Continuous Selections of Solution Sets of Lipschitzean Differential Inclusions

By

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1. Introduction

Recently, Cellina and Ornelas studied in [C], [O], [P-S] and [C-O] the existence of a continuous map \( \xi \rightarrow x_\xi \) such that \( x_\xi \) is a Caratheodory solution of the differential inclusion

\[
\dot{x} \in F(t, x), \quad x(0) = \xi.
\]

They assumed that the right-hand side is Lipschitz continuous with respect to \( x \), with values in \( \mathbb{R}^n \) and \( \xi \) belongs to a compact set.

The purpose of the present paper is to give a generalization of those results. Namely, we consider the Cauchy problems

\[(P_s) \quad \dot{x} \in F(t, x, s), \quad x(0) = \xi(s)\]

where the right-hand side is Lipschitz continuous in \( x \) and lower semicontinuous in \( s \) with values in a separable Banach space. Assuming that the initial data depends continuously on \( s \), we show the existence of a continuous map \( s \rightarrow x_s \), where the \( x_s \) are solutions of \((P_s)\). The proof is based on an argument different from the one used by Cellina and Ornelas; it relies on a selection theorem of Bressan and Colombo [B-C].

Our result contains as a special case the selection theorems due to Antosiewicz and Cellina [An-C], Bressan and Colombo [B-C] and Fręszkowski [F₁].

2. Preliminaries

Denote by \( I \) the interval \([0, 1]\) and by \( \mathcal{L} \) the \( \sigma \)-field of Lebesgue measurable subsets of \( I \). Let \( S \) be a separable metric space and \( X \) a separable Banach space with the norm \( | \cdot | \). \( \mathcal{P}(X) \) will stand for the family of all nonempty closed subsets of \( X \) with the Hausdorff distance \( d_H \) and \( \mathcal{B}(S) \) for the family of Borel

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subsets of $S$.

Denote by $L^1(I, X)$ the Banach space of Bochner integrable functions $u: I \to X$, with the norm $\|u\| = \int_I |u(t)|dt$ and by $AC(I, X)$ the Banach space of absolutely continuous functions $u: I \to X$ with the norm $\|u\|_{AC} = |u(0)| + \|\dot{u}\|$.

A subset $K \subseteq L^1(I, X)$ is called decomposable if for every $u, v \in K$ and any $A \in \mathcal{P}$,

$$u\chi_A + v\chi_{I\setminus A} \in K,$$

where $\chi_A$ stands for the characteristic function of $A$. The family of all nonempty closed and decomposable subsets of $L^1(I, X)$ is denoted by $\mathcal{D}$.

It is known that $K \in \mathcal{D}$ iff there exists a measurable map $F: I \to \mathcal{P}(X)$ such that

$$K = \{u \in L^1(I, X): u(t) \in F(t) \text{ a.e. in } I\},$$

and that $K$ is nonempty iff the function $t \to d(0, F(t))$ is integrable, where $d$ denotes the usual point-to-set distance. For more details on decomposable sets and set-valued maps we refer to [H-U].

A multivalued map $G: S \to \mathcal{P}(X)$ is called lower semicontinuous (l.s.c.) if the set $\{s \in S: G(s) \subseteq C\}$ is closed in $S$ for any closed $C \subseteq X$.

According to [B-C] (Theorem 3) a l.s.c. map $G: S \to \mathcal{D}$ admits a continuous selection, i.e. there exists a continuous map $g: S \to L^1(I, X)$ such that $g(s) \in G(s)$ for all $s \in S$ (see also [F$_1$]).

Consider a map $F: I \times S \to \mathcal{P}(X)$ and set

$$G_F(s) = \{v \in L^1(I, X): v(t) \in F(t, s) \text{ a.e. in } I\}.$$  

The following proposition is a combined version of Proposition 2 and Theorem 3 from [B-C] and Proposition 2 from [F$_2$].

