and combines a closure relation with the notion of institution (see also [16]).

$\sigma(\sigma^{-1}(T)) \subseteq T$, $T \vdash_{\mathcal{L}} \sigma(\bar{\varphi})$. So, $\sigma(\bar{\varphi}) \in T$ because $T \in \mathrm{Th}(\mathcal{L})$; and therefore, $\bar{\varphi} \in \sigma^{-1}(T)$. The shorthand way of expressing the invariance of $\mathrm{Th}(\mathcal{L})$ under σ^{-1} is by the inclusion, $\sigma^{-1}(\mathrm{Th}(\mathcal{L})) \subseteq \mathrm{Th}(\mathcal{L})$, where $\sigma^{-1}(\mathrm{Th}(\mathcal{L})) = \{\sigma^{-1}(T) : T \in \mathrm{Th}(\mathcal{L})\}$ and $\sigma^{-1}(T) = \langle \sigma^{-1}(T_V) : V \in \mathrm{VIS}\rangle$.

A (*visible*) *k-sequent* is a finite sequence $\langle \bar{\varphi}_0 : S_0, \ldots, \bar{\varphi}_{n-1} : S_{n-1}, \bar{\varphi}_n : S_n\rangle$ of (visible) k-formulas that we write in the following form:

$$\frac{\bar{\varphi}_0 : S_0, \ldots, \bar{\varphi}_{n-1} : S_{n-1}}{\bar{\varphi}_n : S_n}. \tag{1}$$

A visible k-formula $\bar{\psi}$ is *directly derivable* from a set Γ of visible k-formulas by a visible k-sequent such as (1), if there is a substitution $h : X \to \mathrm{Te}_\Sigma$ such that $h(\bar{\varphi}_n) = \bar{\psi}$ and $h(\bar{\varphi}_0), \ldots, h(\bar{\varphi}_{n-1}) \in \Gamma$. Given a set AX of visible k-formulas and a set IR of visible k-sequents, we say that $\bar{\psi}$ is *derivable* from Γ by the set AX and the set IR if there is a finite sequence of k-formulas, $\bar{\psi}_0, \ldots, \bar{\psi}_{n-1}$ such that $\bar{\psi}_{n-1} = \bar{\psi}$, and for each $i < n$ either (a) $\bar{\psi}_i \in \Gamma$, or (b) $\bar{\psi}_i$ is a substitution instance of a k-formula in AX or (c) $\bar{\psi}_i$ is directly derivable from $\{\bar{\psi}_j : j < i\}$ by one of the k-sequents in IR. We write $\Gamma \vdash_{\mathrm{AX,IR}} \bar{\psi}$ if $\bar{\psi}$ is derivable from Γ by AX and IR. It is straightforward to show, that a hidden k-logic $\mathcal{L}$ is specifiable iff there exists a (possibly) infinite set of axioms and rules of inference such that, for any visible k-term $\bar{\psi}$ and any set Γ of visible k-terms, $\Gamma \vdash_{\mathcal{L}} \bar{\psi}$ iff $\bar{\psi}$ is derivable from Γ by the given set of axioms and rules.

A visible k-sequent such as (1) is said to be a *valid rule* of $\mathcal{L}$ if $\{\bar{\varphi}_i : i < n\} \vdash_{\mathcal{L}} \bar{\varphi}$, or equivalently, every $T \in \mathrm{Th}(\mathcal{L})$ is closed under (1). We say that (1) is an *admissible rule* if for all $\sigma : X \to \mathrm{Te}$, $\sigma(\bar{\varphi}) \in \mathrm{Thm}(\mathcal{L})$ whenever $\sigma(\bar{\varphi}_i) \in \mathrm{Thm}(\mathcal{L})$, for every $i < n$. Admissible rules in sentential logics have been intensively studied by Rybakov and his collaborators (see [40]). There are admissible rules which are not valid rules. One typical example is the sequent

$$\frac{x.x \approx x}{y \approx z}$$

in the equational theory of semigroups: it is an admissible rule, since $x.x \approx x$ is not a theorem of the theory of semigroups, but it is not a valid rule.

Considering the admissible rules of a logic we define another logic: the *admissible part* of $\mathcal{L}$. The admissible part of $\mathcal{L}$, in symbols $\mathcal{L}^{ad}$, is the logic defined, for every $\Gamma \subseteq \mathrm{Fm}^k(\mathcal{L})$ and every $\bar{\varphi} \in \mathrm{Fm}^k(\mathcal{L})$, in the following way:

$$\Gamma \vdash_{\mathcal{L}^{ad}} \bar{\varphi} \quad \text{if} \quad \frac{\Gamma}{\bar{\varphi}} \text{ is an admissible rule of } \mathcal{L}.$$

It can be proven that $\mathcal{L}^{ad}$ is a hidden logic; however, it may not be specifiable.

The semantics for hidden k-logics is given by considering the set of k-tuples F of a given k-data structure $\mathcal{A} = \langle \mathbf{A}, F \rangle$ as the "truth values" of $\mathcal{A}$. A visible k-formula $\bar{\varphi}{:}V$ is said to be a *semantic consequence* of a set of visible k-formulas Γ in $\mathcal{A}$, in symbols $\Gamma \models_{\mathcal{A}} \bar{\varphi}$, if, for every assignment $h : X \to A$, $h(\bar{\varphi}) \in F_V$ whenever $h(\bar{\psi}) \in F_W$ for every $\bar{\psi}{:}W \in \Gamma$. A visible k-formula $\bar{\varphi}$ is a *valid k-formula*, or simply a *validity* of $\mathcal{A}$, and, conversely, $\mathcal{A}$ is a *model* (or a *correct abstract machine*) of $\bar{\varphi}$, if $\models_{\mathcal{A}} \bar{\varphi}$. A k-sequent such as (1) is a *valid rule*, or simply a *validity* of $\mathcal{A}$, and, conversely, $\mathcal{A}$ is a *model* of the k-sequent, if $\{\bar{\varphi}_0, \ldots, \bar{\varphi}_{n-1}\} \models_{\mathcal{A}} \bar{\varphi}_n$. A visible k-formula $\bar{\varphi}$ is a *semantic consequence* of Γ for an arbitrary class K of k-data structures over Σ, in symbols $\Gamma \models_K \bar{\varphi}$, if $\Gamma \models_{\mathcal{A}} \bar{\varphi}$ for each $\mathcal{A} \in K$. Similarly, a k-formula or rule is a *validity* of K if it is a validity of each member of K. The following theorem states that for any class K of k-data structures $\models_K$ is always a hidden k-logic.

Theorem 7 ([29]) *Let K be a class of k-data structures, all of them over the same hidden signature Σ. Then $\models_K$ is a hidden k-logic.*

$\mathcal{A}$ is a *model* of a hidden k-logic $\mathcal{L}$ if every consequence of $\mathcal{L}$ is a semantic consequence of $\mathcal{A}$, i.e., $\Gamma \vdash_{\mathcal{L}} \bar{\varphi}$ implies $\Gamma \models_{\mathcal{A}} \bar{\varphi}$. The class of all models of $\mathcal{L}$ is denoted by $\mathrm{Mod}(\mathcal{L})$. If $\mathcal{L}$ is a specifiable hidden k-logic, presented by a set of axioms and rules of inference, then $\mathcal{A}$ is a model of $\mathcal{L}$ if and only if every axiom and every rule of inference of $\mathcal{L}$ is a validity of $\mathcal{A}$. The designated filter F of $\mathcal{A}$ is said to be an *$\mathcal{L}$-filter of* $\mathbf{A}$ if $\mathcal{A}$ is a model of $\mathcal{L}$.

The $\mathcal{L}$-filters of the term algebra Te_Σ are just the $\mathcal{L}$-theories. The set of all $\mathcal{L}$-filters of $\mathbf{A}$, denoted by $\mathrm{Fi}_{\mathcal{L}}(\mathbf{A})$, endowed with set-intersection and the join defined, for each $\mathcal{F} \subseteq \mathrm{Fi}_{\mathcal{L}}(\mathbf{A})$, by $\bigvee \mathcal{F} := \bigcap \{G \in \mathrm{Fi}_{\mathcal{L}}(\mathbf{A}) : \bigcup \mathcal{F} \subseteq G\}$, is a complete lattice. It can be shown that the inverse image, by a homomorphism, of an $\mathcal{L}$-filter is always an $\mathcal{L}$-filter. A k-data structure $\mathcal{A} = \langle \mathbf{A}, F \rangle$ is a *behavioral model of* $\mathcal{L}$ (or a *reduced model* in the AAL sense) if it is reduced and a model of $\mathcal{L}$. The class of all behavioral models is denoted by $\mathrm{Mod}^*(\mathcal{L})$. Bidoit et al. call this class of models, in the context of observational logics, *black box semantics* (see [4]). We say that a k-data structure $\mathcal{A} = \langle \mathbf{A}, F \rangle$ is *strictly minimal for* $\mathcal{L}$ if there is a strict surjective homomorphism $h : \langle \mathrm{Fm}(\mathcal{L}), \mathrm{Thm}(\mathcal{L}) \rangle \to \langle \mathbf{A}, F \rangle$, i.e., $h(\mathrm{Fm}(\mathcal{L})) = A$ and $h^{-1}(F) = \mathrm{Thm}(\mathcal{L})$.

The following completeness theorem holds for hidden k-logics[3].

[3] Strictly speaking, this completeness theorem only holds when the models of $\mathcal{L}$ are restricted to k-data structures with a nonempty domain of each sort, but we are assuming that our algebras have nonempty carrier sets for each sort, hence this condition holds.

Theorem 8 ([31]) *For any hidden k-logic $\mathcal{L}$, $\vdash_{\mathcal{L}} = \models_{\mathrm{Mod}(\mathcal{L})} = \models_{\mathrm{Mod}^*(\mathcal{L})}$. That is, for every set of k-formulas Γ and any k-formula $\bar{\varphi}$ the following conditions are equivalent,*

(i) $\Gamma \vdash_{\mathcal{L}} \bar{\varphi}$;
(ii) $\Gamma \models_{\mathrm{Mod}(\mathcal{L})} \bar{\varphi}$;
(iii) $\Gamma \models_{\mathrm{Mod}^*(\mathcal{L})} \bar{\varphi}$.

3.2 Examples

3.2.1 Hidden equational logic

As a consequence of the restriction to visible k-terms in our formalization of hidden k-logics, the non-visible part of our hidden equational logic is truly hidden. Indeed, no representation of the equality predicate between elements of the hidden domains even exists in the object language. When reasoning about hidden data in the object language, only visible properties expressible in the form of conditional equations are allowed. The reason behind this restriction was explained in the introduction. We also consider an equational logic enriched by hidden equality predicates in which some visible axioms may be replaced by hidden ones whose behavioral equivalence has been verified. Technically, this is accomplished by simply modifying the signature and making all sorts visible (in [31], the authors studied the consequences of applying AAL to hidden equational logic). In this approach hidden equational logic is a special class of 2-logics in which a 2-formula $\langle s, t\rangle$ is intended to represent an equation, which we denote by $s \approx t$; similarly, sequents are intended to represent conditional equations.

Definition 9 (Free hidden equational logic) *Let Σ be a hidden signature and* VIS *its set of visible sorts. The* free hidden equational logic *over Σ, in symbols* HEL_Σ*, is the specifiable hidden* 2*-logic presented by the following equations and conditional equations:*

for all $V, W \in$ VIS,

(i) $x{:}V \approx x{:}V$;
(ii) $x{:}V \approx y{:}V \to y{:}V \approx x{:}V$;
(iii) $x{:}V \approx y{:}V\,, y{:}V \approx z{:}V \to x{:}V \approx z{:}V$;
(iv) $s{:}V \approx s'{:}V \to t(x/s){:}W \approx t(x/s'){:}W$ *for every* $t \in \mathrm{Te}_W$, $s, s' \in \mathrm{Te}_V$ *and every* $x \in X_V$.

An *applied hidden equational logic over* Σ (or simply a HEL_Σ) is any hidden 2-logic $\mathcal{L}$ over Σ that satisfies all axioms and rules of inference of the free HEL_Σ (the subscript Σ may be omitted if it is clear from the context). The

expression "hidden equational logic" comes from the fact that the equality predicate is restricted so as to be applied only to visible data elements; the other equality predicates are "hidden". In hidden equational logics a 2-data structure, sometimes called an abstract machine in the context of computer science, is a pair $\mathcal{A} = \langle \mathbf{A}, F \rangle$, where $\mathbf{A}$ is a sorted algebra over Σ and F is a binary relation on A_{VIS}. The set F may be seen as a possible interpretation of the equality in $\mathbf{A}$. If $\mathbf{A}$ is a Σ-algebra, then we can define a $\mathrm{DER}(\Sigma)$-algebra $\mathbf{A}'$ by interpreting each operation symbol t in $\mathrm{DER}(\Sigma)$ as $t^{\mathbf{A}}$. Clearly, a relation $\theta \subseteq A^2$ is a congruence on $\mathbf{A}$ if and only if it is a congruence on $\mathbf{A}'$. Let us define $\mathrm{DER}(\Sigma)^{\mathrm{VIS}}$ as the subsignature of $\mathrm{DER}(\Sigma)$ containing only the operation symbols with visible range (i.e., all the attributes). A VIS-sorted set $F \subseteq A^2_{\mathrm{VIS}}$ is a VIS-*congruence* if $F \cup id_{A_{\mathrm{HID}}}$ is a congruence on $\mathbf{A}' \restriction_{\mathrm{DER}(\Sigma)^{\mathrm{VIS}}}$. It can be proved that a data structure $\mathcal{A} = \langle \mathbf{A}, F \rangle$ is a model of the free HEL_Σ $\mathcal{L}$ if and only if its filter F is a VIS-congruence. The theories of the free HEL_Σ are the VIS-congruences on the term algebra.

In view of the completeness theorem, the extralogical axioms and inference rules correspond to identities and conditional identities of the class of models of $\mathcal{L}$, respectively (see the remarks at the beginning of this subsection). In particular, the visible (unrestricted) conditional equation

$$t_0(\bar{x}) \approx s_0(\bar{x}), \ldots, t_{n-1}(\bar{x}) \approx s_{n-1}(\bar{x}) \rightarrow t_n(\bar{x}) \approx s_n(\bar{x}) \tag{2}$$

is a valid rule of a model $\mathcal{A} = \langle \mathbf{A}, F \rangle$ of the free HEL_Σ if, for every assignment $\bar{a}$ of elements of A to $\bar{x}$ (of the appropriate sorts),

$$t_n^{\mathbf{A}}(\bar{a}) \equiv s_n^{\mathbf{A}}(\bar{a})\ (F) \quad \text{if} \quad t_0^{\mathbf{A}}(\bar{a}) \equiv s_0^{\mathbf{A}}(\bar{a})\ (F), \ldots, t_{n-1}^{\mathbf{A}}(\bar{a}) \equiv s_{n-1}^{\mathbf{A}}(\bar{a})\ (F).$$

A theory of $\mathcal{L}$ is also called an $\mathcal{L}$-*congruence* on the term algebra. For any set E of equations, the theory of $\mathcal{L}$ generated by E, $\mathrm{Cn}_{\mathcal{L}}(E)$, is the smallest $\mathcal{L}$-congruence that contains the pair $\langle t, t' \rangle$ for each equation $t \approx t'$ in E. The conditional equation (2) is a *quasi-identity* of $\mathbf{A}$ if it is a valid rule of $\langle \mathbf{A}, F \rangle$, where $F = id_{A_{\mathrm{VIS}}}$. Models of the free HEL_Σ of the form $\langle \mathbf{A}, id_{A_{\mathrm{VIS}}} \rangle$ are called *equality models*. The class of all equality models of a HEL_Σ $\mathcal{L}$ is denoted by $\mathrm{Mod}^{=}(\mathcal{L})$. Since every equality model is uniquely determined by its algebraic reduct, we shall not be concerned with distinguishing them in the sequel. Thus, for every HEL_Σ $\mathcal{L}$ we identify $\mathrm{Mod}^{=}(\mathcal{L})$ with $\{\, \mathbf{A} : \langle \mathbf{A}, id_{A_{\mathrm{VIS}}} \rangle \in \mathrm{Mod}^{=}(\mathcal{L}) \,\}$.

