On the Number of Invariant Polynomials of Matrix Commutators

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Abstract

We study the possible numbers of nonconstant invariant polynomials of the matrix commutator $XA - AX$, when X varies.

Let F be a field, $A, B \in F^{n \times n}$ and denote by $i(A)$ the number of nonconstant invariant polynomials of A .

In [7], it was proved that, if there exists X such that $A + XBX^{-1}$ is nonderogatory (i.e., $i(A + XBX^{-1}) = 1$), then $i(A) + i(B) \leq n + 1$; and it was conjectured that the converse is true, under rather slight restrictions on F . This conjecture was proved in a theorem [9] that gives necessary and sufficient conditions for the existence of X such that $i(A + XBX^{-1}) \leq t$, where t is a positive integer. Later [11], all the possible values of $i(A + XBX^{-1})$, when X varies, were described, assuming that F is algebraically closed. In general fields, this is an open problem.

The possible numbers of nonconstant invariant polynomials of partially given matrices were also studied in several papers, e. g., [1, 4, 5, 10].

Some properties of the commutator $XA - AX$, when X varies, have already been studied. Suppose that D is a division ring and $A \in D^{n \times n}$. The

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rank of $XA - AX$, when X runs over $D^{n \times n}$, was studied in [2]. The same problem, when X runs over the set of the nonsingular matrices of $D^{n \times n}$, was studied in [8]. The eigenvalues of $XA - AX$, when X runs over $F^{n \times n}$ and also when X runs over the set of the nonsingular matrices of $F^{n \times n}$, were studied in [6].

Given $A \in F^{n \times n}$, the following theorem solves the problem of characterizing the possible values of $i(XA - AX)$, when X varies, assuming that all the irreducible polynomials in $F[x]$ have degree ≤ 2 . In particular, the problem is solved for algebraically closed fields and for the field of real numbers, $\mathbb{R}$. We shall prove Theorem 1 later. Observe that a field F such that all the irreducible polynomials in $F[x]$ have degree ≤ 2 has to be infinite.

From now on, A denotes an $n \times n$ matrix over F , $f_1(x) \mid \cdots \mid f_r(x)$, $r = i(A)$, are the nonconstant invariant polynomials of A , and $t \in \{1, \dots, n\}$. We assume that the invariant polynomials are always monic.

Theorem 1 *Suppose that F is a field such that all the irreducible polynomials in $F[x]$ have degree ≤ 2 . Let $A \in F^{n \times n}$, $t \in \{1, \dots, n\}$. The following conditions are equivalent:*

- (a₁) *There exists a nonsingular matrix $X \in F^{n \times n}$ such that $i(XA - AX) = t$.*
- (b₁) *There exists $X \in F^{n \times n}$ such that $i(XA - AX) = t$.*
- (c₁) *One of the following conditions holds:*
 - (i₁) *$f_r(x)$ is irreducible of degree 2 and t is even.*
 - (ii₁) *$f_r(x)$ is irreducible of degree 2 and $t \leq n/2$.*
 - (iii₁) *$f_r(x)$ is not irreducible of degree 2 and $2i(A) \leq n + t$.*

Corollary 2 *Suppose that F is an algebraically closed field. Then (a₁), (b₁) and the following condition (c₂) are equivalent:*

- (c₂) $2i(A) \leq n + t$.

Given a polynomial $f(x) = x^k - a_{k-1}x^{k-1} - \cdots - a_1x - a_0$, denote by $d(f)$ the degree of f and denote by $C(f)$ the companion matrix

$$\left[\begin{array}{c|ccc} 0 & & & \\ \hline a_0 & a_1 & \cdots & a_{k-1} \end{array} \right].$$

Let $\bar{F}$ be an algebraically closed extension of F . Let

$$R_{\bar{F}}(A) = \min_{\lambda \in \bar{F}} \text{rank}(A - \lambda I_n).$$

Lemma 3 [7] $R_{\bar{F}}(A) = n - i(A)$.

