Received: 30 July 2023 | Revised: 24 January 2024 | Accepted: 10 February 2024

DOI: 10.1002/mma.10002

RESEARCH ARTICLE

WILEY

Modeling blood alcohol concentration using fractional differential equations based on the ψ-Caputo derivative

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Communicated by: J. Vigo-Aguiar

Funding information
Fundação para a Ciência e a Tecnologia (FCT), Grant/Award Number: UIDB/04106/2020, UIDP/04106/2020 and 2022.03091.PTDC

We propose a novel dynamical model for blood alcohol concentration that incorporates ψ-Caputo fractional derivatives. Using the generalized Laplace transform technique, we successfully derive an analytic solution for both the alcohol concentration in the stomach and the alcohol concentration in the blood of an individual. These analytical formulas provide us a straightforward numerical scheme, which demonstrates the efficacy of the ψ-Caputo derivative operator in achieving a better fit to real experimental data on blood alcohol levels available in the literature. In comparison with existing classical and fractional models found in the literature, our model outperforms them significantly. Indeed, by employing a simple yet nonstandard kernel function $\psi(t)$, we are able to reduce the error by more than half, resulting in an impressive gain improvement of 59%.

KEYWORDS

analytic solutions, better fit to real experimental data with a gain improvement of 59%, blood alcohol dynamical model, fractional calculus, generalized Caputo fractional derivatives, mathematical modeling, ψ-Caputo fractional differential equations

MSC CLASSIFICATION

26A33, 34A08, 65L10

1 | INTRODUCTION

Alcohol, a toxic and psychoactive substance known for its addictive properties, has become deeply integrated into many societies. Alcoholic beverages have become a commonplace element of social interactions for a significant portion of the population. This is especially evident in social environments that carry considerable visibility and societal influence, where alcohol often accompanies social gatherings. Regrettably, the detrimental health and social consequences caused or exacerbated by alcohol are frequently overlooked or downplayed. In reality, alcohol consumption is responsible for a staggering three million deaths worldwide each year, while millions more suffer from disabilities and poor health as a result. The harmful use of alcohol accounts for approximately 7.1% of the global burden of disease among males and 2.2% among females. Shockingly, alcohol stands as the primary risk factor for premature mortality and disability among individuals aged 15 to 49, comprising 10% of all deaths within this age group. Moreover, disadvantaged populations, particularly those who are vulnerable, experience disproportionately higher rates of alcohol-related deaths and hospitalizations [1, 2].

Over the past several decades, fractional calculus has captured the attention of researchers across diverse fields of science and engineering [3]. Fractional differential equations, in particular, have emerged as a common tool in various scientific and engineering disciplines [4, 5]. These equations find application in fields such as signal processing, con-

trol theory, diffusion, thermodynamics, biophysics, blood flow phenomena, rheology, electrodynamics, electrochemistry, electromagnetism, continuum mechanics, statistical mechanics, and dynamical systems [6–10].

The ability to model blood alcohol content over time is not only of interest to medical professionals but also holds significant value in comparing metabolic capabilities. Mathematical techniques provide a means not only to model blood concentration but also to analyze the metabolic processes of various endogenous compounds, such as blood glucose levels or administered medications. Recent research [11, 12] has focused on investigating models for blood alcohol concentration. A comparative analysis involving three types of fractional derivatives—Caputo, Atangana–Baleanu–Caputo (ABC), and Caputo–Fabrizio (CF) derivatives—demonstrates that the Caputo and ABC operators are better suited for numerical simulations using real data when compared with classical models employing standard integer-order derivatives [12]. Here, we improve the best results of [11, 12] by using ψ-Caputo fractional derivatives, which allows us to reduce the total square error by more than half, resulting in an impressive gain of 59%. The selection of this generalized-Caputo operator in our work is grounded in several merits that make it a suitable choice for our study due to its versatility, suitability for modeling fractional order systems, and its ability to capture complex dynamics. Indeed, the generalized ψ-Caputo operator is a versatile fractional derivative that provides a unified framework for handling a wide range of real-world problems. It has been successfully applied in various scientific disciplines, including physics, engineering, and mathematical modeling [14]. Its versatility allows us to tackle complex phenomena and systems in a unified manner, being particularly well-suited for modeling systems with memory effects and nonlocal behavior. As we shall see, it allows us to capture the fractional order dynamics of blood alcohol accurately. By using the generalized-Caputo operator, we enhance the fidelity of our model, enabling us to better capture the long-range dependencies and memory effects that play a crucial role in modeling blood alcohol concentration. This improved modeling precision lead us to more accurate predictions and a better understanding of the underlying processes. The choice of the generalized ψ-Caputo operator represents one of the novel aspects of our manuscript compared with previous articles in the field. By using this operator, we contribute to the expanding body of knowledge on fractional calculus and its applications: Our choice allows us to offer a fresh perspective and advance considerably the state of the art.