Example 10 (Flags - revisited) The hidden equational logic of Flags, denoted by $\mathcal{L}_{flags}$, is the hidden equational logic with the hidden sorted signature Σ_{flags} whose axioms are the axioms of Boolean algebra plus the extralogical ones given in Fig. 2. $\diamondsuit$

Extralogical axioms:

$up?(up(F)) \approx true;$
$up?(dn(F)) \approx false;$
$up?(rev(F)) \approx \neg(up?(F))$

Fig. 2. Flags logic.

Axioms:
$x \preceq x;$
Inference rules:

$$\frac{x \preceq y\,, y \preceq z}{x \preceq z};$$

$$\frac{x_0 \preceq y_0, \ldots, x_{n-1} \preceq y_{n-1}}{O(x_0, \ldots, x_{n-1}) \preceq O(y_0, \ldots, y_{n-1})}\,,$$

for any operation symbol O.

Fig. 3. Free inequational logic.

3.2.2 Other hidden k-logics

Example 11 (Free inequational logic) Let Σ be any one-sorted signature. The *free inequational logic* is the one-sorted 2-logic over Σ defined by the axioms and inference rules in Fig. 3. As in the equational case, we use a special symbol to denote the 2-formula $\langle \varphi, \psi \rangle$; namely we write $\varphi \preceq \psi$ for $\langle \varphi, \psi \rangle$. This logic is relevant in the context of ordered universal algebra (see [42]) and abstract algebra. We can generalize the inequational logic to the sorted case and, more generally, to the hidden sorted case in the same way we generalized the equational logic to the hidden equational logic. $\diamondsuit$

Example 12 (Stacks of natural numbers with Booleans) The signature is obtained from the signature of stacks of natural numbers (Fig. 6) by adjoining a new sort *bool*, for the Boolean operation symbols, and one new attribute $eq : nat, nat \to bool$, the equality test for natural numbers. The sort *bool* is the only visible sort. The axioms and inference rules are obtained, roughly speaking, by applying *eq* to each of the axioms and inference rules of the specification of stacks (see Fig. 4). The operation symbol *eq* is called an *equational test function* and the models are called **generalized equality test models**. These models have been studied in [34].

$\diamondsuit$

Axioms:

$eq(x, x)$

$eq(top(pop^n(empty)), zero)$, for all n;

$eq(top(push(x, y)), x)$;

$eq(top(pop^{n+1}(push(x, y))), top(pop^n(y)))$, for all n;

Inference rules:

$$\frac{eq(x, y)}{eq(y, x)} \qquad \frac{eq(x, y)}{eq(s(x), s(y))}$$

$$\frac{eq(x, y), eq(y, z)}{eq(x, z)} \qquad \frac{eq(s(x), s(y))}{eq(x, y)}$$

Fig. 4. Stacks of natural numbers with booleans.

4 Axiomatization of behavioral equivalence

Intuitively, two hidden data elements of the same type are *behaviorally equivalent* if any procedure whose parameter is of this type returns the same visible result when executed with either of the two objects as input. The notion arises from the alternative view of a data structure as a transition system in which the hidden data elements represent states of the system and the operations (called *methods*) that return hidden, as opposed to visible, elements, induce transitions between states. Moreover, it generalizes the equivalence of states in automata theory (two states S_1, S_2 are said to be equivalent if for any input, we go from the state S_1 to a final state if and only if the same happens with the state S_2). Behavioral equivalence has proven to be a useful device to import the techniques and intuitions of transition systems into the algebraic paradigm. The behavioral consequence relation is used to reason effectively about behavioral equivalence. It can be seen as a 2-logic that is not in general specifiable. The basis of the proof theory of behavioral consequence has been coinduction, in some form, in combination with ordinary equational deduction (see [19]). In the following definition, given by Reichel in [37], the intuitive notion of behavioral equivalence is formalized. First, we need to introduce the notion of a context. A *k-context* over Σ is a k-term $\bar{\varphi}(z{:}S, x_0{:}S_0, \ldots, x_{m-1}{:}S_{m-1})$, with a distinguished variable z of sort S and parametric variables $x_0, \ldots, x_{m-1}$. A *visible k-context* is a k-context of a visible sort. The set of all k-contexts over Σ with distinguished variable z of sort S is denoted by $C^k_\Sigma[z{:}S]$. We call the (visible) 1-contexts simply (*visible*) *contexts*. We denote the set of all contexts over Σ by $C_\Sigma[z{:}S]$.

Definition 13 *Let* $\mathbf{A}$ *be a* Σ*-algebra. Two elements* $a, a' \in A_S$ *are said to be* behaviorally equivalent in $\mathbf{A}$*, in symbols* $a \equiv^{\text{beh}}_{\mathbf{A}} a'$*, if, for every visible context* $\varphi(z{:}S, u_0{:}S_0, \ldots, u_{m-1}{:}S_{m-1}) \in C_\Sigma[z{:}S]$ *and for all* $b_0 \in A_{S_0}, \ldots, b_{m-1} \in A_{S_{m-1}}$, $\varphi^{\mathbf{A}}(a, b_0, \ldots, b_{m-1}) = \varphi^{\mathbf{A}}(a', b_0, \ldots, b_{m-1})$.

We generalize the behavioral equivalence relation to k-data structures in the following way:

Definition 14 *Let $\mathcal{A} = \langle \mathbf{A}, F \rangle$ be a k-data structure over a hidden signature Σ. Then, a, $a' \in A_S$ are said to be* behaviorally equivalent *in $\mathcal{A}$, in symbols $a \equiv^{\mathrm{beh}}_{\mathcal{A}} a'$, if, for every visible k-context $\bar{\varphi}(z{:}S,\ x_0{:}S_0, \ldots, x_{m-1}{:}S_{m-1}) \in C^k_\Sigma[z{:}S]_V$ and all $b_0 \in A_{S_0}, \ldots, b_{m-1} \in A_{S_{m-1}}$,*

$$\bar{\varphi}^{\mathbf{A}}(a, b_0, \ldots, b_{m-1}) \in F_V \quad \textit{iff} \quad \bar{\varphi}^{\mathbf{A}}(a', b_0, \ldots, b_{m-1}) \in F_V.$$

It is straightforward to show that $\equiv^{\mathrm{beh}}_{\mathcal{A}}$ is an equivalence relation on A. Moreover, $\equiv^{\mathrm{beh}}_{\mathcal{A}}$ is a congruence on $\mathbf{A}$, in fact it coincides with the Leibniz congruence on $\mathbf{A}$ over F, as the following theorem states. Consequently, it gives an alternative characterization of the Leibniz congruence. This result is well known for the one-sorted case (see for example [6]).

Theorem 15 ([31]) *Let Σ be a hidden signature and let $\mathcal{A} = \langle \mathbf{A}, F \rangle$ be a k-data structure over Σ. Then, $\equiv^{\mathrm{beh}}_{\mathcal{A}} = \Omega(F)$.*

In the hidden equational case we have:

Theorem 16 ([31]) *Let $\mathcal{A} = \langle \mathbf{A}, F \rangle$ be a model of the free HEL_Σ. Then, $a \equiv^{\mathrm{beh}}_{\mathcal{A}} a'$ $(\Omega(F)_S)$ iff, for every visible context $\varphi(z{:}S, u_0{:}S_0, \ldots, u_{m-1}{:}S_{m-1}) \in C_\Sigma[z{:}S]_V$ and all $b_0 \in A_{S_0}, \ldots, b_{m-1} \in A_{S_{m-1}}$,*

$$\varphi^{\mathbf{A}}(a, b_0, \ldots, b_{m-1}) \equiv \varphi^{\mathbf{A}}(a', b_0, \ldots, b_{m-1})\ (F_V).$$

4.1 Protoalgebraic logics

We associate to each sorted algebra $\mathbf{A}$ an operator, called the *Leibniz operator*, from the set of filters on $\mathbf{A}$ to the set of congruences on $\mathbf{A}$. The Leibniz operator maps each filter F into the Leibniz congruence on $\mathbf{A}$ over F . Some classes of hidden k-logics are defined in terms of properties of this operator. The protoalgebraic hidden k-logics are those for which the Leibniz operator on the term algebra, when restricted to the set of theories, is monotonic. The weakly algebraizable hidden k-logics are those for which the Leibniz operator on the term algebra is both monotonic and injective, when restricted to the set of theories. It seems that the protoalgebraic logics form the widest class for which it is possible to obtain interesting algebraic properties.

For each sorted algebra $\mathbf{A}$ and each k we define the *Leibniz k-operator*:

$$\begin{array}{rcl} \Omega^k_{\mathbf{A}} : \mathcal{P}(A^k_{\mathrm{VIS}}) & \to & \mathrm{Con}(\mathbf{A}) \\ F & \mapsto & \Omega_{\mathbf{A}}(F), \end{array}$$

where $\Omega_{\mathbf{A}}(F)$ is the Leibniz congruence on $\mathbf{A}$ over F. Explicit reference to the

dimension k is usually omitted, and if the algebra $\mathbf{A}$ is clear from the context we simply denote the Leibniz operator by Ω. We say that $\Omega_{\mathbf{A}}$ is *injective* if it is an injective mapping, and $\Omega_{\mathbf{A}}$ is said to be *monotonic* if $\forall\ F, G \in \mathcal{P}(A^k_{\mathrm{VIS}})$, $F \subseteq G$ implies $\Omega(F) \subseteq \Omega(G)$. If $\mathcal{F} \subseteq \mathcal{P}(A^k_{\mathrm{VIS}})$, then $\Omega(\mathcal{F})$ denotes the set $\{\Omega(F) : F \in \mathcal{F}\}$. Let $\mathcal{L}$ be a hidden k-logic. We call the pairs, i.e. the equations, in $\Omega(\mathrm{Thm}(\mathcal{L}))$ *behavioral theorems* and the pairs (equations) in $\Omega(T)$ *behavioral consequences of* T.

As a consequence of Lemma 5, we have that $\Omega(\mathrm{Th}(\mathcal{L}))$ is closed under inverse surjective substitutions.

Lemma 17 ([29]) *Let $\mathcal{L}$ be a hidden k-logic. Then, $\Omega(\mathrm{Th}(\mathcal{L}))$ is closed under inverse surjective substitutions.*

Those hidden k-logics with the property that their Leibniz operator on the term algebra is monotonic when restricted to $\mathcal{L}$-theories are called protoalgebraic. They constitute what seems to be the widest class of hidden logics for which a reasonable algebraic theory can be developed. In the context of AAL, protoalgebraic logics were introduced by Blok and Pigozzi in [5].

Definition 18 (Protoalgebraic Logic) *Let $\mathcal{L}$ be a hidden k-logic. We say that $\mathcal{L}$ is a* protoalgebraic hidden k-logic *if Ω is monotonic when restricted to* $\mathrm{Th}(\mathcal{L})$*, i.e., $\forall\ T, G \in \mathrm{Th}(\mathcal{L}), T \subseteq G \Rightarrow \Omega(T) \subseteq \Omega(G)$.*

There are many examples of protoalgebraic logics. The well known propositional and intuitionistic calculus are protoalgebraic. From Theorem 16, it can be shown that every HEL is protoalgebraic (see [29]). Non protoalgebraic sentential logics have been studied individually, since as far as we know there is no subclass of non protoalgebraic logics for which one can develop an interesting algebraic theory. As examples of non protoalgebraic logics we have the inf-sup fragment of the classic propositional calculus, Belnap's Logic and the $\{\vee, \wedge, \neg, \top, \bot\}$-fragment of the intuitionistic propositional calculus (for references and more examples see [14]).

Let $\mathcal{L}$ be a protoalgebraic hidden k-logic. We say that $\mathcal{L}$ is *behaviorally specifiable* if there is a specifiable hidden 2-logic $\mathcal{L}'$, in the same language, such that $\Omega(\mathrm{Th}(\mathcal{L})) = \mathrm{Th}(\mathcal{L}')$. It is shown in [29], Theorem 2.3.2, that for a protoalgebraic hidden k-logic to be behaviorally specifiable it is enough that $\Omega(\mathrm{Th}(\mathcal{L}))$ be closed under intersections, under inverse surjective substitutions and also closed under unions of directed sets.

Next we give an alternative characterization for a hidden k-logic to be protoalgebraic in terms of the Leibniz operator. Let $\mathcal{F}$ be a set of theories. It is straightforward to see that $\bigcap\{\Omega(F) : F \in \mathcal{F}\}$ is compatible with $\bigcap\mathcal{F}$. Hence, $\bigcap\{\Omega(F) : F \in \mathcal{F}\} \subseteq \Omega(\bigcap\mathcal{F})$. The opposite inclusion holds only in the special case of protoalgebraic logics.

Theorem 19 ([29]) *Let $\mathcal{L}$ be a hidden k-logic. Then $\mathcal{L}$ is protoalgebraic if and only if for all $\mathcal{F} \subseteq \mathrm{Th}(\mathcal{L})$, $\Omega(\bigcap \mathcal{F}) = \bigcap \{\Omega(T) : T \in \mathcal{F}\}$.*

Corollary 20 ([29]) *If $\mathcal{L}$ is protoalgebraic then $\Omega(\mathrm{Th}(\mathcal{L}))$ is closed under (arbitrary) intersections.*

The converse of this corollary holds in the case where $\mathcal{L}$-theories are definable by a set of equations in the sense of the following definition. This notion of equational definability generalizes the concept of explicit definability of the truth predicate introduced by Czelakowski. Moreover, in 1-deductive systems, if the data structures in the class K are all reduced, the two notions coincide.

Definition 21 *Let K be a class of k-data structures over a hidden signature Σ. We say that the filters of the k-data structures in K are* equationally definable by a complex sorted set of equations $E = \langle E_V(\bar{x}{:}V) : V \in \mathrm{VIS}\rangle$, *where $E_V = \langle E_{V,S}(\bar{x}{:}V) : S \in \mathrm{SORT}\rangle$ with $E_{V,S}$ being a set of equations of sort S for each $S \in \mathrm{SORT}$ (i.e., $E_{V,S}(\bar{x}{:}V) = \{\delta_i(\bar{x}{:}V) \approx \varepsilon_i(\bar{x}{:}V) : i \in I\}$), if for each data structure $\mathcal{A} = \langle \mathbf{A}, F\rangle \in K$ we have:*

$$F_V = \Big\{\bar{a} \in A_V^k \,:\, \forall S \in \mathrm{SORT}, \forall \langle \delta, \varepsilon\rangle \in E_{V,S}(\bar{x}),\ \delta^{\mathbf{A}}(\bar{a}) \equiv \varepsilon^{\mathbf{A}}(\bar{a})\ (\Omega(F)_S)\Big\}.$$

An immediate property of such a class K is the fact that the Leibniz operator restricted to the $\mathcal{L}$-filters of any data structure in K is injective.

Proposition 22 ([29]) *Let $\mathcal{L}$ be a hidden k-logic. If the class of all models of $\mathcal{L}$ has its filters equationally definable by a sorted set of equations, say $E = \langle E_V(\bar{x}{:}V) : V \in \mathrm{VIS}\rangle$, then, for any sorted algebra $\mathbf{A}$, $\Omega_{\mathbf{A}}$ is injective when restricted to the $\mathcal{L}$-filters of $\mathbf{A}$.*

Theorem 23 ([29]) *Let $\mathcal{L}$ be a hidden k-logic. Assume the class of all models of $\mathcal{L}$, of the form $\langle \mathrm{Fm}(\mathcal{L}), T\rangle$ with T being an $\mathcal{L}$-theory, has its filters equationally definable by a sorted set of equations $E = \langle E_V(\bar{x}{:}V) : V \in \mathrm{VIS}\rangle$. Then, $\mathcal{L}$ is protoalgebraic if and only if $\Omega(\mathrm{Th}(\mathcal{L}))$ is closed under finite intersections.*

4.1.1 Protoequivalence systems

We now consider another metamathematical characterization of protoalgebraicity. It is similar in form to the well-known "Mal'cev conditions" in universal algebra.

