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On a class of N-dimensional anisotropic Sobolev inequalities

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Abstract

In this paper, we study the smallest constant α in the anisotropic Sobolev inequality of the form

$$\|u\|_p^p \le \alpha \|u\|_2^{\frac{2(2N-1)+(3-2N)p}{2}} \|u_x\|_2^{\frac{N(p-2)}{2}} \prod_{k=1}^{N-1} \left\| D_x^{-1}\partial_{y_k}u \right\|_2^{\frac{p-2}{2}}$$

and the smallest constant β in the inequality

$$\|u\|_{p_*}^{p_*} \le \beta \|u_x\|_2^{\frac{2N}{2N-3}} \prod_{k=1}^{N-1} \left\| D_x^{-1}\partial_{y_k}u \right\|_2^{\frac{2}{2N-3}},$$

where $V := (x, y_1, \ldots, y_{N-1}) \in \mathbb{R}^N$ with $N \ge 3$ and $2 < p < p_* = \frac{2(2N-1)}{2N-3}$. These constants are characterized by variational methods and scaling techniques. The techniques used here seem to have independent interests.

MSC: 35J20; 35A30

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1 Introduction

Let $N \ge 2$ and $2 < p \le p_* := \frac{2(2N-1)}{2N-3}$. A classical inequality [1, p. 323] states: there is a positive constant $C > 0$ such that, for any $f \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |f_x|^p \, dV \le C \left(\int_{\mathbb{R}^N} |f_x|^2 \, dV \right)^{\frac{2(2N-1)+(3-2N)p}{4}} \left(\int_{\mathbb{R}^N} |f_{xx}|^2 \, dV \right)^{\frac{N(p-2)}{4}}$$

$$\times \prod_{k=1}^{N-1} \left(\int_{\mathbb{R}^N} |\partial_{y_k}f|^2 \, dV \right)^{\frac{p-2}{4}}, \tag{1.1}$$

where $V := (x, y) \in \mathbb{R}^N$ and $y = (y_1, y_2, \ldots, y_{N-1}) \in \mathbb{R}^{N-1}$. The purpose of the present paper is to characterize the smallest (sharp) positive constant C of (1.1) (see Theorem 2.8 and Theorem 3.8) and the related equations (see (2.1) and (3.2)).

Two special cases of (1.1) have been used to study the solitary waves of the generalized Kadomtsev–Petviashvili equation. For example, when $N = 2$ (at this moment, $V := (x, y_1) \in \mathbb{R}^2$) and $2 < p < 6$, (1.1) in the form

$$\int_{\mathbb{R}^2} |u|^p \, dV \le C \left(\int_{\mathbb{R}^2} u_x^2 \, dV \right)^{\frac{p-2}{2}} \left(\int_{\mathbb{R}^2} \left(D_x^{-1} \partial_{y_1} u \right)^2 dV \right)^{\frac{p-2}{4}} \left(\int_{\mathbb{R}^2} u^2 \, dV \right)^{\frac{6-p}{4}} \tag{1.2}$$

has been used to study the following generalized Kadomtsev–Petviashvili I equation:

$$\varphi_t + \varphi_{xxx} + \varphi^{p-2} \varphi_x = D_x^{-1} \varphi_{y_1 y_1}, \quad (x, y_1) \in \mathbb{R}^2, t > 0. \tag{1.3}$$

de Bouard et al. [6, 7] proved that (1.3) had a solitary wave solution for $2 < p < 6$ and (1.3) did not possess any solitary waves if $p \ge 6$. Stability of solitary wave of (1.3) has been studied in [9] in which (1.2) has played an important role. Chen et al. [4] also used (1.2) to study the Cauchy problem of solutions to the 2-dimensional generalized Kadomtsev–Petviashvili I equation, generalized rotation-modified Kadomtsev–Petviashvili equation and generalized Kadomtsev–Petviashvili coupled with Benjamin–Ono equation.

When $N = 3$ (at this moment, $V := (x, y_1, y_2) \in \mathbb{R}^3$), de Bouard et al. [6, 7] used (1.1) to prove that if $p \ge \frac{10}{3}$ then the following equation:

$$-u + u_{xx} + u^{p-1} = D_x^{-2} u_{y_1 y_1} + D_x^{-2} u_{y_2 y_2}, \quad u \ne 0, \tag{1.4}$$

had no solutions in $Y(3)$, where $Y(3)$ is the closure of $\partial_x(C_0^\infty(\mathbb{R}^3))$ under the norm

$$\|u\|_{Y(3)}^2 = \int_{\mathbb{R}^3} \left(u_x^2 + \left| D_x^{-1} \partial_{y_1} u \right|^2 + \left| D_x^{-1} \partial_{y_2} u \right|^2 + |u|^2 \right) dV.$$

Here we define D_x^{-1}, D_x^{-2} by

$$D_x^{-1} h(x, y) = \int_{-\infty}^x h(s, y) \, ds, \qquad D_x^{-2} h = D_x^{-1} \left(D_x^{-1} h \right).$$

While for $2 < p < \frac{10}{3}$, (1.4) had at least one nonzero solution in $Y(3)$. Observing this previous work, $p_* = 6$ (when $N = 2$) and $p_* = \frac{10}{3}$ (when $N = 3$) seem to be a critical nonlinear exponent, which shares some properties similar to the critical Sobolev exponent $2^* = 2N/(N-2)$ ($N \ge 3$) in the study of semilinear elliptic equations. Recall that the best constant C_S in the Sobolev inequality $\|u\|_{L^{2^*}}^2 \le C_S \|\nabla u\|_{L^2}^2$ is well-known and has been used extensively. But for the smallest constant C in (1.1), few results are known. When $N = 2$ and $2 < p < 6$, the smallest constant C of (1.2) and its applications has been studied in [4]. When $N = 2$ and $p = 6$, the characterization of the smallest constant C of (1.2) and its related properties were studied in [5].

In the present paper, we are interested in the characterization of the smallest constant C of (1.1) in the case of $N \ge 3$. According to the value of $2 < p < p_*$ and $p = p_* = 2(2N-1)/(2N-3)$, the studies were divided into two parts. In the first part, we study (1.1) in the

case of $2 < p < p_*$. At this time, (1.1) is written as the following form:

$$\int_{\mathbb{R}^N} |u|^p \, dV \leq \alpha \left(\int_{\mathbb{R}^N} |u|^2 \, dV \right)^{\frac{2(2N-1)+(3-2N)p}{4}} \left(\int_{\mathbb{R}^N} |u_x|^2 \, dV \right)^{\frac{N(p-2)}{4}}$$

$$\times \prod_{k=1}^{N-1} \left(\int_{\mathbb{R}^N} \left| D_x^{-1} \partial_{y_k} u \right|^2 \, dV \right)^{\frac{p-2}{4}}, \tag{1.5}$$

where $u \in Y_1$ and Y_1 is the closure of $\partial_x (C_0^\infty(\mathbb{R}^N))$ under the norm

$$\|u\|_{Y_1}^2 := \int_{\mathbb{R}^N} \left(u_x^2 + \left| D_x^{-1} \nabla_y u \right|^2 + |u|^2 \right) dV.$$

As before and from now on, $y = (y_1, \ldots, y_{N-1})$,

$$\nabla_y = \left(\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{N-1}} \right), \qquad \left| D_x^{-1} \nabla_y u \right|^2 = \sum_{k=1}^{N-1} \left| D_x^{-1} \partial_{y_k} u \right|^2 \quad \text{and} \quad \triangle_y = \sum_{k=1}^{N-1} \frac{\partial^2}{\partial y_k}.$$

The main result of this part is to prove that the smallest constant α can be represented by N, p and a minimal action solution of

$$-u + u_{xx} + |u|^{p-2} u = D_x^{-2} \triangle_y u, \quad u \neq 0, u \in Y_1.$$

For details, see Theorem 2.5 and Theorem 2.8.

In the second part, we treat (1.1) in the case of $p = p_*$. In this case, (1.1) is written as

$$\int_{\mathbb{R}^N} |u|^{p_*} \, dV \leq \beta \left(\int_{\mathbb{R}^N} |u_x|^2 \, dV \right)^{\frac{N}{2N-3}} \prod_{k=1}^{N-1} \left(\int_{\mathbb{R}^N} \left| D_x^{-1} \partial_{y_k} u \right|^2 \, dV \right)^{\frac{1}{2N-3}}, \tag{1.6}$$

where $u \in Y_0$ and Y_0 is the closure of $\partial_x (C_0^\infty(\mathbb{R}^N))$ under the norm

$$\|u\|_{Y_0}^2 := \int_{\mathbb{R}^N} \left(u_x^2 + \left| D_x^{-1} \nabla_y u \right|^2 \right) dV.$$

The main results of this part are Theorem 3.5 and Theorem 3.8.