The paper is organized as follows. In Section 2, we recall the notions and results from ψ-fractional calculus needed in the sequel. Our contributions are then given in Section 3: We introduce the new ψ-Caputo fractional model and obtain an explicit formula for the exact solution of the problem (Section 3.1); we show how available models and results from the literature can be obtained as particular cases (Sections 3.2.1 and 3.2.2), and we show the accuracy and efficiency of our new model by significantly improving the available results in the literature (Section 3.2.3). We end with Section 4 of conclusion.

2 | PRELIMINARIES

Originally, the ψ-Caputo fractional calculus was introduced by Osler in 1970 [15], being now part of the classical fractional calculus; see Samko et al. [16, Section 18.2] and Kilbas et al. [17, Section 2.5]. Recently, Almeida made a small variation on the Riemann–Liouville operators to get the Caputo versions and popularized the ψ-terminology [18]. Here we recall necessary notions from this calculus and two lemmas that will be useful in the proof of out theoretical result.

Definition 1 (The ψ-Riemann–Liouville fractional integral [16, 17])**.** Let $\alpha > 0$, $f : [a, b] \to \mathbb{R}$ be integrable and $\psi \in C^1([a, b])$ an increasing function such that $\psi'(t) \neq 0$, for all $t \in [a, b]$. The ψ-Riemann–Liouville fractional integral of f of order α is defined as follows:

$$I_{a^+}^{\alpha,\psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s) ds, \ \alpha > 0, \tag{1}$$

where $\Gamma(\alpha)$ is the Gamma function.

Note that for $\psi(t) = t$ and $\psi(t) = \ln(t)$, Equation (1) is reduced to the Riemann–Liouville and Hadamard fractional integrals, respectively.

Definition 2 (The ψ-Riemann–Liouville fractional derivative [16, 17])**.** Let $n \in \mathbb{N}$ and let $\psi, f \in C^n([a, b], \mathbb{R})$ be two functions such that ψ is increasing and $\psi'(t) \neq 0$ for all $t \in [a, b]$. The ψ-Riemann–Liouville fractional derivative of f of order α is given by

$$ {}^{RL}D_{a^+}^{\alpha,\psi} f(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a^+}^{n-\alpha,\psi} f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_a^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1} f(s)ds, \tag{2} $$

where $n = [\alpha] + 1$.

Definition 3 (The ψ-Caputo fractional derivative [19])**.** Let $n \in \mathbb{N}$ and let $\psi, f \in C^n([a,b],\mathbb{R})$ be two functions such that ψ is increasing and $\psi'(t) \neq 0$ for all $t \in [a,b]$. The ψ-Caputo fractional derivative of f of order α is given by

$$ {}^{C}D_{a^+}^{\alpha,\psi} f(t) = I_{a^+}^{n-\alpha,\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1} f_\psi^{[n]}(s)ds, \tag{3} $$

where $n = [\alpha] + 1$ and $f_\psi^{[n]}(t) := \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n f(t)$.

Definition 4 (See [20])**.** The Mittag–Leffler function for one- and two-parameter is defined, respectively, as

$$ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)},\ \alpha \in \mathbb{C},\ Re(\alpha) > 0, \tag{4} $$

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)},\ \alpha,\beta \in \mathbb{C},\ Re(\alpha) > 0. \tag{5} $$

We make use of generalized Laplace transforms.

Definition 5 (See Fahad et al. [21])**.** Let $f : [0,\infty) \to \mathbb{R}$ be a real-valued function and ψ be a nonnegative increasing function such that $\psi(0) = 0$. Then the Laplace transform of f with respect to ψ is defined by

$$ \mathcal{L}_\psi\{f(t)\} = F(s) = \int_0^\infty e^{-s\psi(t)}\psi'(t)f(t)dt $$

for all $s \in \mathbb{C}$ such that this integral converges. Here $\mathcal{L}_\psi$ denotes the Laplace transform with respect to ψ, which is called a generalized Laplace transform.

Definition 6 (See Fahad et al. [21])**.** An n-dimensional function $f : [0,\infty) \to \mathbb{R}$ is said to be of ψ-exponential order $c > 0$ if there exist positive constants M and T such that for all $t > T$,

$$ \|f\|_\infty = \max_{1\le i\le n} \|f_i\|_\infty \le Me^{c\psi(t)}, $$

that is, if $f(t) = O\left(e^{c\psi(t)}\right)$ as $t \to \infty$.

