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Article

Optimizing Variational Problems through Weighted Fractional Derivatives

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Abstract: In this article, we explore a variety of problems within the domain of calculus of variations, specifically in the context of fractional calculus. The fractional derivative we consider incorporates the notion of weighted fractional derivatives along with derivatives with respect to another function. Besides the fractional operators, the Lagrange function depends on extremal points. We examine the fundamental problem, providing the fractional Euler–Lagrange equation and the associated transversality conditions. Both the isoperimetric and Herglotz problems are also explored. Finally, we conclude with an analysis of the variational problem, incorporating fractional derivatives of any positive real order.

Keywords: weighted fractional derivative; fractional calculus of variations; Euler–Lagrange equation

MSC: 26A33; 49K05

1. Introduction

The field of fractional calculus has garnered significant attention and relevance in the past three decades, owing to its demonstrated utility across various scientific and engineering disciplines. Offering a range of potentially valuable tools, it has demonstrated vast applications in modeling complex phenomena that occur in nature. Fractional calculus finds applications across various fields, such as physics [1–3], epidemiology [4–6], engineering [7,8], optimal control [9–11], economics [12,13], and beyond.

It is widely believed that fractional calculus emerged from a query posed in 1695 by L'Hôpital to Leibniz. L'Hôpital asked for a clarification on the interpretation of Leibniz's notation $d^n y/dx^n$ for derivatives of non-integer orders, particularly when $n = 1/2$. In his response, Leibniz remarked that it posed an apparent paradox, yet he hinted at its potential for yielding valuable insights in the future. Subsequent references to fractional derivatives were made by notable mathematicians and physicists such as Euler in 1730, Lagrange in 1772, Laplace in 1812, and a host of others through the 19th and early 20th centuries, each contributing to the development and understanding of this intriguing branch of mathematics.

Liouville authored a series of articles, and this marked the introduction of the first fractional integration operator. Riemann then expanded upon this work, pioneering what is now known as the Riemann–Liouville definition. This period saw unprecedented interest and advancement in the field, until the 1960s, when a revision was made. Many authors, including Caputo, sought a new definition of fractional differentiation to address applied problems. The Grunwald–Letnikov fractional derivative aims to generalize the classical definition of differentiation to accommodate arbitrary derivative orders. In the literature, a variety of definitions for fractional derivatives can be found. Alongside those previously mentioned, additional concepts such as Hadamard, Hilfer, Weyl, Riesz, tempered, distributed-order, and others are also explored.

In the work by Jarad et al. [14], the notion of weighted fractional operators with respect to another function was introduced, accompanied by an in-depth exploration of their

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fundamental properties. Additionally, a modified Laplace transform for these derivatives was developed. Subsequently, Thabet et al. [15] extended these concepts by incorporating a generalized Mittag–Leffler function. Fernandez and Fahad [16] further investigated similar ideas, although with a different differential operator featuring the weight. By exploiting conjugation relationships with classical fractional operators, numerous properties were derived. In [17], a time-fractional diffusion equation involving weighted fractional derivatives was studied. Additionally, variable fractional order and distributed order derivatives were considered. Boundary value problems concerning weighted fractional derivatives were thoroughly examined in the study [18], where the existence of solutions and their stability in the Ulam–Hyers–Rassias sense were established.

Another possible approach to fractional derivatives involves the concept of a derivative with respect to another function [19,20]. Here, an arbitrary function is incorporated into the integral, and for certain selections of this function, we can retrieve classical ones such as Riemann-Liouville, Caputo, or Hadamard. This approach helps mitigate the proliferation of works for different derivatives and, owing to its arbitrariness, allows for a better adaptation of the process dynamics to the fractional derivative.

In this study, we integrated the two aforementioned concepts by exploring derivatives with respect to another function and with weights. Using this derivative, we aimed to investigate various problems within the calculus of variations. This field focuses on optimizing problems, typically around integrals that vary with time, an unknown state function, and its derivative. Instead of the conventional first-order derivative, we employed a fractional derivative. Our goal was to outline the necessary conditions that the optimal curve must fulfill. The Lagrange function we considered depends on time, the state function, both its sided fractional derivatives, and the initial and final points. For related investigations involving initial and final positions within the Lagrange function, we can reference works such as [21–24]. We obtained several necessary optimization conditions for functionals, depending on the variables mentioned earlier. The first one is known in the literature as the Fractional Euler–Lagrange equation. Other cases were considered: where the final time of the integral is also free, when the integration interval of the functional is distinct from that of the fractional derivatives, with an integral constraint in the space of admissible functions, with derivatives of any positive real order, and the generalized Herglotz problem. An interesting question, which remains unanswered, is how to present a generalization of these variational problems, considering a formulation in terms of optimal control.

The structure of this paper is as follows: Section 2 introduces the necessary definitions and a result essential for subsequent discussions. In Section 3, we present the original findings of our study. We commence by investigating the fundamental problem of the calculus of variations, aiming to minimize a functional with dependencies on fractional derivatives ranging from 0 to 1. We then impose an integral-type constraint on the problem, narrowing down the class of admissible functions. The Herglotz variational problem, which is a generalization of the usual problem in the calculus of variations, is addressed. Furthermore, we extend the initial problem by considering fractional derivatives of any positive real order. Finally, we conclude with examples, insights, and a discussion for future research.

2. Some Concepts

We commence by introducing several definitions of weighted fractional operators, as presented in [14]. Consider $n \in \mathbb{N}$ and $\gamma \in (n-1, n)$ to be real and $[t_1, t_2]$ to represent a closed interval of $\mathbb{R}$. Let g and w be two differentiable functions such that $g'(x) > 0$ and $w(x) \neq 0$ for all $x \in [t_1, t_2]$. Here, g serves as the kernel function, while w acts as a weighted function.

Definition 1. *The (left) weighted fractional integral of a function, $u : [t_1, t_2] \to \mathbb{R}$, of order γ, with respect to the kernel, g, is defined as*

$$ {}_w\mathbb{I}^{\gamma,g}_{t_1+}u(x) = \frac{1}{w(x)\Gamma(\gamma)}\int_{t_1}^{x} g'(t)(g(x)-g(t))^{\gamma-1}(wu)(t)\,dt. $$

The (left) weighted Riemann–Liouville fractional derivative of the function u (with order γ and kernel g) is defined as

$$ {}_w\mathbb{D}^{\gamma,g}_{t_1+}u(x) = \frac{1}{w(x)\Gamma(n-\gamma)}\left(\frac{1}{g'(x)}\frac{d}{dx}\right)^{n}\int_{t_1}^{x} g'(t)(g(x)-g(t))^{n-\gamma-1}(wu)(t)\,dt, $$

whereas the weighted Caputo fractional derivative is defined as

$$ {}^{C}_{w}\mathbb{D}^{\gamma,g}_{t_1+}u(x) = \frac{1}{w(x)\Gamma(n-\gamma)}\int_{t_1}^{x} g'(t)(g(x)-g(t))^{n-\gamma-1}\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{n}(wu)(t)\,dt. $$

To simplify the notation, given $k \in \mathbb{N}$ and a function, $u = u(t)$, we write

$$ D^{k}_{g,t}u = \left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{k}u. $$

For instance, suppose $\beta > 0$. Let us define

$$ u(x) = \frac{(g(x)-g(t_1))^{\beta-1}}{w(x)}. $$

Then, (cf. [14] (Propositions 1.3 and 4.3)),

$$ {}_w\mathbb{D}^{\gamma,g}_{t_1+}u(x) = {}^{C}_{w}\mathbb{D}^{\gamma,g}_{t_1+}u(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}\frac{(g(x)-g(t_1))^{\beta-\gamma-1}}{w(x)}. $$

