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A Positive Solution of a Schrödinger-Poisson System with Critical Exponent

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Abstract

We use variational methods to study the existence of at least one positive solution of the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + l(x)\phi u = k(x)|u|^{2^*-2}u + \mu h(x)|u|^{q-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = l(x)u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

under some suitable conditions on the non-negative functions l, k, h and constant $\mu > 0$, where $2 \leq q < 2^*$ (critical Sobolev exponent).

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1 Introduction

In this paper, we study the existence of solutions of the system (1.2) involving a critical growth with the following form

$$\begin{cases} -\Delta u + u + l(x)\phi u = k(x)|u|^{2^*-2}u + \mu h(x)|u|^{q-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = l(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

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where $2 \leq q < 2^*$. We use the standard Mountain Pass Theorem to show the existence of a solution. However, since the nonlinearity involves a critical exponent, the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ $(2 \leq s \leq 6)$ is not compact. This will create great difficulies in the proof of the Palais-Smale condition. We will transform the problem into a nonlocal elliptic equation in $\mathbb{R}^3$ and we also consider the limiting case $q = 2$.

It is known that the Schrödinger-Poisson systems have a strong physical meaning because they appear in quantum mechanics models (see e.g. [6, 9, 22]) and in semiconductor theory (see e.g. [4, 5, 23, 24]). In particular, systems like (1.2) have been introduced in Benci-Fortunato [4, 5] as a model describing solitary waves for the nonlinear stationary Schrödinger equations in three-dimensional space interacting with the electrostatic field which is not a priori assigned. Further applications to superconductors are currently under investigation.

Very recently, Cerami-Vaira [10] studied the existence of positive solutions for the Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + l(x)\phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = l(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

where they considered $f(x,u) = k(x)|u|^{p-2}u$ with $4 < p < 6$ and assumed that $l \in L^2(\mathbb{R}^3)$ and $k : \mathbb{R}^3 \to \mathbb{R}$ are non-negative functions satisfying $\lim_{|x|\to+\infty} l(x) = 0$, $l \not\equiv 0$, $\lim_{|x|\to+\infty} k(x) = k_\infty > 0$ and $k(x) - k_\infty \in L^{6/(6-p)}(\mathbb{R}^3)$.

After Cerami-Vaira [10] many researchers have looked to problem (1.2), such as D'Avenia-Pomponio-Vaira [18], Li-Peng-Wang [21], Sun-Chen-Nieto [27] and Vaira [30], under various assumptions on the non-constant function l. Similar problems continue to attract attention as one can see from the latest works of He-Zou [20] and their references.

Before Cerami-Vaira [10] similar problems to (1.2), with constant function l, had also been widely investigated. We point out the works of Ambrosetti-Ruiz [2], Coclite [12], D'Avenia [17], D'Aprile et al. [13, 14, 15, 16], Ruiz [26] and others. Among of these, Azzollini-Pomponio [3], D'Aprile-Mugnai [14] and Zhao-Zhao [32] dealt with critical exponent case.

There are no existence results about system (1.1) with non-constant function l. In Zhao-Zhao [32], they studied a similar system to (1.1) with function $l = 1$. They established the existence of at least one positive solution for $4 \leq q < 2^*$ and at least one positive radial solution for $2 < q < 4$ with some restrictions on functions k, h and μ. Moreover, note that there was no information about the case where $q = 2$.

The main result, in this work, generalizes some of above results. We consider the following hypotheses (H):

(H_l) $l \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, $l(x) \geq 0$ for any $x \in \mathbb{R}^3$ and $l \not\equiv 0$;

(H_{k_1}) $k(x) \geq 0$ for any $x \in \mathbb{R}^3$;

(H_{k_2}) There exists $x_0 \in \mathbb{R}^3$, $\delta_1 > 0$ and $\rho_1 > 0$ such that $k(x_0) = \max_{\mathbb{R}^3} k(x)$ and $|k(x) - k(x_0)| \leq \delta_1 |x - x_0|^\alpha$ for $|x - x_0| < \rho_1$ with $1 \leq \alpha < 3$;

(H_{h_1}) $h \in L^{6/(6-q)}(\mathbb{R}^3)$ and $h(x) \geq 0$ for any $x \in \mathbb{R}^3$ and $h \not\equiv 0$;

(H_{h_2}) There are $\delta_2 > 0$ and $\rho_2 > 0$ such that $h(x) \geq \delta_2 |x - x_0|^{-\beta}$ for $|x - x_0| < \rho_2$ and $2 - \frac{q}{2} < \beta < 3$, where x_0 is given by (H_{k_2});

(H_{h_μ}) $0 < \mu < \bar{\mu}$ when $2 \le q < 4$; $\mu > 0$ when $4 \le q < 6$, where $\bar{\mu}$ is defined by

$$\bar{\mu} := \mu_h = \inf_{u \in H^1(\mathbb{R}^3)\setminus\{0\}} \left\{ \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)dx \ : \ \int_{\mathbb{R}^3} h(x)|u|^q dx = 1 \right\}.$$

Remark 1.1. The hypotheses (H_{k_1}) and (H_{k_2}) mean that $k \in L^\infty(\mathbb{R}^3)$.

Remark 1.2. The function k, which satisfies a Hölder condition of order α with $1 \le \alpha < 3$ on $H^1(\mathbb{R}^3)$ and achieves its maximum, is a special case of (H_{k_2}).

Remark 1.3. In Lemma 2.3, we show that $\bar{\mu}$ is achieved.

By a solution (u,ϕ) in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ of problem (1.1), we mean that for any $v \in H^1(\mathbb{R}^3)$ it holds

$$\begin{cases} \int_{\mathbb{R}^3} (\nabla u \nabla v + uv + l(x)\phi uv)\, dx = \int_{\mathbb{R}^3} \left(k(x)|u|^{2^*-2}uv + \mu h(x)|u|^{q-2}uv\right) dx, \\ \int_{\mathbb{R}^3} \nabla\phi\nabla v dx = \int_{\mathbb{R}^3} l(x)u^2 v dx. \end{cases}$$

We say the solution is positive if $u(x) > 0$ and $\phi(x) > 0$ for all $x \in \mathbb{R}^3$.