The estimate of the smallest constants α and β is based on variational methods and scaling techniques. Recall that Weinstein [10] used variational methods to estimate the constant C_G in the Gagliardo–Nirenberg interpolation inequality [3],

$$\int_{\mathbb{R}^N} |u|^{q+1} \, dz \leq C_G \left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dz \right)^{\frac{N(q-1)}{4}} \left(\int_{\mathbb{R}^N} |u|^2 \, dz \right)^{\frac{2(q+1)-N(q-1)}{4}}, \quad u \in W^{1,2}(\mathbb{R}^N).$$

This C_G was estimated directly by studying the following minimization problem:

$$C_G^{-1} = \inf \left\{ \frac{\left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dz \right)^{\frac{N(q-1)}{4}} \left(\int_{\mathbb{R}^N} |u|^2 \, dz \right)^{\frac{2(q+1)-N(q-1)}{4}}}{\int_{\mathbb{R}^N} |u|^{q+1} \, dz} : u \in W^{1,2}(\mathbb{R}^N) \backslash \{0\} \right\},$$

due to the compactness embedding of

$$W_{\text{radial}}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{q+1}(\mathbb{R}^N) \quad \text{for } 1 < q < 2^* - 1,$$

where

$$W_{\mathrm{radial}}^{1,2}(\mathbb{R}^N) = \left\{ u \in W^{1,2}(\mathbb{R}^N) : u(x) = u(|x|) \right\},$$

$2^* = 2N/(N-2)$ for $N \geq 3$ and $2^* = +\infty$ for $N = 2$. Weinstein [10] managed to prove the best constant C_G for $N \geq 2$ because the above compactness embedding holds only for $N \geq 2$. However, in the process of studying the best constant α (respectively, β), we cannot use the methods of Weinstein [10] because we are facing anisotropic Sobolev spaces Y_1 (respectively, Y_0). In the present paper, we introduce a new method. The detailed strategy contains three steps, which are given in the next section; and it may have independent interest. In fact, we believe that it can be used to study the smallest constant of other kind of inequalities.

This paper is organized as follows. In Sect. 2, we study the constant α, meanwhile we explain the strategy in detail. In Sect. 3, we use this method to study the smallest constant β under some additional analytic techniques.

Notations Throughout this paper, all integrals are taken over $\mathbb{R}^N$ unless stated otherwise. A function u defined on $\mathbb{R}^N$ is always real-valued. $\| \cdot \|_q$ denotes the L^q norm in $L^q(\mathbb{R}^N)$.

2 The smallest constant α

In this section, we always assume that $2 < p < p_* := \frac{2(2N-1)}{2N-3}$. We introduce a new strategy to estimate α in (1.5). It contains three steps and hence we divide this section into three subsections.

2.1 Minimal action solutions

In this subsection, we prove the existence of the minimal action solutions of the following equation:

$$-u + u_{xx} + |u|^{p-2}u = D_x^{-2}\triangle_y u, \quad u \neq 0, u \in Y_1. \tag{2.1}$$

Define on Y_1 the following functionals:

$$L_1(u) = \int \left(\frac{1}{2}u_x^2 + \frac{1}{2}|D_x^{-1}\nabla_y u|^2 + \frac{1}{2}|u|^2 - \frac{1}{p}|u|^p \right) dV \quad \text{and}$$

$$I_1(u) = \int \left(u_x^2 + |D_x^{-1}\nabla_y u|^2 + |u|^2 - |u|^p \right) dV.$$

Set

$$\Gamma_1 = \left\{ u \in Y_1 : u \neq 0, I_1(u) = 0 \right\} \quad \text{and} \quad d_1 = \inf_{u \in \Gamma_1} L_1(u).$$

Then according to the inequality (1.5), both L_1 and I_1 are well defined and C^1 on Y_1. The following definition is by now standard.

Definition 2.1 An element $v \in Y_1$ is said to be a solution of (2.1) if and only if v is a critical point of L_1, i.e., $L_1'(v) = 0$. Moreover, $v \in Y_1$ is said to be a minimal action solution of (2.1) if $v \neq 0$, $L_1'(v) = 0$ and $L_1(v) \leq L_1(u)$ for any $u \in \Gamma_1$.

The following lemmas will play important roles in what follows.

Lemma 2.2 *For any $u \in Y_1$ and $u \neq 0$, there is a unique $s_u > 0$ such that $s_u u \in \Gamma_1$. Moreover, if $I_1(u) < 0$ then $0 < s_u < 1$.*

Proof For $u \neq 0$ and $s > 0$, we have

$$I_1(su) = \int \left(s^2 u_x^2 + s^2 \left|D_x^{-1}\nabla_y u\right|^2 + s^2|u|^2 - s^p|u|^p\right) dV.$$

Hence from direct computations, we get

$$s_u = \left(\|u\|_{Y_1}^2\right)^{\frac{1}{p-2}} \left(\int |u|^p \, dV\right)^{-\frac{1}{p-2}}.$$

Clearly from the expression of $I_1(u)$, we know that if $I_1(u) < 0$, then $\|u\|_{Y_1}^2 < \int |u|^p \, dV$ and therefore $0 < s_u < 1$. □

Lemma 2.3 *The set Γ_1 is a manifold and there exists $\rho > 0$ such that, for any $u \in \Gamma_1$, $\|u\|_{Y_1} \geq \rho > 0$.*

Proof Firstly, it is observed from Lemma 2.2 that $\Gamma_1 \neq \emptyset$. For any $u \in \Gamma_1$,

$$\left\langle I_1'(u), u \right\rangle_{Y_1} = 2\|u\|_{Y_1}^2 - p\int |u|^p \, dV = (2-p)\int |u|^p \, dV < 0.$$

Hence Γ_1 is a manifold. Secondly, for any $u \in \Gamma_1$, using inequality (1.5) and Young inequality, we know that there is a positive constant C such that

$$\|u\|_{Y_1}^2 = \int |u|^p \, dV \leq C\|u\|_{Y_1}^p.$$

It is deduced that $\|u\|_{Y_1} \geq C^{-\frac{1}{p-2}} := \rho > 0$. The proof is complete. □

Lemma 2.4 *If $v \in \Gamma_1$ and $L_1(v) = d_1$, then v is a critical point of L_1 on Y_1, i.e. $L_1'(v) = 0$.*

Proof By Lagrangian multiplier rule, there is $\theta \in \mathbb{R}$ such that $L_1'(v) = \theta I_1'(v)$. Note that $\langle L_1'(v), v \rangle_{Y_1} = I_1(v) = 0$ and

$$\left\langle I_1'(v), v \right\rangle_{Y_1} = (2-p)\int |u|^p \, dV < 0.$$

One easily obtains $\theta = 0$. Therefore $L_1'(v) = 0$. □

Theorem 2.5 *We see that $d_1 > 0$ and there is a $\phi \in \Gamma_1$ such that $d_1 = L_1(\phi)$. Moreover, ϕ is a minimal action solution of (2.1).*

Proof It is easy to see from Lemma 2.3 that $d_1 > 0$. According to Definition 2.1 and Lemma 2.4, we only need to prove that there is $\phi \in \Gamma_1$ such that $d_1 = L_1(\phi)$.

Let $\{u_n\}_{n\in\mathbb{N}} \subset \Gamma_1$ be a minimizing sequence of d_1, i.e. $u_n \neq 0$, $I_1(u_n) = 0$ and $d_1 + o(1) = L_1(u_n)$. By $I_1(u_n) = 0$ and the anisotropic Sobolev inequality (1.5), we know that $\|u_n\|_{Y_1}$ is bounded. Moreover, Lemma 2.3 implies that $\|u_n\|_{Y_1}$ is uniformly bounded away from zero and we see that

$$\liminf_{n\to\infty} \int |u_n|^p \, dV = \liminf_{n\to\infty} \|u_n\|_{Y_1}^2 > 0.$$

Note that, for any $V \equiv (x, y) \in \mathbb{R}^N$,

$$L_1\big(u(\cdot + x, \cdot + y_1, \dots, \cdot + y_{N-1})\big) = L_1\big(u(\cdot)\big) \quad \text{and}$$
$$I_1\big(u(\cdot + x, \cdot + y_1, \dots, \cdot + y_{N-1})\big) = I_1\big(u(\cdot)\big).$$

We see from the concentration compactness lemma of Lions [8] that there are $V^n \equiv (x^n, y^n) \in \mathbb{R}^N$, where $y^n = (y_1^n, \dots, y_{N-1}^n)$, such that

$$\varphi_n(x, y) := u_n\big(x + x^n, y_1 + y_1^n, \dots, y_{N-1} + y_{N-1}^n\big)$$

satisfies

$$L_1(\varphi_n) = L_1(u_n) \quad \text{and} \quad I_1(\varphi_n) = I_1(u_n) = 0.$$

Moreover, there is $\phi \in Y_1$ and $\phi \neq 0$ such that $\phi_n \rightharpoonup \phi$ weakly in Y_1 and $\varphi_n \to \phi$ a.e. in $\mathbb{R}^N$.