Recalling the concepts of integrals and derivatives with respect to another function, as presented in [19,20], the following relationships are established:

$$ {}_w\mathbb{I}^{\gamma,g}_{t_1+}u(x) = \frac{1}{w(x)}\mathbb{I}^{\gamma,g}_{t_1+}(wu)(x), \quad {}_w\mathbb{D}^{\gamma,g}_{t_1+}u(x) = \frac{1}{w(x)}\mathbb{D}^{\gamma,g}_{t_1+}(wu)(x), $$
$$ \text{and} \quad {}^{C}_{w}\mathbb{D}^{\gamma,g}_{t_1+}u(x) = \frac{1}{w(x)}{}^{C}\mathbb{D}^{\gamma,g}_{t_1+}(wu)(x). \tag{1} $$

Motivated by the preceding relationships, we can introduce versions of right-sided weighted fractional operators:

$$ {}_w\mathbb{I}^{\gamma,g}_{t_2-}u(x) = w(x)\mathbb{I}^{\gamma,g}_{t_2-}\left(\frac{u}{w}\right)(x), \quad {}_w\mathbb{D}^{\gamma,g}_{t_2-}u(x) = w(x)\mathbb{D}^{\gamma,g}_{t_2-}\left(\frac{u}{w}\right)(x), $$
$$ \text{and} \quad {}^{C}_{w}\mathbb{D}^{\gamma,g}_{t_2-}u(x) = w(x)\,{}^{C}\mathbb{D}^{\gamma,g}_{t_2-}\left(\frac{u}{w}\right)(x). \tag{2} $$

To conclude, we establish an integration via a parts formula involving the weighted Caputo fractional derivative.

Theorem 1. *Let u and v be two functions of class C^n, Then,*

$$ \int_{t_1}^{t_2} u(x)\,{}^{C}_{w}\mathbb{D}^{\gamma,g}_{t_1+}v(x)\,dx = \int_{t_1}^{t_2} {}_w\mathbb{D}^{\gamma,g}_{t_2-}\left(\frac{u}{g'}\right)(x)v(x)g'(x)\,dx $$
$$ + \left[\sum_{k=0}^{n-1}(-1)^{k}D^{k}_{g,x}\left({}_w\mathbb{I}^{n-\gamma,g}_{t_2-}\left(\frac{u}{g'}\right)(x)\cdot\frac{1}{w(x)}\right)\cdot D^{n-k-1}_{g,x}(wv)(x)\right]_{t_1}^{t_2} $$

and

$$\int_{t_1}^{t_2} u(x) {}^{C}_{w}\mathbb{D}^{\gamma,g}_{t_2-} v(x)\,dx = \int_{t_1}^{t_2} {}_{w}\mathbb{D}^{\gamma,g}_{t_1+}\left(\frac{u}{g'}\right)(x) v(x) g'(x)\,dx + \left[\sum_{k=0}^{n-1}(-1)^{n-k} D^{k}_{g,x}\left({}_{w}\mathbb{I}^{n-\gamma,g}_{t_1+}\left(\frac{u}{g'}\right)(x)\cdot w(x)\right)\cdot D^{n-k-1}_{g,x}\left(\frac{v}{w}\right)(x)\right]_{t_1}^{t_2}.$$

Proof. The proof follows directly from [19] (Theorem 12), and the relationship between the weighted fractional operators and the fractional derivatives and integrals with respect to another function, as demonstrated in (1) and (2). □

In the most common case, where γ belongs to the interval $(0,1)$, the preceding formulas can be interpreted as:

$$\int_{t_1}^{t_2} u(x) {}^{C}_{w}\mathbb{D}^{\gamma,g}_{t_1+} v(x)\,dx = \int_{t_1}^{t_2} {}_{w}\mathbb{D}^{\gamma,g}_{t_2-}\left(\frac{u}{g'}\right)(x) v(x) g'(x)\,dx + \left[{}_{w}\mathbb{I}^{1-\gamma,g}_{t_2-}\left(\frac{u}{g'}\right)(x)\cdot v(x)\right]_{t_1}^{t_2} \tag{3}$$

and

$$\int_{t_1}^{t_2} u(x) {}^{C}_{w}\mathbb{D}^{\gamma,g}_{t_2-} v(x)\,dx = \int_{t_1}^{t_2} {}_{w}\mathbb{D}^{\gamma,g}_{t_1+}\left(\frac{u}{g'}\right)(x) v(x) g'(x)\,dx - \left[{}_{w}\mathbb{I}^{1-\gamma,g}_{t_1+}\left(\frac{u}{g'}\right)(x)\cdot v(x)\right]_{t_1}^{t_2}. \tag{4}$$

3. Euler–Lagrange-Type Equations

In this section, we explore problems within the calculus of variations, where the Lagrange function depends on both-sided weighted fractional derivatives, along with the initial and final positions of the curve. The aim is to identify the necessary conditions that these optimal curves must satisfy. These conditions, known as the Euler–Lagrange equations in the literature, serve as fundamental tools in our analysis. The central problem is formulated as follows: to find a C^1-class curve, $U : [t_1, t_2] \to \mathbb{R}$, such that functional $\mathcal{F}$, given through (5), achieves its minimum value at U, where

$$\mathcal{F}(u) = \int_{t_1}^{t_2} f(t, u(t_1), u(t_2), u(t), {}^{C}_{w}\mathbb{D}^{\gamma,g}_{t_1+} u(t), {}^{C}_{w}\mathbb{D}^{\gamma,g}_{t_2-} u(t))\,dt. \tag{5}$$

Here, the function $f : [t_1, t_2] \times \mathbb{R}^5 \to \mathbb{R}$ is continuously differentiable. We emphasize that the set of functions is required to belong to the C^1 class to ensure the well-defined nature and continuity of the fractional derivatives. We also remark that, since $u(t_1)$ and $u(t_2)$ are unrestricted, they represent two variables within the optimization problem that we seek to minimize.

Remark 1. *To facilitate notation, we adopt the following conventions. Where clarity permits, we use $f(u)(t)$ instead of the longer expression $f(t, u(t_1), u(t_2), u(t), {}^{C}_{w}\mathbb{D}^{\gamma,g}_{t_1+} u(t), {}^{C}_{w}\mathbb{D}^{\gamma,g}_{t_2-} u(t))$. Additionally, for a function, $f = f(t_1, \ldots, t_n)$, we denote the partial derivative with respect to the variable t_k as $\partial_k f$ for $k = 1, \ldots, n$.*

Remark 2. *We define U as a minimizer of the functional $\mathcal{F}$ if, for every curve $u \in C^1[t_1, t_2]$, we have $\mathcal{F}(U) \leq \mathcal{F}(u)$. On the other hand, U is classified as a local minimizer of $\mathcal{F}$ if $\mathcal{F}(U) \leq \mathcal{F}(u)$ for every curve $u \in C^1[t_1, t_2]$ within a neighborhood of U.*

Theorem 2. *Assume that $U : [t_1, t_2] \to \mathbb{R}$ is a local minimizer of the functional $\mathcal{F}$ given in (5). Then, U satisfies the following three conditions:*

- *The Euler–Lagrange equation: for all $t \in [t_1, t_2]$,*

$$\partial_4 f(U)(t) + {}_{w}\mathbb{D}^{\gamma,g}_{t_2-}\left(\frac{\partial_5 f(U)}{g'}\right)(t) g'(t) + {}_{w}\mathbb{D}^{\gamma,g}_{t_1+}\left(\frac{\partial_6 f(U)}{g'}\right)(t) g'(t) = 0;$$

- *The transversality conditions:*

$$\int_{t_1}^{t_2} \partial_2 f(U)(t)\,dt - {}_w\mathbb{I}_{t_2-}^{1-\gamma,g}\left(\frac{\partial_5 f(U)}{g'}\right)(t_1) + {}_w\mathbb{I}_{t_1+}^{1-\gamma,g}\left(\frac{\partial_6 f(U)}{g'}\right)(t_1) = 0$$

and

$$\int_{t_1}^{t_2} \partial_3 f(U)(t)\,dt + {}_w\mathbb{I}_{t_2-}^{1-\gamma,g}\left(\frac{\partial_5 f(U)}{g'}\right)(t_2) - {}_w\mathbb{I}_{t_1+}^{1-\gamma,g}\left(\frac{\partial_6 f(U)}{g'}\right)(t_2) = 0.$$