Let $\bar{x}$ and $\bar{y}$ be k-variables of the same sort S. By a *pre-protoequivalence system for $\mathcal{L}$* we mean a double VIS-sorted set $\Delta = \langle \Delta_V{:}V \in \mathrm{VIS}\rangle$ where each Δ_V is a globally finite VIS-sorted set

$$\Delta_V(\bar{x}{:}V, \bar{y}{:}V, \widehat{z}{:}\widehat{Q}) = \langle \Delta_{V,R}(\bar{x}{:}V, \bar{y}{:}V, \widehat{z}{:}\widehat{Q}){:}R : R \in \mathrm{VIS}\rangle,$$

where $\Delta_{V,R}(\bar{x}{:}V, \bar{y}{:}V, \widehat{z}{:}\widehat{Q}){:}R$ is a set of k-formulas of visible sort R and whose

variables are the two fixed k-variables $\bar{x}$ and $\bar{y}$, both of sort V, and a finite list $\widehat{z} = \langle z_0{:}Q_0, \ldots, z_{m-1}{:}Q_{m-1}\rangle$ of auxiliary variables of type different from V with at most one variable of each sort.

Definition 24 *Let $\mathcal{L}$ be a hidden k-logic. A pre-protoequivalence system Δ is said to be a* protoequivalence system for $\mathcal{L}$ *if the following consequences hold in $\mathcal{L}$ for each visible sort V.*

(i) $\vdash_{\mathcal{L}} \bar{\delta}(\bar{x}{:}V, \bar{x}{:}V, \widehat{z}{:}\widehat{Q}){:}R$, *for each* $R \in \mathrm{VIS}$ *and each* $\delta(\bar{x}{:}V, \bar{y}{:}V, \widehat{z}{:}\widehat{Q}) \in \Delta_{V,R}(\bar{x}{:}V, \bar{y}{:}V, \widehat{z}{:}\widehat{Q})$;

(ii) $\Delta_V(\bar{x}{:}V, \bar{y}{:}V, \widehat{z}{:}\widehat{Q}),\ \bar{x}{:}V \vdash_{\mathcal{L}} \bar{y}{:}V$. (V-detachment)

A protoequivalence system is said to be *standard* if the sequence of auxiliary variables $\widehat{z}$ is empty. We write the first condition in the above definition in the following abbreviated way: $\vdash_{\mathcal{L}} \Delta_V(\bar{x}{:}V, \bar{x}{:}V, \widehat{z}{:}\widehat{Q})$ (called V-identity).

The following theorem shows that a specifiable and standard hidden k-logic is a protoalgebraic logic if and only if it has a standard protoequivalence system.

Theorem 25 *Let $\mathcal{L}$ be a specifiable hidden k-logic. Then $\mathcal{L}$ is protoalgebraic if and only if it has a protoequivalence system. Moreover, if $\mathcal{L}$ is standard then the protoequivalence system can be taken to be standard.*

PROOF. Assume $\mathcal{L}$ has a protoequivalence system Δ. Let $T, G \in \mathrm{Th}(\mathcal{L})$ such that $T \subseteq G$. It is enough to prove that $\Omega(T)$ is compatible with G. Suppose $\bar{\varphi}{:}V$ and $\bar{\psi}{:}V$ are k-formulas such that $\varphi_i \equiv \psi_i\ (\Omega(T)_V)$ for $i < k$ and $\bar{\varphi} \in G_V$. For each $R \in \mathrm{VIS}$ and each $\bar{\delta}(\bar{x}{:}V, \bar{y}{:}V, \widehat{z}{:}\widehat{Q}) \in \Delta_{V,R}(\bar{x}{:}V, \bar{y}{:}V, \widehat{z}{:}\widehat{Q})$ we have $\bar{\delta}(\bar{\varphi}{:}V, \bar{\psi}{:}V, \widehat{z}{:}\widehat{Q}) \equiv \bar{\delta}(\bar{\varphi}{:}V, \bar{\varphi}{:}V, \widehat{z}{:}\widehat{Q})\ (\Omega^k(T)_R)$.

Since $\bar{\delta}(\bar{\varphi}, \bar{\varphi}, \widehat{z}) \in T_R$, $\Delta_V(\bar{\varphi}, \bar{\psi}, \widehat{z}) \subseteq T \subseteq G$ by the fact that $\Omega(T)$ is compatible with T. Then, since $\bar{\varphi} \in G_V$ and $\Delta_V(\bar{\varphi}, \bar{\psi}, \widehat{z}),\ \bar{\varphi} \vdash_{\mathcal{L}} \bar{\psi}$, by V-detachment, we have $\bar{\psi} \in G_V$. So, $\Omega(T)$ is compatible with G, and hence $\mathcal{L}$ is protoalgebraic.

Assume now that $\mathcal{L}$ is protoalgebraic. For each sort V, let

$$\bar{x} = \langle x_0{:}V, \ldots, x_{k-1}{:}V\rangle \text{ and } \bar{y} = \langle y_0{:}V, \ldots, y_{k-1}{:}V\rangle$$

be fixed k-variables of sort V. Let Φ_V be the VIS-sorted set of formulas defined for each sort $R \in \mathrm{VIS}$ by $\Phi_{V,R} = \{\bar{\varphi} \in \mathrm{Fm}^k_R : \vdash_{\mathcal{L}} \sigma_{\bar{y}\to\bar{x}}(\bar{\varphi})\}$, where $\sigma_{\bar{y}\to\bar{x}}$ is the substitution that takes y_i to x_i ($\sigma(y_i) = x_i$), for each $i < k$, and leaves the remaining variables fixed. We have that Φ_V is a theory. In fact, let $\bar{\varphi} \in \mathrm{Fm}^k_R$ such that $\Phi_V \vdash_{\mathcal{L}} \bar{\varphi}$. Then by substitution invariance, $\sigma_{\bar{y}\to\bar{x}}(\Phi_V) \vdash_{\mathcal{L}} \sigma_{\bar{y}\to\bar{x}}(\bar{\varphi})$. By definition of Φ_V, $\vdash_{\mathcal{L}} \sigma_{\bar{y}\to\bar{x}}(\Phi_V)$. So, $\sigma_{\bar{y}\to\bar{x}}(\bar{\varphi})$ is a theorem and thus $\bar{\varphi}$ belongs to $\Phi_{V,R}$. We are now going to prove that

$$\bar{x} \equiv \bar{y}\ (\Omega(\Phi_V)^k_V). \tag{3}$$

In order to do this we have to show that for every $i < k$ and every visible k-formula $\bar{\varphi}(z{:}V, u_0{:}S_0, \ldots, u_{n-1}{:}S_{n-1}) : R$, with distinguished variable z and parametric variables $u_0, \ldots, u_{n-1}$, and for all parameters $\widehat{\vartheta} = \langle \vartheta_0, \ldots, \vartheta_{n-1} \rangle \in \mathrm{Te}_{S_0} \times \cdots \times \mathrm{Te}_{S_{n-1}}$ we have that

$$\bar{\varphi}(x_i, \vartheta_0, \ldots, \vartheta_{n-1}) \in \Phi_{V,R} \quad \text{iff} \quad \bar{\varphi}(y_i, \vartheta_0, \ldots, \vartheta_{n-1}) \in \Phi_{V,R}. \tag{4}$$

We first note that $\sigma_{y_i \to x_i}(\bar{\varphi}(x_i, \widehat{\vartheta})) = \sigma_{y_i \to x_i}(\bar{\varphi}(y_i, \widehat{\vartheta}))$. Hence, $\sigma_{\bar{y} \to \bar{x}}(\bar{\varphi}(x_i, \widehat{\vartheta})) = \sigma_{\bar{y} \to \bar{x}}(\bar{\varphi}(y_i, \widehat{\vartheta}))$. Then, $\bar{\varphi}(x_i, \widehat{\vartheta}) \in \Phi_{V,R}$ iff $\vdash_{\mathcal{L}} \sigma_{\bar{y} \to \bar{x}}(\bar{\varphi}(x_i, \widehat{\vartheta}))$ iff $\vdash_{\mathcal{L}} \sigma_{\bar{y} \to \bar{x}}(\bar{\varphi}(y_i, \widehat{\vartheta}))$ iff $\bar{\varphi}(y_i, \widehat{\vartheta}) \in \Phi_{V,R}$.

This shows (4). And hence (3) holds. Next we prove that

$$\Phi_V,\ \bar{x}{:}V \vdash_{\mathcal{L}} \bar{y}{:}V. \tag{5}$$

To see this, let T be the $\mathcal{L}$-theory generated by the sorted set of k-formulas obtained from Φ_V by adjoining $\bar{x}$ to $\Phi_{V,V}$. Since $\mathcal{L}$ is protoalgebraic and $\Phi_V \subseteq T$, $\Omega(\Phi_V)$ is compatible with T. But $\bar{x} \in T_V$ and $\bar{x} \equiv \bar{y}\ (\Omega^k(\Phi_V)_V)$ by (3). So $\bar{y} \in T_V$ by compatibility, and this is what we have in (5). Since $\mathcal{L}$ is finitary, there exists a globally finite subset Φ'_V of Φ_V such that

$$\Phi'_V,\ \bar{x}{:}V \vdash_{\mathcal{L}} \bar{y}{:}V. \tag{6}$$

Let τ be a substitution that maps every variable of sort V distinct from the variables $x_0, \ldots, x_{k-1}, y_0, \ldots, y_{k-1}$ to the variable x_0, and, for every sort S different from V, it maps each variable of sort S that occurs in Φ'_V to a fixed but arbitrarily chosen one of them (note that there is only a finite number of such sorts since Φ'_V is globally finite). Let $\widehat{z}$ be the list of these fixed variables. In case Σ is standard, i.e., if there is a ground term of every sort, then τ maps each variable of sort S different from V to a fixed ground term of sort S. Let $\Delta_V(\bar{x}{:}V, \bar{y}{:}V, \widehat{z}) := \tau(\Phi'_V)$, and $\Delta = \langle \Delta_V : V \in \mathrm{VIS} \rangle$. For every $\bar{\varphi} \in \Delta_{V,R}(\bar{x}{:}V, \bar{y}{:}V, \widehat{z})$, $\sigma_{\bar{y} \to \bar{x}}(\tau(\bar{\varphi})) = \tau(\sigma_{\bar{y} \to \bar{x}}(\bar{\varphi}))$ since τ leaves $\bar{x}$ and $\bar{y}$ fixed and maps no variable into y_i but y_i itself. By definition of Φ_V, $\sigma_{\bar{y} \to \bar{x}}(\bar{\varphi})$ is a theorem of $\mathcal{L}$. So $\tau(\sigma_{\bar{y} \to \bar{x}}(\bar{\varphi}))$ is also a theorem since the set of theorems is closed under substitutions. Thus, $\sigma_{\bar{y} \to \bar{x}}(\tau(\bar{\varphi}))$ is a theorem, which implies, again by definition of Φ_V , that $\tau(\bar{\varphi}) \in \Phi_{V,R}$. So $\Delta_V(\bar{x}{:}V, \bar{y}{:}V, \widehat{z}) = \tau(\Phi'_V) \subseteq \Phi_V$. This shows that the V-identity holds. From (6) we get $\tau(\Phi'_V),\ \bar{x} \vdash_{\mathcal{L}} \bar{y}$ since τ leaves $\bar{x}$ and $\bar{y}$ fixed. Equivalently, $\Delta_V(\bar{x}{:}V, \bar{y}{:}V, \widehat{z}),\ \bar{x} \vdash_{\mathcal{L}} \bar{y}$, that is, V-detachment also holds. $\square$

In the one-sorted case (1-deductive systems), when $\mathcal{L}$ is protoalgebraic, many properties of the operator Ω_{Te} when restricted to $\mathrm{Th}(\mathcal{L})$ still hold for $\Omega_{\mathbf{A}}$ when restricted to $\mathrm{Fi}_{\mathcal{L}}(\mathbf{A})$, with $\mathbf{A}$ being an appropriate one-sorted algebra

(e.g. monotonicity and injectivity). This phenomenon is called the *transfer principle* (see [13]). The following theorem shows that the monotonicity of the Leibniz operator transfers from $\mathrm{Th}(\mathcal{L})$ to $\mathrm{Fi}_{\mathcal{L}}(\mathbf{A})$ for any algebra $\mathbf{A}$. In the next section we also show that the definability of the Leibniz congruences on the term algebra over the theories, by equivalence systems, transfers to any $\mathcal{L}$-filter of any data structure $\mathcal{A} = \langle \mathbf{A}, F \rangle$ (see Theorem 28).

Theorem 26 ([29]) *Let $\mathcal{L}$ be a protoalgebraic hidden k-logic. Then, for any algebra $\mathbf{A}$ and any $F, G \in \mathrm{Fi}_{\mathcal{L}}(\mathbf{A})$, $F \subseteq G$ implies $\Omega(F) \subseteq \Omega(G)$.*

4.2 Equivalence systems

Equivalence systems in AAL generalize the notion of equivalence in CPC (the classical propositional calculus). Some k-deductive systems do not have an equivalence symbol. However, there may exist a set of formulas that plays the role of the equivalence symbol. The notion of equivalence system can also be formulated for hidden k-logics and the main results of AAL concerning equivalence systems still hold in this more general context as we will see below. Equivalence systems for hidden k-logics are very different, in form, from the protoequivalence systems since instead of being VIS-sorted sets of k-formulas with two distinguished, visible, k-variables; the equivalence systems are SORT-sorted sets of visible k-formulas with two distinguished ordinary variables and possibly some parameters. We define equivalence systems in a syntactic way and then we prove that, in fact, equivalence systems are exactly the special sorted sets that define the Leibniz congruence of any theory. In fact, we show that equivalence systems define the Leibniz congruence of each filter of any model (see Theorem 28). In Theorem 30, we show that, given a protoequivalence system, we can construct a parameterized equivalence system.

4.2.1 Parameterized equivalence systems

Let $\mathcal{L}$ be a hidden k-logic. A SORT-sorted system of VIS-sorted sets of k-formulas $E = \langle E_S : S \in \mathrm{SORT} \rangle$, with $E_S(x{:}S, y{:}S, \widehat{u}{:}\widehat{Q}) = \langle E_{S,V}(x{:}S, y{:}S, \widehat{u}{:}\widehat{Q}) : V \in \mathrm{VIS} \rangle$, where $E_{S,V}(x{:}S, y{:}S, \widehat{u}{:}\widehat{Q}) \subseteq \mathrm{Fm}^k_V(\mathcal{L})$ is called a *pre-equivalence system with parameters for $\mathcal{L}$*. Let $\Gamma(x, y, \widehat{u}{:}\widehat{Q})$ be a set of k-formulas with parameters. The expression $\widetilde{\forall}\, \widehat{\vartheta}\ \Gamma(x, y, \widehat{\vartheta}{:}\widehat{Q})$ will denote the set of all possible substitution instances of formulas in Γ obtained by substituting arbitrary formulas of the appropriate sort for the parameters $\widehat{u}$, i.e., $\widetilde{\forall}\, \widehat{\vartheta}\ \Gamma(x, y, \widehat{\vartheta}{:}\widehat{Q}) := \{\bar{\varphi}(x, y, \widehat{\vartheta}) : \bar{\varphi} \in \Gamma,\ \widehat{\vartheta} \in \mathrm{Te}_{\widehat{Q}}\}$. We extend this notation to the case of the interpretation of a set of k-formulas in a given k-data structure $\mathcal{A} = \langle \mathbf{A}, F \rangle$ by $\widetilde{\forall}\, \widehat{c}\ \Gamma^{\mathbf{A}}(x, y, \widehat{c}{:}\widehat{Q}) := \{\bar{\varphi}^{\mathbf{A}}(x, y, \widehat{c}) : \bar{\varphi} \in \Gamma,\ \widehat{c} \in A_{\widehat{Q}}\}$.