We shall prove the following theorem.

Theorem 1.4. *Assume the hypotheses* (H) *hold and* $2 \le q < 2^*$. *Then problem (1.1) has at least one positive solution* (u,ϕ_u) *in* $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$.

To prove the result above, we use a combination of techniques, e.g. techniques motivated by Willem [31], to overcome the lack of compactness of the Sobolev embedding, and methods used by Chen-Li-Li [11] and Zhao-Zhao [32], to estimate carefully the energy level.

Notations. Throughout this paper, $L^p \equiv L^p(\mathbb{R}^3)$ $(1 \le p < +\infty)$ is the usual Lebesgue space with the norm $\|u\|_p^p = \int_{\mathbb{R}^3} |u|^p dx$; $L^\infty \equiv L^\infty(\mathbb{R}^3)$ is the space of all essentially bounded functions with the norm $\|u\|_\infty = \operatorname{ess\,sup}|u|$; $H^1 \equiv H^1(\mathbb{R}^3)$ denotes the usual Sobolev space with the norm $\|u\|^2 = \int_{\mathbb{R}^3} \left(|\nabla u|^2 + |u|^2\right) dx$; H^{-1} is the dual space of H^1 and $\langle\cdot,\cdot\rangle \equiv \langle\cdot,\cdot\rangle_{H^{-1}\times H^1}$ is dual bracket; $D^1 \equiv D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|_D^2 = \int_{\mathbb{R}^3} |\nabla u|^2 dx$; $B_\rho(x)$ and B_ρ denote a ball with radius ρ centred at x and 0, respectively in a related space. Let $u^+ = \max\{u,0\}$ and $u^- = \max\{-u,0\}$. We denote strong (weak) convergence for a sequence $(u_n)_{n\in\mathbb{N}}$ and u in a Banach space by $u_n \to u$ ($u_n \rightharpoonup u$), respectively. N is used to denote the dimension, so $N = 3$ if there is no special explanation. The so-called critical Sobolev exponent is denoted by $2^* = \frac{2N}{N-2}$. The symbol C denotes different positive constants and the value of C is allowed to change from line to line and in the same formula.

2 Preliminaries

In this section, we are going to give some preliminary lemmas. Since our methods are variational, first of all, it is necessary to transform the problem (1.1) into a Schrödinger equation with a nonlocal term. In fact, for any $u \in H^1$, denote $L_u(v)$ the linear functional in D^1 by

$$L_u(v) = \int_{\mathbb{R}^3} l(x)u^2 v dx.$$

It follows from the hypothesis (H_l), Hölder and Sobolev inequalities that

$$|L_u(v)| \le \|l\|_\infty \|u\|_{12/5}^2 \|v\|_6 \le C\|l\|_\infty \|u\|_{12/5}^2 \|v\|_D. \tag{2.1}$$

Hence, the Lax-Milgram theorem implies that there exists, for each u in H^1, a unique $\phi_u \in D^1$ such that

$$\int_{\mathbb{R}^3} \nabla\phi_u \nabla v = \int_{\mathbb{R}^3} l(x)u^2 v dx \quad \text{for any } v \in D^1,$$

i.e., ϕ_u is the weak solution of $-\Delta\phi = l(x)u^2$. It holds

$$\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{l(y)u^2(y)}{|x-y|} dy.$$

In particular, we have

$$\|\phi_u\|_D^2 = \int_{\mathbb{R}^3} |\nabla\phi_u|^2 dx = \int_{\mathbb{R}^3} l(x)\phi_u u^2 dx. \tag{2.2}$$

Using (2.1) and (2.2), we obtain

$$\|\phi_u\|_6 \le C\|\phi_u\|_D \le C\|u\|_{12/5}^2 \le C\|u\|^2 \tag{2.3}$$

and

$$\int_{\mathbb{R}^3} l(x)\phi_u(x)u^2(x)dx \le C\|u\|^4.$$

Thus $F : H^1 \to \mathbb{R}$ is well defined with

$$F(u) = \int_{\mathbb{R}^3} l(x)\phi_u(x)u^2(x)dx. \tag{2.4}$$

To give the smoothness of the functional F (about the smoothness, we can find the statement in previous works, but we didn't find complete details), first, it is necessary to introduce the following lemma.

Lemma 2.1. *[25, p.31] Let* $0 < \beta < N$ *and* $f \in L^q(\mathbb{R}^N)$, $g \in L^r(\mathbb{R}^N)$ *with* $\frac{1}{q} + \frac{1}{r} + \frac{\beta}{N} = 2$ *and* $1 < q,\ r < \infty$. *Then*

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|f(x)||g(y)|}{|x-y|^\beta} dxdy \le C(q,r,\beta,N)\|f\|_q \|g\|_r, \quad x,y \in \mathbb{R}^N,$$

where $C(q,r,\beta,N)$ *is a positive constant depending on* q,r,β *and* N.

Lemma 2.2. *If the hypothesis* (H_l) *holds, then* $F \in C^1(H^1,\mathbb{R})$.

Proof. From Lemma 2.1 and hypothesis (H_l) we obtain

$$\begin{aligned}&\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|l(x)u^2(x)||l(y)u(y)v(y)|}{|x-y|}dxdy\\&\le C||u||_{12/5}^2||uv||_{6/5}\le C||u||_{12/5}^2||u||_{12/5}||v||_{12/5}\end{aligned}$$

for any $u,v\in H^1$. Then we may use the Lebesgue Theorem and Fubini Theorem and get

$$\begin{aligned}&\lim_{t\to0}\frac{F(u+tv)-F(u)}{t}\\&=\lim_{t\to0}\int_{\mathbb{R}^3}\frac{l(x)}{t}\left((u+tv)^2\left(\phi_u+2t\int_{\mathbb{R}^3}\frac{l(y)u(y)v(y)}{|x-y|}dy+t^2\phi_v\right)-\phi_u u^2\right)dx\\&=2\int_{\mathbb{R}^3}l(x)\left(u^2(x)\int_{\mathbb{R}^3}\frac{l(y)u(y)v(y)}{|x-y|}dy+u(x)v(x)\int_{\mathbb{R}^3}\frac{l(y)u^2(y)}{|x-y|}dy\right)dx\\&=4\int_{\mathbb{R}^3}l(x)\phi_u uvdx.\end{aligned}$$