Lemma 7 (See Jarad and Abdeljawad [22])**.** *Let $\alpha > 0$, $f \in AC_\psi^n[a,b]$ for any $b > a$, and $f^{[k]}$, $k = 0, 1, \ldots n$, be of $\psi(t)$-exponential order. Then,*

$$ \mathcal{L}_\psi\left\{\left({}^{C}D_{a^+}^{\alpha,\psi} f\right)(t)\right\}(s) = s^\alpha\left[\mathcal{L}_\psi\{f(t)\} - \sum_{k=0}^{n-1} s^{-k-1}\left(f^{[k]}\right)\left(a^+\right)\right]. \tag{6} $$

Lemma 8 (See Jarad and Abdeljawad [22])**.** *Let $Re(\alpha) > 0$ and $|\frac{\lambda}{s^\alpha}| < 1$. Then,*

$$ \mathcal{L}_\psi\{E_\alpha(\lambda(\psi(t)-\psi(a))^\alpha)\} = \frac{s^{\alpha-1}}{s^\alpha-\lambda}, \tag{7} $$

and

$$ \mathcal{L}_\psi\left\{(\psi(t)-\psi(a))^{\beta-1}E_{\alpha,\beta}(\lambda(\psi(t)-\psi(a))^\alpha)\right\} = \frac{s^{\alpha-\beta}}{s^\alpha-\lambda}. \tag{8} $$

3 | MAIN RESULTS

We propose a new blood alcohol model associated with the ψ-Caputo fractional operator (Section 3.1); we obtain its analytical solution (Theorem 11) and show the advantages of our model with respect to the ones available in the literature (Section 3.2).

3.1 | Blood alcohol model and its analytical solution

Alcoholic beverages have been an integral part of various cultures for thousands of years. However, it is important to recognize that alcohol consumption not only leads to disorders but also significantly impacts the incidence of chronic diseases, injuries, and health issues. Understanding the effects of alcohol consumption on health requires considering three key factors: the quality of the alcohol consumed, the volume of alcohol consumed, and the consumption pattern. Examining these categories is crucial as they contribute to both detrimental and beneficial consequences. For instance, epidemiological studies and research conducted on animals have suggested that excessive alcohol consumption can depress cardiac function and cause cardiomyopathy or cardiomyopathy-related injuries. The negative effects of heavy alcohol consumption extend, however, beyond cardiac function, leading to symptoms such as thirst, fatigue, drowsiness, weakness, nausea, dry mouth, headaches, and difficulties with concentration [1, 23–26].

By comprehending the impact of alcohol consumption in terms of quality, volume, and consumption pattern, we can better understand its effects on health and make informed decisions regarding alcohol consumption. Here we propose the following model:

$$\begin{cases} {}^{C}D^{\alpha,\psi}A(t) = -k_1A(t),\ A(0) = A_0, \\ {}^{C}D^{\beta,\psi}B(t) = k_1A(t) - k_2B(t),\ B(0) = B_0, \end{cases} \tag{9}$$

where $A(t)$ and $B(t)$ represent the absorptions of alcohol in stomach and alcohol in the blood at time t, respectively, k_1 and k_2 are nonzero constants, A_0 is the initial absorption of alcohol in the stomach, and B_0 the initial quantity of alcohol in the blood. We obtain the solution for the concentration of alcohol in stomach, $A(t)$, and the concentration of alcohol in the blood, $B(t)$, by employing the generalized Laplace transform technique.

Remark 9. Given the so-called conjugation relations, see Equation (18.26) of Samko et al. [16] or Equations (2.5.7) to (2.5.10) of Kilbas et al. [17], one can easily reduce ψ-Caputo fractional differential equations to classical Caputo fractional differential equations [27].

Remark 10. The main differences between the ψ-Riemann–Liouville and ψ-Caputo fractional derivatives lie in their definitions (cf. Definitions 2 and 3), memory effects, and locality properties. The choice between these two operators depends on the specific mathematical requirements and physical interpretations of the problem at hand. In our case, they allow us to consider the initial conditions $A(0) = A_0$ and $B(0) = B_0$ in (9), which is in agreement with the previous models considered in the literature; see earlier studies [11, 12] and references therein.