It is well-known that A is nonderogatory if and only if $i(A) = 1$. It follows, from the previous lemma, that A is nonderogatory if and only if $R_{\bar{F}}(A) = n - 1$.

Lemma 4 [2, 8] *Suppose that either $F \neq \{0, 1\}$ or $n \neq 2$. Let $\rho \in \{0, \dots, n\}$. The following statements are equivalent:*

- (a₄) *There exists a nonsingular matrix $X \in F^{n \times n}$ such that $\text{rank}(XA - AX) = \rho$.*
- (b₄) *There exists $X \in F^{n \times n}$ such that $\text{rank}(XA - AX) = \rho$.*
- (c₄) *One of the following conditions holds:*
 - (i₄) *$f_r(x)$ is irreducible of degree 2 and ρ is even.*
 - (ii₄) *$f_r(x)$ is irreducible of degree ≥ 3 and $\rho \neq 1$.*
 - (iii₄) *$f_r(x)$ is not irreducible of degree ≥ 2 and $\rho \leq 2R_{\bar{F}}(A)$.*

Remark 1 *In the original papers, Lemma 4 was established, with a slightly different statement, for arbitrary division rings. More precisely, [2] gives a necessary and sufficient condition for (b₄) and [8] gives a necessary and sufficient condition for (a₄).*

Lemma 5 [6] *Let $c_1, \dots, c_n$ be elements of F such that $c_1 + \dots + c_n = 0$. If $2i(A) \leq n$ and $d(f_r) \geq 3$, then there exists a nonsingular matrix $X \in F^{n \times n}$ such that $XA - AX$ has eigenvalues $c_1, \dots, c_n$.*

If $A' \in F^{n \times n}$ is similar to A and $Z \in F^{n \times n}$ is a nonsingular matrix such that $A' = Z^{-1}AZ$, then $R_{\bar{F}}(A) = R_{\bar{F}}(A')$ and, for every $X \in F^{n \times n}$, $XA - AX$ and $(Z^{-1}XZ)A' - A'(Z^{-1}XZ)$ are similar. Therefore, in the proofs of the

following lemmas and in the proof of Theorem 1, we can replace A by any similar matrix and we shall assume, without loss of generality, that

$$A = C(f_1) \oplus \cdots \oplus C(f_r). \quad (1)$$

Assume that A has the form (1). Then, for every $\lambda \in \bar{F}$, $\text{rank}(A - \lambda I_n) \geq n - r$. The equality holds if and only if λ is a root of $f_1(x)$. Note that this argument gives a proof for Lemma 3. The eigenvalues of A are the roots of $f_1(x) \cdots f_r(x)$. We call *primary* eigenvalues of A to the roots of $f_1(x)$.

If A has a primary eigenvalue $\lambda \in F$, then $R_{\bar{F}}(A) = R_{\bar{F}}(A - \lambda I_n)$ and, for every $X \in F^{n \times n}$, $XA - AX = X(A - \lambda I_n) - (A - \lambda I_n)X$. Therefore, in the proofs of the following lemmas and in the proof of Theorem 1, if A has a primary eigenvalue in F , we shall also assume, without loss of generality, that 0 is a primary eigenvalue of A . Note that, in this case, the first column of $C(f_i)$, $i \in \{1, \dots, r\}$, is equal to zero.

Corollary 6 *Suppose that F is an infinite field. Let $c \in F$. If $n \geq 4$, $2i(A) \leq n$ and $d(f_r) \geq 3$, then there exists a nonsingular matrix $X \in F^{n \times n}$ such that $XA - AX$ is nonderogatory and c is eigenvalue of $XA - AX$.*

Proof. As F is infinite and $n \geq 4$, there exist distinct elements $c_1, \dots, c_n \in F$ such that $c_1 = c$ and $c_1 + \cdots + c_n = 0$. According to Lemma 5, there exists a nonsingular matrix $X \in F^{n \times n}$ such that $XA - AX$ has eigenvalues $c_1, \dots, c_n$. Clearly, $XA - AX$ is nonderogatory. ■

The following corollary is also easy to prove.