Proof. Let us assume that the curve U represents an optimal solution to the problem. Now, let us introduce an auxiliary function, $\mu : [t_1, t_2] \to \mathbb{R}$, of class C^1 and consider $\delta \in \mathbb{R}$. Consider the variation $U + \delta\mu$ of the curve μ and the function $\eta(\delta) = \mathcal{F}(U + \delta\mu)$, defined in a neighborhood of zero. Since U is a local minimizer of $\mathcal{F}$, we conclude that $\delta = 0$ is a local minimizer of η. Consequently, $\eta'(0) = 0$, that is, the first variation of the functional is zero at $\delta = 0$. In other words,

$$\begin{aligned}\int_{t_1}^{t_2} &\partial_2 f(U)(t)\mu(t_1) + \partial_3 f(U)(t)\mu(t_2) + \partial_4 f(U)(t)\mu(t) \\ &+ \partial_5 f(U)(t)\,{}^C_w\mathbb{D}_{t_1+}^{\gamma,g}\mu(t) + \partial_6 f(U)(t)\,{}^C_w\mathbb{D}_{t_2-}^{\gamma,g}\mu(t)\,dt = 0. \qquad (6)\end{aligned}$$

By employing the integration via parts formulae, as outlined in (3) and (4), with the final two terms of the integral, we derive that

$$\begin{aligned}\int_{t_1}^{t_2} \partial_5 f(U)(t)\,{}^C_w\mathbb{D}_{t_1+}^{\gamma,g}\mu(t)\,dt = &\int_{t_1}^{t_2} {}_w\mathbb{D}_{t_2-}^{\gamma,g}\left(\frac{\partial_5 f(U)}{g'}\right)(t)g'(t)\mu(t)\,dt \\ &+ \left[{}_w\mathbb{I}_{t_2-}^{1-\gamma,g}\left(\frac{\partial_5 f(U)}{g'}\right)(t)\mu(t)\right]_{t_1}^{t_2}\end{aligned}$$

and

$$\begin{aligned}\int_{t_1}^{t_2} \partial_6 f(U)(t)\,{}^C_w\mathbb{D}_{t_2-}^{\gamma,g}\mu(t)\,dt = &\int_{t_1}^{t_2} {}_w\mathbb{D}_{t_1+}^{\gamma,g}\left(\frac{\partial_6 f(U)}{g'}\right)(t)g'(t)\mu(t)\,dt \\ &- \left[{}_w\mathbb{I}_{t_1+}^{1-\gamma,g}\left(\frac{\partial_6 f(U)}{g'}\right)(t)\mu(t)\right]_{t_1}^{t_2}.\end{aligned}$$

Then, we obtain the following expression:

$$\begin{aligned}&\int_{t_1}^{t_2}\left[\partial_4 f(U)(t) + {}_w\mathbb{D}_{t_2-}^{\gamma,g}\left(\frac{\partial_5 f(U)}{g'}\right)(t)g'(t) + {}_w\mathbb{D}_{t_1+}^{\gamma,g}\left(\frac{\partial_6 f(U)}{g'}\right)(t)g'(t)\right]\mu(t)\,dt \\ &+ \left[\int_{t_1}^{t_2}\partial_2 f(U)(t)\,dt - {}_w\mathbb{I}_{t_2-}^{1-\gamma,g}\left(\frac{\partial_5 f(U)}{g'}\right)(t_1) + {}_w\mathbb{I}_{t_1+}^{1-\gamma,g}\left(\frac{\partial_6 f(U)}{g'}\right)(t_1)\right]\mu(t_1) \\ &+ \left[\int_{t_1}^{t_2}\partial_3 f(U)(t)\,dt + {}_w\mathbb{I}_{t_2-}^{1-\gamma,g}\left(\frac{\partial_5 f(U)}{g'}\right)(t_2) - {}_w\mathbb{I}_{t_1+}^{1-\gamma,g}\left(\frac{\partial_6 f(U)}{g'}\right)(t_2)\right]\mu(t_2) = 0.\end{aligned}$$

As the function μ can assume any value over the interval $[t_1, t_2]$, if we assume the additional condition that $\mu(t_1) = 0$ and $\mu(t_2) = 0$, then

$$\int_{t_1}^{t_2}\left[\partial_4 f(U)(t) + {}_w\mathbb{D}_{t_2-}^{\gamma,g}\left(\frac{\partial_5 f(U)}{g'}\right)(t)g'(t) + {}_w\mathbb{D}_{t_1+}^{\gamma,g}\left(\frac{\partial_6 f(U)}{g'}\right)(t)g'(t)\right]\mu(t)\,dt = 0.$$

As μ is arbitrary elsewhere, using the fundamental lemma of the calculus of variations, we conclude that, for all $t \in [t_1, t_2]$,

$$\partial_4 f(U)(t) + {}_w\mathbb{D}^{\gamma,g}_{t_2-}\left(\frac{\partial_5 f(U)}{g'}\right)(t)g'(t) + {}_w\mathbb{D}^{\gamma,g}_{t_1+}\left(\frac{\partial_6 f(U)}{g'}\right)(t)g'(t) = 0.$$

Consequently,

$$\left[\int_{t_1}^{t_2} \partial_2 f(U)(t)\,dt - {}_w\mathbb{I}^{1-\gamma,g}_{t_2-}\left(\frac{\partial_5 f(U)}{g'}\right)(t_1) + {}_w\mathbb{I}^{1-\gamma,g}_{t_1+}\left(\frac{\partial_6 f(U)}{g'}\right)(t_1)\right]\mu(t_1)$$
$$+ \left[\int_{t_1}^{t_2} \partial_3 f(U)(t)\,dt + {}_w\mathbb{I}^{1-\gamma,g}_{t_2-}\left(\frac{\partial_5 f(U)}{g'}\right)(t_2) - {}_w\mathbb{I}^{1-\gamma,g}_{t_1+}\left(\frac{\partial_6 f(U)}{g'}\right)(t_2)\right]\mu(t_2) = 0.$$

Finally, assuming first that $\mu(t_1) = 0$ and $\mu(t_2) \neq 0$ and then that $\mu(t_1) \neq 0$ and $\mu(t_2) = 0$, we deduce the two remaining necessary conditions that we aimed to prove. □

Remark 3. *If, in the initial problem, $u(t_1)$ and $u(t_2)$ are fixed values, then the dependence of f on $u(t_1)$ and $u(t_2)$ becomes irrelevant. The variation $U + \delta\mu$ must satisfy the conditions $\mu(t_1) = 0$ and $\mu(t_2) = 0$, and thus, we do not obtain the two transversality conditions.*

Remark 4. *It is important to note that these conditions are only necessary, and the mere vanishing of the first variation does not lead to conclusive results. However, if certain convexity assumptions are imposed on the Lagrange function, then the Euler–Lagrange equation indeed becomes a sufficient condition. Also, the result does not assert anything about the existence of a local minimizer of the functional. The Euler–Lagrange equation is merely a necessary condition, and as such, it states that, if such a minimizer exists, then it must satisfy this equation.*

Two immediate results are presented next. In the first one, the final time of the integration interval is free. Consequently, our objective is not only to determine the optimal function U but also to identify the optimal final time.

The functional is defined as

$$\mathcal{F}(u, T) = \int_{t_1}^{T} f(t, u(t_1), u(T), u(t), {}^{C}_{w}\mathbb{D}^{\gamma,g}_{t_1+}u(t), {}^{C}_{w}\mathbb{D}^{\gamma,g}_{t_2-}u(t))\,dt, \tag{7}$$

and we abbreviate the Lagrange function as $f(u, T)(t)$. The functional is defined with the set $C^1[t_1, t_2] \times (t_1, t_2]$.