Theorem 11. *The solution to the system of fractional differential equations (9) is given by*

$$A(t) = A_0E_\alpha\left(-k_1(\psi(t) - \psi(0))^\alpha\right) \tag{10}$$

and

$$\begin{aligned} B(t) = B_0E_\beta\left(-k_2(\psi(t) - \psi(0))^\beta\right) + k_1A_0\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-k_2)^n(-k_1)^m}{\Gamma((n+1)\beta)\Gamma(\alpha m+1)} \\ \times\int_0^t(\psi(t) - \psi(\tau))^{\beta+n\beta-1}(\psi(\tau) - \psi(0))^{\alpha m}\psi'(\tau)d\tau. \end{aligned} \tag{11}$$

Proof. Taking the Laplace transform on the first equation of (9), we get from Lemma 7 that

$$\begin{aligned} \mathcal{L}_\psi\{{}^{C}D^{\alpha,\psi}A(t)\} &= \mathcal{L}_\psi\{-k_1A(t)\}, \\ s^\alpha\left[\mathcal{L}_\psi\{A(t)\} - \sum_{k=0}^{n-1}s^{-k-1}\left(f^{[k]}\right)(0)\right] &= -k_1\mathcal{L}_\psi\{A(t)\}, \end{aligned}$$

$$(s^\alpha + k_1)\mathcal{L}_\psi\{A(t)\} = s^\alpha \sum_{k=0}^{n-1} s^{-k-1}\left(f^{[k]}\right)(0),$$
$$\mathcal{L}_\psi\{A(t)\} = A_0\frac{s^{\alpha-1}}{s^\alpha + k_1},$$

and Equation (10) follows by taking the inverse Laplace transform. Using (10), the second fractional order equation becomes

$${}^C D^{\beta,\psi}B(t) = k_1A_0E_\alpha\left(-k_1(\psi(t)-\psi(0))^\alpha\right) - k_2B(t)$$

and taking the Laplace transform, we get from Lemma 8 that

$$\begin{aligned}
\mathcal{L}_\psi\{{}^C D^{\alpha,\psi}B(t)\} &= \mathcal{L}_\psi\{k_1A_0E_\alpha\left(-k_1(\psi(t)-\psi(0))^\alpha\right) - k_2B(t)\},\\
s^\alpha\left[\mathcal{L}_\psi\{B(t)\} - \sum_{k=0}^{n-1} s^{-k-1}\left(f^{[k]}\right)(0)\right] &= \mathcal{L}_\psi\{k_1A_0E_\alpha\left(-k_1(\psi(t)-\psi(0))^\alpha\right)\} - k_2\mathcal{L}_\psi\{B(t)\},\\
(s^\alpha + k_2)\mathcal{L}_\psi\{B(t)\} - s^{\alpha-1}B_0 &= \mathcal{L}_\psi\{k_1A_0E_\alpha\left(-k_1(\psi(t)-\psi(0))^\alpha\right)\},\\
\mathcal{L}_\psi\{B(t)\} &= B_0\frac{s^{\alpha-1}}{s^\alpha + k_2} + \frac{1}{s^\alpha + k_2}\mathcal{L}_\psi\{k_1A_0E_\alpha\left(-k_1(\psi(t)-\psi(0))^\alpha\right)\},\\
\mathcal{L}_\psi\{B(t)\} &= B_0\mathcal{L}_\psi\{E_\alpha\left(-k_2(\psi(t)-\psi(0))^\alpha\right)\}\\
&\quad + \mathcal{L}_\psi\{(\psi(t)-\psi(0))^{\alpha-1}E_{\alpha,\alpha}\left(-k_2(\psi(t)-\psi(0))^\alpha\right)\}\\
&\quad \times \mathcal{L}_\psi\{k_1A_0E_\alpha\left(-k_1(\psi(t)-\psi(0))^\alpha\right)\}.
\end{aligned}$$

Therefore,

$$\mathcal{L}_\psi\{B(t)\} = B_0\mathcal{L}_\psi\{E_\alpha\left(-k_2(\psi(t)-\psi(0))^\alpha\right) + (\psi(t)-\psi(0))^{\alpha-1}E_{\alpha,\alpha}\left(-k_2(\psi(t)-\psi(0))^\alpha\right) *_\psi A(t)\}$$

and taking the inverse Laplace transform, we get

$$B(t) = B_0E_\alpha\left(-k_2(\psi(t)-\psi(0))^\alpha\right) + (\psi(t)-\psi(0))^{\alpha-1}E_{\alpha,\alpha}\left(-k_2(\psi(t)-\psi(0))^\alpha\right) *_\psi A(t), \tag{12}$$

which proves the intended expression (11). □

Remark 12. Note that the proof of Theorem 11 shows that the double series appearing in Equation (11) can be expressed in terms of the Mittag–Leffler function for two-parameters; see (12).

3.2 | Application

Now an application is provided to support our theoretical model (9). For that, we use blood alcohol levels (BAL) data of a real individual, using our fractional model and showing the important role of fractional differentiation with respect to another function ψ.