Corollary 7 *Suppose that F is an infinite field. Let G be a finite subset of F . If $2i(A) \leq n$ and $d(f_r) \geq 3$, then there exists a nonsingular matrix $X \in F^{n \times n}$ such that $XA - AX$ is nonderogatory and does not have any eigenvalue in G .*

Lemma 8 *Suppose that F is an infinite field. If $C \in F^{2 \times 2}$ is nonderogatory, then the set of the values $\det(XC - CX)$, where X runs over the set of the nonsingular matrices of $F^{2 \times 2}$ such that $XC - CX$ is nonderogatory, is infinite.*

Proof. Without loss of generality, suppose that

$$C = \begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix}. \quad (2)$$

Choose $u \in F \setminus \{0, 1\}$. For every $v \in F$, let

$$X_v = \begin{bmatrix} u-1 & 0 \\ v & u \end{bmatrix}.$$

Then X_v is nonsingular, $X_v C - C X_v$ is nonderogatory and $\det(X_v C - C X_v) = -v^2 - bv + a$. For every $\lambda \in F$, the quadratic equation, in v , $-v^2 - bv + a = \lambda$ has, at most, 2 roots. Therefore $\{-v^2 - bv + a : v \in F\}$ is infinite. $\blacksquare$

Lemma 9 *Suppose that F is an infinite field. Let G be a finite subset of $\bar{F}$. If $d(f_1) = \dots = d(f_r) = 2$, then there exists a nonsingular matrix $X \in F^{n \times n}$ such that $XA - AX$ is nonderogatory and does not have any eigenvalue in G .*

Proof. We have $f_1 = \dots = f_r$. Let $C = C(f_1)$. It follows, from Lemma 8, that there exists an infinite list of nonsingular matrices $X_1, X_2, \dots$, in $F^{2 \times 2}$, such that $D_i := X_i C - C X_i$ is nonderogatory and $\det D_i \neq \det D_j$, $i, j \in \{1, 2, \dots\}$, $i \neq j$. As all the matrices D_i have trace equal to zero, it is easy to deduce that, if $i \neq j$, then D_i and D_j do not have a common eigenvalue. Without loss of generality, we may assume that none of the matrices $D_1, \dots, D_r$ has an eigenvalue in G . Then $XA - AX$, where $X = X_1 \oplus \dots \oplus X_r$, is nonderogatory and does not have any eigenvalue in G . $\blacksquare$

Lemma 10 *Suppose that $F \neq \{0, 1\}$. Let $\rho \in \{2, \dots, n-1\}$. If A is nonderogatory, then there exists a nonsingular matrix $X \in F^{n \times n}$ such that $XA - AX$ is nilpotent and $R_{\bar{F}}(XA - AX) = \rho$.*

Proof. According to a previous assumption, $A = C(f_1)$. If ρ is even, let X be the $n \times n$ matrix with the principal entries, and the entries $(2k+1, 2k-1)$, $k \in \{1, \dots, \rho/2\}$, equal to 1.

If ρ is odd, let $e \in F \setminus \{0, 1\}$ and let X be the $n \times n$ matrix with the principal entries, and the entries $(2k+1, 2k-1)$, $k \in \{1, \dots, (\rho-1)/2\}$, equal to 1 and the entry $(\rho+1, \rho-1)$ equal to e .

For any value of ρ , $XA - AX$ is lower triangular with the principal entries equal to zero, the entries $(i + 1, i)$, $i \in \{1, \dots, \rho\}$, different from zero, and the columns $\rho + 1, \dots, n$ equal to zero. Clearly, $XA - AX$ is nilpotent and $R_{\bar{F}}(XA - AX) = \rho$. $\blacksquare$

Lemma 11 *Suppose that F is an infinite field. Let $\rho \in \{0, \dots, n - 1\}$. Condition (c_4) is equivalent to any of the following conditions:*

- (a_{11}) *There exists a nonsingular matrix $X \in F^{n \times n}$ such that $R_{\bar{F}}(XA - AX) = \text{rank}(XA - AX) = \rho$.*
- (b_{11}) *There exists $X \in F^{n \times n}$ such that $R_{\bar{F}}(XA - AX) = \text{rank}(XA - AX) = \rho$.*

Proof. We only need to prove that (c_4) implies (a_{11}) . This proof is by induction on n . Suppose that (c_4) is satisfied. Note that, if $\rho = 0$, then (a_{11}) is trivial. Thus, suppose that $\rho > 0$. Then (c_4) implies that A is nonscalar.