Theorem 3. *If the pair (U, T) is a local minimizer of the functional (7), then it satisfies the Euler–Lagrange equation outlined in Theorem 2 within the interval $[t_1, T]$, and it also complies with the two transversality conditions, with the equation evaluated at T for the second condition (not at t_2). In addition, it also fulfills the equation $f(U, T)(T) = 0$.*

Proof. In this scenario, we must not only consider a variation of U as $U + \delta\mu$ but also a variation of the optimal time in the form $T + \delta\kappa$, where κ is an arbitrary real number. The first variation is, then, as follows:

$$\int_{t_1}^{T} \partial_2 f(U, T)(t)\mu(t_1) + \partial_3 f(U, T)(t)\mu(T) + \partial_4 f(U, T)(t)\mu(t)$$
$$+ \partial_5 f(U, T)(t)\,{}^{C}_{w}\mathbb{D}^{\gamma,g}_{t_1+}\mu(t) + \partial_6 f(U, T)(t)\,{}^{C}_{w}\mathbb{D}^{\gamma,g}_{t_2-}\mu(t)\,dt + \kappa \cdot f(U, T)(T).$$

Through integration via parts, if we assume that $\kappa = 0$, then the first variation of the functional coincides with that given in Theorem 2, and by following the same procedures, we deduce the necessary conditions proven in that theorem. Thus, we arrive at the equality (for an arbitrary κ) $\kappa \cdot f(U,T)(T) = 0$. Assuming now that $\kappa \neq 0$, we derive the last necessary condition. $\square$

The variational problem involving multiple dependent variables follows a similar structure. Consider the functional

$$\mathcal{F}_n(u) = \int_{t_1}^{t_2} f_n(t, u(t_1), u(t_2), u(t), {}^C_w\mathbb{D}^{\gamma,g}_{t_1+}u(t), {}^C_w\mathbb{D}^{\gamma,g}_{t_2-}u(t))\, dt, \tag{8}$$

where $u = (u_1 \ldots, u_n)$, $u_i \in C^1[t_1,t_2]$ for all $i \in \{1,\ldots,n\}$,

$${}^C_w\mathbb{D}^{\gamma,g}_{t_1+}u(t) = ({}^C_w\mathbb{D}^{\gamma,g}_{t_1+}u_1(t), \ldots, {}^C_w\mathbb{D}^{\gamma,g}_{t_1+}u_n(t)), \quad {}^C_w\mathbb{D}^{\gamma,g}_{t_2-}u(t) = ({}^C_w\mathbb{D}^{\gamma,g}_{t_2-}u_1(t), \ldots, {}^C_w\mathbb{D}^{\gamma,g}_{t_2-}u_n(t)),$$

and $f_n : [t_1,t_2] \times \mathbb{R}^{5n} \to \mathbb{R}$ is continuously differentiable.

Theorem 4. *If $U = (U_1, \ldots, U_n) : [t_1,t_2] \to \mathbb{R}^n$ is a local minimizer of functional (8), then, for all $i \in \{2,\ldots,n+1\}$, the following applies:*

$$\partial_{i+2n} f_n(U)(t) + {}_w\mathbb{D}^{\gamma,g}_{t_2-}\left(\frac{\partial_{i+3n} f_n(U)}{g'}\right)(t)g'(t) + {}_w\mathbb{D}^{\gamma,g}_{t_1+}\left(\frac{\partial_{i+4n} f_n(U)}{g'}\right)(t)g'(t) = 0,$$

for all $t \in [t_1,t_2]$,

$$\int_{t_1}^{t_2} \partial_i f_n(U)(t)\, dt - {}_w\mathbb{I}^{1-\gamma,g}_{t_2-}\left(\frac{\partial_{i+3n} f_n(U)}{g'}\right)(t_1) + {}_w\mathbb{I}^{1-\gamma,g}_{t_1+}\left(\frac{\partial_{i+4n} f_n(U)}{g'}\right)(t_1) = 0,$$

and

$$\int_{t_1}^{t_2} \partial_{i+n} f_n(U)(t)\, dt + {}_w\mathbb{I}^{1-\gamma,g}_{t_2-}\left(\frac{\partial_{i+3n} f_n(U)}{g'}\right)(t_2) - {}_w\mathbb{I}^{1-\gamma,g}_{t_1+}\left(\frac{\partial_{i+4n} f_n(U)}{g'}\right)(t_2) = 0.$$

Proof. If $U = (U_1, \ldots, U_n)$ is an optimal solution to the variational problem, consider n auxiliary functions $\mu_i : [t_1,t_2] \to \mathbb{R}$ for $i \in \{1,\ldots,n\}$, and let $\delta \in \mathbb{R}$. Considering the variation $(U_1 + \delta\mu_1, \ldots, U_n + \delta\mu_n)$ and then computing the first variation of the functional, we can deduce the k-th necessary conditions (for $k = 1,\ldots,n$) by assuming that $\mu_k(t) \neq 0$ and $\mu_i(t) = 0$ for $i \neq k$, for all $t \in [t_1,t_2]$. $\square$

Next, we studied the problem known as the isoperimetric problem. When formulating the variational problem, we assumed that the space of admissible functions must comply with an integral constraint of the form

$$\mathcal{H}(u) = \int_{t_1}^{t_2} h(t, u(t_1), u(t_2), u(t), {}^C_w\mathbb{D}^{\gamma,g}_{t_1+}u(t), {}^C_w\mathbb{D}^{\gamma,g}_{t_2-}u(t))\, dt = \Upsilon. \tag{9}$$

Here, Υ is a fixed real number. Similarly, we assume that the function $h : [t_1,t_2] \times \mathbb{R}^5 \to \mathbb{R}$ is continuously differentiable.

Theorem 5. *Assume that $U : [t_1,t_2] \to \mathbb{R}$ is a local minimizer of the functional $\mathcal{F}$ defined in (5) under the constraint (9). Assume also that one of the three next conditions is not satisfied:*

1. *for all $t \in [t_1,t_2]$,*

$$\partial_4 h(U)(t) + {}_w\mathbb{D}^{\gamma,g}_{t_2-}\left(\frac{\partial_5 h(U)}{g'}\right)(t)g'(t) + {}_w\mathbb{D}^{\gamma,g}_{t_1+}\left(\frac{\partial_6 h(U)}{g'}\right)(t)g'(t) = 0;$$

2.

$$\int_{t_1}^{t_2} \partial_2 h(U)(t)\, dt - {}_w\mathbb{I}^{1-\gamma,g}_{t_2-}\left(\frac{\partial_5 h(U)}{g'}\right)(t_1) + {}_w\mathbb{I}^{1-\gamma,g}_{t_1+}\left(\frac{\partial_6 h(U)}{g'}\right)(t_1) = 0;$$

3.
$$\int_{t_1}^{t_2} \partial_3 h(U)(t)\,dt + {}_w\mathbb{I}_{t_2-}^{1-\gamma,g}\left(\frac{\partial_5 h(U)}{g'}\right)(t_2) - {}_w\mathbb{I}_{t_1+}^{1-\gamma,g}\left(\frac{\partial_6 h(U)}{g'}\right)(t_2) = 0.$$

Then, there exists $\lambda \in \mathbb{R}$, such that U satisfies the following three conditions:

$$\partial_4(f+\lambda h)(U)(t) + {}_w\mathbb{D}_{t_2-}^{\gamma,g}\left(\frac{\partial_5(f+\lambda h)(U)}{g'}\right)(t)g'(t) + {}_w\mathbb{D}_{t_1+}^{\gamma,g}\left(\frac{\partial_6(f+\lambda h)(U)}{g'}\right)(t)g'(t) = 0,$$

for all $t \in [t_1, t_2]$,

$$\int_{t_1}^{t_2} \partial_2(f+\lambda h)(U)(t)\,dt - {}_w\mathbb{I}_{t_2-}^{1-\gamma,g}\left(\frac{\partial_5(f+\lambda h)(U)}{g'}\right)(t_1) + {}_w\mathbb{I}_{t_1+}^{1-\gamma,g}\left(\frac{\partial_6(f+\lambda h)(U)}{g'}\right)(t_1) = 0,$$

and

$$\int_{t_1}^{t_2} \partial_3(f+\lambda h)(U)(t)\,dt + {}_w\mathbb{I}_{t_2-}^{1-\gamma,g}\left(\frac{\partial_5(f+\lambda h)(U)}{g'}\right)(t_2) - {}_w\mathbb{I}_{t_1+}^{1-\gamma,g}\left(\frac{\partial_6(f+\lambda h)(U)}{g'}\right)(t_2) = 0.$$