Given real BAL data along time, consisting of r points, $(t_0, B_0), \ldots, (t_r, B_r)$, we approximate these values by the solution $t \mapsto B(t)$ of our theoretical model. The form B is known, being given by (11), but it depends on ψ and α and β. For each approximation $B(t_i)$ of B_i given by the model, the error is defined as the difference between the exact and the approximated values, that is, by $d_i := B_i - B(t_i)$, $i = 1, \ldots, r$, while the total square error is given by

$$Error = \sum_{i=1}^{r}(d_i)^2. \tag{13}$$

3.2.1 | The classical integer order model

In the particular case, when we chose in our model (9) $\psi(t) = t$ and $\alpha = \beta = 1$, we obtain the classical system of ordinary differential equations that model the blood alcohol level [12, 28, 29]:

$$\begin{cases} \frac{dA(t)}{dt} = -k_1 A(t),\ A(0) = A_0, \\ \frac{dB(t)}{dt} = k_1 A(t) - k_2 B(t),\ B(0) = 0. \end{cases} \tag{14}$$

The solution of problem (14) is given as a direct corollary of our Theorem 11 as

$$A(t) = A_0 e^{-k_1 t}, \tag{15}$$

and

$$B(t) = A_0 \frac{k_1}{k_2 - k_1}(e^{-k_1 t} - e^{-k_2 t}). \tag{16}$$

Ludwin [28] tried to fit the experimental data given in Table 1 using the classical model (14).
Using an Excel solver, Ludwin found the values

$$A_0 \approx 245.8769,\ k_1 \approx 0.109456,\ k_2 \approx 0.017727, \tag{17}$$

for the expression (16) of $B(t)$. The modeling results from Equation (16) with the parameter values (17) are given in Table 2.

The error (13) between the real data of Table 1 and the values of Table 2 obtained by the classical model (14) is 775 $(mg/L)^2$. However, as shown in Qureshi et al. [12], these results can be improved by choosing

$$A_0 = 261.721,\ k_1 = 0.111946,\ k_2 = 0.0186294, \tag{18}$$

for which model (14) gives the values of Table 3, decreasing the error from 775 $(mg/L)^2$ to

$$E_{classical} \approx 496\,(mg/L)^2. \tag{19}$$

Such result can be improved using our fractional model (9). Indeed, by other choices of function ψ and α and β in (9), the solution of our fractional model (9) can be closer to the real data of Table 1 than the solution obtained by the classical model (14). To measure that, we follow Rosales et al. [30] and define the gain $\mathcal{G}$ of the efficiency of our model, comparing the error (19) of the classical model, $E_{classical}$, with the error (13) associated to a particular fractional instance of our model (9):

$$\mathcal{G} = \left| \frac{E_{classical} - E_{fractional}}{E_{classical}} \right| \approx \left| \frac{496 - E_{fractional}}{496} \right|. \tag{20}$$

In percentage, we multiply the value (20) by 100.

TABLE 1 Experimental data for the blood alcohol level (BAL) of a real individual [28].

Time (min)	0	10	20	30	45	80	90	110	170
BAL (mg/L)	0	150	200	160	130	70	60	40	20

TABLE 2 Blood alcohol level (BAL) predicted by the classical theoretical model (14) with the parameter values (17), corresponding to an error (13) of 775 $(mg/L)^2$.

Time (min)	0	10	20	30	45	80	90	110	170
BAL (mg/L)	0.00	147.54	172.95	161.38	129.99	70.99	59.49	41.74	14.41

TABLE 3 Blood alcohol level (BAL) predicted by the classical theoretical model (14) with the parameter values (18), corresponding to an error (13) of 496 $(mg/L)^2$.

Time (min)	0	10	20	30	45	80	90	110	170
BAL (mg/L)	0.00	158.11	182.85	168.62	133.73	70.69	58.69	40.44	13.22

TABLE 4 Blood alcohol level (BAL) predicted by the Caputo theoretical model (21) with the parameter values (22), corresponding to an error (13) of 417 $(mg/L)^2$ and a gain (20) of 16%.