If $I_1(\phi) < 0$, then by Lemma 2.2 there is a $0 < s_\phi < 1$ such that $s_\phi\phi \in \Gamma_1$. Therefore using the Fatou lemma and the fact that $I_1(\varphi_n) = 0$, we obtain

$$d_1 + o(1) = L_1(\varphi_n) = \left(\frac{1}{2} - \frac{1}{p}\right) \int |\varphi_n|^p \, dV \geq \left(\frac{1}{2} - \frac{1}{p}\right) \int |\phi|^p \, dV + o(1)$$
$$= \left(\frac{1}{2} - \frac{1}{p}\right) s_\phi^{-p} \int |s_\phi\phi|^p \, dV + o(1) = s_\phi^{-p} L_1(s_\phi\phi) + o(1)$$

as n large enough. It is deduced from $0 < s_\phi < 1$ that $d_1 > L_1(s_\phi\phi)$ which is a contradiction because of $s_\phi\phi \in \Gamma_1$.

If $I_1(\phi) > 0$, then using Brezis–Lieb lemma [2] one has

$$0 = I_1(\varphi_n) = I_1(\phi) + I_1(v_n) + o(1),$$

where $\varphi_n - \phi := v_n$ in the remaining part of this section. $I_1(\phi) > 0$ implies that

$$\limsup_{n\to\infty} I_1(v_n) < 0. \tag{2.2}$$

From Lemma 2.2 we know that there are $s_n := s_{v_n}$ such that $s_n v_n \in \Gamma_1$. Moreover, we claim that $\limsup_{n\to\infty} s_n \in (0, 1)$. Indeed if $\limsup_{n\to\infty} s_n = 1$, then there is a subsequence $\{s_{n_k}\}$ such that $\lim_{k\to\infty} s_{n_k} = 1$. Therefore from $s_{n_k} v_{n_k} \in \Gamma_1$, one has

$$I_1(v_{n_k}) = I_1(s_{n_k} v_{n_k}) + o(1) = o(1).$$

This contradicts (2.2). Hence $\limsup_{n\to\infty} s_n \in (0,1)$. Since, for n large enough,

$$d_1 + o(1) = \left(\frac{1}{2} - \frac{1}{p}\right)\int |\varphi_n|^p\, dV \geq \left(\frac{1}{2} - \frac{1}{p}\right)\int |v_n|^p\, dV + o(1)$$

$$\geq \left(\frac{1}{2} - \frac{1}{p}\right) s_n^{-p} \int |s_n v_n|^p\, dV + o(1) = s_n^{-p} L_1(s_n v_n) + o(1),$$

one has $d_1 > L_1(s_n v_n)$, which is a contradiction because $s_n v_n \in \Gamma_1$.

Thus $I_1(\phi) = 0$. Next we claim that $\|v_n\|_{Y_1} \to 0$ as $n \to \infty$. We prove this claim by a contradiction. If $\|v_n\|_{Y_1} \not\to 0$ as $n \to \infty$, then there is a subsequence $\{v_{n_k}\}_{k\in\mathbb{N}} \subset \{v_n\}_{n\in\mathbb{N}}$, such that $\|v_{n_k}\|_{Y_1} \to C > 0$ as $k \to \infty$. Using Brezis–Lieb lemma [2], one has

$$0 = I_1(\varphi_{n_k}) = I_1(v_{n_k}) + I_1(\phi) + o(1).$$

Hence $I_1(v_{n_k}) = o(1)$. According to Lemma 2.2, there are $\rho_{n_k} > 0$ such that $I_1(\rho_{n_k} v_{n_k}) = 0$. Moreover, $\rho_{n_k} \to 1$ as $k \to \infty$. Using Brezis–Lieb lemma [2] again, we see that, as n grows large enough,

$$d_1 + o(1) = L_1(\varphi_{n_k}) = L_1(v_{n_k}) + L_1(\phi) + o(1)$$

$$= L_1(\rho_{n_k} v_{n_k}) + L_1(\phi) + o(1)$$

$$\geq d_1 + d_1 + o(1),$$

which is impossible because of $d_1 > 0$. Hence we have proven that $\|v_{n_k}\|_{Y_1} \to 0$ as $k \to \infty$. Therefore $L_1(\varphi_{n_k}) \to L_1(\phi)$ and $d_1 = L_1(\phi)$. □

Next, we give some properties of the minimal action solution ϕ obtained above. These properties seem to be of independent interests and will be very useful in what follows.

Lemma 2.6 *Let ϕ be a minimal action solution of* (2.1). *Then* $I_1(\phi) = 0$,

$$Q_1(\phi) := \int \left(\phi_x^2 + \left|D_x^{-1}\nabla_y\phi\right|^2 - \frac{(2N-1)(p-2)}{2p}|\phi|^p\right) dV = 0 \quad and$$

$$R_1(\phi) := \int \left(\phi_x^2 - \frac{N(p-2)}{2p}|\phi|^p\right) dV = 0.$$

Moreover, we see that

$$\int \left|D_x^{-1}\nabla_y\phi\right|^2 dV = \frac{N-1}{N}\int \phi_x^2\, dV,$$

$$\int |\phi|^p\, dV = \frac{2p}{N(p-2)}\int \phi_x^2\, dV \quad and \tag{2.3}$$

$$\int |\phi|^2\, dV = \frac{(3-2N)p + 2(2N-1)}{N(p-2)}\int \phi_x^2\, dV.$$

Proof Since ϕ is a minimal action solution of (2.1), $L_1'(\phi) = 0$. First, we define

$$\phi_\delta(x,y) = \delta^{\frac{N}{2}}\phi(\delta x, \delta y), \quad y = (y_1,\ldots,y_{N-1}) \in \mathbb{R}^{N-1}.$$

Then, by direct computation, we see that

$$\int (\partial_x \phi_\delta)^2 \, dV = \delta^2 \int \phi_x^2 \, dV; \qquad \int |\phi_\delta|^p \, dV = \delta^{\frac{N(p-2)}{2}} \int |\phi|^p \, dV;$$

$$\int \left| D_x^{-1} \nabla_y \phi_\delta \right|^2 dV = \int \left| D_x^{-1} \nabla_y \phi \right|^2 dV \quad \text{and} \quad \int |\phi_\delta|^2 \, dV = \int |\phi|^2 \, dV.$$

Hence

$$L_1(\phi_\delta) = \frac{\delta^2}{2} \int \phi_x^2 \, dV + \frac{1}{2} \int \left| D_x^{-1} \nabla_y \phi \right|^2 dV + \frac{1}{2} \int |\phi|^2 \, dV - \frac{\delta^{\frac{N(p-2)}{2}}}{p} \int |\phi|^p \, dV.$$

Therefore

$$R_1(\phi) = \frac{\partial L_1(\phi_\delta)}{\partial \delta} \bigg|_{\delta=1} = \left\langle L_1'(\phi), \frac{\partial \phi_\delta}{\partial \delta} \bigg|_{\delta=1} \right\rangle_{Y_1} = 0.$$

Next, we define

$$\phi^\delta(x,y) = \delta^{\frac{2N-1}{2}} \phi(\delta x, \delta^2 y), \quad y = (y_1, \dots, y_{N-1}) \in \mathbb{R}^{N-1}.$$

Then, by direct computation, we obtain

$$\int \left(\partial_x \phi^\delta \right)^2 dV = \delta^2 \int \phi_x^2 \, dV; \qquad \int |\phi^\delta|^p \, dV = \delta^{\frac{(2N-1)(p-2)}{2}} \int |\phi|^p \, dV;$$

$$\int \left| D_x^{-1} \nabla_y \phi^\delta \right|^2 dV = \delta^2 \int \left| D_x^{-1} \nabla_y \phi \right|^2 dV \quad \text{and} \quad \int |\phi^\delta|^2 \, dV = \int |\phi|^2 \, dV.$$

Hence

$$L_1\left(\phi^\delta \right) = \frac{\delta^2}{2} \int \left(\phi_x^2 + \left| D_x^{-1} \nabla_y \phi \right|^2 \right) dV + \frac{1}{2} \int |\phi|^2 \, dV - \frac{\delta^{\frac{(2N-1)(p-2)}{2}}}{p} \int |\phi|^p \, dV.$$

Therefore

$$Q_1(\phi) = \frac{\partial L_1(\phi^\delta)}{\partial \delta} \bigg|_{\delta=1} = \left\langle L_1'(\phi), \frac{\partial \phi^\delta}{\partial \delta} \bigg|_{\delta=1} \right\rangle_{Y_1} = 0.$$

The proof is complete. □

Remark From Lemma 2.6, one also obtains

$$\int \left| D_x^{-1} \nabla_y \phi \right|^2 dV = \frac{(N-1)(p-2)}{(3-2N)p + 2(2N-1)} \int \phi^2 \, dV,$$

$$\int |\phi|^p \, dV = \frac{2p}{(3-2N)p + 2(2N-1)} \int \phi^2 \, dV \quad \text{and} \tag{2.4}$$

$$\int \phi_x^2 \, dV = \frac{N(p-2)}{(3-2N)p + 2(2N-1)} \int \phi^2 \, dV.$$

2.2 Another characterization of the minimal action solutions

In this subsection, we give another characterization of the minimal action solution ϕ of (2.1) obtained in the previous subsection. We emphasize that this characterization will play a key role in the process of estimating α. Define

$$F(u) = \int_{\mathbb{R}^N} \left(u_x^2 + \left| D_x^{-1} \nabla_y u \right|^2 + u^2 \right) dV, \quad u \in Y_1$$

and for $r > 0$ set

$$F_r = \inf \left\{ F(u) : u \in Y_1 \text{ and } \int |u|^p \, dV = r \right\}.$$

Then we have the following proposition.