Suppose that $\rho = 1$. According to Lemma 4, there exists a nonsingular matrix $X \in F^{n \times n}$ such that $\text{rank}(XA - AX) = 1$. Then $R_{\bar{F}}(XA - AX) = 1$.

We have already proved Lemma 11 when $n \leq 2$.

Suppose that $n \geq 3$ and that $\rho \geq 2$. The case $r = 1$ follows, immediately, from Lemma 10. Then, we also suppose that $r \geq 2$.

Case 1. Suppose that $\rho = n - 1$.

Subcase 1.1. Suppose that $d(f_r) \geq 3$. As $r \geq 2$, we have $n \geq 4$. From (c_4) , it follows that $2i(A) \leq n + 1$. If $2i(A) \leq n$, then, according to Corollary 6, there exists a nonsingular matrix $X \in F^{n \times n}$ such that $XA - AX$ is nonderogatory and 0 is eigenvalue of $XA - AX$, that is, (a_{11}) is satisfied, with $\rho = n - 1$.

Now suppose that $2i(A) = n + 1$. Then $f_1(x)$ has degree 1 and $A = [c] \oplus A_0$, where $c \in F$ and $A_0 = C(f_2) \oplus \dots \oplus C(f_r)$. According to Corollary 7, there exists a nonsingular matrix $X_0 \in F^{(n-1) \times (n-1)}$ such that $X_0 A_0 - A_0 X_0$ is nonderogatory and nonsingular. Taking $X = [1] \oplus X_0$, $XA - AX$ is nonderogatory and 0 is eigenvalue of $XA - AX$, that is, (a_{11}) is satisfied.

Subcase 1.2. Suppose that $d(f_r) = 2$. As (c_4) is satisfied, $f_r(x)$ is reducible and, therefore, A has a primary eigenvalue in F . According to a previous assumption, 0 is a primary eigenvalue of A .

Note that, if n is even, then $r = n/2$ and $f_1(x) = \dots = f_r(x)$; and, if n is odd, then $r = (n + 1)/2$, $f_1(x) = x$ and $f_2(x) = \dots = f_r(x)$.

Let $A_1 = C(f_1)$ and $A_2 = C(f_2) \oplus \cdots \oplus C(f_r)$. If n is even, then there exists, as we have already seen, a nonsingular matrix $X_1 \in F^{2 \times 2}$ such that $R_{\bar{F}}(X_1 A_1 - A_1 X_1) = \text{rank}(X_1 A_1 - A_1 X_1) = 1$. If n is odd, let $X_1 = [1] \in F^{1 \times 1}$.

In any case, according to Lemma 9, there exists a nonsingular matrix $X_2 \in F^{n' \times n'}$, where n' is the largest even integer less than n , such that $X_2 A_2 - A_2 X_2$ is nonderogatory and does not have any eigenvalue in common with $X_1 A_1 - A_1 X_1$.

Let $X = X_1 \oplus X_2$. Then $R_{\bar{F}}(X A - A X) = \text{rank}(X A - A X) = n - 1$.