Hence the Gateaux derivative of F on H^1 exists and $\langle\frac{1}{4}F'(u),v\rangle=\int_{\mathbb{R}^3}l(x)\phi_u uvdx$. Let $u_n\to u$ in H^1 and $v\in H^1$, then by (H_l) we obtain

$$\begin{aligned}&||F'(u_n)-F'(u)||_{H^{-1}}=\sup_{||v||=1}|\langle F'(u_n)-F'(u),v\rangle|\\&=4\sup_{||v||=1}\left|\int_{\mathbb{R}^3}l(x)(\phi_{u_n}u_n-\phi_{u_n}u+\phi_{u_n}u-\phi_u u)vdx\right|\\&\le4||l||_\infty\sup_{||v||=1}\left(||\phi_{u_n}||_6||u_n-u||_{12/5}||v||_{12/5}+\int_{\mathbb{R}^3}|\phi_{u_n}-\phi_u||uv|dx\right).\end{aligned}\tag{2.5}$$

It follows from Lemma 2.1 that

$$\begin{aligned}&\int_{\mathbb{R}^3}|\phi_{u_n}-\phi_u||uv|dx\\&=\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|u(x)v(x)||u_n^2(y)-u^2(y)|}{|x-y|}dxdy\\&\le C||u_n^2-u^2||_{6/5}||uv||_{6/5}\le C||u_n^2-u^2||_{6/5}||u||_{12/5}||v||_{12/5}.\end{aligned}$$

From (2.3), (2.5), (2.6) and the fact that $u_n\to u$ in H^1, we obtain

$$||F'(u_n)-F'(u)||_{H^{-1}}\to0.$$

Thus F has a continuous Gateaux derivative on H^1. Therefore $F\in C^1(H^1,\mathbb{R})$. □

Let's introduce the Euler functional of the problem (1.1) as $I:H^1\to\mathbb{R}$ defined by

$$I(u)=\frac{1}{2}||u||^2+\frac{1}{4}F(u)-\int_{\mathbb{R}^3}\left(\frac{1}{2^*}k(x)|u^+|^{2^*}+\frac{\mu}{q}h(x)|u^+|^q\right)dx.\tag{2.6}$$

By Lemma 2.2 we know that the functional I is of class C^1 and its critical points are weak solutions of (1.1).

To prove Theorem 1.4, we still need some other preliminary lemmas.

Lemma 2.3. *Assume that the hypothesis* (H_l) *holds. Then* F *is a weakly continuous functional.*

Proof. Suppose $u_n \rightharpoonup u$ in H^1. Since $u_n \to u$ in L^2_{loc}, going if necessary to a subsequence, we can assume that

$$u_n \to u \text{ a.e. in } \mathbb{R}^3 \quad \text{and} \quad \phi_{u_n} \to \phi_u \text{ a.e. in } \mathbb{R}^3.$$

In fact, the last statement is true since, by (H_l) and Hölder inequality, we have

$$\begin{aligned} \left|\phi_{u_n}(x) - \phi_u(x)\right| &\leq \frac{1}{4\pi}\int_{\mathbb{R}^3} \left|l(y)\|u_n^2(y) - u^2(y)\right| \frac{1}{|x-y|}dy \\ &\leq C\|u_n^2 - u^2\|_{L^2(B_R(x))}\left(\int_{|x-y|\leq R} \frac{1}{|x-y|^2}dy\right)^{1/2} \\ &+C\|u_n^2 - u^2\|_{L^{4/3}(B_R^c(x))}\left(\int_{|x-y|>R} \frac{1}{|x-y|^4}dy\right)^{1/4} \\ &\leq C\|u_n^2 - u^2\|_{L^2(B_R(x))} + CR^{-\frac{1}{4}}\|u_n^2 - u^2\|_{L^{4/3}(B_R^c(x))} \\ &\to 0, \end{aligned} \tag{2.7}$$

as $n \to \infty$ and $R \to \infty$. Then $\phi_{u_n}u_n^2 \to \phi_u u^2$ a.e. on $\mathbb{R}^3$. Moreover, the sequence $(\phi_{u_n}u_n^2)_{n\in\mathbb{N}}$ is bounded in L^2, since

$$\int_{\mathbb{R}^3}\left(\phi_{u_n}u_n^2\right)^2 dx \leq \left(\int_{\mathbb{R}^3}\phi_{u_n}^6 dx\right)^{1/3}\left(\int_{\mathbb{R}^3}u_n^6 dx\right)^{2/3} = \|\phi_{u_n}\|_6^2\|u_n\|_6^4 \leq C\|u_n\|^6.$$

Hence $\phi_{u_n}u_n^2 \rightharpoonup \phi_u u^2$ in L^2. By (H_l) we have

$$F(u_n) = \int_{\mathbb{R}^3} l(x)\phi_{u_n}u_n^2 dx \to \int_{\mathbb{R}^3} l(x)\phi_u u^2 dx = F(u).$$

We have proved that F is weakly continuous. □

Lemma 2.4. *Assume the hypothesis* (H_l) *holds. Let* $u_n \rightharpoonup u$ *in* H^1*, then*

$$F(u_n - u) = F(u_n) - F(u) + o(1).$$

Proof. Since (H_l) holds, from the proof of [32, Lemma 2.1], the result follows. □

From a similar proof as in [31, Lemma 2.13], we obtain the next result.

Lemma 2.5. *If the hypothesis* (H_{h_1}) *holds and* $2 \leq q < 6$*, then the functional*

$$\psi_h : H^1 \to \mathbb{R} : u \mapsto \int_{\mathbb{R}^3} h(x)|u|^q dx$$

is weakly continuous.