Proposition 2.7 *Let ϕ be a minimal action solution of* (2.1). *Then ϕ is a minimizer of F_r with $r = \int |\phi|^p \, dV$.*

Proof Since ϕ is a minimal action solution of (2.1), we know that $L_1(\phi) \leq L_1(u)$ for any $u \in Y_1$ with $u \neq 0$ and $I_1(u) = 0$. Denote

$$F_{r_0} = \inf \left\{ F(u) : u \in Y_1 \text{ and } \int |u|^p \, dV = \int |\phi|^p \, dV \right\}.$$

One immediately has $F(\phi) \geq F_{r_0}$.

Next, we will prove that, for any $u \in Y_1$ with $\int |u|^p \, dV = \int |\phi|^p \, dV$,

$$F(\phi) \leq F(u).$$

For any $\mu > 0$, $I_1(\mu u) = \mu^2 F(u) - \mu^p \int |u|^p \, dV$. Hence

$$\mu_0 = \left(F(u) \right)^{\frac{1}{p-2}} \left(\int |u|^p \, dV \right)^{-\frac{1}{p-2}}$$

is such that $I_1(\mu_0 u) = 0$. The fact that $\mu_0 u \neq 0$ implies that

$$L_1(\phi) \leq L_1(\mu_0 u) = \frac{1}{2} \mu_0^2 F(u) - \frac{1}{p} \mu_0^p \int |u|^p \, dV$$

$$= \left(\frac{1}{2} - \frac{1}{p} \right) \left(F(u) \right)^{\frac{p}{p-2}} \left(\int |u|^p \, dV \right)^{-\frac{2}{p-2}}$$

$$= \left(\frac{1}{2} - \frac{1}{p} \right) \left(F(u) \right)^{\frac{p}{p-2}} \left(\int |\phi|^p \, dV \right)^{-\frac{2}{p-2}}.$$

Since $\int |\phi|^p \, dV = F(\phi)$ and $L_1(\phi) = (\frac{1}{2} - \frac{1}{p})F(\phi)$, one deduces that

$$\left(F(\phi) \right)^{\frac{p}{p-2}} \leq \left(F(u) \right)^{\frac{p}{p-2}},$$

which implies that $F(\phi) \leq F(u)$. Since u is chosen arbitrarily, one has $F(\phi) \leq F_{r_0}$. Combining this with $F(\phi) \geq F_{r_0}$, one has $F(\phi) = F_{r_0}$ and hence ϕ is a minimizer of F_r with $r = \int |\phi|^p \, dV$. □

2.3 Estimate of the smallest constant α

In this subsection, we estimate the constant α of (1.5). To simplify the notation, we denote $T = (3 - 2N)p + 2(2N - 1)$. Consider the following minimization problem:

$$\alpha^{-1} = \inf_{u \neq 0, u \in Y_1} J_1(u), \tag{2.5}$$

where

$$J_1(u) = \left(\int |u|^2 \, dV \right)^{\frac{T}{4}} \left(\int u_x^2 \, dV \right)^{\frac{N(p-2)}{4}} \prod_{k=1}^{N-1} \left(\int \left| D_x^{-1} \partial_{y_k} u \right|^2 \, dV \right)^{\frac{p-2}{4}} \left(\int |u|^p \, dV \right)^{-1}.$$

We have the following theorem.

Theorem 2.8 *Let $2 < p < p_*$ and $T = (3 - 2N)p + 2(2N - 1)$. Then*

$$\begin{aligned}
\alpha^{-1} &= \left(\frac{T}{2p} \right)^{\frac{T}{4}} \left(\frac{N^N (p-2)^{2N-1}}{(2p)^{2N-3} T^2} \right)^{\frac{p-2}{4}} \left(\int |\phi|^2 \, dV \right)^{\frac{p-2}{2}} \\
&= \left(\frac{T}{2p} \right)^{\frac{T}{4}} \left(\frac{N^N (p-2)^{2N-1}}{(2p)^{2N-3} T^2} \right)^{\frac{p-2}{4}} \left(\frac{T}{p-2} d_1 \right)^{\frac{p-2}{2}},
\end{aligned} \tag{B}$$

where ϕ is the minimal action solution of (2.1) obtained in the above and $d_1 = L_1(\phi)$.

Remark From Theorem 2.8, we know that α^{-1} can be exactly expressed by N, p and the minimal action solution ϕ of (2.1). Even though we do not know if the minimal action solution of (2.1) is unique or not, the second equality of (B) implies that α^{-1} is independent of the choice of the minimal action solution ϕ.

Proof of Theorem 2.8 The proof is divided into three steps. In the first two steps, we prove the first equality of (B). In the third step, we prove the second equality of (B).

Step 1. In this step, we prove that

$$\alpha^{-1} \geq \left(\frac{T}{2p} \right)^{\frac{T}{4}} \left(\frac{N^N (p-2)^{2N-1}}{(2p)^{2N-3} T^2} \right)^{\frac{p-2}{4}} \left(\int |\phi|^2 \, dV \right)^{\frac{p-2}{2}}.$$

For any $u \in Y_1$ and $u \neq 0$, denote $u \equiv u(x, y_1, \ldots, y_{N-1})$. We define

$$w(x, y) = \lambda u(\mu x, \xi_1 y_1, \ldots, \xi_{N-1} y_{N-1}),$$

where $\lambda, \mu, \xi_1, \ldots, \xi_{N-1}$ are $N + 1$ positive parameters which will be determined later. From direct computation, one obtains

$$
\int |D_x^{-1} \partial_{y_k} w|^2 \, dV = \lambda^2 \mu^{-3} \left(\prod_{j \neq k} \xi_j^{-1} \right) \xi_k \int |D_x^{-1} \partial_{y_k} u|^2 \, dV, \quad k = 1, \ldots, N-1;
$$

$$
\int |w|^p \, dV = \lambda^p \mu^{-1} \left(\prod_{k=1}^{N-1} \xi_k^{-1} \right) \int |u|^p \, dV;
$$

$$
\int w_x^2 \, dV = \lambda^2 \mu \left(\prod_{k=1}^{N-1} \xi_k^{-1} \right) \int u_x^2 \, dV;
$$

$$
\int w^2 \, dV = \lambda^2 \mu^{-1} \left(\prod_{k=1}^{N-1} \xi_k^{-1} \right) \int u^2 \, dV.
$$

(2.6)

Here $\lambda, \mu, \xi_1, \ldots, \xi_{N-1}$ are determined by the following $N + 1$ equations:

$$
\lambda^2 \mu^{-3} \left(\prod_{j \neq k} \xi_j^{-1} \right) \xi_k \int |D_x^{-1} \partial_{y_k} u|^2 \, dV
$$

$$
= \frac{p-2}{(3-2N)p + 2(2N-1)} \int \phi^2 \, dV,
$$

(2.7)

where for $k = 1, \ldots, N-1$, (2.7) is denoted by $(2.7)_k$;

$$
\lambda^p \mu^{-1} \left(\prod_{k=1}^{N-1} \xi_k^{-1} \right) \int |u|^p \, dV = \frac{2p}{(3-2N)p + 2(2N-1)} \int \phi^2 \, dV,
$$

(2.8)

$$
\lambda^2 \mu \left(\prod_{k=1}^{N-1} \xi_k^{-1} \right) \int u_x^2 \, dV = \frac{N(p-2)}{(3-2N)p + 2(2N-1)} \int \phi^2 \, dV.
$$

(2.9)