**Proposition 2.1.** Assume $F: I \times S \to \mathcal{P}(X)$ is $\mathcal{L} \otimes \mathcal{B}(S)$ measurable and l.s.c. in $s$. Then the map $s \to G_F(s)$ given by (2.2) is l.s.c. from $S$ into $\mathcal{D}$ iff there exists a continuous $\beta: S \to L^1(I, R)$ such that for every $s \in S$

$$\beta(s)(t) \geq d(0, F(t, s)) \text{ a.e. in } I.$$

**Proof.** The necessity is obvious since if $g(\cdot)$ is a continuous selection of $G_F(\cdot)$ then $\beta(s)(t) = |g(s)(t)|$ satisfies (2.3).

In order to prove that (2.3) is also sufficient, let $C \subseteq L^1(I, X)$ be an arbitrary closed set and let $s_n \rightarrow s_0$ be such that $G_F(s_n) \subseteq C$. Take any $v_0 \in G_F(s_0)$ and consider measurable selections $v_n(t)$ of $t \rightarrow F(t, s_n)$ such that

$$|v_n(t) - v_0(t)| < d(v_0(t), F(t, s_n)) + \frac{1}{n} \text{ a.e. in } I.$$
The existence of such \( v_n \) follows from Proposition 2 in [B-C]. Let us notice that since for every \( t \) the map \( s \rightarrow F(t, s) \) is l.s.c. then for every \( x \in X \)

\[
(2.5) \quad s \rightarrow d(x, F(t, s)) \text{ is u.s.c.}
\]

Therefore from (2.4) we obtain that

\[
(2.6) \quad u_n(t) \rightarrow v_0(t) \text{ a.e. in } I.
\]

We show that \( v_n \rightarrow v_0 \) in \( L^1(I, X) \). From (2.4) we have

\[
(2.7) \quad |v_n(t) - v_0(t)| < |v_0(t)| + \beta(s_n)(t) + \frac{1}{n} \text{ a.e. in } I.
\]

Denote by \( a_n(t) \) the right-hand side of (2.7) and observe that the sequence \( a_n(\cdot) \) is strongly convergent in \( L^1(I, R) \). Thus it is bounded in \( L^1(I, R) \) and uniformly integrable, so the same holds for the sequence of functions \( t \rightarrow |v_n(t) - v_0(t)| \). Therefore, \( v_n \rightarrow v_0 \) in \( L^1(I, X) \), because of (2.6).

Since \( C \) is closed and \( v_n \in C, v_0 \in C \) as well. But \( v_0 \) is an arbitrary point of \( G_F(s_0) \), hence \( G_F(s_0) \subseteq C \), that was to be proved.

Theorem 3 and Proposition 4 in [B-C] or Proposition 2.2 and Theorem 1 in [F.] imply:

**Proposition 2.2.** Consider a l.s.c. multivalued map \( G : S \rightarrow \mathcal{D} \) and assume that \( \phi : S \rightarrow L^1(I, X) \) and \( \psi : S \rightarrow L^1(I, R) \) are continuous maps and for every \( s \in S \) the set

\[
H(s) = \text{cl}\{u \in G(s) : |u(t) - \phi(s)(t)| < \psi(s)(t) \text{ a.e. in } I\}
\]

is nonempty. Then the map \( H : S \rightarrow \mathcal{D} \) is l.s.c., so it admits a continuous selection. (\( \text{cl} \) stands for the closure).

Consider a map \( F : I \times X \times S \rightarrow \mathcal{P}(X) \). We shall assume the following hypotheses on \( F \):

(H1): \( F \) is \( \mathcal{L} \otimes \mathcal{B}(X \times S) \) measurable.

(H2): There exists a map \( s \rightarrow k(\cdot, s) \) continuous from \( S \) into \( L^1(I, R) \) such that \( k(t, s) > 0 \) and for any \( s \in S \) and \( x, y \in X \),

\[
d_H(F(t, x, s), F(t, y, s)) \leq k(t, s)|x - y| \text{ a.e. in } I.
\]

(H3): For any \( (t, x) \) the map \( s \rightarrow F(t, x, s) \) is l.s.c.