Case 2. Suppose that $\rho < n - 1$. Let $\delta = d(f_1)$, $\rho_1 = \min\{\delta - 1, \rho\}$, $\rho_2 = \rho - \rho_1$, $A_1 = C(f_1)$, $A_2 = C(f_2) \oplus \cdots \oplus C(f_r)$. Then $R_{\bar{F}}(A_1) = \delta - 1$ and $R_{\bar{F}}(A_2) = R_{\bar{F}}(A) - \delta + 1$. If $\delta = 1$, we have $2R_{\bar{F}}(A_2) = 2R_{\bar{F}}(A) \geq \rho = \rho_2$. If $\delta > 1$ and $\rho_1 = \delta - 1$, we have $2R_{\bar{F}}(A_2) \geq n - \delta > \rho - \delta + 1 = \rho_2$. If $\rho_1 = \rho$, then $\rho_2 = 0$ and we also have $2R_{\bar{F}}(A_2) \geq \rho_2$.

If the induction assumption can be used, then there exist nonsingular matrices $X_1 \in F^{\delta \times \delta}$, $X_2 \in F^{(n-\delta) \times (n-\delta)}$ such that $R_{\bar{F}}(X_1 A_1 - A_1 X_1) = \text{rank}(X_1 A_1 - A_1 X_1) = \rho_1$ and $R_{\bar{F}}(X_2 A_2 - A_2 X_2) = \text{rank}(X_2 A_2 - A_2 X_2) = \rho_2$. Let $X = X_1 \oplus X_2$. Clearly, $R_{\bar{F}}(X A - A X) = \text{rank}(X A - A X) = \rho$.

Now suppose that the induction assumption cannot be used in the previous argument. Then one, at least, of the following conditions is satisfied:

(a'_{11}) $f_1(x)$ is irreducible and $\rho_1 = 1$.

(b'_{11}) $f_r(x)$ is irreducible and $\rho_2 = 1$.

Subcase 2.1. Suppose that (a'_{11}) is satisfied.

Subcase 2.1.1. Suppose that $\rho < n - 2$. Note that, if $\rho_1 = \rho$, then (c_4) implies that $f_r(x)$ is reducible; and that, if $\rho_1 = \delta - 1 < \rho$ and ρ is odd, then (c_4) also implies that $f_r(x)$ is reducible. In any situation, according to the induction assumption, there exists a nonsingular matrix $X_2 \in F^{(n-\delta) \times (n-\delta)}$ such that $R_{\bar{F}}(X_2 A_2 - A_2 X_2) = \text{rank}(X_2 A_2 - A_2 X_2) = \rho$. Let $X = I_\delta \oplus X_2$. Then $R_{\bar{F}}(X A - A X) = \text{rank}(X A - A X) = \rho$.

Subcase 2.1.2. Suppose that $\rho = n - 2$. Then $1 = \rho_1 = \delta - 1 < \rho$. According to Corollary 7 or Lemma 9, there exists a nonsingular matrix $X_2 \in F^{(n-2) \times (n-2)}$ such that $X_2 A_2 - A_2 X_2$ is nonderogatory and 0 is not eigenvalue of $X_2 A_2 - A_2 X_2$. Let $X = I_\delta \oplus X_2$. Then $R_{\bar{F}}(X A - A X) = \text{rank}(X A - A X) = \rho$.

Subcase 2.2. Suppose that (a'_{11}) is false and that (b'_{11}) is satisfied. Then $1 < \rho_1 = \delta - 1 < \rho$.

Firstly, suppose that $\rho_1 > 2$. According to the induction assumption, there exist nonsingular matrices $X_1 \in F^{\delta \times \delta}$ and $X_2 \in F^{(n-\delta) \times (n-\delta)}$ such that $R_{\bar{F}}(X_1 A_1 - A_1 X_1) = \text{rank}(X_1 A_1 - A_1 X_1) = \rho - 2$ and $R_{\bar{F}}(X_2 A_2 - A_2 X_2) = \text{rank}(X_2 A_2 - A_2 X_2) = 2$. Let $X = X_1 \oplus X_2$. Then $R_{\bar{F}}(X A - A X) = \text{rank}(X A - A X) = \rho$.