Lemma 2.6. *Suppose the hypothesis* (H_{h_1}) *holds and* $2 \leq q < 4$*. Then the following infimum*

$$\bar{\mu} := \mu_h = \inf_{u\in H^1\setminus\{0\}}\left\{\int_{\mathbb{R}^3}(|\nabla u|^2 + u^2)dx \ : \ \int_{\mathbb{R}^3} h(x)|u|^q dx = 1\right\} \tag{2.8}$$

is achieved.

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset H^1$ be a minimizing sequence such that

$$\int_{\mathbb{R}^3} h(x)|u_n|^q dx = 1 \quad \text{and} \quad \int_{\mathbb{R}^3} (|\nabla u_n|^2 + u_n^2)dx \to \mu_h, \quad \text{as } n \to \infty.$$

So $(u_n)_{n\in\mathbb{N}}$ is bounded in H^1. Then there exists a subsequence satisfying $u_n \rightharpoonup u$ in H^1. Since $h \in L^{6/(6-q)}$, by Lemma 2.5, we have

$$\int_{\mathbb{R}^3} h(x)|u_n|^q dx \to \int_{\mathbb{R}^3} h(x)|u|^q dx. \quad \text{Hence } \int_{\mathbb{R}^3} h(x)|u|^q dx = 1.$$

Then, by the weakly lower semi-continuous property of the norm, we get

$$\mu_h = \liminf_{n\to\infty} \int_{\mathbb{R}^3} (|\nabla u_n|^2 + u_n^2)dx \geq \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)dx \geq \mu_h.$$

Thus the infimum μ_h is achieved. □

Lemma 2.7. *Suppose the hypotheses* (H_l), (H_{k_1}), (H_{h_1}) *and* (H_{h_μ}) *hold. Then* $I(0) = 0$ *and*
(I_1) *there are constants* $\rho, \alpha > 0$ *such that* $I|_{\partial B_\rho} \geq \alpha$; *and*
(I_2) *there is* $\bar{u} \in H^1 \setminus \bar{B}_\rho$ *such that* $I(\bar{u}) < 0$.

Proof. It is clear from the definition of I that $I(0) = 0$. To prove (I_1) and (I_2), we consider $2 \leq q < 4$ and $4 \leq q < 6$ respectively. First, for $2 \leq q < 4$, we have $0 < \mu < \bar{\mu}$ by (H_{h_μ}). It follows from (H_{k_1}), Lemma 2.6 and Sobolev inequality that

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}F(u) - \frac{1}{2^*}\int_{\mathbb{R}^3} k(x)|u^+|^{2^*}dx - \frac{\mu}{q}\int_{\mathbb{R}^3} h(x)|u^+|^q dx \\ &\geq \frac{1}{2}\|u\|^2 - C\|u\|^{2^*} - \frac{\mu}{q\bar{\mu}}\|u\|^2 = \|u\|^2\left(\frac{1}{2} - \frac{\mu}{q\bar{\mu}} - C\|u\|^{2^*-2}\right). \end{aligned}$$

Set $\rho = \|u\|$, small enough such that $C\rho^{2^*-2} \leq \frac{1}{2}(\frac{1}{2} - \frac{\mu}{q\bar{\mu}})$. Hence we have

$$I(u) \geq \frac{1}{2}\left(\frac{1}{2} - \frac{\mu}{q\bar{\mu}}\right)\rho^2. \tag{2.9}$$

Take $\alpha = \frac{1}{2}(\frac{1}{2} - \frac{\mu}{q\bar{\mu}})\rho^2$. Then we get the result (I_1). By (2.3) and the fact that $\mu h(x) \geq 0$, for fixed u_0 with $\|u_0\| = 1$ and $\text{supp}(u_0) \subset \text{supp}(k)$, we have

$$I(tu_0) \leq t^{2^*}\left(\frac{1}{2t^4}\|u_0\|^2 + \frac{C}{4t^2}\|u_0\|^4 - \frac{C}{2^*}\int_{\mathbb{R}^3} k(x)|u_0^+|^{2^*}dx\right).$$

Let t be large enough such that $t > \rho$ and

$$\frac{1}{2t^4}\|u_0\|^2 + \frac{C}{4t^2}\|u_0\|^4 - \frac{C}{2^*}\int_{\mathbb{R}^3} k(x)|u_0^+|^{2^*}dx < 0.$$

Take $\bar{u} = tu_0$. Then (I_2) follows.

Next, we consider $4 \le q < 6$, so $\mu > 0$ by (H_{h_μ}). Since (H_{k_1}) and (H_{h_1}) hold, the Hölder inequality and Sobolev inequality implies that

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}F(u) - \frac{1}{2^*}\int_{\mathbb{R}^3} k(x)|u^+|^{2^*}dx - \frac{\mu}{q}\int_{\mathbb{R}^3} h(x)|u^+|^q dx \\ &\ge \frac{1}{2}\|u\|^2 - C\|u\|^{2^*} - \frac{\mu}{q}\|h\|_{\frac{6}{6-q}}\|u\|_6^q \\ &\ge \|u\|^2\left(\frac{1}{2} - C\|u\|^{2^*-2} - C\|u\|^{q-2}\right) \end{aligned}$$

for each $\mu > 0$ fixed. Hence (I_1) follows from the similar estimate with (2.9). The proof of (I_2) is the same to the case $2 \le q < 4$. □

3 The proof of Theorem 1.4

To prove Theorem 1.4, we will apply the Mountain Pass Theorem to find a solution of problem (1.1) and then prove that it is a positive solution. Let us first recall (one of the versions of) the Mountain Pass Theorem.

Mountain Pass Theorem [1]. Let E be a real Banach space and $I \in C^1(E,R)$. Suppose $I(0) = 0$ and

(I_1) there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} \ge \alpha$; and

(I_2) there is $\bar{u} \in E \setminus \bar{B_\rho}$ such that $I(\bar{u}) < 0$. If I satisfies the $(PS)_c$-condition, where c is defined as

$$c = \inf_{g\in\Gamma} \max_{u\in g[0,1]} I(u) \ \text{ with } \Gamma = \{g \in C([0,1],E) \ : \ g(0) = 0, g(1) = \bar{u}\}. \tag{3.1}$$

Then I possesses a critical value $c \ge \alpha$.