(H4): For any continuous map \( s \rightarrow \gamma(\cdot, s) \) from \( S \) into \( AC(I, X) \), there exists a continuous map \( \beta : S \rightarrow L^1(I, R) \) such that for every \( s \in S \)

\[
(2.8) \quad \beta(s)(t) \geq d(\gamma(t, s), F(t, y(t, s), s)) \text{ a.e. in } I.
\]

Notice that due to (H2) and Proposition 2.1 the assumption (H4) may be
replaced by the equivalent condition:

\[(H4_0): \text{ There exists a continuous map } \beta_0: S \to L^1(I, \mathbb{R}) \text{ such that for any } s \in S \]

\[\beta_0(s)(t) \geq d(0, F(t, 0, s)) \text{ a.e. in } I.\]

Indeed, it easily follows from the inequality

\[(2.9) \ d(\dot{y}(t, s), F(t, y(t, s), s)) \leq |\dot{y}(t, s)| + d(0, F(t, 0, s)) + k(t, s)|y(t, s)| \text{ a.e. in } I.\]

Let us remark that from Proposition 2.1 and (H3) it follows that (H4) is also equivalent to the condition:

\[(H4') : \text{ For any continuous map } s \to y(\cdot, s) \text{ from } S \text{ into } AC(I, X), \text{ the map }\]

\[G_y(\cdot) \text{ defined by }\]

\[G_y(s) = \{v \in L^1(I, X) : v(t) \in F(t, y(t, s), s) \text{ a.e. in } I\}\]

is l.s.c. from \(S\) into \(\mathcal{D}\).

Similarly (H4_0) holds iff

\[(H4'_0): \text{ The map } G_0(\cdot) \text{ defined by }\]

\[(2.10) \ G_0(s) = \{v \in L^1(I, X) : v(t) \in F(t, 0, s) \text{ a.e. in } I\}\]

is l.s.c. from \(S\) into \(\mathcal{D}\).

Indeed, if \(s \to y(\cdot, s)\) is continuous from \(S\) into \(AC(I, X)\), then the map \(s \to F(t, y(t, s), s) - \dot{y}(t, s)\) is l.s.c. and \(\mathcal{L} \otimes \mathcal{B}(S)\) measurable, so we can apply Proposition 2.1.

3. Main result

Let \(F: I \times X \times S \to \mathcal{P}(X)\) and consider the following Cauchy problems

\[(P_s) \quad \dot{x} \in F(t, x, s), \quad x(0) = \xi(s),\]

where \(\xi: S \to X\) is a continuous function.

For given \(s\), by a solution of \((P_s)\) we mean a function \(x \in AC(I, X)\) with \(x(0) = \xi(s)\) such that

\[\dot{x}(t) \in F(t, x(t), s) \text{ a.e..}\]

The main result of this paper is the following

**Theorem 3.1.** Suppose \(F\) satisfies \((H1), \ldots, (H4)\). Then for any continuous map \(s \to y(\cdot, s)\) from \(S\) into \(AC(I, X)\) and \(s \to \beta(s) = \beta_y(s)\) from \(S\) into \(L^1(I, \mathbb{R})\) satisfying (2.8) and for every \(\epsilon > 0\), there exists a function \(x: I \times X\) such that

(a) For every \(s\) the function \(t \to x(t, s)\) is a solution of \((P_s)\).

(b) The map \(s \to x(\cdot, s)\) is continuous from \(S\) into \(AC(I, X)\).

(c) For every \(s \in S\)
Lipschitzian Differential Inclusions

\[ |\dot{y}(t, s) - \dot{x}(t, s)| \leq \epsilon + \epsilon k(t, s)e^{m(t, s)} + k(t, s)|y(0, s) - \xi(s)|e^{m(t, s)} + k(t, s) \int_0^t \beta(s) e^{m(t, s) - m(\tau, s)} d\tau + \beta(s) \] a.e. in I.

(d) For all \((t, s) \in I \times S\)

\[ |[y(t, s) - x(t, s)] - [y(0, s) - \xi(s)]| \leq \epsilon e^{m(t, s)} + |y(0, s) - \xi(s)|(e^{m(t, s)} - 1) + \int_0^t \beta(s) e^{m(t, s) - m(\tau, s)} d\tau, \]

where \(m(t, s) = \int_0^s k(t, s) d\tau\).