Time (min)	0	10	20	30	45	80	90	110	170
BAL (mg/L)	0.000	157.733	184.469	169.800	131.700	67.879	57.162	41.883	20.730

3.2.2 | The Caputo fractional order model

In the particular case, when we chose in our model (9) $\psi(t) = t$ with $\alpha, \beta \in (0, 1)$, we obtain the standard Caputo system of fractional differential equations studied in Almeida et al. [11]:

$$\begin{cases} {}^{C}D^{\alpha}A(t) = -k_1A(t),\ A(0) = A_0, \\ {}^{C}D^{\beta}B(t) = k_1A(t) - k_2B(t),\ B(0) = 0. \end{cases} \tag{21}$$

The solution of (21) is also a direct consequence of our Theorem 11, which gives

$$A(t) = A_0E_{\alpha}(-k_1t^{\alpha}),$$

and

$$B(t) = A_0 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-k_2)^n(-k_1)^m}{\Gamma(n\beta + \beta + m\alpha + 1)} t^{n\beta+\beta+m\alpha}.$$

Using the real experimental data of blood alcohol level of Table 1, Almeida et al. [11] used a numerical optimization approach based on the least squares approximation to determine the orders α and β of the fractional Caputo operator that better describes the real data. They proved that the Caputo fractional model (21) fits better the available data when compared with the classical one given by (14). Moreover, in 2019, Qureshi et al. [12] considered not only the Caputo fractional operator, which has a singular kernel, but also nonsingular kernels: They investigated the use of the ABC and the CF kernels to fractionalize the classical model. It has been shown in Qureshi et al. [12] that the fractional versions based on ABC and CF operators are not able to improve the accuracy of the results obtained by the Caputo model (21). The current state of the art is thus given by the fractional model (21) with the parameters

$$A_0 \approx 991.085,\ k_1 \approx 0.0287362,\ k_2 \approx 0.0843802,\ \alpha \approx 0.881521,\ \beta = 1, \tag{22}$$

found via the least squares error minimization technique [12], for which the Caputo model (21) gives the values of Table 4.

The values of Table 4 lead to an error of 417 $(mg/L)^2$, which represents a gain of 16% with respect to the best fitting of the classical model. As we shall see, we can, however, improve this state of the art by using our general ψ-Caputo model (9) with an appropriate function ψ different from the identity.

3.2.3 | The ψ-Caputo fractional order model

We now consider an application of our fractional differential system with ψ-Caputo fractional derivatives to the blood alcohol concentration involving the two linked absorption processes $A(t)$ and $B(t)$, first in the stomach and then in the blood. Precisely, we provide a different function $\psi(t)$ for which the solutions of the fractional model (9) models better the given real data of Table 1 when compared with the ones studied in the literature.

Recall that for any choice of $\psi(t)$ one can always reduce our ψ-fractional system to a classical Caputo system: According with Remark 9, ψ-Caputo fractional problems are just Caputo fractional problems. Here we show that one does not need to use nontrivial functions ψ to improve the state of the art. Indeed, in comparison to existing classical and fractional models found in the literature, we outperform them significantly by employing a simple yet nonstandard kernel function $\psi(t)$, reducing the error by more than half, resulting in an impressive gain improvement of 59%.

Let $\psi(t) = a_1 + a_2 t$, $a_1, a_2 \in \mathbb{R}$. It follows from (11) that

$$B(t) = k_1 A_0 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-k_2)^n (-k_1)^m}{\Gamma((n+1)\beta + \alpha m + 1)} (a_2 t)^{\alpha m + \beta(n+1)} \tag{23}$$

or, equivalently, in terms of a naturally emerging bivariate Mittag–Leffler function,

$$B(t) = k_1 A_0 (a_2 t)^{\beta} E_{\alpha,\beta,\beta+1}\left(-k_1 (a_2 t)^{\alpha}, -k_2 (a_2 t)^{\beta}\right),$$

where this E is the naturally emerging bivariate Mittag–Leffler function introduced in 2020 [13].

To obtain the best possible values for the parameters a_1 and a_2 that define $\psi(t)$ and the best values of α and β, we have used the free and open source GNU Octave high-level programming language and the `lsqcurvefit` routine of the optimization package, which solves nonlinear data fitting problems in the least squares sense. Precisely, we developed the GNU Octave code of Listing 1.

Listing 1: GNU Octave code used for Section 3.2.3.

```
clear, clc;
pkg load optim
t   = [0 10 20 30 45 80 90 110 170];
BAL = [0 150 200 160 130 70 60 40 20];
t0 = t(1);
p0= [991.085 0.0287362 0.0843802 0.881521 1 1];
Error = @(R,M) sum((R-M).^2);
% psi-Fractional case  with psi(t)=a1+a2.*t
B = @(k,t) arrayfun(@(t) Spsi(t0,t,k),t);
B1= B(p0,t)
printf('Total cpu time: %f seconds\n', cputime-cput);
Error(BAL,B1);
cput = cputime;
p = lsqcurvefit(B,p0,t,BAL);
%printf('Total cpu time: %f seconds\n', cputime-cput);
Bp01 = B(p,t);
% Best values from paper of S. Qureshi et al. 2019
BAL01= [0 157.733 184.469 169.80 131.70 67.879 57.162 41.883 20.730];
figure
plot(t,Bp01,'r')
ylabel('Blood alcohol level (mg/l)')
xlabel('Time (minutes)')
hold on
plot(t,BAL01,'--g')
hold on
plot(t,BAL,"o")
legend({'Fractional psi(x)=a1+a2t','Fractional psi(x)=x','real data'})
hold off
```

Using our Octave code of Listing 1, we obtained the parameter values given in Table 5.