Since Lemma 2.7 shows that the functional I has the Mountain Pass geometry, to apply this theorem to the functional I with $E \equiv H^1$, it is enough to prove that the Palais-Smale condition holds at the level c (the $(PS)_c$-condition for short), which means that every sequence $(u_n)_{n\in\mathbb{N}} \subset H^1$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$ in H^{-1} implies that $(u_n)_{n\in\mathbb{N}}$ possesses a convergent subsequence in H^1.

Lemma 3.1. *Assume (H_l), (H_{k_1}), (H_{h_1}) and (H_{h_μ}) hold. Then the functional I satisfies the $(PS)_c$ -condition for $c \in \left(0, \frac{1}{N}\mathcal{S}^{\frac{N}{2}}\|k\|_\infty^{-\frac{N-2}{2}}\right)$, where $\mathcal{S}$ denotes the best Sobolev constant defined by*

$$\mathcal{S} = \inf_{u\in D^1\setminus\{0\}} \frac{\int_{\mathbb{R}^3}|\nabla u|^2 dx}{\left(\int_{\mathbb{R}^3}|u|^{2^*}dx\right)^{2/2^*}}. \tag{3.2}$$

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a $(PS)_c$ -sequence of I at the level $c \in \left(0, \frac{1}{N}\mathcal{S}^{\frac{N}{2}}\|k\|_\infty^{-\frac{N-2}{2}}\right)$, i.e.,

$$I(u_n) \to c \text{ and } I'(u_n) \to 0 \text{ in } H^{-1}. \tag{3.3}$$

Step 1. We consider $2 \le q < 4$, so we get $0 < \mu < \bar{\mu}$ by (H_{h_μ}). Then by the Sobolev inequality, Lemma 2.6 and $k(x) \ge 0$ for any $x \in \mathbb{R}^3$, for large n we have

$$\begin{aligned} c+1+\|u_n\| &\ge I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle \\ &= \frac{1}{4}\|u_n\|^2 + \left(\frac{1}{4} - \frac{1}{2^*}\right)\int_{\mathbb{R}^3} k(x)|u_n^+|^{2^*} dx + \left(\frac{\mu}{4} - \frac{\mu}{q}\right)\int_{\mathbb{R}^3} h(x)|u_n^+|^q dx \\ &\ge \frac{1}{4}\|u_n\|^2 + \left(\frac{1}{4} - \frac{1}{2^*}\right)\int_{\mathbb{R}^3} k(x)|u_n^+|^{2^*} dx + \left(\frac{1}{4} - \frac{1}{q}\right)\frac{\mu}{\bar{\mu}}\|u_n\|^2 \\ &\ge \left(\frac{1}{4} + \left(\frac{1}{4} - \frac{1}{q}\right)\frac{\mu}{\bar{\mu}}\right)\|u_n\|^2, \end{aligned} \tag{3.4}$$

which implies $(u_n)_{n\in\mathbb{N}}$ is bounded in H^1, since $0 < \mu < \bar{\mu}$ and $2 \le q < 4$. Passing if necessary to a subsequence, we can assume that

$$u_n \rightharpoonup u \quad \text{in } H^1, \quad u_n \to u \quad \text{a.e. in } \mathbb{R}^3,$$

$$\nabla u_n \rightharpoonup \nabla u \quad \text{in } L^2, \quad \text{and} \quad u_n \rightharpoonup u \quad \text{in } L^2.$$

Let us define $w_n = k(x)|u_n^+|^{N+2/N-2}$ and $w = k(x)|u^+|^{N+2/N-2}$. Since $(u_n)_{n\in\mathbb{N}}$ is bounded in L^{2^*} and $k \in L^\infty$, then w_n is bounded in $L^{2N/N+2}$ and so $w_n \rightharpoonup w$ in $L^{2N/N+2}$. Note that for any $v \in H^1$, we have $v \in L^{2N/N-2}$, $\nabla v \in L^2$ and $v \in L^2$. Hence

$$\int_{\mathbb{R}^3} w_n v dx \to \int_{\mathbb{R}^3} w v dx, \text{ i.e., } \int_{\mathbb{R}^3} k(x)|u_n^+|^{2^*-1} v dx \to \int_{\mathbb{R}^3} k(x)|u^+|^{2^*-1} v dx, \tag{3.5}$$

and

$$\int_{\mathbb{R}^3} (\nabla u_n \nabla v + u_n v)\, dx \to \int_{\mathbb{R}^3} (\nabla u \nabla v + uv)\, dx. \tag{3.6}$$

From the proof of Lemma 2.3 and Lemma 2.5 we also have

$$\int_{\mathbb{R}^3} h(x)|u_n^+|^{q-1} v dx \to \int_{\mathbb{R}^3} h(x)|u^+|^{q-1} v dx, \tag{3.7}$$

and

$$\int_{\mathbb{R}^3} l(x)\phi_{u_n} u_n v dx \to \int_{\mathbb{R}^3} l(x)\phi_u u v dx. \tag{3.8}$$

Combining (3.5)–(3.8), for $u_n \rightharpoonup u$ in H^1, we obtain

$$\begin{aligned} \langle I'(u_n), v\rangle &= \int_{\mathbb{R}^3} (\nabla u_n \nabla v + u_n v) dx + \int_{\mathbb{R}^3} l(x)\phi_{u_n} u_n v dx \\ &- \int_{\mathbb{R}^3} k(x)|u_n^+|^{2^*-1} v dx - \mu \int_{\mathbb{R}^3} h(x)|u_n^+|^{q-1} v dx \\ &\to \int_{\mathbb{R}^3} (\nabla u \nabla v + uv) dx + \int_{\mathbb{R}^3} l(x)\phi_u u v dx - \int_{\mathbb{R}^3} k(x)|u^+|^{2^*-1} v dx \\ &- \mu \int_{\mathbb{R}^3} h(x)|u^+|^{q-1} v dx = \langle I'(u), v\rangle. \end{aligned} \tag{3.9}$$