Remark. denote by \(\mathcal{R}(s)\) the closed subset of \(AC(I, X)\) consisting of all solutions of \((P_s)\). Theorem 3.1 provides the existence of a continuous selection of the map \(\mathcal{R}\). This implies the selection theorems due to Antosiewicz and Cellina [An-C], Bressan and Colombo [B-C] and Fryszkowski [F_1].

Proof of Theorem 3.1. We may assume that for any \((t, s) \in I \times S\)

\[ y(t, s) = 0 \quad \text{and} \quad \xi(s) = 0. \]

In fact, denote by

\[ \tilde{F}(t, z, s) = F(t, z + y(t, s) - y(0, s) + \xi(s), s) - \dot{y}(t, s) \]

and consider the problem

\[(\tilde{P}_s) \quad \dot{z} \in \tilde{F}(t, z, s), \quad z(0) = 0.\]

Now the function

\[ x(t, s) = z(t, s) + y(t, s) - y(0, s) + \xi(s) \]

is a desired solution of \((\tilde{P}_s)\), whenever \(z\) satisfies (a), ..., (d) for \((\tilde{P}_s)\) with

\[ \tilde{\beta}(s) = \beta(s) + |\dot{y}(t, s)| + k(t, s)|\xi(s) - y(0, s)| \geq d(0, \tilde{F}(t, 0, s)) \] a.e. in I.

Fix \(\epsilon > 0\), set \(\epsilon_n = ((n + 1)/(n + 2))\epsilon\) and put

\[ \beta_n(s) = \int_0^s \beta(u) \frac{(m(t, s) - m(u, s))^{n-1}}{(n-1)!} du + \frac{m(t, s)^{n-1}}{(n-1)!} \epsilon_n. \]

We shall construct a Cauchy sequence of successive approximations \(x_n(t, s),\)

\[ x_n(\cdot, s) \in AC(I, X), \] such that for all \(n \geq 0\), \(x_n(0, s) = 0\) and

(i) \(s \rightarrow x_n(\cdot, s)\) are continuous,

(ii) \(\dot{x}_{n+1}(t, s) \in F(t, x_n(t, s), s)\) a.e. in I,

(iii) \(|\dot{x}_{n+1}(t, s) - \dot{x}_n(t, s)| \leq k(t, s)\beta_n(s)\) a.e. in I,

where, for simplicity, \(k(t, s)\beta_0(s)\) is understood as \(\beta(s) + \epsilon_0\).
Remark that repeating for any \( s \) the calculations provided in [Au-C], (formula (14), page 122) we can conclude that

\[
(3.1) \quad \int_0^t k(u, s) \beta_n(s)(u) du = \int_0^t \beta(s)(u) \frac{(m(t, s) - m(u, s))^n}{n!} du + \frac{m(t, s)^n}{n!} \epsilon_n < \beta_{n+1}(s)(t) \text{ a.e. in } I.
\]

Therefore, from (iii), we also have

\[
(3.2) \quad |x_{n+1}(t, s) - x_n(t, s)| < \beta_{n+1}(s)(t) \text{ a.e. in } I.
\]

Set \( x_0(t, s) = 0 \) and denote by

\[
G_0(s) = \{ v \in L^1(I, X): v(t) \in F(t, x_0(t, s), s) \text{ a.e. in } I \}.
\]

Consider the map \( H_0 \) defined by

\[
H_0(s) = \text{cl}\{ v \in G_0(s): |v(t)| < \beta(s)(t) + \epsilon_0 \}.
\]

Proposisin 2.2 applied to \( H_0 \) implies the existence of a continuous map \( h_0: S \to L^1(I, X) \) such that

\[
h_0(s)(t) \in F(t, x_0(t, s), s) \text{ a.e. in } I
\]

and

\[
|h_0(s)(t)| \leq \beta(s)(t) + \epsilon_0.
\]

Define

\[
x_1(t, s) = \int_0^t h_0(s)(\tau) d\tau
\]

and notice that

\[
|x_1(t, s) - x_0(t, s)| \leq \int_0^t |h_0(s)(\tau) d\tau| < \int_0^t \beta(s)(\tau) d\tau + \epsilon_0 < \beta_1(s)(t) \text{ a.e. in } I.
\]