Definition 27 *Let $\mathcal{L}$ be a hidden k-logic. We say that a pre-equivalence sys-*

tem $E = \langle E_S(x{:}S, y{:}S, \widehat{u}{:}\widehat{Q}) : S \in \mathrm{SORT}\rangle$ *is a* parameterized equivalence system for $\mathcal{L}$ *if the following conditions hold*[4] *:*

(i) $\vdash_{\mathcal{L}} \widetilde{\forall}\, \widehat{\vartheta} E_S(x{:}S, x{:}S, \widehat{\vartheta}{:}\widehat{Q})$; (parameterized S-identity)
(ii) $\widetilde{\forall}\, \widehat{\vartheta}\; E_S(x{:}S, y{:}S, \widehat{\vartheta}{:}\widehat{Q}) \vdash_{\mathcal{L}} \widetilde{\forall}\, \widehat{\vartheta} E_S(y{:}S, x{:}S, \widehat{\vartheta}{:}\widehat{Q})$;
(iii) $\widetilde{\forall}\, \widehat{\vartheta}\; E_S(x{:}S, y{:}S, \widehat{\vartheta}{:}\widehat{Q}), \widetilde{\forall}\, \widehat{\vartheta} E_S(y{:}S, z{:}S, \widehat{\vartheta}{:}\widehat{Q}) \vdash_{\mathcal{L}} \widetilde{\forall}\, \widehat{\vartheta} E_S(x{:}S, z{:}S, \widehat{\vartheta}{:}\widehat{Q})$;
(iv) $\widetilde{\forall}\, \widehat{\vartheta}\; E_{S_0}(x_0{:}S_0, y_0{:}S_0, \widehat{\vartheta}{:}\widehat{Q}), \ldots, \widetilde{\forall}\, \widehat{\vartheta} E_{S_{n-1}}(x_{n-1}{:}S_{n-1}, y_{n-1}{:}S_{n-1}, \widehat{\vartheta}{:}\widehat{Q}) \vdash_{\mathcal{L}}$

$$\widetilde{\forall}\, \widehat{\vartheta} E_{S_n}(O(x_0, \ldots, x_{n-1}){:}S_n, O(y_0, \ldots, y_{n-1}){:}S_n, \widehat{\vartheta}{:}\widehat{Q}),$$

for each operation symbol O *of type* $S_0, \ldots, S_{n-1} \to S_n$*;*
(parameterized S-replacement)
(v) *for every* $V \in \mathrm{VIS}$, (parameterized V-detachment)

$$\widetilde{\forall}\, \widehat{\vartheta}\; E_V(x_0{:}V, y_0{:}V, \widehat{\vartheta}{:}\widehat{Q}), \ldots, \widetilde{\forall}\, \widehat{\vartheta} E_V(x_{k-1}{:}V, y_{k-1}{:}V, \widehat{\vartheta}{:}\widehat{Q}),\ \bar{x} \vdash_{\mathcal{L}} \bar{y}.$$

There are several hidden k-logics which admit equivalence systems, even without parameters. (At the end of this section, some examples will be presented.)

A parameterized equivalence system E such that for each sort S, E_S is globally finite, is called a *finite parameterized equivalence system.* We say that a hidden k-logic $\mathcal{L}$ is *parameterized equivalential* if it has a parameterized equivalence system; $\mathcal{L}$ is called *parameterized finitely equivalential* if it has a finite parameterized equivalence system. In the context of AAL, equivalential logics were first introduced by Prucnal and Wroński (see [36]) and later studied in detail by Czelakowski in [12]. In section 4.3, we discuss different kinds of equivalence systems and we relate the corresponding classes of logics with closure properties of the class of behavioral models. For parameterized equivalential logics, the Leibniz congruence on the underlying algebra over the designated filter can be characterized by using the parameterized equivalence system, in the following way:

Theorem 28 *Let* $\mathcal{L}$ *be a hidden* k*-logic and* $E = \langle E_S(x{:}S, y{:}S, \widehat{u}{:}\widehat{Q}) : S \in \mathrm{SORT}\rangle$ *a pre-equivalence system. Then the following conditions are equivalent:*

(i) *E is a parameterized equivalence system for $\mathcal{L}$;*
(ii) *For every* $T \in \mathrm{Th}(\mathcal{L})$, $\Omega(T)_S = \{\, \langle t, t'\rangle \in \mathrm{Te}^2_S : T \vdash_{\mathcal{L}} \widetilde{\forall}\, \widehat{\vartheta}\; E_S(t, t', \widehat{\vartheta})\}$;
(iii) *For every* $\mathcal{A} = \langle \mathbf{A}, F\rangle \in \mathrm{Mod}(\mathcal{L})$, $\Omega(F)_S = \{\, \langle a, b\rangle \in A^2_S : \widetilde{\forall}\, \widehat{c}\; E^{\mathbf{A}}_S(a, b, \widehat{c}) \subseteq F\}$.

[4] In case of HEL's the equivalence system may be defined only using the hidden part since the visible part can be taken always as $E_V = \{x{:}V \approx y{:}V\}$ for each visible sort V (see [29]).

PROOF. (i) $\Rightarrow$ (iii) Let θ be defined for each $S \in \mathrm{SORT}$ as the relation on A_S, $\theta_S = \{\langle a, b\rangle \in A_S^2 : \widetilde{\forall}\ \widehat{c}\ E_S^{\mathbf{A}}(a, b, \widehat{c}) \subseteq F\}$. From the definition of a parameterized equivalence system we can show that θ is a congruence compatible with F. In fact, the proof of reflexivity is straightforward. To prove symmetry, let $a, b \in A_S$. Suppose that

$$a \equiv b\ (\theta_S), \tag{7}$$

that is $E_S(a, b, \widehat{c}) \subseteq F$, for every $\widehat{c} \in A_{\widehat{Q}}$. Let $\widehat{c} \in A_{\widehat{Q}}$ and $h : X \to A$ be an assignment such that $h(x) = a$, $h(y) = b$ and $h(u_i) = c_i$. By definition of parameterized equivalence system, taking the special tuple of terms $\widehat{u}$ for $\widehat{\vartheta}$ we have, $\widetilde{\forall}\ \widehat{\vartheta}\ E_S(x, y, \widehat{\vartheta}) \vdash_{\mathcal{L}} \bar{\delta}(y, x, \widehat{u})$, for every $\bar{\delta} \in E_{S,V}(y, x, \widehat{u})$. By (7), $h(\bar{\delta}) \in F$, for every $\bar{\delta} \in \widetilde{\forall}\ \widehat{\vartheta}\ E_S(x, y, \widehat{\vartheta})$. Hence, by Theorem 8, $h(\bar{\delta}(y, x, \widehat{u})) = \bar{\delta}^{\mathbf{A}}(b, a, \widehat{c}) \in F_V$, for every $\bar{\delta} \in E_{S,V}(y, x, \widehat{u})$, i.e., $(b, a) \in \theta$. By a similar argument we can prove that θ is transitive and a congruence.

To prove that θ is compatible with F, let $\bar{a}, \bar{b} \in A_V^k$ such that $\bar{a} \equiv \bar{b}\ (\theta_V^k)$. Suppose that $\bar{a} \in F_V$. By definition of parameterized equivalence system, we have that $\bigcup_{i<k} \widetilde{\forall}\ \widehat{\vartheta}\ E_S(x_i, y_i, \widehat{\vartheta}), \bar{x} \vdash_{\mathcal{L}} \bar{y}$. Then by taking the assignment $h : X \to A$, such that $h(\bar{x}) = \bar{a}$ and $h(\bar{y}) = \bar{b}$, and applying again Theorem 8 and the hypothesis: $\bar{a} \equiv \bar{b}\ (\theta_V^k)$ and $\bar{a} \in F_V$, we get that $h(\bar{y}) = \bar{b} \in F_V$. To see that θ is the largest congruence compatible with F, let θ' be any congruence compatible with F. Assume that $a \equiv b\ (\theta'_S)$. Then for all $\bar{\delta}(x, y, \widehat{u}) \in E_{S,V}$ and all $\widehat{c} \in A_{\widehat{Q}}^k$, $\delta_i^{\mathbf{A}}(a, a, \widehat{c}) \equiv \delta_i^{\mathbf{A}}(a, b, \widehat{c})\ (\theta'_S)$, $i < k$. Since $\bar{\delta}^{\mathbf{A}}(a, a, \widehat{c}) \in F_V$, by compatibility, we have that $\bar{\delta}^{\mathbf{A}}(a, b, \widehat{c}) \in F_V$. Hence, $a \equiv b\ (\theta_S)$. Therefore, θ is the largest congruence compatible with F, which shows that $\theta = \Omega(F)$.

(iii) $\Rightarrow$ (ii) It is straightforward. Take $\langle \mathbf{A}, F\rangle$ to be the k-data structure $\langle \mathrm{Fm}(\mathcal{L}), T\rangle$ and then apply (iii).

(ii) $\Rightarrow$ (i) Suppose that (ii) holds. The properties of $\Omega(T)$ as a congruence relation compatible with T translate directly into the properties that specify E as an equivalence system with parameters. For example, suppose that O is an operation symbol of type $S_0, \ldots, S_{n-1} \to S_n$. Let $T \in \mathrm{Th}(\mathcal{L})$ and for each $i < n$, let $\varphi_i, \psi_i \in \mathrm{Te}_{S_i}$ such that $\widetilde{\forall}\ \widehat{\vartheta}\ E_{S_i}(\varphi_i, \psi_i, \widehat{\vartheta}) \subseteq T$. Then $\varphi_i \equiv \psi_i\ (\Omega(T)_{S_i})$, $i < n$. Thus $O(\varphi_0, \ldots, \varphi_{n-1}) \equiv O(\psi_0, \ldots, \psi_{n-1})\ (\Omega(T)_{S_n})$ and thus $\widetilde{\forall}\ \widehat{\vartheta}\ E_{S_n}(O(\varphi_0, \ldots, \varphi_{n-1}), O(\psi_0, \ldots, \psi_{n-1}), \widehat{\vartheta}) \subseteq T$. Since this is true for all $T \in \mathrm{Th}(\mathcal{L})$, the parameterized S-replacement holds. $\square$

Corollary 29 ([29]) *Let $\mathcal{L}$ be a parameterized equivalential hidden k-logic and let $E = \langle E_S(x{:}S, y{:}S, \widehat{u}{:}\widehat{Q}) : S \in \mathrm{SORT}\rangle$ be a parameterized equivalence system for $\mathcal{L}$. Let $E' = \langle E'_S(x{:}S, y{:}S, \widehat{v}{:}\widehat{P}) : S \in \mathrm{SORT}\rangle$ be a pre-equivalence system with parameters for $\mathcal{L}$. Then, the following are equivalent:*

(i) *E' is a parameterized equivalence system for $\mathcal{L}$;*

(ii) *For every sort S and every $\varphi, \psi \in \mathrm{Te}_S$,*

$$\widetilde{\forall}\, \widehat{\vartheta}\; E_S(\varphi{:}S, \psi{:}S, \widehat{\vartheta}{:}\widehat{Q}) \;{}_{\mathcal{L}}\!\dashv\!\vdash_{\mathcal{L}}\; \widetilde{\forall}\, \widehat{\xi}\; E'_S(\varphi{:}S, \psi{:}S, \widehat{\xi}{:}\widehat{P}),$$

where $\Gamma \;{}_{\mathcal{L}}\!\dashv\!\vdash_{\mathcal{L}}\; \Upsilon$ means that, $\Gamma \vdash_{\mathcal{L}} \bar{\delta}$ for every $\bar{\delta} \in \Upsilon$, and $\Upsilon \vdash_{\mathcal{L}} \bar{\gamma}$ for every $\bar{\gamma} \in \Gamma$.

Specifiable protoalgebraic logics may be characterized in terms of equivalence systems.

Theorem 30 *Let $\mathcal{L}$ be a specifiable hidden k-logic. Then $\mathcal{L}$ is protoalgebraic if and only if $\mathcal{L}$ has a parameterized equivalence system.*

PROOF. Suppose that E is an equivalence system with parameters for $\mathcal{L}$. By Theorem 28, E defines the Leibniz congruence over each theory of $\mathcal{L}$. Let $T, G \in \mathrm{Th}(\mathcal{L})$ such that $T \subseteq G$. Let $\varphi, \psi \in \mathrm{Te}_S$. Suppose that $\varphi \equiv \psi\ (\Omega(T)_S)$. Then $\widetilde{\forall}\, \widehat{\vartheta}\; E_S(\varphi, \psi, \widehat{\vartheta}) \subseteq T \subseteq G$. Hence, $\varphi \equiv \psi\ (\Omega(G)_S)$. So $\Omega(T) \subseteq \Omega(G)$, and $\mathcal{L}$ is protoalgebraic.

Assume now that $\mathcal{L}$ is protoalgebraic. Let $\Delta = \langle \Delta_R(\bar{x}{:}R, \bar{y}{:}R, \widehat{z}{:}\widehat{P}) : R \in \mathrm{VIS}\rangle$ be a protoequivalence system for $\mathcal{L}$ (we know that such a system exists from Theorem 25). For each pair of sorts S, V with $V \in \mathrm{VIS}$, take $E_{S,V}(x{:}S, y{:}S, \widehat{u}{:}\widehat{Q}, \widehat{z}{:}\widehat{P})$ to be the union of all sets of k-formulas of the form

$$\Delta_{R,V}(\bar{\nu}(x{:}S, \widehat{u}{:}\widehat{Q}){:}R, \bar{\nu}(y{:}S, \widehat{u}{:}\widehat{Q}){:}R, \widehat{z}{:}\widehat{P}),$$

where R ranges over all visible sorts and $\bar{\nu}(x{:}S, \widehat{u}{:}\widehat{Q}){:}R$ ranges over all k-formulas of sort R whose distinguished variable is $x{:}S$. Take $E_S = \langle E_{S,V} : V \in \mathrm{VIS}\rangle$ and $E = \langle E_S : S \in \mathrm{SORT}\rangle$. We are going to prove that E defines the Leibniz congruences on the tem algebra over the theories of $\mathcal{L}$.

Let $T \in \mathrm{Th}(\mathcal{L})$ and let $\varphi, \psi \in \mathrm{Te}_S$. Assume that

$$\widetilde{\forall}\, \widehat{\vartheta}\, \widetilde{\forall}\, \widehat{\xi}\; E_S(\varphi, \psi, \widehat{\vartheta}, \widehat{\xi}) \subseteq T, \tag{8}$$

i.e., for every visible sort R, every $\bar{\nu}(x{:}S, \widehat{u}{:}\widehat{Q}) \in \mathrm{Te}^k_R$, and any choice of parameters $\widehat{\vartheta} \in \mathrm{Te}_{\widehat{Q}}$, and $\widehat{\xi} \in \mathrm{Te}_{\widehat{P}}$,

$$\Delta_R(\bar{\nu}(\varphi, \widehat{\vartheta}), \bar{\nu}(\psi, \widehat{\vartheta}), \widehat{\xi}{:}\widehat{P}) \subseteq T. \tag{9}$$

It follows that also

$$\Delta_R(\bar{\nu}(\psi, \widehat{\vartheta}), \bar{\nu}(\varphi, \widehat{\vartheta}), \widehat{\xi}{:}\widehat{P}) \subseteq T. \tag{10}$$

To show this, consider any $\bar{\nu}(x{:}S, \widehat{u}{:}\widehat{Q}){:}R \in \mathrm{Te}^k_R$ and any $\bar{\delta} \in \Delta_{R,V}$ and define

$$\tau(x{:}S, y{:}S, \widehat{u}{:}\widehat{Q}, \widehat{z}{:}\widehat{P}) := \bar{\delta}(\bar{\nu}(x, \widehat{u}), \bar{\nu}(y, \widehat{u}), \widehat{z}{:}\widehat{P}).$$

By considering x to be the distinguished variable of τ and $y, \widehat{u}, \widehat{z}$ as parametric variables, by (8) we have that

$$\widetilde{\forall}\, \widehat{\vartheta}\, \widetilde{\forall}\, \widehat{\xi}\, \widetilde{\forall}\widehat{\upsilon}\ \Delta_V(\tau(\varphi, \varphi, \widehat{\vartheta}, \widehat{\xi}), \tau(\psi, \varphi, \widehat{\vartheta}, \widehat{\xi}), \widehat{\upsilon}) \subseteq T.$$

On the other hand, it follows from V-identity that $\tau(\varphi, \varphi, \widehat{\vartheta}, \widehat{\xi}) \subseteq T_V$. However, by V-detachment we have

$$\tau(\varphi, \varphi, \widehat{\vartheta}, \widehat{\xi}),\ \Delta_V(\tau(\varphi, \varphi, \widehat{\vartheta}, \widehat{\xi})\tau(\psi, \varphi, \widehat{\vartheta}, \widehat{\xi}), \widehat{\upsilon}) \vdash_{\mathcal{L}} \tau(\psi, \varphi, \widehat{\vartheta}, \widehat{\xi}).$$

Hence, $\tau(\psi, \varphi, \widehat{\vartheta}, \widehat{\xi}) \in T_V$, which implies, by definition of τ, that

$$\bar{\delta}(\bar{\nu}(\psi, \widehat{\vartheta}), \bar{\nu}(\varphi, \widehat{\vartheta}), \widehat{\xi}) \in T_V, \text{ for every } \bar{\delta} \in \Delta_{R,V}, \text{ i.e, (10) holds.}$$

By R-detachment, we conclude from (9) and (10) that

$$\bar{\nu}(\varphi, \widehat{\vartheta}) \in T_R \quad \text{iff} \quad \bar{\nu}(\psi, \widehat{\vartheta}) \in T_R.$$

Therefore, by the characterization of $\Omega(T)$ (see Theorem 28), $\varphi \equiv \psi\ (\Omega(T)_S)$.

