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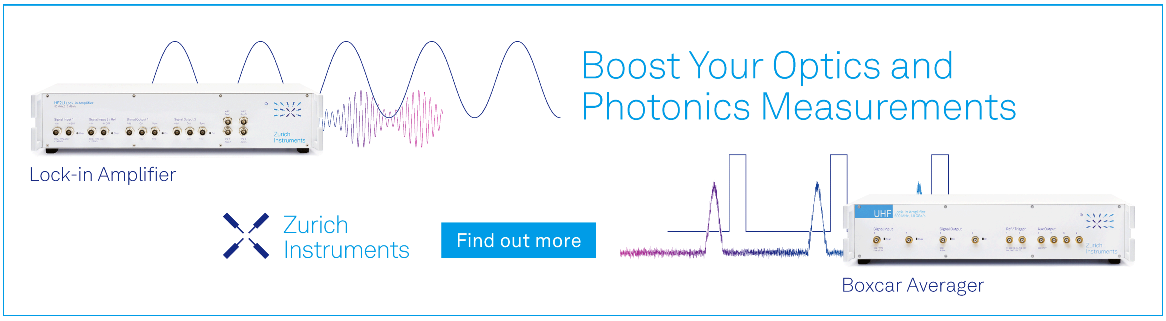

Frontier Analysis on Optimal Inspection Times for Reinforced Concrete Structures

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Abstract. Reinforced concrete is one of the most predominant materials in infrastructures throughout the world, hence management of the structures maintenance is an important subject of study in order to slow down the degradation process and to extend a structure lifespan. This work focus on a numerical model to find the best inspection planning for concrete structures under carbonation-induced corrosion, decomposed in three steps: (a) estimate the probability density function for critical failure time, (b) determine the optimal times for inspections sequences maximising its detectability; (c) multi-objective optimisation via frontier analysis to have a final relative ranking of inspection sequences. In short, this work answer the questions: when the inspections should be done and what are the best inspections for a given efficiency metric, under a set of parameters characterising the degradation process and inspection capabilities. Our formulation extend the models in the literature (e.g. detectability function) and its seems that the multidirectional efficiency analysis algorithm used here is better suited for real situations.

MATHEMATICAL FORMULATION

The degradation of infrastructures requires an analysis highly governed by a mixture of stochastic processes and optimisation problems; hence their study require good mathematical models and accurate numerical techniques to obtain the results. Its importance in structure maintenance and cost reduction despite its high complexity makes inspection planning an active subject of research with great impact in budget and structure lifespan. This research addresses the capability of each inspection method (detectability) and the probability of early detection of damage, see [1, 2, 3, 4]; a new formulation, developed from other studies, are proposed to give a decision-making model more suitable for real cases; and answer the questions: when the inspections should be done and what are the best inspections for a given efficiency metric, under a set of parameters characterising the degradation process and inspection capabilities.

In what follows, we formulate the steps of the algorithm and briefly describe the models proposed to attain our aims. Theoretical results are kept to the minimum for presentation reasons.

§1. **Probability density function for the time failure (Step 1).** In this step, we obtain the probability function $\psi_{T_f}(t)$ for the time of failure T_f to attain a fixed critical damage η_{cr}.

§1.1 *Damage Degree.* Let $D_0 \in (0, +\infty)$ be the initial bar diameter (cm), $D(t) \in [0, D_0]$ the rebar diameter (cm), $V_{corr} \in [0, 1]$ the corrosion rate (cm/year), $T_{icorr} \in [0, T)$ the time of corrosion initiation (years), and $T \in (0, +\infty)$ is the structure lifespan. The degradation mechanism is represented as a function measuring the corrosion damage degree $\eta(t)$, at the time t, as follows

$$\eta(t) = \begin{cases} 0 & \text{, if } t \le T_{icorr} \\ \frac{D_0 - D(t)}{D_0} & \text{, if } t > T_{icorr} \end{cases} \quad \text{with} \quad D(t) = D_0 - 2\, V_{corr}\, (t - T_{icorr}). \tag{1}$$

