An elementary proof of a converse mean value theorem

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Abstract

We present a new converse mean value theorem, with a rather elementary proof.

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1 Introduction

The goal of this note is to present a simple proof for a converse mean value theorem. Given a differentiable function $f : [a, b] \to \mathbb{R}$ and a point $c \in]a, b[$, are there reals $a_1, b_1 \in]a, b[$ such that $f(b_1) - f(a_1) = f'(c)(b_1 - a_1)$? Sufficient conditions for the above converse to hold are established in [1]. However, there appear to be a gap in the proof. They claim that, since f is differentiable at $x = c_i$ then $f'(c_i) = \lim_{x,y\to c_i} (f(x) - f(y))/(x - y)$. The following example shows that this can be false.

Let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then f is differentiable at $x = 0$ and $f'(0) = 0$. However, given $N \in \mathbb{N}$, let $k = 2N + 1$. Then

$$\frac{f\left(\dfrac{1}{k\pi} + \dfrac{1}{2(k\pi)^2}\right) - f\left(\dfrac{1}{k\pi}\right)}{\dfrac{1}{2(k\pi)^2}} = \frac{\dfrac{4(k\pi)^2 + 4k\pi + 1}{4(k\pi)^4} \sin \dfrac{2(k\pi)^2}{2k\pi + 1} - 0}{\dfrac{1}{2(k\pi)^2}}$$

$$= \frac{4(k\pi)^2 + 4k\pi + 1}{2(k\pi)^2} \sin\left(k\pi - \frac{k\pi}{2k\pi + 1}\right)$$

$$= \frac{4(k\pi)^2 + 4k\pi + 1}{2(k\pi)^2} (-1)^{k+1} \sin\left(\frac{k\pi}{2k\pi + 1}\right) = \frac{4(k\pi)^2 + 4k\pi + 1}{2(k\pi)^2} \sin \frac{k\pi}{2k\pi + 1}$$

(since $k = 2N + 1$) and so the limit is $2 \sin \dfrac{1}{2}$ if $N \to \infty$.

2 The theorem

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function and differentiable in $]a, b[$. Given $c \in]a, b[$, let $k_0 > 0$ be such that $]c - k_0, c + k_0[\subseteq]a, b[$. If, for all $k \in]0, k_0[$,*

1. *$f'(c-k) < f'(c) < f'(c+k)$ then there exist $a_1, b_1 \in]a, b[$ with $c \in]a_1, b_1[$ and $f(b_1) - f(a_1) = f'(c)(b_1 - a_1)$.*
2. *$f'(c-k) \leq f'(c) \leq f'(c+k)$ then there exist $a_1, b_1 \in]a, b[$ with $c \in [a_1, b_1]$ and $f(b_1) - f(a_1) = f'(c)(b_1 - a_1)$.*

Proof. First we prove 1. Define the function $g(x) = f(x) - f'(c)x, x \in [a, b]$. Then g is differentiable and $g'(x) = f'(x) - f'(c)$. By the hypothesis, for all $k \in]0, k_0[$,

$$g'(c) = 0\,,\; g'(c+k) > 0 \;\text{ and }\; g'(c-k) < 0$$

and so g has a strict local minimum at c. Therefore we have two possible cases:

(a) $g(c) < g(c - \frac{k_0}{2}) \leq g(c + \frac{k_0}{2})$

(b) $g(c) < g(c + \frac{k_0}{2}) \leq g(c - \frac{k_0}{2})$

If the first one occurs, then by the intermediate value theorem, there exists $b_1 \in]c, c + \frac{k_0}{2}]$ with $g(b_1) = g(c - \frac{k_0}{2})$. If we take $a_1 = c - \frac{k_0}{2}$ then $c \in]a_1, b_1[$ and $g(a_1) = g(b_1)$, which is equivalent to $f(b_1) - f(a_1) = f'(c)(b_1 - a_1)$. The second case is analogous.

To proof the second part of the theorem, observe that in this case

$$g'(c) = 0\,,\; g'(c+k) \geq 0 \;\text{ and }\; g'(c-k) \leq 0$$

and so c is a local minimum of g. Therefore one of the two following cases holds

(a) $g(c) \leq g(c - \frac{k_0}{2}) \leq g(c + \frac{k_0}{2})$

(b) $g(c) \leq g(c + \frac{k_0}{2}) \leq g(c - \frac{k_0}{2})$

If (a) is true, there exists $b_1 \in [c, c + \frac{k_0}{2}]$ with $g(a_1) = g(b_1)$, where $a_1 = c - \frac{k_0}{2}$ and $c \in [a_1, b_1]$. Analogous for (b).

□

References

[1] J. Tong and P.A. Braza, *A converse of the mean value theorem*, Amer. Math. Monthly 106 (1997), pp. 939–942.