Conversely, if $\varphi \equiv \psi\ (\Omega(T)_S)$, then for every pair of visible sorts R, V, every $\bar{\delta}(\bar{x}{:}R, \bar{y}{:}R, \widehat{z}{:}\widehat{P}){:}V \in \Delta_{R,V}$, every $\bar{\nu}(x{:}S, \widehat{u}{:}\widehat{Q}){:}R \in \mathrm{Te}^k_R$, and any choice of parameters $\widehat{\vartheta}{:}\widehat{Q} \in \mathrm{Te}_{\widehat{Q}}$ and $\widehat{\xi}{:}\widehat{P} \in \mathrm{Te}_{\widehat{P}}$

$$\bar{\delta}(\bar{\nu}(\varphi, \widehat{\vartheta}), \bar{\nu}(\psi, \widehat{\vartheta}), \widehat{\xi}) \equiv \bar{\delta}(\bar{\nu}(\varphi, \widehat{\vartheta}), \bar{\nu}(\varphi, \widehat{\vartheta}), \widehat{\xi})\ (\Omega^k(T)_V).$$

By parametric V-identity, $\bar{\delta}(\bar{\nu}(\varphi, \widehat{\vartheta}), \bar{\nu}(\varphi, \widehat{\vartheta}), \widehat{\xi}) \in T_V$. So by compatibility of $\Omega(T)$ with T, $\bar{\delta}(\bar{\nu}(\varphi, \widehat{\vartheta}), \bar{\nu}(\psi, \widehat{\vartheta}), \widehat{\xi}) \in T_V$. Thus, for every $V \in \mathrm{VIS}$,

$$\Delta_{R,V}(\bar{\nu}(\varphi, \widehat{\vartheta}), \bar{\nu}(\psi, \widehat{\vartheta}), \widehat{\xi}) \subseteq T_V, \text{ i.e., } \Delta_R(\bar{\nu}(\varphi, \widehat{\vartheta}), \bar{\nu}(\psi, \widehat{\vartheta}), \widehat{\xi}) \subseteq T.$$

Since this inclusion holds for every visible sort R, every $\bar{\nu}(x{:}S, \widehat{u}{:}\widehat{Q}){:}R \in \mathrm{Te}^k_R$, and every choice of parameters $\widehat{\vartheta} \in \mathrm{Te}_{\widehat{Q}}$, we finally conclude that $\widetilde{\forall}\, \widehat{\vartheta}\, \widetilde{\forall}\, \widehat{\xi}\ E_S(\varphi, \psi, \widehat{\vartheta}, \widehat{\xi}) \subseteq T$. $\square$

4.2.2 Equivalence systems without parameters

When there are no parametric variables, the definition of an equivalence system takes the following simpler form. A pre-equivalence system $E = \langle E_S : S \in \mathrm{SORT}\rangle$ without parameters is said to be an *equivalence system (without parameters)* if the following conditions hold:

(i) $\vdash_{\mathcal{L}} E_S(x{:}S, x{:}S)$; ($S$-identity)

(ii) $E_S(x{:}S, y{:}S) \vdash_{\mathcal{L}} E_S(y{:}S, x{:}S)$;

(iii) $E_S(x{:}S, y{:}S),\ E_S(y{:}S, z{:}S) \vdash_{\mathcal{L}} E_S(x{:}S, z{:}S)$;

(iv) $E_{S_0}(x_0{:}S_0, y_0{:}S_0), \ldots, E_{S_{n-1}}(x_{n-1}{:}S_{n-1}, y_{n-1}{:}S_{n-1}) \vdash_{\mathcal{L}}$ (S-replacement)
$E_{S_n}(O(x_0, \ldots, x_{n-1}){:}S_n, O(y_0, \ldots, y_{n-1}){:}S_n)$,
for each operation symbol O of type $S_0, \ldots, S_{n-1} \to S_n$;

(v) for every $V \in \mathrm{VIS}$,

$$E_V(x_0{:}V, y_0{:}V), \ldots, E_V(x_{k-1}{:}V, y_{k-1}{:}V),\ \bar{x} \vdash_{\mathcal{L}} \bar{y}. \qquad (V\text{-detachment})$$

An equivalence system without parameters E such that, for each sort S, E_S is globally finite is called a *finite equivalence system.* We say that a hidden k-logic $\mathcal{L}$ is *equivalential* if it has an equivalence system without parameters; $\mathcal{L}$ is called *finitely equivalential* if it has a finite equivalence system. These definitions of equivalential and finitely equivalential logics can be regarded as their characterization by "Mal'cev conditions".

The next proposition shows that for finitely equivalential logics, each equivalence system without parameters contains a finite equivalence system.

Proposition 31 *Let $\mathcal{L}$ be a specifiable hidden k-logic. If $\mathcal{L}$ is a finitely equivalential logic then each equivalence system for $\mathcal{L}$ contains a finite equivalence system.*

PROOF. Let E be an equivalence system for $\mathcal{L}$ and E' be a finite equivalence system for $\mathcal{L}$ (we know that such E' exists since $\mathcal{L}$ is finitely equivalential). From Corollary 29, we have that for each sort S, E_S and E'_S are interderivable, that is, $E_S \ {}_{\mathcal{L}}\!\dashv\!\vdash_{\mathcal{L}} E'_S$. Since $\mathcal{L}$ is specifiable (i.e., finitary), for every $\delta \in E'_{S,V}$ there is a finite subset of E_S, say E^δ_S such that $E^\delta_S \vdash_{\mathcal{L}} \delta$. Then, taking $E^0_{S,V} := \bigcup\{E^\delta_S : \delta \in E'_{S,V}\}$, we have that $E^0_S \vdash_{\mathcal{L}} E'_S$. Moreover, since E' is a finite equivalence system, E'_S is globally finite and consequently E^0_S is also globally finite. Clearly, we have that $E'_S \vdash_{\mathcal{L}} E^\delta_S$. So, $E^0 = \langle E^0_S{:}S \in \mathrm{SORT}\rangle$ is a finite equivalence system contained in E. $\square$

A theory T of a hidden k-logic is a *Leibniz theory*, if for every $G \in \mathrm{Th}(\mathcal{L})$, $\Omega(T) = \Omega(G)$ implies that $T \subseteq G$, i.e., $T = \bigcap\{G \in \mathrm{Th}(\mathcal{L}) : \Omega(G) = \Omega(T)\}$. The Leibniz operator is said to *preserve the union of* (*upward*) *directed sets of $\mathcal{L}$-theories* if $\Omega(\bigcup X) = \bigcup \Omega(X)$, for every (upward) directed $X \subseteq \mathrm{Th}(\mathcal{L})$.

Theorem 32 *Let $\mathcal{L}$ be a hidden k-logic. If the Leibniz operator preserves unions of directed sets of $\mathcal{L}$-theories, then $\Omega(\mathrm{Th}(\mathcal{L}))$ is an algebraic closed set system over $\mathrm{Con}(\mathrm{Fm}(\mathcal{L}))$.*

PROOF. Assume Ω preserves unions of directed subsets of $\mathrm{Th}(\mathcal{L})$. Let $T, U \in Th(\mathcal{L})$ such that $T \subseteq U$. Then $\{T, U\}$ is directed, so $\Omega(T) \cup \Omega(U) = \Omega(T \cup U) = \Omega(U)$, i.e., $\Omega(T) \subseteq \Omega(U)$. Hence, $\mathcal{L}$ is protoalgebraic, and then by Corollary 20, $\Omega(\mathrm{Th}(\mathcal{L}))$ is closed under intersections. Let $Y \subseteq \Omega(\mathrm{Th}(\mathcal{L}))$ be directed. For each $\theta \in Y$, define M_θ to be the set $\bigcap\{T \in \mathrm{Th}(\mathcal{L}) : \Omega(T) = \theta\}$. Then $\{M_\theta : \theta \in Y\}$ is also directed, since the mapping $\theta \mapsto M_\theta$ is an order-isomorphism between $\Omega(\mathrm{Th}(\mathcal{L}))$ and the set of Leibniz theories of $\mathcal{L}$. Thus $\bigcup Y = \bigcup\{\theta : \theta \in Y\} = \bigcup\{\Omega(M_\theta) : \theta \in Y\} = \Omega(\bigcup\{M_\theta : \theta \in Y\})$. $\square$

In the next theorem we characterize finitely equivalential logics in terms of the Leibniz operator.

Theorem 33 *Let $\mathcal{L}$ be a protoalgebraic and specifiable hidden k-logic over a standard signature Σ. $\mathcal{L}$ is finitely equivalential if and only if for every upward directed set X of $\mathcal{L}$-theories $\Omega(\bigcup X) = \bigcup \Omega(X)$.*

PROOF. Assume that E is a finite equivalence system for $\mathcal{L}$, and let $X \subseteq \mathrm{Th}(\mathcal{L})$ be directed. Then for every sort S and all $\varphi, \psi \in \mathrm{Te}_S$, $\varphi \equiv \psi$ $(\Omega(\bigcup X)_S)$ iff $E_S(\varphi, \psi) \subseteq \bigcup X$ iff $E_S(\varphi, \psi) \subseteq T$ for some $T \in X$. Therefore, $\Omega(\bigcup X) = \bigcup \Omega(X)$.

Conversely, assume that $\Omega(\bigcup X) = \bigcup \Omega(X)$ for every directed $X \subseteq \mathrm{Th}(\mathcal{L})$. From Theorem 32, $\Omega(\mathrm{Th}(\mathcal{L}))$ is an algebraic closed set system.

Let us recall that M_θ is the set $\bigcap \{T \in \mathrm{Th}(\mathcal{L}) : \Omega(T) = \theta\}$. We claim that,

If θ is finitely generated as an $\Omega(\mathrm{Th}(\mathcal{L}))$-set, then M_θ is finitely generated as an $\mathcal{L}$-theory.

Actually, let X be the set of all finitely generated theories included in M_θ. X is obviously upward directed and $\bigcup X = M_\theta$. Thus $\theta = \Omega(M_\theta) = \Omega(\bigcup X) = \bigcup \Omega(X)$. Since $\Omega(X)$ is directed and θ is finitely generated, we have that $\theta = \Omega(T)$ for some $T \in X$. Thus, $M_\theta = T$ since M_θ is the smallest $\mathcal{L}$-theory whose Leibniz congruence is θ. So M_θ is finitely generated.

Let S be any sort and x, y distinct variables of sort S. Let θ be the $\Omega(\mathrm{Th}(\mathcal{L}))$-set generated by $\{x, y\}$. Then by the result above, M_θ is finitely generated as an $\mathcal{L}$-theory. But M_θ is also generated as an $\mathcal{L}$-theory by the infinite set $\widetilde{\forall}\ \widehat{\vartheta}\ E_S(x, y, \widehat{\vartheta})$ for some parameterized equivalence system (there is such an equivalence system by Theorem 30 since we are assuming that $\mathcal{L}$ is protoalgebraic). Thus there is a globally finite subset E'_S of $\widetilde{\forall}\ \widehat{\vartheta}\ E_S(x, y, \widehat{\vartheta})$ that also generates M_θ. In particular we have

$$E'_S \vdash_{\mathcal{L}} E_S(x{:}S, y{:}S, \widehat{u}{:}\widehat{Q}). \tag{11}$$

Moreover, E'_S will contain only a finite number of variables, say $\upsilon_0{:}P_0, \ldots, \upsilon_{n-1}{:}P_{n-1}$ different from x and y which, since X_S is countable for each sort S, we can also assume them to be different from all the parametric variables $\widehat{u}{:}\widehat{Q}$. Thus, E'_S may be written as $E'_S(x, y, \upsilon_0, \ldots, \upsilon_{n-1})$. Let $E''(x, y) = E'_S(x, y, \nu_0, \ldots, \nu_{n-1})$, where ν_i is a ground term of sort P_i for each $i < n$. Consider any $\varphi, \psi \in \mathrm{Te}_S$ and any choice of parameters $\widehat{\vartheta} \in \mathrm{Te}_{\widehat{Q}}$. Let σ be any substitution such that $\sigma(x) = \varphi$, $\sigma(y) = \psi$, $\sigma(\upsilon_i) = \nu_i$ for each $i < n$, and $\sigma(\widehat{u}) = \widehat{\vartheta}$. By applying σ to both sides of (11) we obtain, by the substitution invariance of $\mathcal{L}$, that $E''_S(\varphi, \psi) \vdash_{\mathcal{L}} E_S(\varphi, \psi, \widehat{\vartheta})$. Since this holds for every

choice of parameters $\widehat{\vartheta}{:}\widehat{Q}$ we get $E''_S(\varphi,\psi) \vdash_{\mathcal{L}} \widetilde{\forall}\,\widehat{\vartheta} E_S(\varphi,\psi,\widehat{\vartheta})$, and since the consequence in the opposite direction obviously holds, we finally have that

$$E''_S(\varphi,\psi) \;{}_{\mathcal{L}}\!\dashv\!\vdash_{\mathcal{L}} \widetilde{\forall}\,\widehat{\vartheta} E_S(\varphi,\psi,\widehat{\vartheta}) \text{ for all } \varphi,\psi \in \mathrm{Te}_S. \tag{12}$$

Take $E'' = \langle E''_S(x{:}S, y{:}S) : S \in \mathrm{SORT}\rangle$. Then it follows from (12) and the fact that E is an equivalence system for $\mathcal{L}$ with parameters, using Corollary 29, that E'' is a finite equivalence system for $\mathcal{L}$ without parameters. $\square$

Corollary 34 *Let $\mathcal{L}$ be a specifiable hidden k-logic. If $\mathcal{L}$ is finitely equivalential then $\mathcal{L}$ is behaviorally specifiable.*

PROOF. Suppose that $\mathcal{L}$ is finitely equivalential. Then, it has a finite equivalence system which obviously is a protoequivalence system. Hence, $\mathcal{L}$ is protoalgebraic. From Corollary 20 and Lemma 17, $\Omega(\mathrm{Th}(\mathcal{L}))$ is closed under intersections and under inverse surjective substitutions, respectively. From Theorem 33, $\Omega(\mathrm{Th}(\mathcal{L}))$ is also closed under unions of directed sets. Therefore, $\Omega(\mathrm{Th}(\mathcal{L}))$ is the set of theories of a specifiable hidden 2-logic (see [29] and the remarks preceding Theorem 19). This shows that $\mathcal{L}$ is behaviorally specifiable, since it is protoalgebraic. $\square$

In [29], we define the behavioral equivalence $\models^{\mathrm{beh}}_{\mathcal{L}}$ as $\models_{\mathrm{Mod}^*(\mathcal{L})}$, and it is well known that for any class K of k-data structures $\models_K$ is finitary if and only if K is a quasivariety. Thus, a necessary and sufficient condition for a hidden k-logic to be behaviorally specifiable is that the class of the algebraic reducts of the behavioral models is a quasivariety, that is the class can be axiomatized by a set of conditional equations. Moreover, the axiomatization gives directly the presentation by axioms and inference rules for the behavioral logic of $\mathcal{L}$. There are examples of hidden k-logics which are not finitely equivalential, but the class of the algebraic reducts of their behavioral models is a quasivariety. An example of such logics is the K^{MP}, the smallest normal modal logic that satisfies *modus ponens* (see [24]). Therefore, the converse of the previous corollary is not true in general. It is still an open problem to find, in terms of equivalence systems, necessary and sufficient conditions for a hidden k-logic to be behaviorally specifiable. In [31], the authors studied this question for hidden equational logics and they showed that the converse of this corollary holds for hidden equational logics.