Suppose we have defined the functions \( x_0, \ldots, x_n \) satisfying (i), (ii) and (iii). Observe that

\[
d(\dot{x}_n(t, s), F(t, x_n(t, s), s)) \leq d_h(F(t, x_{n-1}(t, s), s), F(t, x_n(t, s), s)) \\
\leq k(t, s)|x_n(t, s) - x_{n-1}(t, s)|.
\]

The latter and (3.2) yield

\[
(3.3) \quad d(\dot{x}_n(t, s), F(t, x_n(t, s), s)) < k(t, s)\beta_n(s)(t) \text{ a.e. in } I.
\]
Denote

\[ G_n(s) = \{ v \in L^1(I, X) : v(t) \in F(t, x_n(t, s), s) \text{ a.e. in } I \} \]

and consider the map

(3.4) \[ H_n(s) = \text{cl}\{ v \in G_n(s) : |v(t) - \dot{x}_n(t, s)| < k(t, s)\beta_n(s)(t) \text{ a.e. in } I \}. \]

\( H_n(s) \) is nonempty because of (3.3). By Proposition 2.2 there exists a continuous map \( h_n : S \rightarrow L^1(I, X) \) such that

\[ h_n(s)(t) \in F(t, x_n(t, s), s) \text{ a.e. in } I \]

and

\[ |h_n(s)(t) - \dot{x}_n(t, s)| \leq k(t, s)\beta_n(s)(t) \text{ a.e. in } I. \]

Define

\[ x_{n+1}(t, s) = \int_0^t h_n(s)(\tau)d\tau. \]

Clearly, \( x_{n+1} \) satisfies (i), (ii) and (iii).

From (iii) and (3.1) we obtain that

(3.5) \[ \|x_{n+1}(\cdot, s) - x_n(\cdot, s)\|_{\text{AC}} \leq \beta_{n+1}(s)(1). \]

The right-hand side of (3.4) can be estimated by

\[ \beta_{n+1}(s)(1) \leq \int_0^1 \beta(s)(t) \frac{\|k(\cdot, s)\|^n}{n!} dt + \frac{m(1, s)^n}{n!} \varepsilon_{n+1}. \]

Therefore

\[ \beta_{n+1}(s)(1) \leq \frac{\|k(\cdot, s)\|^n}{n!} (\|\beta(s)\| + \varepsilon), \]

since

\[ m(t, s) - m(u, s) = \int_u^t k(\tau, s) d\tau \leq \|k(\cdot, s)\|, \]

and

\[ m(1, s) = \|k(\cdot, s)\|. \]

Hence we have

(3.6) \[ \|x_{n+1}(\cdot, s) - x_n(\cdot, s)\|_{\text{AC}} \leq \frac{\|k(\cdot, s)\|^n}{n!} (\|\beta(s)\| + \varepsilon). \]
The functions \( s \to \| \beta(s) \|_{AC} \) and \( s \to \| k(\cdot, s) \|_{AC} \) are continuous. Therefore, (3.6) implies that for every \( s \), the sequence \( \{ x_n(\cdot, s') \} \) satisfies the Cauchy condition uniformly in \( s' \) on some neighbourhood of \( s \). Hence \( s \to x(\cdot, s) \) where \( x(t, s) = \lim x_n(t, s) \) is continuous from \( S \) into \( AC(I, X) \). To see that the function \( t \to x(t, s) \) is a solution of (P) it is enough to notice that

\[
d(\hat{x}_{n+1}(t, s), F(t, x(t, s), s)) \leq k(t, s)|x_n(t, s) - x(t, s)|.
\]

We shall now prove (c) and (d).

