On the continuity of functions

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Abstract. Some theorems on continuity are presented. First we will prove that every convex function \( f : \mathbb{R}^n \to \mathbb{R} \) is continuous using nonstandard analysis methods. Then we prove that if the image of every compact (resp. convex) is compact (resp. convex), then the function is continuous.

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1 Sufficient Conditions for Continuity

The purpose of this paper is to present some results on continuity. Now let us introduce some terminology. In what follows, if \( E \) is a (standard) set, \( ^*E \) will denote its nonstandard extension. If \( (E,|\cdot|) \) is a normed space and \( x, y \in ^*E \), we say that \( x \approx y \) if \( x - y \) is infinitesimal, \( i.e., \) if \( |x - y| < r \) for all positive real \( r \in \mathbb{R} \); if \( x \) is standard and \( x \approx y \), we say that \( y \) is near-standard and write \( x = st(y) \). For further details, the reader is referred to [3], [4], [5] or [6].

Definition 1 Let \( E \) be a linear space and consider a function \( f : E \to \mathbb{R} \). The function \( f \) is called convex if

\[
(1.1) \quad f((1 - \lambda)x_1 + \lambda x_2) \leq (1 - \lambda)f(x_1) + \lambda f(x_2) \quad (\text{Jensen’s inequality})
\]

for all \( x_1, x_2 \in E \) and \( \lambda \in [0,1] \).

Theorem 1 Let \( (E,|\cdot|) \) be a normed space and \( f : E \to \mathbb{R} \) a convex function. If \( f(^*S^1) \subseteq \text{fin}(^*\mathbb{R}) \), where \( S^1 \) denotes the unit sphere in \( E \) and \( \text{fin}(^*\mathbb{R}) \) the set of finite hyperreals, then \( f \) is continuous.

Proof. Fix any \( x_0 \in E \). Without any loss of generality, we may assume that \( x_0 = 0 \) and \( f(x_0) = 0 \) (simply replace \( f \) by the convex function \( g(x) := f(x + x_0) - f(x_0) \)). Then given \( 0 \approx \epsilon \in ^*E, \epsilon \neq 0 \), we have that

1. \( f(\epsilon) \approx 0 \) because
\( f(\epsilon) = f(\frac{(1 - |\epsilon|)0 + |\epsilon| \cdot \frac{\epsilon}{|\epsilon|}}{1 - |\epsilon|}0 + |\epsilon| \cdot f(\frac{\epsilon}{|\epsilon|}) = 0. \)

2. \( f(\epsilon) \gtrsim 0 \) because

\[
0 = \frac{1}{1 + |\epsilon|} \epsilon + \frac{|\epsilon|}{1 + |\epsilon|} \cdot \frac{-\epsilon}{|\epsilon|}
\]

and so

\[
0 \leq \frac{1}{1 + |\epsilon|} f(\epsilon) + \frac{|\epsilon|}{1 + |\epsilon|} f\left(\frac{-\epsilon}{|\epsilon|}\right) \Rightarrow f(\epsilon) \geq -|\epsilon| \cdot f\left(\frac{-\epsilon}{|\epsilon|}\right) \approx 0.
\]

We conclude then that \( f(\epsilon) \approx 0. \)

We will now see the special case when \( E \) is a finite dimensional space. First we need the following result due to Michel Goze (see [1] or [2]):

**Theorem 2** Let \( M \in \ast \mathbb{R}^n \) be an infinitesimal vector. Then there are non-null infinitesimals \( \epsilon_1, \ldots, \epsilon_k \in \ast \mathbb{R} \) and standard vectors \( V_1, \ldots, V_k \in \mathbb{R}^n \), for some \( k \leq n \), with

\[
M = \epsilon_1 V_1 + \epsilon_1 \epsilon_2 V_2 + \ldots + \epsilon_1 \epsilon_2 \ldots \epsilon_k V_k.
\]

With this we can prove the well known theorem:

**Theorem 3** Every convex function \( f : \mathbb{R}^n \to \mathbb{R} \) is continuous.

**Proof.** Again we assume that \( x_0 = 0 \) and \( f(x_0) = 0 \). Fix any \( \epsilon \approx 0 \) and write \( \epsilon = \epsilon_1 V_1 + \epsilon_1 \epsilon_2 V_2 + \ldots + \epsilon_1 \epsilon_2 \ldots \epsilon_k V_k \). We can also assume that all the infinitesimals \( \epsilon_i \) are positive (replacing \( V_i \) by \( -V_i \) if necessary).