In this section we characterize, by algebraic properties, some classes of logics defined by syntactical properties of their equivalence systems; namely, the properties of having a parameterized equivalence system, a finite parameterized equivalence system, an equivalence system and a finite equivalence system. The characterizations are established using closure properties of the class of behavioral models, as well as by properties of the Leibniz operator. Moreover, some of those characterizations only hold under the assumption that the signature must be standard. Otherwise we only get necessary conditions. Namely, in Theorem 38 and Theorem 39 the signature Σ of the hidden k-logic $\mathcal{L}$ has to be standard [5] (this is one of the main difficulties in dealing with heterogeneous systems). The following theorem will be useful in obtaining such characterizations. It is an immediate consequence of applying to behavioral models the fact that parameterized equivalence systems define the Leibniz congruence over the filters (recall that behavioral models are reduced k-data structures $\mathcal{A} = \langle \mathbf{A}, F \rangle$, i.e., $\Omega(F) = id_A$). The theorem gives an axiomatization of the class of behavioral models which is finite when $\mathcal{L}$ has a finite parameterized equivalence system.

Theorem 35 *Let $\mathcal{L}$ be a parameterized finitely equivalential hidden k-logic with E its finite parameterized equivalence system. Then, a model*[6] *$\mathcal{A}$ of $\mathcal{L}$ is reduced if and only if for each $S \in \mathrm{SORT}$, it satisfies the following Horn sentence,*

$$\forall x, y \; [\; (\forall \widehat{u} \bigwedge \{D_V(\bar{\delta}(x{:}S, y{:}S, \widehat{u}{:}\widehat{Q})) : \\ \bar{\delta} \in E_{S,V}(x{:}S, y{:}S, \widehat{u}{:}\widehat{Q}),\; V \in \mathrm{VIS}\}\;) \to x \approx y] \; . \quad (13)$$

Moreover, if $\mathcal{L}$ is finitely equivalential, then (13) *is in fact a universal Horn sentence. In this case,* $\mathrm{Mod}^*(\mathcal{L})$ *is closed under the formation of data substructures and filtered products.*

PROOF. Let $\mathcal{A} = \langle \mathbf{A}, F \rangle$ be a behavioral model of $\mathcal{L}$. Let $a, b \in A_S$ and $\widehat{c} \in A_{\widehat{Q}}$ such that for all $\bar{\delta} \in E_{S,V}$, $\bar{\delta}^{\mathbf{A}}(a, b, \widehat{c}) \in D_V^{\mathbf{A}} = F_V$. Since E is a finite parameterized equivalence system for $\mathcal{L}$ we have that $a \equiv b \; (\Omega(F)_S)$. Since $\mathcal{A}$ is reduced, $a = b$. That is, $\mathcal{A}$ satisfies the first-order formula (13).

[5] The condition that Σ must be standard is not too restrictive, since each $\mathcal{L}$ has a standard conservative extension (see [30]).

[6] Here we consider the models as first order structures in the expanded language $\mathcal{L}_H$ obtained by adding the new k-relational symbol D which will be interpreted as the filter F (see [29]).

Conversely, suppose that $\mathcal{A}$, considered as a first-order structure in the language $\mathcal{L}_H$, satisfies the first-order formula (13). Let $a, b \in A_S$ such that $a \equiv b$ $(\Omega(F)_S)$. Then, since E is a parameterized equivalence system, from Theorem 28 we obtain that for all $\bar{\delta} \in E_{S,V}$ and every $\widehat{c} \in A_{\widehat{Q}}$, $\bar{\delta}^{\mathbf{A}}(a, b, \widehat{c}) \in D_V^{\mathbf{A}} = F_V$. So, $a = b$. That is, $\mathcal{A}$ is reduced.

The second part of this theorem is a consequence of the multisorted version of the Quasivariety Theorem (see Theorem 5.3.24 in [32]). $\square$

The next theorem provides a semantic characterization of protoalgebraic logics among all members in the class of specifiable hidden k-logics.

Theorem 36 *Let $\mathcal{L}$ be a specifiable hidden k-logic. $\mathcal{L}$ is protoalgebraic if and only if* $\mathrm{Mod}^*(\mathcal{L})$ *is closed under subdirect products.*

PROOF. Suppose that $\mathcal{L}$ is protoalgebraic. Then by Theorem 30 it has a parameterized equivalence system with parameters, say $E = \langle E_S(x{:}S, y{:}S, \widehat{u}{:}\widehat{Q}){:}\ S \in \mathrm{SORT}\rangle$. Let $\mathcal{B}_i = \langle \mathbf{B}_i, G_i\rangle$, $i \in I$, be a family of behavioral models of $\mathcal{L}$ and let $\mathcal{A} = \langle \mathbf{A}, F\rangle$ be a subdirect product of $\prod_{i\in I} \mathcal{B}_i$. We recall that the projection $\pi_i : A \twoheadrightarrow B_i$ is onto, for every $i \in I$. Let $a, b \in A_S$. Suppose that $a \equiv b$ $(\Omega(F)_S)$. Since $\mathcal{A}$ is a model of $\mathcal{L}$, by Theorem 28, $E_S^{\mathbf{A}}(a, b, \widehat{c}) \subseteq F$, for all $\widehat{c} \in A_{\widehat{Q}}$. Hence, by the definition of subdirect product, $E_S^{\mathbf{B}_i}(a(i), b(i), \widehat{c}(i)) \subseteq G_i$, for each $i \in I$ and all $\widehat{c}(i) \in A_{\widehat{Q}}$. Since, for every sort S, $(\pi_i)_S : A_S \to (B_i)_S$ is onto, for every i, $\widehat{c}(i)$ ranges over all $\widehat{d}_i \in (B_i)_{\widehat{Q}}$, as $\widehat{c}$ ranges over $A_{\widehat{Q}}$. Hence, we have $E_S^{\mathbf{B}_i}(a(i), b(i), \widehat{d}_i) \subseteq G_i$, for each $i \in I$ and all $\widehat{d}_i \in (B_i)_{\widehat{Q}}$. So $a(i) \equiv b(i)$ $(\Omega(G_i)_S)$. Since each $\mathcal{B}_i$ is reduced, $a(i) = b(i)$, for every $i \in I$. Hence $a = b$.

To prove the converse, let $\mathbf{A}$ be an algebra and let F and G be two filters of A such that $F \subseteq G$. Let $\theta := \Omega(F) \cap \Omega(G)$. We define $\mathcal{B} := \langle \mathbf{A}/\theta, F/\theta\rangle$. We have that $\mathcal{B}$ is isomorphic to a subdirect product of $\mathcal{A}_1 = \langle \mathbf{A}/\Omega(F), F/\Omega(F)\rangle$ and $\mathcal{A}_2 = \langle \mathbf{A}/\Omega(G), G/\Omega(G)\rangle$, by the mapping $h(a/\theta) := (i_1(a), i_2(b))$ (where i_1 and i_2 are the canonical morphisms from $\mathbf{A}$ into $\mathbf{A}/\Omega(F)$ and $\mathbf{A}/\Omega(G)$, respectively). Both $\mathcal{A}_1$ and $\mathcal{A}_2$ are behavioral models of $\mathcal{L}$. Hence, $\mathcal{B}$ has to be reduced as well. This means that θ is the largest congruence of A compatible with F, i.e., $\Omega(F) = \theta$. Therefore $\Omega(F) \subseteq \Omega(G)$. $\square$

The class of behavioral models of a parameterized finitely equivalential specifiable logic is closed under subdirect products and ultraproducts. Moreover, in the standard case this property characterizes this class.

Theorem 37 *Let $\mathcal{L}$ be a specifiable hidden k-logic over a hidden signature with a finite number of sorts. $\mathcal{L}$ is parameterized finitely equivalential if and only if* $\mathrm{Mod}^*(\mathcal{L})$ *is closed under subdirect products and ultraproducts.*

PROOF. Suppose $\mathcal{L}$ is parameterized finitely equivalential. By Theorem 35, we know that the class of behavioral models $\mathrm{Mod}^*(\mathcal{L})$ is axiomatized by a set of special Horn sentences. Hence, it is closed under ultraproducts (see Theorem 5.3.24 in [32]). By the previous theorem $\mathrm{Mod}^*(\mathcal{L})$ is closed under subdirect products.

Suppose now that the class $\mathrm{Mod}^*(\mathcal{L})$ is closed under ultraproducts and subdirect products. Then, by Theorems 30 and 36, there is a parameterized equivalence system for $\mathcal{L}$, say E. We are going to identify the sorted set E_S with the unsorted set $\bigcup_{V\in\mathrm{VIS}} E_{S,V}$. Let $I = \langle I_S : S \in \mathrm{SORT}\rangle$ be a sorted set of indices such that $S \neq S' \Rightarrow I_S \cap I_{S'} = \emptyset$, and, for every S, $\bigcup_{V\in\mathrm{VIS}} E_{S,V} = \{\bar{\delta}_i(x{:}S, y{:}S, \widehat{u}{:}\widehat{Q}) : i \in I_S\}$. Note that, by hypothesis (the set of sorts is finite) we have that any subset J of I is globally finite if and only if, for each sort S, J_S is finite. We are going to show that there is a globally finite sorted subset of E that is a parameterized finite equivalence system for $\mathcal{L}$. By way of contradiction, assume that this property fails.

For each globally finite subset J of I, we choose a model $\mathcal{A}_J = \langle \mathbf{A}_J, F_J\rangle$ of $\mathcal{L}$ for which the set Φ_J, defined by
$(\Phi_J)_S := \Big\{\langle a, b\rangle \in (A_J)^2_S : \forall\, \widehat{c} \in (A_J)_{\widehat{Q}}\, (\bar{\delta}_i)^{\mathbf{A}_J}(a, b, \widehat{c}) \in F_J,\ \text{for all } i \in J \cap I_S\Big\}$,
contains $\Omega(F_J)$ strictly (i.e. $\Omega(F_J) \subsetneq \Phi_J$). Note that this model always exists by our assumption. Moreover, we can assume $\mathcal{A}_J$ reduced, by taking its reduction if necessary. Consider now the lattice $\mathcal{I} = \langle \mathcal{P}_{\mathcal{GF}}(I), \cap, \cup\rangle$, where $\mathcal{P}_{\mathcal{GF}}(I)$ is the set of all globally finite sorted subsets of I. Take $\mathcal{U}$ to be any ultrafilter on $\mathcal{I}$ that contains the subsets of $\mathcal{P}_{\mathcal{GF}}(I)$ of the form:

$$\widehat{J} := \{K \in \mathcal{P}_{\mathcal{GF}}(I) : J \subseteq K\}, \text{ for each } J \in \mathcal{P}_{\mathcal{GF}}(I).$$

We define the following ultraproduct

$$\mathcal{B} = \Big(\textstyle\prod_{J\in\mathcal{P}_{\mathcal{GF}}(I)} \mathcal{A}_J\Big)\big/\mathcal{U}.$$

We denote the filter of $\mathcal{B}$ by $D^{\mathcal{U}}_{\prod A_J}$. We know that for every $J \in \mathcal{P}_{\mathcal{GF}}(I)$ there are a sort S_J and a pair $\langle a^{S_J}_J, b^{S_J}_J\rangle \in (A_J)^2_{S_J}$ such that $a^{S_J}_J \not\equiv b^{S_J}_J\ (\Omega(F_J)_{S_J})$ (i.e., $a^{S_J}_J \neq b^{S_J}_J$) and $\langle a^{S_J}_J, b^{S_J}_J\rangle \in (\Phi_J)_{S_J}$. Take $d_S \in (\prod_{J\in\mathcal{P}_{\mathcal{GF}}(I)} A_J)_S$ fixed and define $f_S, g_S \in (\prod_{J\in\mathcal{P}_{\mathcal{GF}}(I)} A_J)_S$ in the following way:

$$f_S(J) = \begin{cases} a^{S_J}_J, & \text{if } S = S_J; \\ d_S(J), & \text{otherwise.} \end{cases} \qquad g_S(J) = \begin{cases} b^{S_J}_J, & \text{if } S = S_J; \\ d_S(J), & \text{otherwise.} \end{cases}$$

Note that f_S and g_S are defined in such a way that for every $J \in \mathcal{P}_{\mathcal{GF}}(I)$ $f_S(J) \equiv g_S(J)$ (Φ_J). We claim that:

Claim. *There exists $S \in$ SORT such that $f_S \not\equiv g_S$ $(\mathcal{U})$.*

In fact, suppose that for every $S \in$ SORT, $f_S \equiv g_S$ $(\mathcal{U})$, i.e, $\{J : f_S(J) = g_S(J)\} \in \mathcal{U}$. Then, $\bigcap_{S \in \text{SORT}} \{J : f_S(J) = g_S(J)\} \in \mathcal{U}$ because we are assuming that the number of sorts is finite. But $\bigcap_{S \in \text{SORT}} \{J : f_S(J) = g_S(J)\} = \emptyset$. So, such a sort S must exist. Let us call it H.

Let $i_H \in I_H$. Define $K \subseteq I$ such that $K_H = \{i_H\}$ and $K_S = \emptyset$, for any sort S different from H.

Since $f_H(K) \equiv g_H(K)$ (Φ_K), $\forall\, \widehat{c} \in (A_K)_{\widehat{Q}}\, \bar{\delta}^{\mathbf{A}_K}_{i_H}(f_H(K), g_H(K), \widehat{c}) \in (F_K)_V$. Let $L \subseteq \mathcal{P}_{\mathcal{GF}}(I)$ be the following set

$$L := \{J \in \mathcal{P}_{\mathcal{GF}}(I) : \forall\, \widehat{c} \in (A_J)_{\widehat{Q}}\, \bar{\delta}^{\mathbf{A}_J}_{i_H}(f_H(J), g_H(J), \widehat{c}) \in (F_J)_V\}.$$

Obviously $K \in L$. Moreover, $\widehat{K} \subseteq L$ and by definition of $\mathcal{U}$, $\widehat{K} \in \mathcal{U}$. Thus, $L \in \mathcal{U}$. Therefore,

$\langle \forall\, \widehat{c} \in (A_J)_{\widehat{Q}}\, \bar{\delta}^{\mathbf{A}_J}_{i_H}(f_H(J), g_H(J), \widehat{c}) : J \in \mathcal{P}_{\mathcal{GF}}(I)\rangle \subseteq D^{\mathcal{U}}_{\prod \mathcal{A}_J}$, for each $i_H \in I_H$.

This implies that $\bar{\delta}^{\mathbf{A}_J}_{i_H}(f_H(J), g_H(J), \widehat{c})/\mathcal{U} = \bar{\delta}^{\mathbf{B}}_{i_H}(f_H/\mathcal{U}, g_H/\mathcal{U}, \widehat{c}/\mathcal{U}) \in D^{\mathcal{U}}_{\prod \mathcal{A}_J}/\mathcal{U}$. Since E is a parameterized equivalence system, $\langle f_H/\mathcal{U}, g_H/\mathcal{U}\rangle \in \Omega_B(D^{\mathcal{U}}_{\prod \mathcal{A}_J}/\mathcal{U})$. And finally, since by hypothesis we have that $\text{Mod}^*(\mathcal{L})$ is closed under ultraproducts, $\mathcal{B}$ is reduced. Thus, $f_H/\mathcal{U} = g_H/\mathcal{U}$, i.e., $f_H \equiv g_H$ $(\mathcal{U})$, which contradicts the choice of H. Hence, we can conclude that some globally finite subset of E must be a parameterized finite equivalence system itself. $\square$

Theorem 38 *Let $\mathcal{L}$ be a specifiable hidden k-logic over a hidden signature Σ. If $\mathcal{L}$ is equivalential then* $\text{Mod}^*(\mathcal{L})$ *is closed under data substructures and products. Moreover, if Σ is standard then the converse holds.*

PROOF. Suppose that $\mathcal{L}$ is equivalential. Then it is parameterized equivalential. Therefore, by Theorem 30, it is protoalgebraic and, hence, by Theorem 36, $\text{Mod}^*(\mathcal{L})$ is closed under subdirect products and, in particular, under products. Let E be an equivalence system for $\mathcal{L}$ and $\mathcal{A} = \langle \mathbf{A}, F\rangle \in \text{Mod}^*(\mathcal{L})$ and $\mathcal{B} = \langle \mathbf{B}, G\rangle$ be a data substructure of $\mathcal{A}$. For all $b, b' \in B_S$, we have that $b \equiv b'(\Omega(G)_S)$ iff $E^{\mathbf{B}}_S(b, b') \subseteq G$. Since $\mathcal{B} \subseteq \mathcal{A}$, then $E^{\mathbf{B}}_S(b, b') = E^{\mathbf{A}}_S(b, b')$. From the fact that $G = F \cap B^2$, we have that $E^{\mathbf{B}}_S(b, b') \subseteq G$ iff $E^{\mathbf{A}}_S(b, b') \subseteq F$. So, $b \equiv b'(\Omega(G)_S)$ iff $b \equiv b'(\Omega(F)_S)$ iff $b = b'$. Therefore, $\mathcal{B}$ is reduced.