Next, we solve $\lambda, \mu, \xi_1, \ldots, \xi_{N-1}$ from $(2.7)_k$, where $k = 1, \ldots, N-1$, (2.8) and (2.9). Firstly, (2.8) and (2.9) imply that

$$
\mu^2 = \frac{N(p-2)}{2p} \lambda^{p-2} \int |u|^p \, dV \left(\int u_x^2 \, dV \right)^{-1}.
$$

(2.10)

Using (2.8) and $(2.7)_k$, one gets

$$
\lambda^{p-2} \mu^2 \xi_k^{-2} \int |u|^p \, dV = \frac{2p}{p-2} \int |D_x^{-1} \partial_{y_k} u|^2 \, dV, \quad k = 1, \ldots, N-1.
$$

(2.11)

It is now deduced from (2.10) and (2.11) that

$$
\lambda^{2(p-2)} \left(\int |u|^p \, dV \right)^2 \left(\int u_x^2 \, dV \right)^{-1} \left(\int |D_x^{-1} \partial_{y_k} u|^2 \, dV \right)^{-1} \xi_k^{-2}
$$

$$
= \frac{1}{N} \left(\frac{2p}{p-2} \right)^2.
$$

(2.12)

Using (2.9), (2.10) and (2.12), we obtain

$$
\lambda^4 \mu^2 \prod_{k=1}^{N-1} \xi_k^{-2} \left(\int u_x^2 \, dV \right)^2 = \left(\frac{N(p-2)}{(3-2N)p + 2(2N-1)} \int \phi^2 \, dV \right)^2 .
$$

Hence

$$
\lambda^{4+(p-2)(3-2N)} \left(\int u_x^2 \, dV \right)^N \left(\int |u|^p \, dV \right)^{3-2N} \prod_{k=1}^{N-1} \int \left| D_x^{-1} \partial_{y_k} u \right|^2 \, dV
$$

$$
= \frac{N^N (p-2)^2}{((3-2N)p + 2(2N-1))^2} \left(\frac{p-2}{2p} \right)^{2N-3} \left(\int \phi^2 \, dV \right)^2 .
$$

Secondly, remember the notation $T = (3-2N)p + 2(2N-1)$, the above equality can be written as

$$
\lambda^T \left(\int u_x^2 \, dV \right)^N \left(\int |u|^p \, dV \right)^{3-2N} \prod_{k=1}^{N-1} \int \left| D_x^{-1} \partial_{y_k} u \right|^2 \, dV
$$

$$
= \frac{N^N (p-2)^{2N-1}}{(2p)^{2N-3} T^2} \left(\int \phi^2 \, dV \right)^2 .
$$

Therefore

$$
\lambda = \left(\frac{N^N (p-2)^{2N-1}}{(2p)^{2N-3} T^2} \right)^{\frac{1}{T}} \left(\int \phi^2 \, dV \right)^{\frac{2}{T}} \left(\int u_x^2 \, dV \right)^{\frac{-N}{T}}
$$

$$
\times \left(\int |u|^p \, dV \right)^{\frac{2N-3}{T}} \prod_{k=1}^{N-1} \left(\int \left| D_x^{-1} \partial_{y_k} u \right|^2 \, dV \right)^{\frac{-1}{T}} . \tag{2.13}
$$

It is deduced from (2.6), (2.8) and (2.13) that

$$
\int w^2 \, dV = \frac{2p}{T} \lambda^2 \int u^2 \, dV \left(\lambda^p \int |u|^p \, dV \right)^{-1} \int |\phi|^2 \, dV
$$

$$
= \frac{2p}{T} \int |\phi|^2 \, dV \int u^2 \, dV \left(\int |u|^p \, dV \right)^{-1} \lambda^{2-p}
$$

$$
= \frac{2p}{T} \left(\frac{N^N (p-2)^{2N-1}}{(2p)^{2N-3} T^2} \right)^{\frac{2-p}{T}} \left(\int |\phi|^2 \, dV \right)^{\frac{2(2N-1)-(2N-1)p}{T}}
$$

$$
\times \left(\int u_x^2 \, dV \right)^{\frac{-N(2-p)}{T}} \int u^2 \, dV \left(\int |u|^p \, dV \right)^{\frac{-4}{T}} \prod_{k=1}^{N-1} \left(\int \left| D_x^{-1} \partial_{y_k} u \right|^2 \, dV \right)^{\frac{p-2}{T}} .
$$

According to Proposition 2.7, ϕ is a minimizer of F_{r_0} with $r_0 = \int |\phi|^p \, dV$. Hence we obtain from $\int |w|^p \, dV = \int |\phi|^p \, dV \ F(w) \geq F(\phi)$. Using the definition of w and (2.4), one immediately has

$$
\int w^2 \, dV \geq \int \phi^2 \, dV .
$$

It is deduced that

$$\left(\int u_x^2 \, dV\right)^{\frac{-N(2-p)}{T}} \int u^2 \, dV \left(\int |u|^p \, dV\right)^{\frac{-4}{T}} \prod_{k=1}^{N-1} \left(\int \left|D_x^{-1}\partial_{y_k}u\right|^2 \, dV\right)^{\frac{p-2}{T}}$$

$$\geq \frac{T}{2p}\left(\frac{N^N(p-2)^{2N-1}}{(2p)^{2N-3}T^2}\right)^{\frac{p-2}{T}}\left(\int |\phi|^2 \, dV\right)^{1 - \frac{2(2N-1)-(2N-1)p}{T}}.$$

Therefore

$$J_1(u) \geq \left(\frac{T}{2p}\right)^{\frac{T}{4}}\left(\frac{N^N(p-2)^{2N-1}}{(2p)^{2N-3}T^2}\right)^{\frac{p-2}{4}}\left(\int |\phi|^2 \, dV\right)^{\frac{p-2}{2}}.$$

Since u is arbitrary, we deduce that

$$\alpha^{-1} \geq \left(\frac{T}{2p}\right)^{\frac{T}{4}}\left(\frac{N^N(p-2)^{2N-1}}{(2p)^{2N-3}T^2}\right)^{\frac{p-2}{4}}\left(\int |\phi|^2 \, dV\right)^{\frac{p-2}{2}}.$$

Step 2. We prove that

$$\alpha^{-1} \leq \left(\frac{T}{2p}\right)^{\frac{T}{4}}\left(\frac{N^N(p-2)^{2N-1}}{(2p)^{2N-3}T^2}\right)^{\frac{p-2}{4}}\left(\int |\phi|^2 \, dV\right)^{\frac{p-2}{2}}.$$

In the first place, using mean value inequality, one has

$$\prod_{k=1}^{N-1}\left(\int \left|D_x^{-1}\partial_{y_k}\phi\right|^2 \, dV\right)^{\frac{p-2}{4}} = \left(\prod_{k=1}^{N-1}\int \left|D_x^{-1}\partial_{y_k}\phi\right|^2 \, dV\right)^{\frac{p-2}{4}}$$

$$\leq \left[\left(\frac{1}{N-1}\sum_{k=1}^{N-1}\int \left|D_x^{-1}\partial_{y_k}\phi\right|^2 \, dV\right)^{N-1}\right]^{\frac{p-2}{4}}$$

$$= \left(\frac{p-2}{T}\int \phi^2 \, dV\right)^{\frac{(N-1)(p-2)}{4}}.$$

In the second place, using the fact that $\phi \neq 0$, $\phi \in Y_1$ and (2.4), one obtains immediately

$$J_1(\phi) = \left(\int \phi^2 \, dV\right)^{\frac{T}{4}}\left(\int \phi_x^2 \, dV\right)^{\frac{N(p-2)}{4}}$$

$$\times \left(\int |\phi|^p \, dV\right)^{-1}\prod_{k=1}^{N-1}\left(\int \left|D_x^{-1}\partial_{y_k}\phi\right|^2 \, dV\right)^{\frac{p-2}{4}}$$

$$\leq \left(\int \phi^2 \, dV\right)^{\frac{T}{4}}\left(\frac{N(p-2)}{T}\int \phi^2 \, dV\right)^{\frac{N(p-2)}{4}}$$

$$\times \left(\frac{2p}{T}\int |\phi|^2 \, dV\right)^{-1}\left(\frac{p-2}{T}\int \phi^2 \, dV\right)^{\frac{(N-1)(p-2)}{4}}$$

$$= \left(\frac{T}{2p}\right)^{\frac{T}{4}}\left(\frac{N^N(p-2)^{2N-1}}{(2p)^{2N-3}T^2}\right)^{\frac{p-2}{4}}\left(\int |\phi|^2 \, dV\right)^{\frac{p-2}{2}}.$$

From *Step 1.* and *Step 2.*, we get the first equality of (B).