Now suppose that $\rho_1 = 2$. According to Corollary 7, there exists a nonsingular matrix $X_1 \in F^{3 \times 3}$ such that $X_1 A_1 - A_1 X_1$ is nonderogatory and 0 is not eigenvalue of $X_1 A_1 - A_1 X_1$. Let $X = X_1 \oplus I_{n-3}$. Then $R_{\bar{F}}(X A - A X) = \text{rank}(X A - A X) = 3 = \rho$. $\blacksquare$

Now suppose that $C \in F^{2 \times 2}$ is a matrix of the form (2) and that the characteristic polynomial of C is irreducible. According to [3], $K = F[C]$ is a field and, according to [2], there exists $B \in F^{2 \times 2}$ such that $U = BC - CB$ is nonsingular and

$$\{XC - CX \mid X \in F^{2 \times 2}\} = UK = KU.$$

Let $S = [S_{i,j}] \in \bar{F}^{2p \times 2p}$, where the blocks $S_{i,j}$ are of size 2×2 . We shall say that S is a $\star$ -matrix if $S_{i,j} \in UK$, whenever $i \leq j$.

Lemma 12 *With the previous notation, suppose that $b \neq 0$. Let $S = [S_{i,j}] \in \bar{F}^{2p \times 2p}$, $S_{i,j} \in \bar{F}^{2 \times 2}$, be a $\star$ -matrix and $\lambda \in \bar{F} \setminus \{0\}$. Then $\text{rank}(S - \lambda I_{2p}) \geq p$.*

Proof. By induction on p . Firstly, note that a commutator $XC - CX$ is not scalar, unless it is the zero matrix. Therefore, the lemma is true when $p = 1$.

Suppose that $p \geq 2$. Suppose that $S_{1,j} = 0$, for every $j \in \{2, \dots, p\}$. Considering S as a $p \times p$ matrix with entries in $\bar{F}^{2 \times 2}$, let S_0 be the principal submatrix obtained from S by deleting the first row and the first column. Using the induction assumption,

$$\text{rank}(S - \lambda I_{2p}) \geq \text{rank}(S_{1,1} - \lambda I_2) + \text{rank}(S_0 - \lambda I_{2p-2}) \geq p.$$

Now suppose that $S_{1,j} \neq 0$, for some $j \in \{2, \dots, p\}$. Choose the maximum $v \in \{2, \dots, p\}$ such that $S_{1,v} \neq 0$. Consider $S - \lambda I_{2p}$ as a $p \times p$ matrix with entries in $\bar{F}^{2 \times 2}$. For each $w \in \{2, \dots, v-1\}$, subtract the v th column multiplied by $S_{1,v}^{-1} S_{1,w}$ from the w th column. It is not hard to see that the matrix obtained has the form $S' - \lambda I_{2p}$, where $S' = [S'_{i,j}]$, $S'_{i,j} \in \bar{F}^{2 \times 2}$, is a $\star$ -matrix. Clearly, $\text{rank}(S - \lambda I_{2p}) = \text{rank}(S' - \lambda I_{2p})$. If $p = 2$, then

$\text{rank}(S' - \lambda I_4) \geq \text{rank } S'_{1,2} = 2$. Suppose that $p \geq 3$. Considering S' as a matrix with entries in $\bar{F}^{2 \times 2}$, let S_0 be the principal submatrix obtained from S' by deleting the first and the v th rows and columns. Using the induction assumption,

$$\text{rank}(S' - \lambda I_{2p}) \geq \text{rank}(S_0 - \lambda I_{2p-4}) + \text{rank } S'_{1,v} \geq p. \quad \blacksquare$$

Lemma 13 *Suppose that $f_r(x)$ is irreducible of degree 2. If $\rho < n/2$ and ρ is odd, then there is no matrix $X \in F^{n \times n}$ such that $R_{\bar{F}}(XA - AX) = \rho$.*

Proof. Suppose that $R_{\bar{F}}(XA - AX) = \rho$, where $X \in F^{n \times n}$, $\rho < n/2$ and ρ is odd. Take $\lambda \in \bar{F}$ such that $\text{rank}(XA - AX - \lambda I_n) = \rho$. As $\rho < n/2$, it can be deduced that $\lambda \in F$.