By adding the inequalities (iii) for all \( n \), we obtain that

\[
|\hat{x}_{n+1}(t, s)| \leq \beta(s)(t) + \sum_{i=1}^{n} |\hat{x}_{i+1}(t, s) - \hat{x}_{i}(t, s)| + \varepsilon_0
\]

\[
\leq \beta(s)(t) + k(t, s) \int_0^t \beta(s)(u) \left[ \sum_{i=1}^{n} \frac{(m(t, s) - m(u, s))^i}{(i-1)!} \right] du
\]

\[
+ \varepsilon k(t, s) \left[ \sum_{i=1}^{n} \frac{m(t, s)^{i-1}}{(i-1)!} \right] + \varepsilon.
\]

Similarly, by adding (3.2) we get

\[
|x_{n+1}(t, s)| \leq \sum_{i=0}^{n} |x_{i+1}(t, s) - x_{i}(t, s)|
\]

\[
\leq \int_0^t \beta(s)(u) \left[ \sum_{i=0}^{n} \frac{(m(t, s) - m(u, s))^i}{i!} \right] du + \varepsilon \left[ \sum_{i=0}^{n} \frac{m(t, s)^i}{i!} \right].
\]

So, by passing to the limit and using the identity \( e^{-m(t,s)} + \int_0^t k(u, s)e^{-m(u,s)} du = 1 \) we obtain (c) and (d). This ends the proof.

4. Properties of the solution sets

In what follows we assume that \( F: I \times X \times S \to \mathcal{P}(X) \) satisfies (H1), \ldots, (H4). Denote by \( \mathcal{R}(s) \) the closed subset of \( AC(I, X) \) consisting of all solutions of (P). From Theorem 3.1 we already know that \( s \to \mathcal{R}(s) \) admits a continuous selection. Now we shall provide some other properties of this map.

Theorem 4.1. Fix \( s_0 \in S \) and \( x_0 \in \mathcal{R}(s_0) \). Then there exists a continuous selection \( r: S \to AC(I, X) \) of \( \mathcal{R} \) such that \( r(s_0) = x_0 \).

Proof. Using the same argument as at the beginning of the proof of Theorem 3.1 we may assume that \( x_0 = 0 \) so we have

\[
0 \in F(t, 0, s_0).
\]

Consider the map \( F_0: I \times X \times S \to \mathcal{P}(X) \) defined by
\[ F_\ast(t, x, s) = \begin{cases} F(t, x, s) & \text{if } s \neq s_0 \\ \{0\} & \text{if } s = s_0. \end{cases} \]

Clearly, \( F_\ast \) satisfies (H1), (H2) and (H3). We claim that also (H4) holds and moreover one can choose a continuous \( \beta_\ast: S \to L^1(I, R) \) with
\[ \beta_\ast(s_0) = 0. \]

From the definition of \( F_\ast \), we see that
\[ d(0, F_\ast(t, 0, s)) = d(0, F(t, 0, s)). \]

Consider
\[ P(s) = \text{cl}\{v \in L^1(I, R): v(t) > d(0, F_\ast(t, 0, s)) \text{ a.e. in } I\} \]
and notice that
\[ 0 \in P(s_0). \]

By Proposition 2.1, \( P(\cdot) \) is l.s.c. from \( S \) into \( \mathcal{D} \). Therefore it admits a continuous selection \( \beta_\ast(\cdot) \) such that \( \beta_\ast(s_0) = 0 \). This proves the claim.

Repeating the same construction as in the proof of Theorem 3.1, we see that \( 0 \in H_n(s_0) \) for any \( n, H_n(s) \) as in (3.4). So, we can always choose a continuous selection \( h_n \) of \( H_n \) such that \( h_n(s_0) = 0 \). Hence, the sequence of approximate solutions \( x_n(t, s) \) is such that for all \( n, x_n(\cdot, s_0) = 0 \) and the same holds for the limit. This completes the proof.

Denote by \( r_{s_0, x_0} \) a selection of \( \mathcal{R} \) such that
\[ r_{s_0, x_0}(s_0) = x_0. \]

Clearly, for every \( s \in S \)
\[ \mathcal{R}(s) = \{r_{s_0, x_0}(s): s_0 \in S, x_0 \in \mathcal{R}(s_0)\}. \]

**Theorem 4.2.** The map \( \mathcal{R}: S \to \mathcal{P}(AC(I, X)) \) is l.s.c. and admits a continuous selection. Moreover, if \( S \) is compact, then there exists a countable family \( \{r_n\} \) of selections of \( \mathcal{R} \) such that
\[ \mathcal{R}(s) = \text{cl}\{r_n(s): n \in N\}. \]

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