Step 3. Now we prove the second equality of (B). Since $d_1 = L_1(\phi)$, we obtain from (2.4) that

$$
\begin{aligned}
d_1 &= \int \left(\frac{1}{2}\phi_x^2 + \frac{1}{2}\phi^2 + \frac{1}{2}\left|D_x^{-1}\nabla_{y_k}\phi\right|^2 - \frac{1}{p}|\phi|^p \right) dV \\
&= \frac{1}{2}\left(\frac{N(p-2)}{T} + \frac{(N-1)(p-2)}{T} + \frac{1}{2} \right) \int |\phi|^2\, dV - \frac{1}{p} \cdot \frac{2p}{T} \int |\phi|^2\, dV \\
&= \frac{p-2}{T} \int |\phi|^2\, dV.
\end{aligned}
$$

Combining this with the first equality of (B), one gets the second equality of (B).　　□

3 Estimate of the smallest constant β

In this section, we study the smallest constant β in (1.6). We use variational methods and the ideas from the previous section. Observing the proofs in the previous section, we find that it is very important to do the scaling and solve λ, μ and ξ_k, where $k = 1, 2, \dots, N-1$, from $N + 1$ equations; see $(2.7)_k$, (2.8) and (2.9). However, as we will see below, in the process of estimating β, we still need to solve $N + 1$ positive parameters, but we only have N equations; see (3.7) and $(3.6)_k$, where $k = 1, 2, \dots, N-1$. Hence we need to do the scaling and investigate the parameters carefully. Keeping the notation p_* in mind, we consider the following minimization problem:

$$
\beta^{-1} = \inf_{u \neq 0, u \in Y_0} J_0(u), \tag{3.1}
$$

where

$$
J_0(u) = \left(\int u_x^2\, dV \right)^{\frac{N}{2N-3}} \prod_{k=1}^{N-1} \left(\int \left|D_x^{-1}\partial_{y_k}u\right|^2 dV \right)^{\frac{1}{2N-3}} \left(\int |u|^{p_*}\, dV \right)^{-1}
$$

and Y_0 is the closure of $\partial_x(C_0^\infty(\mathbb{R}^N))$ under the norm

$$
\|u\|_{Y_0}^2 := \int_{\mathbb{R}^N} \left(u_x^2 + \left|D_x^{-1}\nabla_y u\right|^2 \right) dV.
$$

The following related equation is useful in what follows:

$$
u_{xx} + |u|^{p_*-2}u = D_x^{-2}\triangle_y u, \quad u \neq 0, u \in Y_0. \tag{3.2}
$$

Define on Y_0 the functionals

$$
L_0(u) = \int \left(\frac{1}{2}u_x^2 + \frac{1}{2}\left|D_x^{-1}\nabla_y u\right|^2 - \frac{1}{p_*}|u|^{p_*} \right) dV \quad \text{and}
$$

$$
I_0(u) = \int \left(u_x^2 + \left|D_x^{-1}\nabla_y u\right|^2 - |u|^{p_*} \right) dV.
$$

Set

$$
\Gamma_0 = \left\{ u \in Y_0 : u \neq 0, I_0(u) = 0 \right\} \quad \text{and} \quad d_0 = \inf_{u \in \Gamma_0} L_0(u).
$$

Then according to inequality (1.6), both L_0 and I_0 are well defined and C^1 on Y_0.

Definition 3.1 An element $v \in Y_0$ is said to be a solution of (3.2) if and only if v is a critical point of L_0, i.e., $L_0'(v) = 0$. Moreover, $v \in Y_0$ is said to be a minimal action solution of (3.2) if $v \neq 0$, $L_0'(v) = 0$ and $L_0(v) \leq L_0(u)$ for any $u \in \Gamma_0$.

Lemma 3.2 *For any $u \in Y_0$ and $u \neq 0$, there is a unique $s_u > 0$ such that $s_u u \in \Gamma_0$. Moreover, if $I_0(u) < 0$ then $0 < s_u < 1$.*

Lemma 3.3 *The set Γ_0 is a manifold and there exists $\rho > 0$ such that, for any $u \in \Gamma_0$, $\|u\|_{Y_0} \geq \rho > 0$.*

Lemma 3.4 *If $v \in \Gamma_0$ and $L_0(v) = d_0$, then v is a critical point of L_0 on Y_0, i.e. $L_0'(v) = 0$.*

Remark The proofs of Lemma 3.2, Lemma 3.3 and Lemma 3.4 are similar to the proofs of Lemma 2.2, Lemma 2.3 and Lemma 2.4. We omit the details here.

Theorem 3.5 *We see that $d_0 > 0$ and there is a $\psi \in \Gamma_0$ such that $d_0 = L_0(\psi)$. Moreover, ψ is a minimal action solution of (3.2).*

Remark The proof of Theorem 3.5 follows lines similar to the proof of Theorem 2.5. We emphasize that in the proof of Theorem 2.5, the functionals L_1 and I_1 only have invariance under translations, i.e., for any $V \in \mathbb{R}^N$, $L_1(u(\cdot + V)) = L_1(u(\cdot))$ and $I_1(u(\cdot + V)) = I_1(u(\cdot))$. But, in the case $p = p_*$, the functionals L_0 and I_0 not only have invariance under translation, but also have invariance under dilation; see below (*IUD*) for details. Hence, we give a detailed proof of Theorem 3.5.

Proof of Theorem 3.5 By Lemma 3.3 we know that $d_0 > 0$. According to Definition 3.1 and Lemma 3.4, we only need to prove that there is $\psi \in \Gamma_0$ such that $d_0 = L_0(\psi)$.

Let $\{u_n\}_{n \in \mathbb{N}} \subset \Gamma_0$ be a minimizing sequence of d_0, i.e., $u_n \neq 0$, $I_0(u_n) = 0$ and $d_0 + o(1) = L_0(u_n)$ for n large enough. Then it is easy to see from $I_0(u_n) = 0$ that $\|u_n\|_{Y_0}$ is uniformly bounded with respect to n. Moreover, Lemma 3.3 implies that $\|u_n\|_{Y_0}$ is uniformly bounded away from zero and we see that

$$\liminf_{n \to \infty} \int |u_n|^{p_*} \, dV = \liminf_{n \to \infty} \|u_n\|_{Y_0}^2 > 0.$$

Note that, for any $V \equiv (x, y_1, \ldots, y_{N-1}) \in \mathbb{R}^N$,

$$L_0\big(u(\cdot + x, \cdot + y_1, \ldots, \cdot + y_{N-1})\big) = L_0\big(u(\cdot)\big) \quad \text{and}$$
$$I_0\big(u(\cdot + x, \cdot + y_1, \ldots, \cdot + y_{N-1})\big) = I_0\big(u(\cdot)\big).$$

Moreover, for any $\lambda > 0$, denoting

$$u^\lambda(x, y) := \lambda u\big(\lambda^{\frac{2}{2N-3}} x, \lambda^{\frac{4}{2N-3}} y_1, \ldots, \lambda^{\frac{4}{2N-3}} y_{N-1}\big),$$

we have

$$L_0\big(u^\lambda\big) = L_0(u) \quad \text{and} \quad I_0\big(u^\lambda\big) = I_0(u). \tag{IUD}$$

We obtain from the concentration compactness lemma of Lions [8] that there are γ_n and $V^n \equiv (x^n, y_1^n, \ldots, y_{N-1}^n) \in \mathbb{R}^N$ such that $\psi_n(x, y) := \gamma_n u_n(\gamma_n^{\frac{2}{2N-3}}(x + x^n), \gamma_n^{\frac{4}{2N-3}}(y_1 + y_1^n), \ldots, \gamma_n^{\frac{4}{2N-3}}(y_{N-1} + y_{N-1}^n))$ satisfies

$$L_0(\psi_n) = L_0(u_n) \quad \text{and} \quad I_0(\psi_n) = I_0(u_n).$$

Additionally, there is $\psi \in Y_0$ such that $\psi_n \rightharpoonup \psi$ weakly in Y_0, $\psi_n \to \psi$ a.e. in $\mathbb{R}^N$ and $\psi \neq 0$.

If $I_0(\psi) < 0$, then by Lemma 3.2 there is a s_ψ such that $0 < s_\psi < 1$ and $s_\psi \psi \in \Gamma_0$. Therefore using the Fatou lemma and $I_0(\psi_n) = I_0(u_n) = 0$, we obtain

$$d_0 + o(1) = L_0(\varphi_n) = \left(\frac{1}{2} - \frac{1}{p_*}\right) \int |\psi_n|^{p_*} dV \geq \frac{1}{2N - 1} \int |\psi|^{p_*} dV + o(1)$$

$$= \frac{1}{2N - 1} s_\psi^{-p_*} \int |s_\psi \psi|^{p_*} dV + o(1) = s_\psi^{-p_*} L_0(s_\psi \psi) + o(1).$$

It is now deduced from $0 < s_\psi < 1$ that $d_0 > L_0(s_\psi \psi)$, which is a contradiction because of $s_\psi \psi \in \Gamma_0$.