According to [6], the eigenvalues of $XA - AX$ can be joined in pairs so that the sum of the two eigenvalues of each pair is 0. Therefore $2\lambda = 0$. Assuming that F has characteristic different from 2, we have $\lambda = 0$ and $\text{rank}(XA - AX) = \rho$, what contradicts Lemma 4.

Suppose that the companion matrix of $f_r(x)$ has the form (2).

Suppose that $b \neq 0$. As A has the form $C \oplus \cdots \oplus C$, $XA - AX$ is a $\star$ -matrix. According Lemmas 4 and 12, the equality $\text{rank}(XA - AX - \lambda I_n) = \rho$ is impossible.

Finally, suppose that F has characteristic 2 and that $b = 0$. Take

$$Y = \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \in F^{n \times n}.$$

Then $YA - AY = \lambda I_n$ and $(X - Y)A - A(X - Y) = XA - AX - \lambda I_n$ has rank ρ , what contradicts Lemma 4. $\blacksquare$

Bearing in mind Lemma 3, it is clear that Theorem 1 follows, immediately, from the following lemma.

Lemma 14 *Let F be a field such that all the irreducible polynomials in $F[x]$ have degree ≤ 2 . Let $A \in F^{n \times n}$, $\rho \in \{0, \dots, n-1\}$.*

The following statements are equivalent:

- (a₁₄) *There exists a nonsingular matrix $X \in F^{n \times n}$ such that $R_{\bar{F}}(XA - AX) = \rho$.*
- (b₁₄) *There exists $X \in F^{n \times n}$ such that $R_{\bar{F}}(XA - AX) = \rho$.*

(c_{14}) One of the following conditions holds:

- (i_{14}) $f_r(x)$ is irreducible of degree 2 and ρ is even.
- (ii_{14}) $f_r(x)$ is irreducible of degree 2 and $\rho \geq n/2$.
- (iii_{14}) $f_r(x)$ is not irreducible of degree 2 and $\rho \leq 2R_{\bar{F}}(A)$.

Proof. Suppose that (b_{14}) is satisfied. Suppose that $f_r(x)$ is irreducible of degree 2. According to Lemma 13, ρ is even or $\rho \geq n/2$. Now suppose that $f_r(x)$ is not irreducible of degree 2. Then, using Lemma 4,

$$\rho = R_{\bar{F}}(XA - AX) \leq \text{rank}(XA - AX) \leq 2R_{\bar{F}}(A).$$

Conversely, suppose that (c_{14}) is satisfied, in order to prove (a_{14}). If one of the conditions (i_{14}), (iii_{14}) is satisfied, then, according to Lemma 11, (a_{14}) holds. Now suppose that $f_r(x)$ is irreducible of degree 2 and $\rho \geq n/2$. Let $s := n - \rho \leq n/2 = r$. We have $A = C \oplus \cdots \oplus C$, where $C = C(f_1) = \cdots = C(f_r)$. Let $Y \in F^{2 \times 2}$ be a nonsingular matrix such that $YC - CY$ is nonderogatory. Let $\lambda_1, \lambda_2 \in \bar{F}$ be the eigenvalues of $YC - CY$. Let $X_1 = Y \oplus \cdots \oplus Y \in F^{2s \times 2s}$.

If $s = r$, let $X = X_1$.

If $s < r$, let $A_1 = C(f_1) \oplus \cdots \oplus C(f_s)$, $A_2 = C(f_{s+1}) \oplus \cdots \oplus C(f_r)$. According to Lemma 9, there exists a nonsingular matrix $X_2 \in F^{2s \times 2s}$ such that $X_2A_2 - A_2X_2$ is nonderogatory and does not have any eigenvalue in $\{\lambda_1, \lambda_2\}$. Let $X = X_1 \oplus X_2$.

For any value of s , it is not hard to deduce that

$$\rho = R_{\bar{F}}(XA - AX) = \text{rank}(XA - AX - \lambda_1 I_n). \quad \blacksquare$$

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