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AN EXOTIC EXAMPLE OF A TENSOR PRODUCT OF A CM ELLIPTIC CURVE AND A WEIGHT 1 FORM

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Abstract. The representation obtained as a tensor product of a rational elliptic curve with a weight 1 modular form is in general an irreducible four-dimensional representation. However, there are some instances where such representation is reducible. In this short article we give an exotic way to obtain such a reducible representation.

Introduction

Let E be a rational elliptic curve, and let f be a weight 1 modular form. Given p a rational prime, let $\rho_{E,p}$ be the p-adic Galois representation attached to E and $\rho_{f,p}$ the 2-dimensional p-adic Galois representation attached to f by Deligne–Serre (see [2]). The Galois representation $\rho_{E,p} \otimes \rho_{f,p}$ is a rational 4-dimensional representation, which is generally irreducible. However, there are some instances when such representation decomposes as a direct sum of two 2-dimensional ones. A first such instance occurs when E has complex multiplication by an imaginary quadratic field K and f corresponds to $\operatorname{Ind}_{G_K}^{G_{\mathbb{Q}}} \chi$ for χ a character of G_K (as explained in Lemma 1.1). In this article, we present a related situation, where the form f is obtained as the induction of a quadratic character of a real quadratic extension L.

The explanation behind our example is that dihedral weight 1 modular forms can have more "endomorphisms" than expected, in the sense that they can be obtained as induction of characters of different quadratic fields. Then even when our form does not seem to belong to the previously described situation (the fact that L is real quadratic implies it does not match the CM field), it does indeed.

The example appeared while studying the $\mathbb{Q}$-curve attached to a trivial solution of the Diophantine equation $x^4 - dy^2 = z^p$ (as described in [4]).

1. The involved objects

If E is a rational elliptic curve, and K is a number field, let E_K denote the base change of E to K.

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Lemma 1.1. *Let E be a rational elliptic curve with complex multiplication by an order in an imaginary quadratic field K, and let χ be a character of K. Then $\operatorname{Ind}_{G_K}^{G_{\mathbb{Q}}}(\rho_{E_K,p}\otimes\chi)$ decomposes as a direct sum of two irreducible 2-dimensional representations.*

Proof. Since E is an elliptic curve with complex multiplication, there exists a Hecke character θ of infinity type $(1,0)$ such that $\rho_{E,p}=\operatorname{Ind}_{G_K}^{G_{\mathbb{Q}}}\theta_p$, where θ_p is the p-adic character attached to θ via class field theory. Let τ be an element of $G_{\mathbb{Q}}$ giving the non-trivial element of $\operatorname{Gal}(K/\mathbb{Q})$. By induction-restriction, $\rho_{E_K,p}=\theta_p\oplus\theta_p^\tau$ (where $\theta_p^\tau(\sigma)=\theta_p(\tau\sigma\tau^{-1})$), hence $\operatorname{Ind}_{G_K}^{G_{\mathbb{Q}}}(\rho_{E_K,p}\otimes\chi)=\operatorname{Ind}_{G_K}^{G_{\mathbb{Q}}}(\theta_p\otimes\chi)\oplus\operatorname{Ind}_{G_K}^{G_{\mathbb{Q}}}(\theta_p^\tau\otimes\chi)$ as claimed. $\square$

If we replace the field K of complex multiplication by another quadratic extension L, in general, the representation $\operatorname{Ind}_{G_L}^{G_{\mathbb{Q}}}\rho_{E_L,p}\otimes\chi$ will be an irreducible 4-dimensional representation, except for some very exceptional situations related to dihedral weight 1 modular forms. Let E be the elliptic curve 256-a1

$$E: y^2=x^3+x^2-13x-21.$$

It has complex multiplication by $\mathbb{Z}[\sqrt{-2}]$, hence its L-series matches that of a Hecke character θ of infinity type $(1,0)$ and conductor $(\sqrt{-2})^5$ over the imaginary quadratic field $K=\mathbb{Q}(\sqrt{-2})$. Equivalently, for any prime p, the Galois representation $\rho_{E,p}$ matches $\operatorname{Ind}_{G_K}^{G_{\mathbb{Q}}}\theta_p$, where θ_p is the p-adic character attached to θ via class field theory.