Proof. In this case, we first ensure the existence of an infinite family of variations that satisfy the integral constraint (9). To achieve this, we consider variations with dependencies on two functions, μ_1 and u_2, as well as on two real numbers, δ_1 and δ_2. Defining

$$\mathrm{f}(\delta_1,\delta_2) = \mathcal{F}(U + \delta\mu_1 + \delta\mu_2) \quad \text{and} \quad \mathrm{h}(\delta_1,\delta_2) = \mathcal{H}(U + \delta\mu_1 + \delta\mu_2) - \Upsilon,$$

and computing $\partial_2 h(0,0)$ in a similar manner as done in the proof of Theorem 2, we arrive at

$$\begin{aligned}\partial_2 h(0,0) &= \int_{t_1}^{t_2}\left[\partial_4 h(U)(t) + {}_w\mathbb{D}_{t_2-}^{\gamma,g}\left(\frac{\partial_5 h(U)}{g'}\right)(t)g'(t) + {}_w\mathbb{D}_{t_1+}^{\gamma,g}\left(\frac{\partial_6 h(U)}{g'}\right)(t)g'(t)\right]\mu_2(t)\,dt \\ &\quad + \left[\int_{t_1}^{t_2}\partial_2 h(U)(t)\,dt - {}_w\mathbb{I}_{t_2-}^{1-\gamma,g}\left(\frac{\partial_5 h(U)}{g'}\right)(t_1) + {}_w\mathbb{I}_{t_1+}^{1-\gamma,g}\left(\frac{\partial_6 h(U)}{g'}\right)(t_1)\right]\mu_2(t_1) \\ &\quad + \left[\int_{t_1}^{t_2}\partial_3 h(U)(t)\,dt + {}_w\mathbb{I}_{t_2-}^{1-\gamma,g}\left(\frac{\partial_5 h(U)}{g'}\right)(t_2) - {}_w\mathbb{I}_{t_1+}^{1-\gamma,g}\left(\frac{\partial_6 h(U)}{g'}\right)(t_2)\right]\mu_2(t_2).\end{aligned}$$

Therefore, we conclude that there exists a function, μ_2, such that $\partial_2 h(0,0) \neq 0$. Since $h(0,0) = 0$, according to the Implicit Function Theorem, there exists a function, $\sigma :]-r, r[\to \mathbb{R}$, such that $\sigma(0) = 0$ and $h(\delta, \sigma(\delta)) = 0$ for every $\delta \in]-r, r[$, thus establishing the desired result.

We next proceeded to deduce the three necessary conditions of the theorem. To do so, let us observe that we can formulate the variational problem as a finite-dimensional problem with two variables: minimizing f, subject to the constraint $h \equiv 0$. Since we previously established that $\nabla h(0,0) \neq (0,0)$, applying the method of Lagrange multipliers ensures the existence of a real number, λ, such that $\nabla(f+\lambda h)(0,0) = (0,0)$. By setting $\partial_1(f+\lambda h)(0,0) = (0,0)$, we arrive at the desired conditions. □

Remark 5. *In the formulation of Theorem 5, if we remove the condition that U does not satisfy one of the three conditions presented in this theorem, a similar result can still be proven. In this scenario, we ensure the existence of two real numbers, λ and κ, and by substituting $(f+\lambda h)$ with $(\kappa f + \lambda h)$,* *we still arrive at the same conclusions. Indeed, if the conditions of Theorem 5 are satisfied, we choose* $\kappa = 1$ *and* λ*, as prescribed according to Theorem 5. Otherwise, if these conditions are not met, we set* $\kappa = 0$ *and select any arbitrary* λ.

In the preceding results, the integration interval of the functional coincides with the integration intervals of the sided fractional derivatives. We can extend these findings by considering the integration interval of the functional as a subset of the integral $[t_1, t_2]$. To achieve this, let $t_1 < T_1 < T_2 < t_2$, and consider the functional

$$\mathcal{F}_T(u) = \int_{T_1}^{T_2} f(t, u(T_1), u(T_2), u(t), {}^{C}_{w}\mathbb{D}^{\gamma,g}_{t_1+}u(t), {}^{C}_{w}\mathbb{D}^{\gamma,g}_{t_2-}u(t))\, dt. \tag{10}$$

Theorem 6. *Suppose* $U : [t_1, t_2] \to \mathbb{R}$ *represents a local minimizer of the functional* $\mathcal{F}_T$ *as defined in (10). In such a case,* U *fulfills under the subsequent conditions:*

- *The Euler–Lagrange equations: for all* $t \in [t_1, T_1]$,

$$ {}_{w}\mathbb{D}^{\gamma,g}_{T_2-}\left(\frac{\partial_5 f(U)}{g'}\right)(t)g'(t) - {}_{w}\mathbb{D}^{\gamma,g}_{T_1-}\left(\frac{\partial_5 f(U)}{g'}\right)(t)g'(t) = 0,$$

for all $t \in [T_1, T_2]$,

$$\partial_4 f(U)(t) + {}_{w}\mathbb{D}^{\gamma,g}_{T_2-}\left(\frac{\partial_5 f(U)}{g'}\right)(t)g'(t) + {}_{w}\mathbb{D}^{\gamma,g}_{T_1+}\left(\frac{\partial_6 f(U)}{g'}\right)(t)g'(t) = 0,$$

and for all $t \in [T_2, t_2]$,

$$ {}_{w}\mathbb{D}^{\gamma,g}_{T_1+}\left(\frac{\partial_6 f(U)}{g'}\right)(t)g'(t) - {}_{w}\mathbb{D}^{\gamma,g}_{T_2+}\left(\frac{\partial_6 f(U)}{g'}\right)(t)g'(t) = 0;$$

- *The transversality conditions:*

$$ {}_{w}\mathbb{I}^{1-\gamma,g}_{T_1-}\left(\frac{\partial_5 f(U)}{g'}\right)(t_1) - {}_{w}\mathbb{I}^{1-\gamma,g}_{T_2-}\left(\frac{\partial_5 f(U)}{g'}\right)(t_1) = 0,$$

$$\int_{T_1}^{T_2} \partial_2 f(U)(t)\, dt - {}_{w}\mathbb{I}^{1-\gamma,g}_{T_1-}\left(\frac{\partial_5 f(U)}{g'}\right)(T_1) + {}_{w}\mathbb{I}^{1-\gamma,g}_{T_1+}\left(\frac{\partial_6 f(U)}{g'}\right)(T_1) = 0,$$

$$\int_{T_1}^{T_2} \partial_3 f(U)(t)\, dt + {}_{w}\mathbb{I}^{1-\gamma,g}_{T_2-}\left(\frac{\partial_5 f(U)}{g'}\right)(T_2) - {}_{w}\mathbb{I}^{1-\gamma,g}_{T_2+}\left(\frac{\partial_6 f(U)}{g'}\right)(T_2) = 0,$$

and

$$ {}_{w}\mathbb{I}^{1-\gamma,g}_{T_2+}\left(\frac{\partial_6 f(U)}{g'}\right)(t_2) - {}_{w}\mathbb{I}^{1-\gamma,g}_{T_1+}\left(\frac{\partial_6 f(U)}{g'}\right)(t_2) = 0.$$

Proof. Given $\mu : [t_1, t_2] \to \mathbb{R}$, the first variation of the functional is

$$\int_{T_1}^{T_2} \partial_2 f(U)(t)\mu(T_1) + \partial_3 f(U)(t)\mu(T_2) + \partial_4 f(U)(t)\mu(t) + \partial_5 f(U)(t){}^{C}_{w}\mathbb{D}^{\gamma,g}_{t_1+}\mu(t) + \partial_6 f(U)(t){}^{C}_{w}\mathbb{D}^{\gamma,g}_{t_2-}\mu(t)\, dt.$$