On the other hand, by the fact $I'(u_n) \to 0$ in H^{-1}, we get that $\langle I'(u_n), v\rangle \to 0$ for any $v \in H^1$. So $\langle I'(u), v\rangle = 0$ for any $v \in H^1$, i.e.

$$-\Delta u + u + l(x)\phi_u u = k(x)|u^+|^{2^*-1} + \mu h(x)|u^+|^{q-1}. \tag{3.10}$$

In particular, $\langle I'(u), u\rangle = 0$ and then from Lemma 2.6 and $k(x) \geq 0$ we obtain

$$\begin{aligned} I(u) &= \frac{1}{4}\langle I'(u), u\rangle + \frac{1}{4}\|u\|^2 + \left(\frac{1}{4} - \frac{1}{2^*}\right)\int_{\mathbb{R}^3} k(x)|u^+|^{2^*} dx + \left(\frac{\mu}{4} - \frac{\mu}{q}\right)\int_{\mathbb{R}^3} h(x)|u^+|^q dx \\ &\geq \left(\tfrac{1}{4} + \left(\tfrac{1}{4} - \tfrac{1}{q}\right)\tfrac{\mu}{\bar{\mu}}\right)\|u\|^2 \geq 0. \end{aligned} \tag{3.11}$$

Let $v_n = u_n - u$ and so $v_n \rightharpoonup 0$ in H^1. Hence, using the given hypotheses, the Brézis-Lieb Lemma [7] implies that

$$\|u_n\|^2 = \|v_n\|^2 + \|u\|^2 + o(1),$$

$$\int_{\mathbb{R}^3} k(x)|u_n^+|^{2^*} dx = \int_{\mathbb{R}^3} k(x)|v_n^+|^{2^*} dx + \int_{\mathbb{R}^3} k(x)|u^+|^{2^*} dx + o(1),$$

$$\int_{\mathbb{R}^3} h(x)|u_n^+|^q dx = \int_{\mathbb{R}^3} h(x)|v_n^+|^q dx + \int_{\mathbb{R}^3} h(x)|u^+|^q dx + o(1),$$

and hence by Lemma 2.4 we have

$$I(u_n) = I(u) + \frac{1}{2}\|v_n\|^2 + \frac{1}{4}F(v_n) - \frac{1}{2^*}\int_{\mathbb{R}^3} k(x)|v_n^+|^{2^*} dx - \frac{1}{2}\int_{\mathbb{R}^3} h(x)|v_n^+|^q dx + o(1),$$

and

$$\langle I'(u_n), u_n\rangle = \langle I'(u), u\rangle + \|v_n\|^2 + F(v_n) - \int_{\mathbb{R}^3} k(x)|v_n^+|^{2^*} dx - \mu\int_{\mathbb{R}^3} h(x)|v_n^+|^q dx + o(1).$$

Therefore it follows from Lemma 2.3, Lemma 2.5 and the hypotheses $I(u_n) \to c$ and $I'(u_n) \to 0$ in H^{-1} that

$$c = \lim_{n\to\infty} I(u_n) = I(u) + \lim_{n\to\infty} \frac{1}{2}\|v_n\|^2 - \lim_{n\to\infty} \frac{1}{2^*}\int_{\mathbb{R}^3} k(x)|v_n^+|^{2^*} dx, \tag{3.12}$$

and

$$\langle I'(u), u\rangle + \lim_{n\to\infty} \|v_n\|^2 - \lim_{n\to\infty}\int_{\mathbb{R}^3} k(x)|v_n^+|^{2^*} dx = 0. \tag{3.13}$$

Using (3.10) and (3.13) we obtain

$$\|v_n\|^2 - \int_{\mathbb{R}^3} k(x)|v_n^+|^{2^*} dx \to -\langle I'(u), u\rangle = 0.$$

Now we may assume that

$$\|v_n\|^2 \to b \quad \text{and} \int_{\mathbb{R}^3} k(x)|v_n^+|^{2^*} dx \to b.$$

By Sobolev's inequality we have

$$\|v_n\|^2 \geq \int_{\mathbb{R}^3} |\nabla v_n|^2 dx \geq \mathcal{S}\left(\int_{\mathbb{R}^3} |v_n^+|^{2^*} dx\right)^{2/2^*},$$

which means that

$$\int_{\mathbb{R}^3} k(x)|v_n^+|^{2^*} dx \leq \|k\|_\infty \int_{\mathbb{R}^3} |v_n^+|^{2^*} dx \leq \|k\|_\infty \left(\mathcal{S}^{-1}\|v_n\|^2\right)^{2^*/2},$$

i.e., $b \le \|k\|_\infty \left(\mathcal{S}^{-1}b\right)^{2^*/2}$. So we get that $b = 0$ or $b \ge \mathcal{S}^{\frac{N}{2}} \|k\|_\infty^{-\frac{N-2}{2}}$. Assume $b \ge \mathcal{S}^{\frac{N}{2}} \|k\|_\infty^{-\frac{N-2}{2}}$. Then combining (3.11) and (3.12), we obtain

$$c \ge \frac{1}{2}b - \frac{1}{2^*}b = \frac{1}{N}b \ge \frac{1}{N}\mathcal{S}^{\frac{N}{2}} \|k\|_\infty^{-\frac{N-2}{2}},$$

which contradicts the fact that $c < \frac{1}{N}\mathcal{S}^{\frac{N}{2}} \|k\|_\infty^{-\frac{N-2}{2}}$. Hence $b = 0$.