Conversely, assume that Σ is standard and suppose that $\text{Mod}^*(\mathcal{L})$ is closed under products and data substructures and hence closed under subdirect products. Thus, by Theorem 36, $\mathcal{L}$ is protoalgebraic and, hence, by Theorem 30, it is a parameterized equivalential hidden k-logic. So, $\mathcal{L}$ has a parameterized

equivalence system, say

$$E = \langle E_S(x{:}S, y{:}S, \widehat{u}) : S \in \mathrm{SORT}\rangle.$$

Let us define E' to be the system of sorted sets such that for each unrestricted sort S and each visible sort V, $E'_{S,V}(x{:}S, y{:}S)$ is the set of all formulas obtained by replacing the parameters in each $\bar{\delta} \in E_{S,V}$ by all possible terms generated only by the variables x or y (for each sort S there is at least one such term that contains no variables other than x and y since we assume Σ standard), i.e., $E'_{S,V}(x{:}S, y{:}S) := \{\bar{\delta}(x{:}S, y : S, \widehat{\tau}{:}\widehat{Q}) : \bar{\delta} \in E_{S,V},\ \tau_i \in \mathrm{Te}\{x, y\}_{Q_i}\}$.

We are going to prove that E' is an equivalence system, by showing that it defines the Leibniz congruence over any filter of an arbitrary model of $\mathcal{L}$. Let $\mathcal{A} = \langle \mathbf{A}, F\rangle$ be a model of $\mathcal{L}$ and take any $a, b \in A_S$. Let $\mathbf{B}$ be the subalgebra of $\mathbf{A}$ generated by $\{a, b\}$. Since Σ is standard, $B_S \neq \emptyset$, for every $S \in$ SORT. Moreover, each element of B is of the form $\tau^{\mathbf{A}}(a, b)$, for some $\tau(x, y) \in \mathrm{Te}\{x, y\}_{Q_i}$. Clearly, $\langle \mathbf{B}/(\Omega(F) \cap B^2), (F \cap B^k)/(\Omega(F) \cap B^2)\rangle$ is isomorphic to a data substructure of $\mathcal{A}/\Omega(F)$. Consequently, it is reduced since $\mathcal{A}/\Omega(F) = \langle \mathbf{A}/\Omega(F), F/\Omega(F)\rangle$ is reduced and by hypothesis $\mathrm{Mod}^*(\mathcal{L})$ is closed under data substructures. So, $\Omega(F) \cap B^2$ is the largest congruence on B compatible with $F \cap B^k$, which means that $\Omega(F \cap B^k) = \Omega(F) \cap B^2$. So, we have

$$\begin{aligned}
a \equiv b\ (\Omega(F)_S) \ \ &\text{iff}\ \ a \equiv b\ \Big((\Omega(F) \cap B^2)_S\Big)\\
&\text{iff}\ \ a \equiv b\ \Big(\Omega(F \cap B^k)_S\Big)\\
&\text{iff}\ \ E^{\mathbf{B}}_S(a, b, \widehat{c}) \subseteq F \cap B^k,\ \forall\, \widehat{c} \in B_{\widehat{Q}}\\
&\text{iff}\ \ E^{\mathbf{B}}_S(a, b, \widehat{\tau}^{\mathbf{B}}(a, b)) \subseteq F \cap B^k,\ \forall\, \widehat{\tau} \in \mathrm{Te}\{x, y\}_{\widehat{Q}}\\
&\text{iff}\ \ E^{\mathbf{A}}_S(a, b, \widehat{\tau}^{\mathbf{A}}(a, b)) \subseteq F,\ \forall\, \widehat{\tau} \in \mathrm{Te}\{x, y\}_{\widehat{Q}}\\
&\text{iff}\ \ (E'_S)^{\mathbf{A}}(a, b) \subseteq F.
\end{aligned}$$

Thus, by Theorem 28, E' is an equivalence system. □

Theorem 39 *Let $\mathcal{L}$ be a specifiable hidden k-logic over a hidden signature Σ. If $\mathcal{L}$ is finitely equivalential then* $\mathrm{Mod}^*(\mathcal{L})$ *is closed under data substructures, products and ultraproducts. Moreover, if Σ is standard and has only a finite number of sorts, the converse is also true.*

PROOF. The first part holds by Theorem 35, since the class of behavioral models is axiomatized by a set of universal Horn formulas. Suppose now that $\mathrm{Mod}^*(\mathcal{L})$ is closed under data substructures, products and ultraproducts. Then, by the previous theorem, there is an equivalence system E for $\mathcal{L}$. Since $\mathrm{Mod}^*(\mathcal{L})$ is closed under subdirect products and ultraproducts, using an ultraproduct argument, as we did in the proof of Theorem 37, there is a subset of E which is a finite equivalence system for $\mathcal{L}$. So, $\mathcal{L}$ is finitely equivalential. □

4.4 Examples

4.4.1 Unsorted hidden logics

Sentential logics The classical propositional calculus (CPC) is the best example to illustrate the meaning of an equivalence system since, in this case, the equivalence system is the CPC-equivalence, that is, the set $\{x \leftrightarrow y\}$.

There is much work concerning the study of the existence of equivalence systems in particular deductive systems. We refer to the book [13] and the paper [12] by Czelakowski, and the paper [35] by Pigozzi for further references. $\diamondsuit$

Free inequational logic (revisited). The free inequational logic is an example of an unsorted equivalential 2-logic (see Example 11). The equivalence system is simply the set $\{x \preceq y, y \preceq x\}$. $\diamondsuit$

4.4.2 Hidden equational logic

The classes of equivalential, finitely parameterized equivalential and the finitely equivalential HEL's are pairwise distinct. First we show that there are HEL's which are not equivalential and then we study more examples of HEL's; namely an equivalential and a finitely equivalential logic.

A non equivalential HEL**.** Every HEL is protoalgebraic and thus parameterized equivalential by Theorem 30, but not every HEL is equivalential. Clearly a HEL without hidden sorts is equivalential. First we state a lemma which is useful in proving that an equality model is a behavioral model. It will be used in the proof of Theorem 41. The proof of this lemma is straightforward and will be omitted.

Lemma 40 ([29]) *Let $\mathcal{L}$ be a* HEL *and $\mathcal{A} = \langle \mathbf{A}, id_{A_{\mathrm{VIS}}} \rangle \in \mathrm{Mod}(\mathcal{L})$. $\mathcal{A}$ is a behavioral model of $\mathcal{L}$ if, for every hidden sort H, A_H has only one element or, there is a (visible) context $\varphi(z{:}H, \widehat{x}{:}\widehat{Q}) \in C_\Sigma[z{:}H]$ such that $\forall\, \widehat{x}\ \varphi^{\mathbf{A}}(z{:}H, \widehat{x}{:}\widehat{Q})$ is injective, as a mapping, on the argument H (i.e., $\forall a_1, a_2 \in A_H\ (\forall \widehat{b} \in A_{\widehat{Q}}\ \varphi^{\mathbf{A}}(a_1, \widehat{b}) = \varphi^{\mathbf{A}}(a_2, \widehat{b})) \Rightarrow a_1 = a_2$).*

Theorem 41 *Any free hidden equational logic $\mathcal{L}$ over a hidden signature Σ with* VIS $\neq \emptyset$, HID $\neq \emptyset$ *and having at least one attribute (i.e., an operation symbol of visible range) with at least one argument of visible sort and at least one argument of hidden sort, fails to be equivalential.*

PROOF. Let g be an attribute in Σ having one argument of visible sort and one of hidden sort. We can assume, without loss of generality, that g is of

type $H, V, S_0 \ldots, S_{n-1} \to V'$, for some $H \in$ HID and some $V, V' \in$ VIS and $S_0 \ldots, S_{n-1} \in$ SORT.

Consider now two 2-data structures over $\mathcal{L}$, $\mathcal{A} = \langle \mathbf{A}, id_{A_V} \rangle$ and $\mathcal{B} = \langle \mathbf{B}, id_{B_V} \rangle$, where $A_V = A_{V'} = \{1, 2\}$, $A_H = A_V^2$, $B_V = B_{V'} = \{1, 2, 3\}$ and $B_H = B_V^2$. For the remaining sorts the carrier sets are each a one-element set, i.e., $A_S = B_S = \{\star\}$. We interpret the symbol g in $\mathbf{B}$ in a way that satisfies the following condition: $g^{\mathbf{B}}(a, b, c_0, \ldots, c_{n-1}) = g^{\mathbf{B}}(a, b, c'_0, \ldots, c'_{n-1})$, for all $a \in B_H, b \in B_V, c_0, \ldots, c_{n-1}, c'_0, \ldots, c'_{n-1} \in B_{\widehat{Q}}$.

In this way we can consider $g^{\mathbf{B}}$ as a mapping from $B_H \times B_V$ into B'_V. By the same reason, we assume that $g^{\mathbf{A}}$ is also a mapping from $A_H \times A_V$ into A'_V. Let the interpretation of g in $\mathbf{A}$ and in $\mathbf{B}$ be given by Fig. 5. For the remaining operation symbols we take the trivial interpretations, i.e, they are interpreted as constant operations. It can be shown that $\mathcal{A} \subseteq \mathcal{B}$. Obviously, $\mathcal{A}$ and $\mathcal{B}$ are models of $\mathcal{L}$.

$g^{\mathcal{B}}$	$\mathbf{1}$	$\mathbf{2}$	3
$(\mathbf{1}, \mathbf{1})$	$\mathbf{1}$	$\mathbf{1}$	1
$(\mathbf{1}, \mathbf{2})$	$\mathbf{1}$	$\mathbf{2}$	1
$(\mathbf{2}, \mathbf{1})$	$\mathbf{1}$	$\mathbf{2}$	2
$(\mathbf{2}, \mathbf{2})$	$\mathbf{2}$	$\mathbf{2}$	2
$(1, 3)$	1	2	3
$(2, 3)$	1	3	1
$(3, 1)$	1	3	2
$(3, 2)$	1	3	3
$(3, 3)$	3	3	3

Fig. 5. Interpretation of the operation symbol g.

By applying the previous lemma to this case we have $\mathcal{B} \in \mathrm{Mod}^*(\mathcal{L})$ i.e., $\Omega(id_{B_V}) = id_B$. On the other hand, we have that

$$g^{\mathbf{A}}(\langle 2, 1 \rangle, 1) = g^{\mathbf{A}}(\langle 2, 1 \rangle, 1) \text{ and } g^{\mathbf{A}}(\langle 2, 1 \rangle, 2) = g^{\mathbf{A}}(\langle 2, 1 \rangle, 2).$$

Moreover, by induction on the complexity of the contexts we can show that for any (visible) context $\varphi(z{:}H, \widehat{u} : \widehat{Q})$ we have that $\varphi^{\mathbf{A}}(\langle 2, 1 \rangle, \widehat{b}) = \varphi^{\mathbf{A}}(\langle 2, 1 \rangle, \widehat{b})$. Then, $\langle 2, 1 \rangle \equiv \langle 2, 1 \rangle$ $(\Omega(id_{A_V})_H)$. Hence, $\mathcal{A} \notin \mathrm{Mod}^*(\mathcal{L})$. Thus, $\mathrm{Mod}^*(\mathcal{L})$ is not closed under data substructures. Therefore, by Theorem 38, $\mathcal{L}$ is not equivalential. $\square$

4.4.3 Stacks

Let us recall the specification of stacks over the hidden signature $\mathcal{L}_{stacks}$ (see [29]).

SORT : *stack*, *nat*
VIS : *nat*
Operation symbols:

$zero : \ \to nat$	$top : stack \to nat$
$empty : \ \to stack$	$pop : stack \to stack$
$s : nat \to nat$	$push : nat, stack \to stack$

Extralogical axioms:

$top(pop^n(empty)) \approx zero$, for all $n \geq 0$;
$top(push(x, y)) \approx x$;
$top(pop^{n+1}(push(x, y))) \approx top(pop^n(y))$, for all $n \geq 0$;

Extralogical inference rule: $\dfrac{s(x) \approx s(y)}{x \approx y}$

Fig. 6. Stacks logic.

It is not difficult to show that $\{top(pop^n(x)) \approx top(pop^n(y)) : n \in \mathbb{N}\}$, with visible part being $\{x{:}nat \approx y{:}nat\}$, is an equivalence system for $\mathcal{L}_{stacks}$.

We claim that $\mathcal{L}_{stacks}$ is not finitely equivalential.

Claim. *$\mathcal{L}_{stacks}$ is not finitely equivalential.*

In fact, suppose that $\mathcal{L}_{stacks}$ is finitely equivalential. Then from Proposition 31 there is a finite subset of the equivalence system $\{top(pop^n(x)) \approx top(pop^n(y)) : n \in \mathbb{N}\}$ which is a finite equivalence system itself, say

$$\{top(pop^n(x)) \approx top(pop^n(y)) : n \in I\},$$

with I a finite subset of $\mathbb{N}$. Let n_0 be the largest element in I, and take the following two stacks $s = \langle y, x_0, \ldots, x_{n_0} \rangle$ and $s' = \langle y', x_0, \ldots, x_{n_0} \rangle$, of length $n_0 + 1$ with $y \neq y'$, of the standard model $\mathcal{S}$ of stacks. Thus, for any $n \in I$, $top^{\mathbf{S}}((pop^n)^{\mathbf{S}}(s)) = top^{\mathbf{S}}((pop^n)^{\mathbf{S}}(s'))$. However s and s' are not behaviorally equivalent since

$$top^{\mathbf{S}}((pop^{n_0+1})^{\mathbf{S}}(s)) \neq top^{\mathbf{S}}(((pop^{n_0+1})^{\mathbf{S}}(s')). \qquad \diamondsuit$$

4.4.4 Flags

Similarly to the previous example we can prove that $\{up?(x) \approx up?(y)\}$, with visible part being the obvious one, is a finite equivalence system for $\mathcal{L}_{flags}$. So the flags logic is a finitely equivalential logic. $\diamondsuit$

4.4.5 Sets

In the specification of sets given in Fig. 7, denoted by $\mathcal{L}_{sets}$, the situation is more complicated, since we have two visible sorts and more operation symbols (for the details see [29]). But, it can be proved that $\mathcal{L}_{sets}$ has a finite parameterized equivalence system which is

$$E_{set,nat}(x,y) = E_{nat,bool}(x,y) = E_{bool,nat}(x,y) = \varnothing;$$

$$E_{set,bool}(x,y,n{:}nat) = \{in(n,x) \approx in(n,y)\};$$

$$E_{nat,nat}(x,y) = E_{bool,bool}(x,y) = \{x \approx y\}.$$

SORT : $set,\ elt,\ bool$
VIS : $bool,\ elt.$
Operation symbols:

$empty : \quad \to set;$	$true : \quad \to bool;$
$in : elt, set \to bool;$	$false : \quad \to bool;$
$\cup : set, set \to set;$	$\neg : bool \to bool;$
$add : elt, set \to set;$	$\wedge : bool, bool \to bool;$
$neg : set \to set;$	$\vee : bool, bool \to bool.$
$\cap : set, set \to set;$	

Extralogical axioms:
$in(n, empty) \approx false;$
$in(n, \cup(x,x')) \approx in(n,x) \ \vee \ in(n,x');$
$in(n, neg(x)) \approx \neg(in(n,x));$
$in(n, \cap(x,x')) \approx in(n,x) \ \wedge \ in(n,x');$

Extralogical inference rules:

$$\frac{in(z,x) \approx in(z,y)}{in(z, add(n,x)) \approx in(z, add(n,y))};$$

$$\frac{m \approx n}{in(z, add(m,x)) \approx in(z, add(n,x))}.$$

Fig. 7. Sets logic.