If $I_0(\psi) > 0$, then using the Brezis–Lieb lemma [2] one has

$$0 = I_0(\psi_n) = I_0(\psi) + I_0(v_n) + o(1),$$

where we denote $\psi_n - \psi$ by v_n in the rest of this section. This and $I_0(\psi) > 0$ imply that

$$\limsup_{n \to \infty} I_0(v_n) < 0. \tag{3.3}$$

From Lemma 3.2 we know that there are $s_n := s_{v_n}$ such that $s_n v_n \in \Gamma_0$. Moreover, we claim that $\limsup_{n \to \infty} s_n \in (0, 1)$. Indeed if $\limsup_{n \to \infty} s_n = 1$, then there is a subsequence $\{s_{n_k}\}$ such that $\lim_{k \to \infty} s_{n_k} = 1$. Therefore from $s_{n_k} v_{n_k} \in \Gamma_0$ one has

$$I_0(v_{n_k}) = I_0(s_{n_k} v_{n_k}) + o(1) = o(1).$$

This contradicts (3.3). Hence $\limsup_{n \to \infty} s_n \in (0, 1)$. Since

$$d_0 + o(1) = \left(\frac{1}{2} - \frac{1}{p_*}\right) \int |\psi_n|^{p_*} dV$$

$$\geq \frac{1}{2N - 1} \int |v_n|^{p_*} dV + o(1)$$

$$= \frac{1}{2N - 1} s_n^{-p_*} \int |s_n v_n|^{p_*} dV + o(1)$$

$$= s_n^{-p_*} L_0(s_n v_n) + o(1),$$

one deduces that $d_0 > L_0(s_n v_n)$, which is a contradiction because of $s_n v_n \in \Gamma_0$.

Thus $I_0(\phi) = 0$. Now similar to the proof of Theorem 2.5, we obtain $\|v_{n_k}\|_{Y_0} \to 0$ as $k \to \infty$. Therefore $L_0(\psi_{n_k}) \to L_0(\psi)$ and $d_0 = L_0(\psi)$.   □

Next we give some properties of the minimal action solution ψ of (3.2).

Lemma 3.6 *Let ψ be a minimal action solution of* (3.2). *Then $I_0(\phi) = 0$ and*

$$R_0(\phi) := \int \left(\psi_x^2 - \frac{N}{2N-1} |\psi|^{p_*} \right) dV = 0.$$

Moreover, we see that

$$\int \left| D_x^{-1} \nabla_y \phi \right|^2 dV = \frac{N-1}{N} \int \phi_x^2 \, dV \quad and$$

$$\int |\phi|^{p_*} \, dV = \frac{2N-1}{N} \int \phi_x^2 \, dV.$$
(3.4)

Proof The proof is similar to Lemma 2.6. We omit the details here. □

Next, we give another characterization of the minimal action solutions ψ of (3.2). For $u \in Y_0$, define

$$K(u) = \int \left(u_x^2 + \left| D_x^{-1} \nabla_y u \right|^2 \right) dV$$

and for $r > 0$ set

$$K_r = \inf \left\{ K(u) : u \in Y_0 \text{ and } \int |u|^{p_*} \, dV = r \right\}.$$

Then we have the following proposition.

Proposition 3.7 *Let ψ be a minimal action solution of* (3.2). *Then ψ is a minimizer of K_r with $r = \int |\psi|^{p_*} \, dV$.*

Proof The proof is similar to Proposition 2.7. We omit the details. □

Now we are in a position to study the smallest constant β in (1.6).

Theorem 3.8 *Let ψ be the minimal action solution of* (3.2) *obtained in Theorem 3.5 and $d_0 = L_0(\psi)$. The smallest constant β in* (1.6) *is*

$$\beta^{-1} = (2N-1)^{-1} N^{\frac{N-2}{2N-3}} \left(\int \psi_x^2 \, dV \right)^{\frac{2}{2N-3}}$$

$$= (2N-1)^{-1} N^{\frac{N}{2N-3}} d_0^{\frac{2}{2N-3}}.$$

Remark From Theorem 3.8, β is unique, since the minimum d_0 is unique. We point out that β^{-1} is independent of the choice of the minimal action solution ψ of (3.2), although we do not know the uniqueness of the minimal action solution. In fact the uniqueness of the minimal action solution of (3.2) was an open problem.

Proof of Theorem 3.8 The proof is divided into three steps.

Step 1. In this step, we prove that

$$\beta^{-1} \geq (2N-1)^{-1} N^{\frac{N-2}{2N-3}} \left(\int \psi_x^2 \, dV \right)^{\frac{2}{2N-3}}.$$

For any $u \in Y_0$ and $u \neq 0$, denote $u \equiv u(x, y)$ with $y = (y_1, \ldots, y_{N-1})$. Define

$$w(x, y) = \lambda u(\mu x, \xi_1 y_1, \ldots, \xi_{N-1} y_{N-1}),$$

where $\lambda, \mu, \xi_1, \ldots, \xi_{N-1}$ are positive parameters which will be determined later. Then, by direct computations, we see that

$$\int \left| D_x^{-1} \partial_{y_k} w \right|^2 dV = \lambda^2 \mu^{-3} \left(\prod_{j \neq k} \xi_j^{-1} \right) \xi_k \int \left| D_x^{-1} \partial_{y_k} u \right|^2 dV, \quad k = 1, \ldots, N-1;$$

$$\int |w|^{p_*} dV = \lambda^{p_*} \mu^{-1} \left(\prod_{k=1}^{N-1} \xi_k^{-1} \right) \int |u|^{p_*} dV; \tag{3.5}$$

$$\int w_x^2 dV = \lambda^2 \mu \left(\prod_{k=1}^{N-1} \xi_k^{-1} \right) \int u_x^2 dV.$$

In here $\lambda, \mu, \xi_1, \ldots, \xi_{N-1}$ are determined by the following N equations:

$$\lambda^2 \mu^{-3} \left(\prod_{j \neq k} \xi_j^{-1} \right) \xi_k \int \left| D_x^{-1} \partial_{y_k} u \right|^2 dV = \frac{1}{N} \int \psi_x^2 dV, \tag{3.6}$$

where for $k = 1, \ldots, N-1$, (3.6) is denoted by $(3.6)_k$;

$$\lambda^{p_*} \mu^{-1} \left(\prod_{k=1}^{N-1} \xi_k^{-1} \right) \int |u|^{p_*} dV = \frac{2N-1}{N} \int \psi_x^2 dV. \tag{3.7}$$

(*Note:* we need to solve $N + 1$ variables only from N equations.)

In the first place, from $(3.6)_1$ and (3.7), one gets

$$\lambda^{2-p_*} \mu^{-2} \xi_1^2 \int \left| D_x^{-1} \partial_{y_1} u \right|^2 dV = \frac{1}{2N-1} \int |u|^{p_*} dV. \tag{3.8}$$

In the second place, from $(3.6)_k$, one gets

$$\xi_k^{-1} = \left(\int \left| D_x^{-1} \partial_{y_k} u \right|^2 dV \right)^{\frac{1}{2}} \left(\int \left| D_x^{-1} \partial_{y_1} u \right|^2 dV \right)^{-\frac{1}{2}} \xi_1^{-1}, \quad k = 2, \ldots, N-1.$$

It is now deduced from $(3.6)_1$ and the expression of ξ_k^{-1} ($k = 2, \ldots, N-1$) that

$$\lambda^2 \mu^{-3} \xi_1^{3-N} \left(\int \left| D_x^{-1} \partial_{y_1} u \right|^2 dV \right)^{2-\frac{N}{2}} \prod_{k=2}^{N-1} \left(\int \left| D_x^{-1} \partial_{y_k} u \right|^2 dV \right)^{\frac{1}{2}} = \frac{1}{N} \int \phi_x^2 dV. \tag{3.9}$$

Equations (3.8) and (3.9) imply that

$$\lambda^{3(2-p_*)-4} \xi_1^{6-2(3-N)} \left(\int \left| D_x^{-1} \partial_{y_1} u \right|^2 dV \right)^{3-(4-N)} \prod_{k=2}^{N-1} \left(\int \left| D_x^{-1} \partial_{y_k} u \right|^2 dV \right)^{-1}$$

$$= N^2 \left(\frac{1}{2N-1} \right)^3 \left(\int |u|^{p_*} dV \right)^3 \left(\int \psi_x^2 dV \right)^{-2}. \tag{3.10}$$