To apply the integration via parts formulae (3) and (4), we consider the relations

$$\int_{T_1}^{T_2} \partial_5 f(U)(t){}^{C}_{w}\mathbb{D}^{\gamma,g}_{t_1+}\mu(t)\, dt = \int_{t_1}^{T_2} \partial_5 f(U)(t){}^{C}_{w}\mathbb{D}^{\gamma,g}_{t_1+}\mu(t)\, dt - \int_{t_1}^{T_1} \partial_5 f(U)(t){}^{C}_{w}\mathbb{D}^{\gamma,g}_{t_1+}\mu(t)\, dt$$

and

$$\int_{T_1}^{T_2} \partial_6 f(U)(t) {}^{C}_{w}\mathbb{D}^{\gamma,g}_{t_2-}\mu(t)\,dt = \int_{T_1}^{t_2} \partial_6 f(U)(t) {}^{C}_{w}\mathbb{D}^{\gamma,g}_{t_2-}\mu(t)\,dt - \int_{T_2}^{t_2} \partial_6 f(U)(t) {}^{C}_{w}\mathbb{D}^{\gamma,g}_{t_2-}\mu(t)\,dt.$$

By employing fractional integration with parts to the four integrals on the right-hand side and subsequently reordering the terms, we attain the intended conclusion. □

In 1930, Herglotz [25] introduced a generalized variational principle, expanding upon the classical variational principle. As remarked in [26,27], Herglotz's principle provides a versatile tool for describing nonconservative processes, even in cases where the Lagrangian is autonomous, a capability not achievable through classical methods. The problem involves discovering two functions, related via a differential equation, with the objective of minimizing the value of one of them at the final point. It can be demonstrated that, under specific conditions, the Herglotz problem simplifies to a classical variational problem, thereby representing an extension of the fundamental problem in the calculus of variations.

The problem is formulated as follows: finding two functions, $u \in C^2[t_1, t_2]$ and $v \in C^1[t_1, t_2]$, such that $v(t_2)$ achieves a minimum value, while the pair (u, v) satisfies the system

$$\begin{cases} v'(t) = f_H(t, u(t_1), u(t_2), u(t), {}^{C}_{w}\mathbb{D}^{\gamma,g}_{t_1+}u(t), {}^{C}_{w}\mathbb{D}^{\gamma,g}_{t_2-}u(t), v(t)), & t \in [t_1, t_2] \\ v(t_1) = v_1, \quad v_1 \in \mathbb{R}, \end{cases} \tag{11}$$

where the function $f_H : [t_1, t_2] \times \mathbb{R}^6 \to \mathbb{R}$ is of class C^1. Given that the function f_H depends on both u and v, we will abbreviate it as $f_H(u, v)$. Additionally, it is worth noting that the function v depends not only on the variable t but also on the function u. Thus, we represent it as $v[u](t)$ when we need to refer to this dependence.

Theorem 7. *Suppose that functions U and V are a solution to Herglotz's problem as related via system* (11)*. Introduce the function $E : [t_1, t_2] \to \mathbb{R}$ as follows:*

$$E(t) = \exp\left(-\int_{t_1}^{t} \partial_7 f_H(U, V)(s)\,ds\right).$$

Then, the pair (U, V) satisfies the following three conditions:

- *for all $t \in [t_1, t_2]$,*

$$E(t)\partial_4 f_H(U,V)(t) + {}_{w}\mathbb{D}^{\gamma,g}_{t_2-}\left(\frac{E \cdot \partial_5 f_H(U,V)}{g'}\right)(t)g'(t) + {}_{w}\mathbb{D}^{\gamma,g}_{t_1+}\left(\frac{E \cdot \partial_6 f_H(U,V)}{g'}\right)(t)g'(t) = 0;$$

- *and*

$$\int_{t_1}^{t_2} E(t)\partial_2 f_H(U,V)(t)\,dt - {}_{w}\mathbb{I}^{1-\gamma,g}_{t_2-}\left(\frac{E \cdot \partial_5 f_H(U,V)}{g'}\right)(t_1) + {}_{w}\mathbb{I}^{1-\gamma,g}_{t_1+}\left(\frac{E \cdot \partial_6 f_H(U,V)}{g'}\right)(t_1) = 0$$

and

$$\int_{t_1}^{t_2} E(t)\partial_3 f_H(U,V)(t)\,dt + {}_{w}\mathbb{I}^{1-\gamma,g}_{t_2-}\left(\frac{E \cdot \partial_5 f_H(U,V)}{g'}\right)(t_2) - {}_{w}\mathbb{I}^{1-\gamma,g}_{t_1+}\left(\frac{E \cdot \partial_6 f_H(U,V)}{g'}\right)(t_2) = 0.$$

Proof. Since (U,V) is a solution pair of the optimization problem, let us consider an arbitrary function, $\mu \in C^2[t_1,t_2]$, and a real number, δ. We define the function $\xi : [t_1,t_2] \to \mathbb{R}$ with the formula

$$\xi(t) = \frac{d}{d\delta} V[U+\delta\mu](t)\Big|_{\delta=0}, \quad \text{for } t \in [t_1,t_2].$$

Since $v(t_1)$ is a fixed value, we have $\xi(t_1)=0$. On the other hand, since V achieves an extremum at $t=t_2$, we also have $\xi(t_2)=0$. Additionally,

$$\begin{aligned}
\xi'(t) \quad &= \frac{d}{d\delta}\frac{d}{dt} V[U+\delta\mu](t)\Big|_{\delta=0} = \frac{d}{d\delta} f_H(U+\delta\mu, V[U+\delta\mu])(t)\Big|_{\delta=0} \\
&= \partial_2 f_H(U,V)(t)\mu(t_1) + \partial_3 f_H(U,V)(t)\mu(t_2) + \partial_4 f_H(U,V)(t)\mu(t) \\
&\quad + \partial_5 f_H(U,V)(t) {}^C_w\mathbb{D}^{\gamma,g}_{t_1+}\mu(t) + \partial_6 f_H(U,V)(t) {}^C_w\mathbb{D}^{\gamma,g}_{t_2-}\mu(t) + \partial_7 f_H(U,V)(t)\xi(t).
\end{aligned}$$

We derive the ordinary differential equation

$$\begin{aligned}
\xi'(t) - \partial_7 f_H(U,V)(t)\xi(t) &= \partial_2 f_H(U,V)(t)\mu(t_1) + \partial_3 f_H(U,V)(t)\mu(t_2) \\
&+ \partial_4 f_H(U,V)(t)\mu(t) + \partial_5 f_H(U,V)(t) {}^C_w\mathbb{D}^{\gamma,g}_{t_1+}\mu(t) + \partial_6 f_H(U,V)(t) {}^C_w\mathbb{D}^{\gamma,g}_{t_2-}\mu(t),
\end{aligned}$$

whose solution satisfies the following relationship:

$$\begin{aligned}
E(t_2)\xi(t_2) - E(t_1)\xi(t_1) = \int_{t_1}^{t_2} E(t)\Big[&\partial_2 f_H(U,V)(t)\mu(t_1) + \partial_3 f_H(U,V)(t)\mu(t_2) \\
&+ \partial_4 f_H(U,V)(t)\mu(t) + \partial_5 f_H(U,V)(t) {}^C_w\mathbb{D}^{\gamma,g}_{t_1+}\mu(t) + \partial_6 f_H(U,V)(t) {}^C_w\mathbb{D}^{\gamma,g}_{t_2-}\mu(t)\Big]\, dt.
\end{aligned}$$