$\diamondsuit$

These examples, together with some others discussed in [29], allow us to draw a picture that illustrates the relations between those classes of logics (Fig. 8).

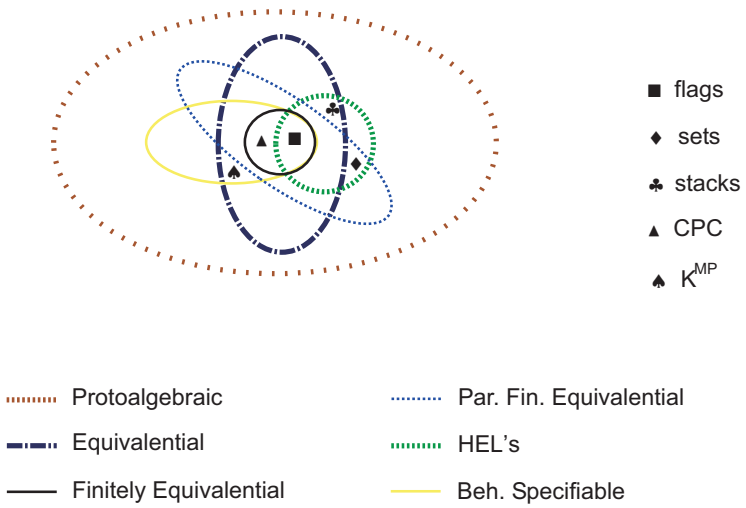

Fig. 8. Hierarchy of hidden k-logics.

4.5 Further topic: Definability of the set of behavioral theorems

Computer scientists are very often specially interested in determining whether a given equation is a behavioral theorem (see [9], [19] and [26], where the focus is on determining whether a given equation is a behavioral theorem). It seems to us that in the context of AAL this problem has never been considered. Here, we develop an introductory theory for this problem and we hope to develop it further in the near future. We will briefly discuss when there is a sorted set of visible k-formulas that defines the set of behavioral theorems in a natural way. In that case, the set of formulas is called a *defining set for the behavioral theorems*. We show that such a defining set for the behavioral theorems could be syntactically characterized by making use of admissible rules instead of valid rules. Finally, we show that the defining set for the behavioral theorems defines the Leibniz congruence on $\mathbf{A}$ over each filter F of any strictly minimal model $\langle \mathbf{A}, F\rangle$.

Definition 42 *Let $\mathcal{L}$ be a hidden k-logic and E a pre-equivalence system for $\mathcal{L}$. We say that the set of behavioral theorems of $\mathcal{L}$ is defined by E if*

$$\Omega(\mathrm{Thm}(\mathcal{L}))_S = \{\, \langle t, t'\rangle \in \mathrm{Te}^2_S : \text{ for all } \widehat{\vartheta} \in \mathrm{Te}_{\widehat{Q}},\ \vdash_{\mathcal{L}} E_S(t, t', \widehat{\vartheta})\}.$$

We call E a defining set (with parameters) for the behavioral theorems.

Since the set of theorems is substitution invariant we can characterize a defining set for the behavioral theorems by replacing the condition in the Definition 42 by a condition which only requires that $\vdash_{\mathcal{L}} E_S(t, t', \widehat{u})$, for some tuple of variables $\widehat{u}$ such that $u_i \notin Var\{t\} \cup Var\{t'\}$.

As an immediate consequence of Theorem 28 we have that if $\mathcal{L}$ is a parameterized equivalential hidden k-logic with E being its parameterized equivalence system, then E is also a defining set for the behavioral theorems of $\mathcal{L}$. Instead

of defining the Leibniz congruence on the term algebra over every theory, defining sets for behavioral theorems define only the Leibniz congruence over the set of behavioral theorems.

The next theorem shows that the defining sets for the behavioral theorems can be characterized as the pre-equivalence systems which define the Leibniz congruence on $\mathbf{A}$ over F for any strictly minimal model $\langle \mathbf{A}, F\rangle$.

Theorem 43 ([29]) *Let $\mathcal{L}$ be a hidden k-logic and $E = \langle E_S(x{:}S, y{:}S, \widehat{u}{:}\widehat{Q}) : S \in \mathrm{SORT}\rangle$ a pre-equivalence system. Then, the behavioral theorems of $\mathcal{L}$ are definable by E if and only if for each strictly minimal model $\langle \mathbf{A}, F\rangle$ of $\mathcal{L}$,*

$$\Omega(F)_S = \{\, \langle a, b\rangle \in A^2_S : \forall \widehat{c} \in A_{\widehat{Q}}\ E^{\mathbf{A}}_S(a, b, \widehat{c}) \subseteq F\}.$$

As was mentioned above, each parameterized equivalence system is always a defining set for the behavioral theorems. Moreover, the defining set of the behavioral theorems without parameters may be syntactically characterized by using the admissible part $\mathcal{L}^{ad}$ of $\mathcal{L}$ (the proof can be found in [29]).

Theorem 44 ([29]) *Let $\mathcal{L}$ be a hidden k-logic and E a pre-equivalence system for $\mathcal{L}$ without parameters. Then, E is a defining set, without parameters, for the behavioral theorems of $\mathcal{L}$ if and only if the following conditions hold:*

(i) $\vdash_{\mathcal{L}^{ad}} E_S(x{:}S, x{:}S)$; (admissible S-identity)

(ii) $E_S(x{:}S, y{:}S) \vdash_{\mathcal{L}^{ad}} E_S(y{:}S, x{:}S)$;

(iii) $E_S(x{:}S, y{:}S),\ E_S(y{:}S, z{:}S) \vdash_{\mathcal{L}^{ad}} E_S(x{:}S, z{:}S)$;

(iv) $E_{S_0}(x_0{:}S_0, y_0{:}S_0), \ldots, E_{S_{n-1}}(x_{n-1}{:}S_{n-1}, y_{n-1}{:}S_{n-1}) \vdash_{\mathcal{L}^{ad}}$
$E_{S_n}(O(x_0, \ldots, x_{n-1}){:}S_n, O(y_0, \ldots, y_{n-1}){:}S_n)$,
for each operation symbol O of type $S_0, \ldots S_{n-1} \to S_n$;
(admissible S-replacement)

(v) *for every $V \in \mathrm{VIS}$, $E_V(x_0{:}V, y_0{:}V) \ldots E_V(x_{k-1}{:}V, y_{k-1}{:}V),\ \bar{x} \vdash_{\mathcal{L}^{ad}} \bar{y}$.*
(admissible V-detachment)

Conclusions and related work

This paper is a step forward in the growth of the generalized theory of the AAL introduced by the author and Pigozzi in [31]. This generalized framework allows for the introduction of multisorts and accommodates the dichotomy "visible vs hidden" within the standard AAL. This theory has been already applied to the OO paradigm (see [29–31]). In the present work, we discuss the axiomatization of the class of behavioral models via abstract algebraic logic. This wide analysis of such axiomatization is achieved using properties

of the equivalence systems and properties of the Leibniz operator; the latter constitutes the main tool in our approach.

We characterize some classes of hidden k-logics by properties of their Leibniz operator, by closure properties of the class of their behavioral models, and by properties of equivalence systems. This is displayed in the hierarchy diagram of hidden k-logics we present in Fig. 8.

Next we will discuss some results that are related to the theory developed in this paper.

Hidden algebras. Hidden algebras were introduced by Goguen in [20] and further developed in [19,21], in order to generalize many-sorted algebras to give an algebraic semantics for the object oriented paradigm. When hidden algebras first appeared, they were considered over restricted signatures. These were assumed to have the visible part fixed, in the sense that all sorted algebras over it have the same visible part. Usually, this visible part was a standard algebra such as the natural numbers or the two-element Boolean algebra. This is called *fixed-data semantics*. Another restriction which is sometimes assumed in order to apply coalgebraic methods and results to the study of behavioral equivalence is the requirement that the methods and the attributes have exactly one hidden argument. This semantics is called *monadic semantics*. The behavioral aspects of modern software make hidden algebras more suitable than standard algebras for abstract machine implementation in practice. Consequently, there has been an increasing development in this field. Goguen and his collaborators, in the last fifteen years, have been improving their theory and applying it in more general settings. Now almost all of the results may be established for *polyadic loose-data semantics*. Polyadic loose-data semantics allows any kind of operation symbols and, in order to have more freedom to choose an adequate implementation, the visible part of the algebras is no longer fixed: it may be any sorted algebra in which the requirements (axioms) of the given specification are valid. However, some authors are interested in applying coalgebraic methods, and then they have to restrict their signatures to the monadic fixed-data semantics. Malcolm [27] shows that behavioral equivalence may also be formulated in the context of coalgebra.

Behavioral equivalence and hidden logic. Two terms are said to be behaviorally equivalent if and only if they cannot be distinguished by any visible context. This is the primitive notion of behavioral equivalence due to Reichel [37]. The idea of looking at the satisfaction relation between hidden terms as behavioral equivalence was also introduced by Reichel in the 80's [37] and it seems to be the correct way of interpreting equality between hidden terms. Since then, it has been adopted and generalized by many people. The most significant contributions have been made by Goguen, Bidoit, Bouhoula and their associates [9,18].

Generalizations of the notion of behavioral equivalence have been considered. Goguen et al. consider Γ-*behavioral equivalence*, where Γ is a subset of the set of all operation symbols in the signature. Γ-behavioral equivalence is defined analogously to ordinary behavioral equivalence, but making use only of the contexts built from the operation symbols in Γ. It can be proved that the Γ-behavioral equivalence is the largest Γ-congruence with the identity as the visible part. Thus, coinduction methods, based on this fact, may still be formulated for this more general notion. We should emphasize that our approach can be extended in order to accommodate such contexts by changing the definition of hidden equational logic; namely, by replacing condition (iv) in Definition 9 by: (iv') $s{:}V \approx s'{:}V \rightarrow t(x/s){:}W \approx t(x/s'){:}W$, for every $t \in (\mathrm{Te}_\Gamma)_W$, any $s, s' \in \mathrm{Te}_V$ and every $x \in X_V$. The Leibniz congruence $\Omega(F)$ has to be redefined as the largest Γ-congruence (Γ-congruence here means a relation compatible with the interpretations of the operation symbols in Γ) compatible with F. Clearly, we also have to adapt all the notions and results if we intend to develop a parallel theory to ours based on this generalized notion of context.

Some authors also require that each context contains only one occurrence of the distinguished variable z. However, we do not need to impose such a requirement because, by considering the ordinary contexts, one generates exactly the same behavioral equivalence, the Leibniz congruence (this requirement is needed to deal with the models as coalgebras in the fixed-data semantics, see [19,39]).

In this more general case, some interesting questions concerning Γ-behavioral equivalence may arise, such as the study of the compatibility of some operation symbols outside of Γ with respect to Γ-behavioral equivalence. This problem has been studied by Diaconescu et al. [15] and Bidoit et al. [3]. On the other hand, Bidoit and Hennicker [4] generalize this notion by endowing hidden algebras with a binary relation, that may be partial. As a particular case we can apply their algebraic approach to the behavioral setting by considering their algebras together with Γ-behavioral equivalence.

Various notions of behavioral logics have been considered. The most important are *hidden logic* by Goguen et al. [19] and *observational logic* by Bidoit et al. [2,22]. There is also another observational logic due to Padawitz [33], called *swinging types logic*, but it is similar to the observational logic of Bidoit et al. (see `http://ls5-www.cs.uni-dortmund.de/~peter/Swinging.html` for more details). Hidden logic is a variant of the equational logic in which some part of the specification is visible and another is hidden. The formulas are just equations and the satisfaction relation is taken behaviorally. Observational logic is different from hidden logic but both are based on behavioral equivalence, i.e., indistinguishability under contexts. Observational logic was introduced by Bidoit and Hennicker (see [22,23]) to formalize behavioral validity

(correctness). Tarski's satisfaction relation of first-order formulas (with equality) is considered as a "behavioral satisfaction relation" which is determined, in a natural way, by the family of congruence relations (possibly partial) with which each algebra is provided. This relation is called *behavioral equality* (see also [25]). The behavioral satisfaction relation is just defined by considering the equality symbol interpreted as the behavioral equality. First-order theories are generalized to the so-called *behavioral theories* where the equality symbol is interpreted as the behavioral equality. In [2] the authors develop a method for proving behavioral theorems whenever an axiomatization of the behavioral equality is provided. This is based on reducing behavioral satisfaction to ordinary satisfaction. Consequently, any proof system for first-order logic can be used to prove the behavioral validity, with respect to a given behavioral equality, of first-order formulas.

Protoalgebraic logics. Protoalgebraic deductive systems were introduced by Blok and Pigozzi [5]. They are primitively defined as deductive systems that have a protoequivalence system. The equivalence between the two concepts in the general setting of k-deductive systems was noted by Blok and Pigozzi in [7]. They also did an exhaustive study of the class of behavioral models (called reduced models in the context of deductive systems) for specifiable protoalgebraic logics.

Axiomatization of the behavioral equivalence. The class of parameterized finitely equivalential hidden k-logics has not been much investigated, even in the one-sorted case; we only found a short reference concerning this kind of logics in [13]. However, the notion is of some interest in behavioral reasoning in the context of hidden equational logics.

Bidoit and Hennicker considered a special class of models, called *black box semantics* which coincides with our class of behavioral models. They did not develop a theory of closure properties of this class. They have only shown that there is always an axiomatization, possibly with infinitary first-order formulas. For this class of algebras such an axiomatization is called *axiomatization of behavioral equality*. Moreover, they gave a complex sufficient and necessary condition for such infinite axiomatization to be replaced by a finitary one. Hence the class of behavioral models may be finitely axiomatizable by first-order formulas. This condition is called the *observability kernel condition*. It is based on the fact that, in some specific cases, we do not need all contexts to define the behavioral equivalence. It may happen that there is a finite set of contexts which defines the behavioral equivalence. These contexts are called *crucial observable contexts* (see [1,2]). The observability kernel condition provides a technique that allows us to reduce infinitary characterizations of behavioral equality to finitary ones. However, there are interesting examples, as the one used to specify stacks, for which this condition is not satisfied (see [1]). That is, Bidoit and Hennicker only consider observational logics

of the two following kinds: the finitely axiomatizable and those which are always axiomatized by infinitary first-order formulas. In our approach, the axiomatizability of behavioral equivalence ramifies into more cases. Namely, the finitely axiomatized ones are split into two classes: the finitely equivalential and the parameterized finitely equivalential hidden k-logics. Relating to this setting, we also consider two other kinds of finitely axiomatizable classes: the parameterized equivalential and the equivalential hidden k-logics. We give characterizations of these axiomatizations by means of the closure operations and by properties of the Leibniz operator. As far as we know, our approach is new in the context of hidden logics (observational logics and hidden equational logics) and provides this interesting hierarchy of hidden k-logics. A similar hierarchy for the homogeneous case was established by Blok and Pigozzi [7]. However, they did not consider the class of parameterized finitely equivalential logics.

In [38] some results are presented concerning axiomatizations for the class of behavioral models of hidden equational logics. Namely, the author shows that a class of equality models (algebras) is defined by a set of equations if and only if the class of equality models is closed under coproducts, quotients, morphisms and representative inclusions. However, the results are in the context of monadic semantics and fixed-data-semantics and there is no discussion concerning the parameters those equations may have.

Wolter, in [43], showed that there is no finitary axiomatization of the observable behavior of stacks. In [41], Schoett investigated another example, the specification of counters; he showed that it also does not admits a finitary axiomatization. More recently, this topic was studied by Buss and Roşu; they investigated the incompleteness of the behavioral logic (see [8]). They also discussed the complexity of the behavioral satisfaction problem. Related to this matter, in Subsection 4.4.3, by using Proposition 31, we present a different, simpler proof that the specification of stacks is not finitely equivalential. There are significant differences between our approach and the Schoett's and Wolter's approaches to this matter; since we use here properties of the equivalence systems, we think that our approach can be more easily generalized to other specifications.

Acknowledgements. The author wishes to thank Don Pigozzi and Isabel Ferreirim for fruitful discussions concerning this work. He also gratefully thanks the anonymous referees who made important suggestions to improve this work.

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