The Blood alcohol level (11) associated with the values of Table 5 are given in Table 6,
for which we decrease the error of 417 $(\text{mg/L})^2$ for the best model in the literature to a total error of less than 202 $(\text{mg/L})^2$. This corresponds to a gain of more than 59% with respect to the classical model.

In Figure 1, we plot the real data of Table 1 with the curves obtained with $\psi(t) = t$ (Section 3.2.2) and $\psi(t) = a_1 + a_2 t$ (Section 3.2.3).

TABLE 5 Optimal parameter values for the ψ-Caputo model (9) with $\psi(t) = a_1 + a_2 t$.

A_0	k_1	k_2	α	β	a_1	a_2
1270.679	0.0217903	0.1330596	1.012392	1.288845	$\forall a_1 \in \mathbb{R}$	0.621767

TABLE 6 Blood alcohol level (BAL) predicted by the new ψ-Caputo model (9) with the parameter values of Table 5, corresponding to an error (13) of less than 202 mg/L^2 and a gain (20) of more than 59%.

Time (min)	0	10	20	30	45	80	90	110	170
BAL (mg/L)	0.000	152.569	193.826	169.420	123.541	70.156	60.340	44.502	17.518

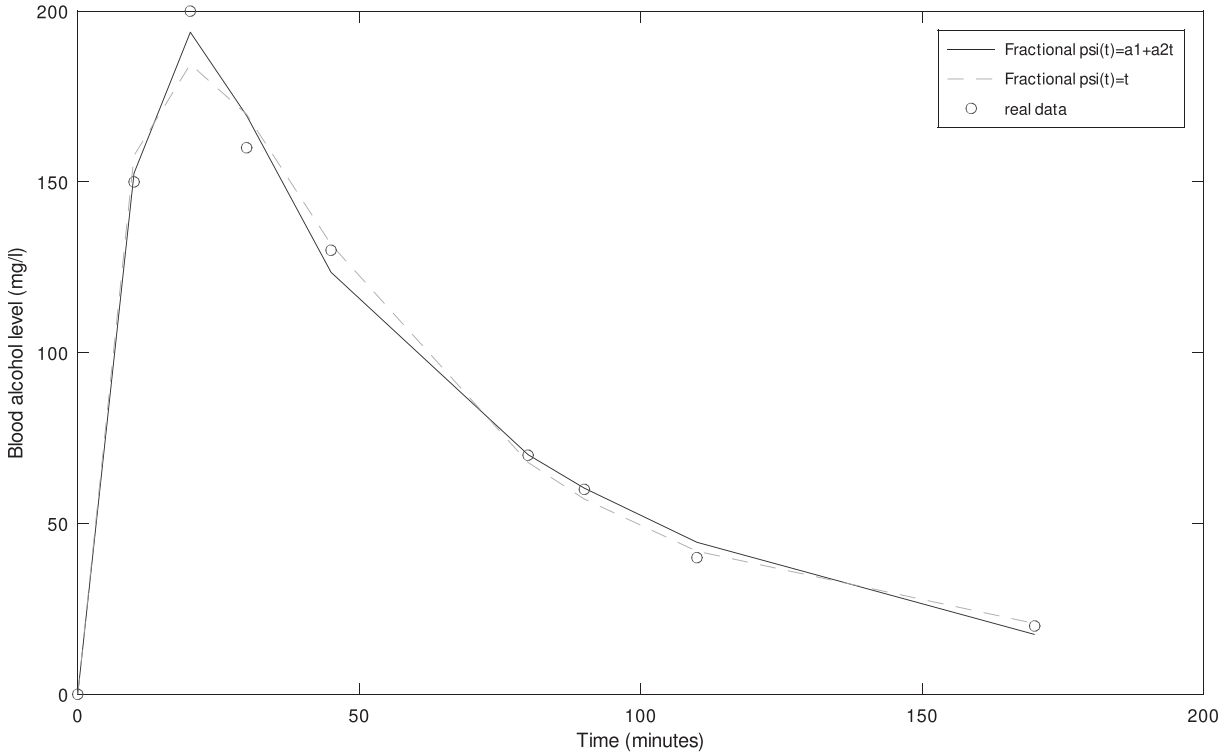

FIGURE 1 Blood alcohol level comparison between the real data of Table 1 and the predictions obtained from the best fractional models (9) with $\psi(t) = t$ (Caputo) and $\psi(t) = 0.621767t$.