Recalling that $\xi(t_1)=0$ and $\xi(t_2)=0$, and applying formulae (3) and (4), we arrive at:

$$\begin{aligned}
\int_{t_1}^{t_2} \Big[E(t)\partial_4 f_H(U,V)(t) &+ {}_w\mathbb{D}^{\gamma,g}_{t_2-}\left(\frac{E\cdot\partial_5 f_H(U,V)}{g'}\right)(t)g'(t) \\
&+ {}_w\mathbb{D}^{\gamma,g}_{t_1+}\left(\frac{E\cdot\partial_6 f_H(U,V)}{g'}\right)(t)g'(t)\Big]\mu(t)\, dt \\
&+ \Big[\int_{t_1}^{t_2} E(t)\partial_2 f_H(U,V)(t)\, dt - {}_w\mathbb{I}^{1-\gamma,g}_{t_2-}\left(\frac{E\cdot\partial_5 f_H(U,V)}{g'}\right)(t_1) \\
&+ {}_w\mathbb{I}^{1-\gamma,g}_{t_1+}\left(\frac{E\cdot\partial_6 f_H(U,V)}{g'}\right)(t_1)\Big]\mu(t_1) \\
&+ \Big[\int_{t_1}^{t_2} E(t)\partial_3 f_H(U,V)(t)\, dt + {}_w\mathbb{I}^{1-\gamma,g}_{t_2-}\left(\frac{E\cdot\partial_5 f_H(U,V)}{g'}\right)(t_2) \\
&- {}_w\mathbb{I}^{1-\gamma,g}_{t_1+}\left(\frac{E\cdot\partial_6 f_H(U,V)}{g'}\right)(t_2)\Big]\mu(t_2) = 0.
\end{aligned}$$

Based on the arbitrariness of the function μ, we deduce the intended result, thereby concluding the proof of the theorem. □

In our previous investigations, we solely focused on the case where the order of fractional derivatives falls within the interval $(0,1)$. Now, we consider the broader context, encompassing cases where the derivative order is any positive real number. To do so, we fix a positive integer, n, and consider a sequence denoted as $(\gamma_i)_i$, where $i \in \{1,\ldots,n\}$. Furthermore, for each $i \in \{1,\ldots,n\}$, $i-1 < \gamma_i < i$. We seek to find a C^n-class curve

$U : [t_1, t_2] \to \mathbb{R}$ such that the functional $\mathcal{F}_n$, defined using Equation (12), attains its minimum value on U. Functional $\mathcal{F}_n$ is given via the integral

$$\mathcal{F}_n(u) = \int_{t_1}^{t_2} f_n(t, u(t_1), u(t_2), u(t), {}^{C}_{w}\mathbb{D}^{\gamma_1,g}_{t_1+}u(t), \ldots, {}^{C}_{w}\mathbb{D}^{\gamma_n,g}_{t_1+}u(t), {}^{C}_{w}\mathbb{D}^{\gamma_1,g}_{t_2-}u(t), \ldots, {}^{C}_{w}\mathbb{D}^{\gamma_n,g}_{t_2-}u(t))\, dt, \quad (12)$$

where $f_n : [t_1, t_2] \times \mathbb{R}^{2n+3} \to \mathbb{R}$ is a continuously differentiable function. Once more, we employ the abbreviation $f_n(U)(t)$ for brevity when there is no risk of confusion.

Theorem 8. *If $U : [t_1, t_2] \to \mathbb{R}$ is a local minimizer of the functional $\mathcal{F}_n$, as is (12), then it must fulfill the following three conditions.*

- *The Euler–Lagrange equation: for all $t \in [t_1, t_2]$,*

$$\partial_4 f_n(U)(t) + \sum_{i=1}^{n}\left[{}_{w}\mathbb{D}^{\gamma_i,g}_{t_2-}\left(\frac{\partial_{i+4} f_n(U)}{g'}\right)(t)g'(t) + {}_{w}\mathbb{D}^{\gamma_i,g}_{t_1+}\left(\frac{\partial_{i+n+4} f_n(U)}{g'}\right)(t)g'(t)\right] = 0;$$

- *The transversality conditions: at $t = t_1$, the following holds:*

$$\int_{t_1}^{t_2} \partial_2 f_n(U)(t)\, dt + \sum_{i=1}^{n}\left[(-1)^i D^{i-1}_{g,t}\left({}_{w}\mathbb{I}^{i-\gamma_i,g}_{t_2-}\left(\frac{\partial_{i+4} f_n(U)}{g'}\right)(t)\frac{1}{w(t)}\right)w(t) + D^{i-1}_{g,t}\left({}_{w}\mathbb{I}^{i-\gamma_i,g}_{t_1+}\left(\frac{\partial_{i+n+4} f_n(U)}{g'}\right)(t)w(t)\right)\frac{1}{w(t)}\right] = 0,$$

and at $t = t_2$,

$$\int_{t_1}^{t_2} \partial_3 f_n(U)(t)\, dt - \sum_{i=1}^{n}\left[(-1)^i D^{i-1}_{g,t}\left({}_{w}\mathbb{I}^{i-\gamma_i,g}_{t_2-}\left(\frac{\partial_{i+4} f_n(U)}{g'}\right)(t)\frac{1}{w(t)}\right)w(t) + D^{i-1}_{g,t}\left({}_{w}\mathbb{I}^{i-\gamma_i,g}_{t_1+}\left(\frac{\partial_{i+n+4} f_n(U)}{g'}\right)(t)w(t)\right)\frac{1}{w(t)}\right] = 0.$$

Furthermore, at $t = t_2$ and $t = t_1$, and for each $j \in \{2, \ldots, n\}$, the following conditions hold:

$$\sum_{i=j}^{n}(-1)^{i-j} D^{i-j}_{g,t}\left({}_{w}\mathbb{I}^{i-\gamma_i,g}_{t_2-}\left(\frac{\partial_{i+4} f_n(U)}{g'}\right)(t)\frac{1}{w(t)}\right) = 0$$

and

$$\sum_{i=j}^{n} D^{i-j}_{g,t}\left({}_{w}\mathbb{I}^{i-\gamma_i,g}_{t_1+}\left(\frac{\partial_{i+n+4} f_n(U)}{g'}\right)(t)w(t)\right) = 0.$$

Proof. Assuming the curve U represents an optimal solution to the problem, let us introduce an auxiliary function, $\mu : [t_1, t_2] \to \mathbb{R}$, of class C^n and take $\delta \in \mathbb{R}$. Considering the variation $U + \delta\mu$ of the curve μ and the function $\delta \mapsto \mathcal{F}(U + \delta\mu)$, we can infer that the first variation of this function is zero at $\delta = 0$, implying,

$$\int_{t_1}^{t_2} \partial_2 f_n(U)(t)\mu(t_1) + \partial_3 f_n(U)(t)\mu(t_2) + \partial_4 f_n(U)(t)\mu(t) + \sum_{i=1}^{n}\left[\partial_{i+4} f_n(U)(t)\,{}^{C}_{w}\mathbb{D}^{\gamma_i,g}_{t_1+}\mu(t) + \partial_{i+n+4} f_n(U)(t)\,{}^{C}_{w}\mathbb{D}^{\gamma_i,g}_{t_2-}\mu(t)\right] dt = 0. \quad (13)$$

By employing integration via parts (refer to (3) and (4)) and reorganizing the terms, we can derive the following:

$$
\begin{aligned}
&\int_{t_1}^{t_2}\left[\partial_4 f_n(U)(t)+\sum_{i=1}^{n}\left[{}_{w}\mathbb{D}_{t_2-}^{\gamma_i,g}\left(\frac{\partial_{i+4}f_n(U)}{g'}\right)(t)g'(t)+{}_{w}\mathbb{D}_{t_1+}^{\gamma_i,g}\left(\frac{\partial_{i+n+4}f_n(U)}{g'}\right)(t)g'(t)\right]\right]\\
&\times\mu(t)\,dt+\left[\int_{t_1}^{t_2}\partial_2 f_n(U)(t)\,dt+\sum_{i=1}^{n}\left[(-1)^i D_{g,t}^{i-1}\left({}_{w}\mathbb{I}_{t_2-}^{i-\gamma_i,g}\left(\frac{\partial_{i+4}f_n(U)}{g'}\right)(t)\frac{1}{w(t)}\right)w(t)\right.\right.\\
&\qquad\left.\left.+D_{g,t}^{i-1}\left({}_{w}\mathbb{I}_{t_1+}^{i-\gamma_i,g}\left(\frac{\partial_{i+n+4}f_n(U)}{g'}\right)(t)w(t)\right)\frac{1}{w(t)}\right]_{t=t_1}\right]\mu(t_1)\\
&+\left[\int_{t_1}^{t_2}\partial_3 f_n(U)(t)\,dt-\sum_{i=1}^{n}\left[(-1)^i D_{g,t}^{i-1}\left({}_{w}\mathbb{I}_{t_2-}^{i-\gamma_i,g}\left(\frac{\partial_{i+4}f_n(U)}{g'}\right)(t)\frac{1}{w(t)}\right)w(t)\right.\right.\\
&\qquad\left.\left.+D_{g,t}^{i-1}\left({}_{w}\mathbb{I}_{t_1+}^{i-\gamma_i,g}\left(\frac{\partial_{i+n+4}f_n(U)}{g'}\right)(t)w(t)\right)\frac{1}{w(t)}\right]_{t=t_2}\right]\mu(t_2)\\
&+\left[\sum_{i=2}^{n}\left[(-1)^{i-2}D_{g,t}^{i-2}\left({}_{w}\mathbb{I}_{t_2-}^{i-\gamma_i,g}\left(\frac{\partial_{i+4}f_n(U)}{g'}\right)(t)\frac{1}{w(t)}\right)\right]D_{g,t}^{1}(w\mu)(t)\right]_{t_1}^{t_2}\\
&+\left[\sum_{i=2}^{n}\left[D_{g,t}^{i-2}\left({}_{w}\mathbb{I}_{t_1+}^{i-\gamma_i,g}\left(\frac{\partial_{i+n+4}f_n(U)}{g'}\right)(t)w(t)\right)\right]D_{g,t}^{1}\left(\frac{\mu}{w}\right)(t)\right]_{t_1}^{t_2}\\
&+\left[\sum_{i=3}^{n}\left[(-1)^{i-3}D_{g,t}^{i-3}\left({}_{w}\mathbb{I}_{t_2-}^{i-\gamma_i,g}\left(\frac{\partial_{i+4}f_n(U)}{g'}\right)(t)\frac{1}{w(t)}\right)\right]D_{g,t}^{2}(w\mu)(t)\right]_{t_1}^{t_2}\\
&-\left[\sum_{i=3}^{n}\left[D_{g,t}^{i-3}\left({}_{w}\mathbb{I}_{t_1+}^{i-\gamma_i,g}\left(\frac{\partial_{i+n+4}f_n(U)}{g'}\right)(t)w(t)\right)\right]D_{g,t}^{2}\left(\frac{\mu}{w}\right)(t)\right]_{t_1}^{t_2}\\
&+\ldots+\left[{}_{w}\mathbb{I}_{t_2-}^{n-\gamma_n,g}\left(\frac{\partial_{n+4}f_n(U)}{g'}\right)(t)\frac{1}{w(t)}D_{g,t}^{n-1}(w\mu)(t)\right]_{t_1}^{t_2}\\
&+\left[(-1)^n\,{}_{w}\mathbb{I}_{t_1+}^{n-\gamma_n,g}\left(\frac{\partial_{2n+4}f_n(U)}{g'}\right)(t)w(t)D_{g,t}^{n-1}\left(\frac{\mu}{w}\right)(t)\right]_{t_1}^{t_2}=0.
\end{aligned}
$$

Given that μ can assume any value, and the function w satisfies $w(t) \neq 0$ for all $t \in [t_1, t_2]$, we deduce that the functions $w\mu$ and μ/w are likewise arbitrary. Consequently, we have established the desired theorem. $\square$

4. Examples

In this section, we provide two examples to illustrate our results. Let us consider an arbitrary kernel, $g : [t_1, t_2] \to \mathbb{R}$, a weighted function, $w : [t_1, t_2] \to \mathbb{R}$, and a fractional order, $\gamma \in (0, 1)$. Given the function

$$\overline{U}(x) = \frac{(g(x) - g(t_1))^2}{w(x)},$$

we have

$${}_{w}^{C}\mathbb{D}_{t_1+}^{\gamma,g}\overline{U}(x) = \frac{2}{\Gamma(3-\gamma)}\frac{(g(x) - g(t_1))^{2-\gamma}}{w(x)}.$$

As a first example, consider the functional:

$$\mathcal{F}(u) = \int_{t_1}^{t_2} \left[\left({}_{w}^{C}\mathbb{D}_{t_1+}^{\gamma,g}\, u(t) - \frac{2}{\Gamma(3-\gamma)} \frac{(g(t)-g(t_1))^{2-\gamma}}{w(t)} \right)^4 + t^2 - t + 5 + (u(t_1))^5 + (u(t_2) - \overline{U}(t_2))^3 \right] dt. \quad (14)$$

If $U : [t_1, t_2] \to \mathbb{R}$ is a local minimizer of the functional (14), then it must satisfy the following conditions (see Theorem 2).

- The Euler–Lagrange equation: for all $t \in [t_1, t_2]$,

$${}_{w}\mathbb{D}_{t_2-}^{\gamma,g} \left(\frac{4\left({}_{w}^{C}\mathbb{D}_{t_1+}^{\gamma,g}\, u - \frac{2}{\Gamma(3-\gamma)} \frac{(g-g(t_1))^{2-\gamma}}{w} \right)^3}{g'} \right)(t) g'(t) = 0;$$

- The transversality conditions:

$$\int_{t_1}^{t_2} 5(u(t_1))^4\, dt - {}_{w}\mathbb{I}_{t_2-}^{1-\gamma,g} \left(\frac{4\left({}_{w}^{C}\mathbb{D}_{t_1+}^{\gamma,g}\, u - \frac{2}{\Gamma(3-\gamma)} \frac{(g-g(t_1))^{2-\gamma}}{w} \right)^3}{g'} \right)(t_1) = 0$$

  and

$$\int_{t_1}^{t_2} 3(u(t_2) - \overline{U}(t_2))^2\, dt + {}_{w}\mathbb{I}_{t_2-}^{1-\gamma,g} \left(\frac{4\left({}_{w}^{C}\mathbb{D}_{t_1+}^{\gamma,g}\, u - \frac{2}{\Gamma(3-\gamma)} \frac{(g-g(t_1))^{2-\gamma}}{w} \right)^3}{g'} \right)(t_2) = 0.$$

It can be easily verified that the function $u = \overline{U}$ satisfies all the necessary conditions mentioned.

For a second example, let us consider the functional

$$\mathcal{F}(u) = \int_{t_1}^{t_2} \left[\left({}_{w}^{C}\mathbb{D}_{t_1+}^{\gamma,g}\, u(t) \right)^2 + \left(\frac{2}{\Gamma(3-\gamma)} \frac{(g(t)-g(t_1))^{2-\gamma}}{w(t)} \right)^2 + t^2 - t + 5 + (u(t_1))^5 + (u(t_2) - \overline{U}(t_2))^3 \right] dt, \quad (15)$$

subject to the integral constraint

$$\int_{t_1}^{t_2} \left[{}_{w}^{C}\mathbb{D}_{t_1+}^{\gamma,g}\, u(t) \cdot \frac{2}{\Gamma(3-\gamma)} \frac{(g(t)-g(t_1))^{2-\gamma}}{w(t)} \, dt = \Upsilon, \right.$$

where

$$\Upsilon = \int_{t_1}^{t_2} \left(\frac{2}{\Gamma(3-\gamma)} \frac{(g(t)-g(t_1))^{2-\gamma}}{w(t)} \right)^2 dt.$$

Considering $\lambda = -2$ and $u = \overline{U}$, the necessary conditions given in Theorem 5 are satisfied.

5. Conclusions

In this article, we investigated optimization problems within the framework of fractional derivatives with respect to another function and incorporating weights. The inclusion of extremal points as variables in the problem enabled us to derive the respective transversality conditions. Employing variational methods, we derived a necessary condition in the form of a fractional differential equation for the problem. Additionally, we explored isoperimetric problems and those involving higher-order derivatives.

As a potential future research direction, we aim to investigate optimal control problems wherein the differential equation depends on this type of fractional derivative. Our goal would be to deduce the equivalent of Pontryagin's minimum principle. Another significant challenge would involve exploring efficient numerical methods to address these problems. Solutions are described using fractional differential equations, which are generally analytically challenging to solve, except in rare cases. Hence, finding numerical approximations would be crucial to effectively resolving these issues.

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