Let $L=\mathbb{Q}(\sqrt{3})$ and let χ be the quadratic character attached to the quadratic extension $L[\sqrt{1+\sqrt{3}}]$ of conductor $\mathfrak{p}_2^5$, where $\mathfrak{p}_2$ is the unique (ramified) prime ideal of L dividing 2. Then by [3, Theorem 8.2] $\operatorname{Ind}_{G_L}^{G_{\mathbb{Q}}}\chi$ is the weight 1 modular form f (384.1.h.b) whose level has valuation $1\cdot(1+0)$ at 3 and $1\cdot(2+5)$ at 2 (hence its level equals $3\cdot 2^7$) and Nebentypus δ, the quadratic character of conductor 24 corresponding to the imaginary quadratic field $\mathbb{Q}(\sqrt{-6})$. The form f has attached a 2-dimensional complex representation (see [2]) with finite image; in particular, for any prime number p it has attached a 2-dimensional p-adic representation $\rho_{f,p}$ of $G_{\mathbb{Q}}$ into $\mathrm{GL}_2(\overline{\mathbb{Q}_p})$. Let E_L denote the base change of E to L. The 4-dimensional representation $\rho_{E,p}\otimes\rho_{f,p}$ matches $\operatorname{Ind}_{G_L}^{G_{\mathbb{Q}}}\rho_{E_L\otimes\chi,p}$. Although the character χ is not induced from a rational character (via the norm map) and $\rho_{E_L,p}$ is absolutely irreducible, the representation $\operatorname{Ind}_{G_L}^{G_{\mathbb{Q}}}\rho_{E_L\otimes\chi,p}$ is reducible due to the rare properties of the weight 1 form f.

Lemma 1.2. *Let $D_4=\langle\rho,s:\rho^4=s^2=1,\rho s=s\rho^3\rangle$ be the dihedral group of eight elements. Consider the index 2 subgroups*

$$C_1=\langle s,\rho^2\rangle,\qquad C_2=\langle\rho\rangle,\qquad C_3=\langle\rho s,\rho^2\rangle,$$

and the characters $\chi_i: C_i\to\{\pm1\}$ given on generators by $\chi_1(s)=1=\chi_3(\rho s)$, $\chi_1(\rho^2)=\chi_3(\rho^2)=-1$ and $\chi_2(\rho)=\sqrt{-1}$. Then

$$\operatorname{Ind}_{C_1}^{D_4}\chi_1\simeq\operatorname{Ind}_{C_2}^{D_4}\chi_2\simeq\operatorname{Ind}_{C_3}^{D_4}\chi_3. \tag{1.1}$$

Proof. The proof follows either from the fact that D_4 has a unique 2-dimensional irreducible representation, or otherwise, it can easily be checked by looking at the traces on conjugacy classes (or with a computational system such as Magma [1]). $\square$

Remark 1.3. There are actually two different characters on each subgroup C_i making (1.1) hold. They are characterized by the fact that their value at ρ^2 is -1, while their value at the other generator (if any) might be ± 1.

Theorem 1.4. *The representation* $\operatorname{Ind}_{G_L}^{G_{\mathbb{Q}}} \rho_{E_L \otimes \chi, p} = V_1 \oplus V_2$, *where each* V_i *is an irreducible representation corresponding to a weight* 2 *modular form of level* $3 \cdot 2^8$ *and Nebentypus* δ_K *(of conductor* 12*).*

Proof. The extension $L[\sqrt{1+\sqrt{3}}]/\mathbb{Q}$ is not Galois; its Galois group corresponds to an extension M corresponding to the decomposition field of $x^8 - 20x^6 + 160x^4 - 600x^2 + 4356$, which contains the imaginary quadratic fields $\mathbb{Q}(\sqrt{-2})$. Consider the following field diagram:

$$\begin{tikzcd}
 & M \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
 & \mathbb{Q}(\sqrt{1+\sqrt{3}}) \arrow[d, dash] & \\
\mathbb{Q}(\sqrt{-6}) \arrow[dr, dash] & L = \mathbb{Q}(\sqrt{3}) \arrow[d, dash] & \mathbb{Q}(\sqrt{-2}) = K \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

The Galois group $\operatorname{Gal}(M/\mathbb{Q})$ equals D_4 and the representation attached to f is precisely the unique irreducible 2-dimensional representation of it. An easy computation proves that $\mathbb{Q}(\sqrt{-6}) = M^{\langle \rho \rangle}$, while the other two quadratic fields correspond to the fixed field of $C_1 = \langle s, \rho^2 \rangle$ and $C_3 = \langle \rho s, \rho^2 \rangle$ (there is no canonical choice of such subgroups). If we choose the rotation so that L is the fixed field of $C_1 = \langle s, \rho^2 \rangle$, then the previous lemma implies that

$$\rho_f = \operatorname{Ind}_{G_L}^{G_{\mathbb{Q}}} \chi_1 \simeq \operatorname{Ind}_{G_K}^{G_{\mathbb{Q}}} \chi_3.$$

Via the identification $\operatorname{Gal}(M/\mathbb{Q}) \simeq D_4$, the character χ_3 is a quadratic character of K of conductor $\mathfrak{p}_3 \cdot (\sqrt{-2})^4$, where $\mathfrak{p}_3$ is one of the primes of K dividing 3. The result now follows from Lemma 1.1 while [3, Theorem 8.2] gives the conductor and Nebentypus statement. $\square$

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