Step 2. For $4 \le q < 6$ and $\mu > 0$, we obtain that

$$\begin{aligned} c + 1 + \|u_n\| &\ge I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle \\ = \frac{1}{4}\|u_n\|^2 + \left(\frac{1}{4} - \frac{1}{2^*}\right)\int_{\mathbb{R}^3} k(x)|u_n^+|^{2^*} dx &+ \left(\frac{\mu}{4} - \frac{\mu}{q}\right)\int_{\mathbb{R}^3} h(x)|u_n^+|^q dx \ge \frac{1}{4}\|u_n\|^2, \end{aligned}$$

which implies that $(u_n)_{n\in\mathbb{N}}$ is bounded in H^1. To finish this step, we just need to replace (3.4) in Step 1 by the above inequality. The rest of the proof is similar to Step 1, so we omit it here. □

Lemma 3.2. *Suppose the hypotheses* (H) *hold. Then* $c < \frac{1}{N}\mathcal{S}^{\frac{N}{2}} \|k\|_\infty^{-\frac{N-2}{2}}$.

Proof. The idea here is to find a path in Γ such that the maximum of the functional I at this path is strictly less than $\frac{1}{N}\mathcal{S}^{\frac{N}{2}} \|k\|_\infty^{-(N-2)/2}$. To construct this path, we need the extremal function u_{ε,x_0} for the embedding $D^1 \hookrightarrow L^6$, where

$$u_{\varepsilon,x_0} = C\frac{\varepsilon^{1/4}}{(\varepsilon + |x - x_0|^2)^{1/2}}.$$

Here C is a normalizing constant and x_0 is given in (H_{k_2}). Let $\varphi \in C_0^\infty$ be such that $0 \le \varphi \le 1, \varphi|_{B_{R_2}} \equiv 1$ and supp $\varphi \subset B_{2R_2}$ for some $R_2 > 0$. Set $v_\varepsilon = \varphi u_{\varepsilon,x_0}$ and then $v_\varepsilon \in H^1$ with $v_\varepsilon(x) \ge 0$ for each $x \in \mathbb{R}^3$. The following asymptotic estimates hold if ε is small enough (see Brézis-Nirenberg [8]):

$$\|\nabla v_\varepsilon\|_2^2 = k_1 + O(\varepsilon^{\frac{1}{2}}), \quad \|v_\varepsilon\|_{2^*}^2 = k_2 + O(\varepsilon), \tag{3.14}$$

$$\|v_\varepsilon\|_s^s = \begin{cases} O(\varepsilon^{\frac{s}{4}}) & s \in [2,3), \\ O(\varepsilon^{\frac{s}{4}}|\ln \varepsilon|) & s = 3, \\ O(\varepsilon^{\frac{6-s}{4}}) & s \in (3,6), \end{cases} \tag{3.15}$$

with $k_1/k_2 = \mathcal{S}$, and $2 \le s < 2^*$. We know the path $tv_\varepsilon \in \Gamma$. For the rest, we will prove

$$\max_{t\ge 0} I(tv_\varepsilon) < \frac{1}{N}\mathcal{S}^{\frac{N}{2}} \|k\|_\infty^{-(N-2)/2} \tag{3.16}$$

for small ε. Since $I(tv_\varepsilon) \to -\infty$ as $t \to \infty$, there exists $t_\varepsilon > 0$ such that $I(t_\varepsilon v_\varepsilon) = \max_{t\ge 0} I(tv_\varepsilon)$. Also by Lemma 2.7, $\max_{t\ge 0} I(tv_\varepsilon) \ge \alpha > 0$. Then we have $I(t_\varepsilon v_\varepsilon) \ge \alpha > 0$. Thus from the continuity of I, we may assume that there exists some positive t_0 such that $t_\varepsilon \ge t_0 > 0$.

Moreover from $I(tv_\varepsilon) \to -\infty$ as $t \to \infty$ and $I(t_\varepsilon v_\varepsilon) \geq \alpha > 0$, we get that there exists T_0 such that $t_\varepsilon \leq T_0$. Hence $t_0 \leq t_\varepsilon \leq T_0$. Let $I(t_\varepsilon v_\varepsilon) = A(\varepsilon) + B(\varepsilon) + C(\varepsilon)$, where

$$A(\varepsilon) = \frac{t_\varepsilon^2}{2} \int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx - \frac{t_\varepsilon^{2^*}}{2^*} \int_{\mathbb{R}^3} k(x_0)|v_\varepsilon|^{2^*} dx,$$

$$B(\varepsilon) = \frac{t_\varepsilon^{2^*}}{2^*} \int_{\mathbb{R}^3} k(x_0)|v_\varepsilon|^{2^*} dx - \frac{t_\varepsilon^{2^*}}{2^*} \int_{\mathbb{R}^3} k(x)|v_\varepsilon|^{2^*} dx,$$

and

$$C(\varepsilon) = \frac{t_\varepsilon^2}{2} \int_{\mathbb{R}^3} |v_\varepsilon|^2 dx + \frac{t_\varepsilon^4}{4} F(v_\varepsilon) - \frac{t_\varepsilon^2 \mu}{2} \int_{\mathbb{R}^3} h(x)|v_\varepsilon|^q dx,$$

since $v_\varepsilon^+ = v_\varepsilon$. First, we claim that

$$A(\varepsilon) \leq \frac{1}{N} \mathcal{S}^{\frac{N}{2}} ||k||_\infty^{-\frac{N-2}{2}} + C\varepsilon^{1/2}. \tag{3.17}$$

Indeed, let $g(t) = \frac{t^2}{2} \int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx - \frac{t^{2^*}}{2^*} \int_{\mathbb{R}^3} k(x_0)|v_\varepsilon|^{2^*} dx$. It is clear that $g(t)$ achieves its maximum value at some T_ε. So

$$0 = g'(T_\varepsilon) = T_\varepsilon \int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx - T_\varepsilon^{2^*-1} \int_{\mathbb{R}^3} k(x_0)|v_\varepsilon|^{2^*} dx.$$

That is,

$$T_\varepsilon = \left(\frac{\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx}{\int_{\mathbb{R}^3} k(x_0)|v_\varepsilon|^{2^*} dx} \right)^{\frac{1}{2^*-2}}.$$