Combining this with the fact of $p_* = 2(2N-1)/(2N-3)$, one obtains

$$
\lambda^{\frac{-8N}{2N-3}} \xi_1^{2N} \left(\int \left| D_x^{-1} \partial_{y_1} u \right|^2 dV \right)^{N-1} \prod_{k=2}^{N-1} \left(\int \left| D_x^{-1} \partial_{y_k} u \right|^2 dV \right)^{-1}
$$
$$
= \frac{N^2}{(2N-1)^3} \left(\int |u|^{p_*} dV \right)^3 \left(\int \psi_x^2 \, dV \right)^{-2}. \tag{3.11}
$$

Note that (3.8) can be written as

$$
\lambda^{-\frac{8N}{2N-3}} \mu^{-4N} \xi_1^{4N} \left(\int \left| D_x^{-1} \partial_{y_1} u \right|^2 dV \right)^{2N} = \frac{1}{(2N-1)^{2N}} \left(\int |u|^{p_*} dV \right)^{2N}. \tag{3.12}
$$

We obtain from (3.11) and (3.12)

$$
\mu^{4N} \xi_1^{-2N} \left(\int \left| D_x^{-1} \partial_{y_1} u \right|^2 dV \right)^{-N-1} \prod_{k=2}^{N-1} \left(\int \left| D_x^{-1} \partial_{y_k} u \right|^2 dV \right)^{-1}
$$
$$
= N^2 (2N-1)^{2N-3} \left(\int |u|^{p_*} dV \right)^{3-2N} \left(\int \psi_x^2 \, dV \right)^{-2}. \tag{3.13}
$$

Therefore

$$
\mu^4 \xi_1^{-2} = N^{\frac{2}{N}} (2N-1)^{\frac{2N-3}{N}} \left(\int |u|^{p_*} dV \right)^{\frac{3-2N}{N}} \left(\int \psi_x^2 \, dV \right)^{\frac{-2}{N}}
$$
$$
\times \left(\int \left| D_x^{-1} \partial_{y_1} u \right|^2 dV \right)^{\frac{N+1}{N}} \prod_{k=2}^{N-1} \left(\int \left| D_x^{-1} \partial_{y_k} u \right|^2 dV \right)^{\frac{1}{N}}. \tag{3.14}
$$

Writing (3.12) as

$$
\lambda^{-\frac{4}{2N-3}} = \frac{1}{(2N-1)} \mu^2 \xi_1^{-2} \int |u|^{p_*} dV \left(\int \left| D_x^{-1} \partial_{y_1} u \right|^2 dV \right)^{-1},
$$

we deduce from (3.7) and (3.5) that

$$
\int w_x^2 \, dV = \lambda^2 \mu \prod_{k=1}^{N-1} \xi_k^{-1} \int u_x^2 \, dV
$$
$$
= \lambda^2 \mu \int u_x^2 \, dV \left[\frac{2N-1}{N} \int \psi_x^2 \, dV \lambda^{-p_*} \mu \left(\int |u|^{p_*} dV \right)^{-1} \right]
$$
$$
= \lambda^{2-p_*} \mu^2 \int u_x^2 \, dV \left(\int |u|^{p_*} dV \right)^{-1} \frac{2N-1}{N} \int \psi_x^2 \, dV \quad \text{(using (3.8))}
$$
$$
= \mu^4 \xi_1^{-2} \frac{1}{N} \int \psi_x^2 \, dV \int u_x^2 \, dV \left(\int \left| D_x^{-1} \partial_{y_1} u \right|^2 dV \right)^{-1}
$$
$$
= \frac{1}{N} \int \psi_x^2 \, dV \int u_x^2 \, dV \left(\int \left| D_x^{-1} \partial_{y_1} u \right|^2 dV \right)^{-1}
$$
$$
\times N^{\frac{2}{N}} (2N-1)^{\frac{2N-3}{N}} \left(\int |u|^{p_*} dV \right)^{\frac{3-2N}{N}} \left(\int \psi_x^2 \, dV \right)^{\frac{-2}{N}}
$$

$$\times \left(\int |D_x^{-1}\partial_{y_1} u|^2 \, dV \right)^{\frac{N+1}{N}} \prod_{k=2}^{N-1} \left(\int |D_x^{-1}\partial_{y_k} u|^2 \, dV \right)^{\frac{1}{N}}$$

$$= N^{\frac{2}{N}-1}(2N-1)^{\frac{2N-3}{N}} \left(\int \psi_x^2 \, dV \right)^{1-\frac{2}{N}} \int u_x^2 \, dV$$

$$\times \prod_{k=1}^{N-1} \left(\int |D_x^{-1}\partial_{y_k} u|^2 \, dV \right)^{\frac{1}{N}} \left(\int |u|^{p*} \, dV \right)^{\frac{3-2N}{N}}.$$

From the definition of w and the fact that ψ is a minimizer of K_r with $r = \int |\psi|^{p*} \, dV$, we get

$$\int w_x^2 \, dV \geq \int \psi_x^2 \, dV,$$

which implies that

$$\int u_x^2 \, dV \left(\int |u|^p \, dV \right)^{\frac{3-2N}{N}} \prod_{k=1}^{N-1} \left(\int |D_x^{-1}\partial_{y_k} u|^2 \, dV \right)^{\frac{1}{N}}$$

$$\geq N^{1-\frac{2}{N}}(2N-1)^{\frac{2N-3}{N}} \left(\int \psi_x^2 \, dV \right)^{\frac{2}{N}}.$$

Therefore

$$J_0(u) \geq N^{\frac{N-2}{2N-3}}(2N-1)^{-1} \left(\int \psi_x^2 \, dV \right)^{\frac{2}{2N-3}}.$$

Since $u \neq 0$ and $u \in Y_0$ is chosen arbitrarily, we get

$$\beta^{-1} \geq (2N-1)^{-1} N^{\frac{N-2}{2N-3}} \left(\int \psi_x^2 \, dV \right)^{\frac{2}{2N-3}}.$$

Step 2. In this step, we prove that

$$\beta^{-1} \leq (2N-1)^{-1} N^{\frac{N-2}{2N-3}} \left(\int \psi_x^2 \, dV \right)^{\frac{2}{2N-3}}.$$

Since $\psi \neq 0$ and $\psi \in Y_0$, we obtain from Lemma 3.6 and the mean value inequality that

$$J_0(\psi) = \left(\int \psi_x^2 \, dV \right)^{\frac{N}{2N-3}} \left(\int |\psi|^{p*} \, dV \right)^{-1} \prod_{k=1}^{N-1} \left(\int |D_x^{-1}\partial_{y_k} \psi|^2 \, dV \right)^{\frac{1}{2N-3}}$$

$$\leq \left(\int \psi_x^2 \, dV \right)^{\frac{N}{2N-3}} \left(\frac{2N-1}{N} \int \psi_x^2 \, dV \right)^{-1} \left(\frac{1}{N-1} \sum_{k=1}^{N-1} \int |D_x^{-1}\partial_{y_k} \psi|^2 \, dV \right)^{\frac{N-1}{2N-3}}$$

$$= \left(\int \psi_x^2 \, dV \right)^{\frac{N}{2N-3}} \left(\frac{2N-1}{N} \int \psi_x^2 \, dV \right)^{-1} \left(\frac{1}{N} \int \psi_x^2 \, dV \right)^{\frac{N-1}{2N-3}}$$

$$= (2N-1)^{-1} N^{\frac{N-2}{2N-3}} \left(\int \psi_x^2 \, dV \right)^{\frac{2}{2N-3}}.$$

Therefore

$$\beta^{-1} \le (2N-1)^{-1} N^{\frac{N-2}{2N-3}} \left(\int \psi_x^2 \, dV \right)^{\frac{2}{2N-3}}.$$

Combining *Step 1.* and *Step 2.*, we get β^{-1} as stated in the theorem.

Step 3. We prove that

$$\beta^{-1} = (2N-1)^{-1} N^{\frac{N}{2N-3}} d_0^{\frac{2}{2N-3}}.$$

Indeed, since $d_0 = L_0(\psi)$, we obtain from Lemma 3.6 that

$$
\begin{aligned}
d_0 &= \int \left(\frac{1}{2} \psi_x^2 \, dV + \frac{1}{2} \left| D_x^{-1} \nabla_{y_k} \psi \right|^2 - \frac{1}{p_*} |\phi|^{p_*} \right) dV \\
&= \left(\frac{1}{2} - \frac{1}{p_*} \cdot \frac{2N-1}{N} + \frac{1}{2} \cdot \frac{N-1}{N} \right) \int \psi_x^2 \, dV \\
&= \frac{1}{N} \int \psi_x^2 \, dV.
\end{aligned}
$$

Combining this with the first equality of β^{-1} in the statement of Theorem 3.8, we get the second equality of β^{-1} in Theorem 3.8. □

4 Conclusions

In this paper, we not only estimated the smallest constant in a general N-dimensional anisotropic Sobolev inequality in the subcritical case; we also gave an estimate of the smallest constant for N-dimensional anisotropic Sobolev inequality in the critical case.

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