TABLE 7 Comparison between integer-order, Caputo and ψ-Caputo models.

$\psi(t)$	α	β	A_0	k_1	k_2	Error
Integer order	1	1	245.8769	0.109456	0.017727	775.2226
$\psi(t) = t$ (Caputo)	0.979	0.979	991.085	0.030960	0.08887702	1383,82052
$\psi(t) = (t + 0.5)^{0.97}$	0.972	0.972	850.085	0.030960	0.08887702	756.9497

If one only uses $\psi(t) = a_1 + a_2t$, then it is not really ψ-fractional calculus, but only a constant multiple of classical fractional calculus. Indeed, putting $\psi(t) = a_1 + a_2t$ in Equation (1), it is clear that the ψ-Riemann–Liouville fractional integral to order alpha is simply a_2^α times the Riemann–Liouville fractional integral to order α. Similarly, from Equations (2) and (3), it is clear that the ψ-fractional derivative to order α (Riemann–Liouville or Caputo) is simply a_2^α times the fractional derivative to order α (Riemann–Liouville or Caputo) when $\psi(t) = a_1 + a_2t$. Therefore, in this case, all of our ψ-Caputo fractional models are just Caputo fractional models – constant multiples do not change the shape of the problem. To demonstrate the usefulness of ψ-Caputo fractional calculus, we end with an example where we use an actual ψ-Caputo model, comparing it with the models available in the literature. As one can see from Table 7, the choice $\psi(t) = (t + 0.5)^{0.97}$ is enough to improve the results published in the literature.

4 | CONCLUSION

We have introduced a novel blood alcohol concentration model that captures the dynamics using a fractional differential equation featuring the ψ-Caputo fractional derivative. The utilization of the ψ-Caputo operator ensures an optimal curve fitting by allowing for the selection of a specific kernel ψ based on the particular data being studied. By considering ψ as a

first-degree polynomial, our results demonstrate significant improvement compared with existing literature. Specifically, the total square error, as shown in Equation (13), is reduced from 496 $(\text{mg/L})^2$ using the classical model with ordinary differential equations to 417 $(\text{mg/L})^2$ with the Caputo fractional model. However, with our ψ-Caputo model, employing $\psi(t) = 0.621767t$, we achieve a remarkable reduction in the total error to just 202 $(mg/L)^2$, resulting in a substantial gain of 59%.

In summary, the key points of advantages of our research are as follows:

- We provide a novel dynamical model for blood alcohol concentration;
- We successfully derive an analytic solution for both the alcohol concentration in the stomach and the alcohol concentration in the blood of an individual;
- We prove analytical formulas that provide a straightforward numerical scheme;
- We improved the state of the art with a better fit to real experimental data on blood alcohol levels;
- Our model outperforms available ones significantly: We are able to reduce the error by more than half, resulting in an impressive gain improvement of 59%.

Given the good results obtained, for future work, we plan to investigate the usefulness of generalized ψ-Caputo operators in other contexts, for example, with respect to the respiratory syncytial virus infection [31, 32].

AUTHOR CONTRIBUTIONS

Om Kalthoum Wanassi: Conceptualization; methodology; software; investigation; validation; formal analysis; visualization; writing—original draft; writing—review and editing. **Delfim F. M. Torres:** Conceptualization; methodology; software; investigation; validation; formal analysis; supervision; visualization; writing—original draft; writing—review and editing.

ACKNOWLEDGEMENTS

The authors are supported by the Portuguese Foundation for Science and Technology (FCT) through the Center for Research and Development in Mathematics and Applications (CIDMA), grants UIDB/04106/2020 and UIDP/04106/2020, and within the project "Mathematical Modelling of Multi-scale Control Systems: Applications to Human Diseases" (CoSysM3), reference 2022.03091.PTDC, financially supported by national funds (OE) through FCT/MCTES. They are grateful to three anonymous referees for several suggestions and comments that helped them to improve the paper.

CONFLICT OF INTEREST STATEMENT

The authors have no conflicts of interest to disclose.

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How to cite this article: O. K. Wanassi and D. F. M. Torres, *Modeling blood alcohol concentration using fractional differential equations based on the ψ-Caputo derivative*, Math. Meth. Appl. Sci. **47** (2024), 7793–7803, DOI 10.1002/mma.10002.