Therefore, from (3.14), we have

$$g(T_\varepsilon) = \sup_{t \geq 0} g(t) = \frac{1}{N} \frac{\left(\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx \right)^{N/2}}{\left(\int_{\mathbb{R}^N} k(x_0)|v_\varepsilon|^{2^*} dx \right)^{N-2/2}} = \frac{1}{N} \mathcal{S}^{\frac{N}{2}} ||k||_\infty^{-\frac{N-2}{2}} + C\varepsilon^{1/2}.$$

Then (3.17) follows. Secondly, we claim that $B(\varepsilon) \leq C\varepsilon^{1/2}$. In fact, since $t_0 \leq t_\varepsilon \leq T_0$ and $k \in L^\infty$, by the definition of v_ε, (H_{k_2}) and using a change of variables with $1 \leq \alpha < 3$, we have

$$\begin{aligned}
B(\varepsilon) &= \frac{t_\varepsilon^{2^*}}{2^*} \int_{\mathbb{R}^3} (k(x_0) - k(x))|v_\varepsilon|^{2^*} dx \\
&\leq C\delta_1 \int_{|x-x_0|<\rho_1} \frac{|x-x_0|^\alpha \varepsilon^{3/2}}{(\varepsilon + |x-x_0|^2)^3} dx + C \int_{|x-x_0| \geq \rho_1} \frac{\varepsilon^{3/2}}{(\varepsilon + |x-x_0|^2)^3} dx \\
&\leq C\delta_1 \varepsilon^{\frac{3}{2}} \int_0^{\rho_1} \frac{r^{2+\alpha}}{(\varepsilon + r^2)^3} dr + C\varepsilon^{\frac{3}{2}} \int_{\rho_1}^\infty r^{-4} dr \\
&= C\delta_1 \varepsilon^{\frac{\alpha}{2}} \int_0^{\rho_1 \varepsilon^{-\frac{1}{2}}} \frac{\rho^{2+\alpha}}{(1+\rho^2)^3} d\rho + C\rho_1^{-3} \varepsilon^{3/2} \\
&\leq C\delta_1 \varepsilon^{\frac{\alpha}{2}} + C\varepsilon^{3/2} \leq C\varepsilon^{\frac{1}{2}}.
\end{aligned}$$

So we have proved our claim. Therefore, to finish the proof, it is enough to show

$$\lim_{\varepsilon \to 0^+} \frac{C(\varepsilon)}{\varepsilon^{1/2}} = -\infty. \tag{3.18}$$

Actually, from the definition of v_ε, (H_{h_2}) and for any ε such that $0<\varepsilon\le\rho_2^2$, it follows that

$$\begin{aligned}\int_{\mathbb{R}^3} h(x)|v_\varepsilon|^q dx &\ge C\delta_2\int_{|x-x_0|<\rho_2}\frac{|x-x_0|^{-\beta}\varepsilon^{q/4}}{(\varepsilon+|x-x_0|^2)^{q/2}}dx+\int_{|x-x_0|\ge\rho_2}h(x)|v_\varepsilon|^q dx\\ &\ge C\delta_2\varepsilon^{q/4}\int_0^{\rho_2}\frac{r^2}{r^\beta(\varepsilon+r^2)^{q/2}}dr\\ &= C\delta_2\varepsilon^{\frac{3}{2}-\frac{q}{4}-\frac{\beta}{2}}\int_0^{\rho_2\varepsilon^{-\frac{1}{2}}}\frac{\rho^2}{\rho^\beta(1+\rho^2)^{q/2}}d\rho\\ &\ge C\delta_2\varepsilon^{\frac{3}{2}-\frac{q}{4}-\frac{\beta}{2}}\int_0^1\frac{\rho^2}{2^q\rho^\beta}d\rho = C\varepsilon^{\frac{3}{2}-\frac{q}{4}-\frac{\beta}{2}}.\end{aligned}$$

Therefore, by the fact that $t_0\le t_\varepsilon\le T_0$ and hypothesis (H_l), we have

$$\begin{aligned}C(\varepsilon) &= \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^3}|v_\varepsilon|^2dx+\frac{t_\varepsilon^4}{4}F(v_\varepsilon)-\frac{t_\varepsilon^2\mu}{2}\int_{\mathbb{R}^3}h(x)|v_\varepsilon|^q dx\\ &\le C\|v_\varepsilon\|_2^2+C\|v_\varepsilon\|_{12/5}^4-\mu C\varepsilon^{\frac{3}{2}-\frac{q}{4}-\frac{\beta}{2}}\\ &\le C\varepsilon^{\frac{1}{2}}+C\varepsilon-\mu C\varepsilon^{\frac{3}{2}-\frac{q}{4}-\frac{\beta}{2}}.\end{aligned}$$

It follows from $2-\frac{q}{2}<\beta<3$ that for fixed μ we have

$$\frac{C(\varepsilon)}{\varepsilon^{1/2}}\le C+C\varepsilon^{\frac{1}{2}}-\mu C\varepsilon^{1-\frac{q}{4}-\frac{\beta}{2}}\to-\infty,\ \text{as}\ \varepsilon\to 0.$$

So we prove the claim (3.18). Therefore (3.16) follows. □

Proof of Theorem 1.4. It follows from Lemma 3.1 and Lemma 3.2 that the functional I satisfies the $(PS)_c$ -condition at the level c defined by (3.1). And by Lemma 2.7, the functional I has the Mountain Pass geometry. Hence the functional I has a critical value $c>0$. That is, there exists a nontrivial $u\in H^1$ such that $I'(u)=0$, which means that (u,ϕ_u) is the nontrivial solution of system (1.1).

Since $0=\langle I'(u),u^-\rangle=\|u^-\|^2+\int_{\mathbb{R}^3}l(x)\phi_u|u^-|^2dx\ge\|u^-\|^2$, then $u\ge 0$ in $\mathbb{R}^3$. By standard arguments as in DiBenedetto [19] and Tolksdorf [28], we have that $u\in L^\infty$ and $u\in C^{1,\gamma}_{loc}$ with $0<\gamma<1$. Furthermore, by Harnack's inequality (see Trudinger [29]), $u(x)>0$ for any $x\in\mathbb{R}^3$. Thus (u,ϕ_u) is a positive solution of system (1.1). □

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