1. \( f(\epsilon) \gtrsim 0 \):

\[
f(\epsilon) = f\left(\frac{(1 - \epsilon_1)0 + \epsilon_1 (V_1 + \epsilon_2 V_2 + \epsilon_2 \epsilon_3 V_3 + \ldots + \epsilon_2 \epsilon_3 \ldots \epsilon_k V_k)}{1 - \epsilon_1}\right) \leq (1 - \epsilon_1) f(0) + \epsilon_1 f(V_1 + \epsilon_2 V_2 + \epsilon_2 \epsilon_3 V_3 + \ldots + \epsilon_2 \epsilon_3 \ldots \epsilon_k V_k).
\]

It is enough to prove that \( f(V_1 + \epsilon_2 V_2 + \epsilon_2 \epsilon_3 V_3 + \ldots + \epsilon_2 \epsilon_3 \ldots \epsilon_k V_k) \) is bounded from above:

\[
f(V_1 + \epsilon_2 V_2 + \epsilon_2 \epsilon_3 V_3 + \ldots + \epsilon_2 \epsilon_3 \ldots \epsilon_k V_k) = f\left(\frac{(1 - \epsilon_2) V_1 + \epsilon_2 (V_1 + V_2 + \epsilon_3 V_3 + \ldots + \epsilon_3 \ldots \epsilon_k V_k)}{1 - \epsilon_2}\right) \leq \]
To see that \( f(V_1 + V_2 + \epsilon_3 V_3 + \ldots + \epsilon_k V_k) \) is bounded above, we have

\[
(1 - \epsilon) f(V_1 + V_2 + \epsilon_3 V_3 + \ldots + \epsilon_k V_k) = 
\]

\[
((1 - \epsilon_3) f(V_1 + V_2) + \epsilon_3 f(V_1 + V_2 + V_3 + \epsilon_4 V_4 + \ldots + \epsilon_k V_k)) \leq 
\]

\[
(1 - \epsilon_3) f(V_1 + V_2) + \epsilon_3 f(V_1 + V_2 + V_3 + \epsilon_4 V_4 + \ldots + \epsilon_k V_k). 
\]

Repeating this process we obtain

\[
\]

\[
(1.9) f(V_1 + V_2 + \ldots + \epsilon_k V_k) \leq (1 - \epsilon_k) f(V_1 + V_2 + \ldots + V_{k-1}) + \epsilon_k f(V_1 + V_2 + \ldots + V_k) 
\]

which is bounded from above.

2. \( f(\epsilon) \gtrsim 0 \):

Since

\[
0 = \frac{1}{1 + \epsilon_1} \epsilon + \frac{\epsilon_1}{1 + \epsilon_1} \cdot \frac{-\epsilon}{\epsilon_1} 
\]

we obtain

\[
0 \leq \frac{1}{1 + \epsilon_1} f(\epsilon) + \frac{\epsilon_1}{1 + \epsilon_1} f\left(\frac{-\epsilon}{\epsilon_1}\right) \Rightarrow f(\epsilon) \gtrsim -\epsilon_1 f\left(\frac{-\epsilon}{\epsilon_1}\right). 
\]

If

\[
(1.12) f\left(\frac{-\epsilon}{\epsilon_1}\right) = f(-\epsilon_1 - \epsilon_2 V_2 - \ldots - \epsilon_k V_k) 
\]

is bounded from above then \( f(\epsilon) \gtrsim 0 \). Replacing \( V_i \) by \( W_i := -V_i \), with the same calculations as presented before, we conclude the desired.

For our next result, we need the following: If \( A \) is a compact set, then for each \( a \in ^*A \), there exists \( st(a) \) and \( st(a) \in A \).

**Theorem 4** Let \((E, | \cdot |)\) be a finite-dimensional normed space, \((F, T)\) a Hausdorff linear topological space and \( f : E \to F \) a function. If the image of every compact subspace of \( E \) is compact in \( F \) and the image of every convex subspace of \( E \) is convex in \( F \), then \( f \) is continuous.

**Proof.** Fix \( x \in E \) and \( y \in ^*E \) with \( y \approx x \). For every \( n \in \mathbb{N} \), the closed ball \( \overline{B}_{1/n}(x) \) is compact and convex, so \( F_n := f\left( \overline{B}_{1/n}(x) \right) \) is also compact and convex. Besides this, we have for each \( n \in \mathbb{N} \)

\[
(1.13) x, y \in \overline{B}_{1/n}(x) \Rightarrow f(x), f(y) \in F_n \Rightarrow f(x), st(f(y)) \in F_n. 
\]
So there exists
\[ x_n \in \overline{B_{1/n}(x)} \text{ with } f(x_n) = \frac{1}{n}f(x) + \left(1 - \frac{1}{n}\right)st(f(y)). \]

Since \( \lim x_n = x \), the set \( A := \{x\} \cup \{x_n|n \in \mathbb{N}\} \) is compact and so \( f(A) = \{f(x_n)|n \in \mathbb{N}\} \) is also compact. Consequently, \( f(x) = st(f(y)) \).

As a consequence, we have:

**Theorem 5** Let \( f : \mathbb{R} \to \mathbb{R} \) be a function. If the image of every compact subset of \( \mathbb{R} \) is compact and the image of every connected subset of \( \mathbb{R} \) is connected, then \( f \) is continuous.

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