§1.2 *Time to Attain η_{cr} Failure and Probability Density Function $\psi_{T_f}(t)$.* Considering the problem under the structural reliability theory, the probability of failure is obtained as $P_f = P[g(\eta_{(t)}) \le \eta_{cr}]$, where $g(\eta_{(t)})$ is the limit state function

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that defines the failure of concrete due to corrosion damage degree. Hence, from (1), the limit state function can be expressed by $\eta(T_f) = \eta_{cr}$, so $T_f = \frac{D_0\,\eta_{cr}}{2\,V_{corr}} + T_{icorr}$ where η_{cr} is the critical damage degree, usually having the value 0.25 in the literature. In practice, no deterministic values are realistic in practical applications, so D_0, V_{corr} and T_{icorr} are assumed to be random variables, usually following a lognormal distribution. In this way, the probability density function (PDF) of T_{icorr} can be calculated by a Monte Carlo method or by applying a numerical method to

$$\psi_{T_f}(t) = \frac{2}{\eta_{cr}} \int_{-\infty}^{+\infty} \int \psi_{icorr}(t-\zeta)\,\psi_{D_0,V_{corr}}(\zeta\,\theta,\theta)\,|\theta|\,d\theta\,d\zeta,$$

where $\psi_{D_0,V_{corr}}$ is the join PDF (e.g. by using the composition of an algorithm for numerical calculating convolutions with an algorithm for the product of pdf's).

§2. **Optimal inspection sequence times (Step 2).** This step associate to each inspection sequence a set of times, where the sequence has the maximum probability of detection before failure. So, answer the question which are the best time to do the inspections.

§2.1 *Inspection Methods and their Detectability Function.* Suppose we may use a inspection techniques θ from a given set $\theta \in \Theta = \{A, B, C, \ldots\}$. The quality of an inspection technique θ is usually characterised by a probability of detection function (i.e. detectability of θ) which depends on $\eta^{\theta}_{0.5}$ and σ^{θ}, where $\eta^{\theta}_{0.5}$ is the damage intensity at which the inspection technique has a 50% probability of detection, and σ^{θ} is the standard deviation. The detectability function of θ may be modelled in several ways depending on the deterioration mechanism and building structure. For corrosion damage the most common formula in the literature (see [2]) is the following left hand side expression

$$P^{\theta}_{D}(\eta) = \begin{cases} 0 & ,\text{if } \eta \le \eta^{\theta}_{min}, \\ \Phi\left(\frac{\eta-\eta^{\theta}_{0.5}}{\sigma^{\theta}}\right) & ,\text{if } \eta^{\theta}_{min} < \eta \le \eta^{\theta}_{max}, \\ 1 & ,\text{if } \eta > \eta^{\theta}_{max}. \end{cases} \quad \longrightarrow \quad P^{\theta}_{D}(\eta) = \begin{cases} p_1(\eta) & ,\text{if } \eta \le \eta^{\theta}_{min}, \\ \Phi\left(\frac{\eta-\eta^{\theta}_{0.5}}{\sigma^{\theta}}\right) & ,\text{if } \eta^{\theta}_{min} < \eta \le \eta^{\theta}_{max}, \\ p_2(\eta) & ,\text{if } \eta > \eta^{\theta}_{max}, \end{cases} \tag{2}$$

where Φ is the standard normal cumulative distribution function, $\eta \in [0, +\infty)$ is the damage degree and $\eta^{\theta}_{min}, \eta^{\theta}_{max}$ are given values. Note that the function P^{θ}_{D} is not continuous, since $0 < \Phi(x) < 1$ for $x \in \mathbb{R}$, $\lim_{x\to-\infty} \Phi(x) = 0$, and $\lim_{x\to+\infty} \Phi(x) = 1$. When there is no experimental based values for the parameters $\eta^{\theta}_{0.5}$ and σ^{θ}, it's common to consider $\sigma^{\theta} = 0.1\eta^{\theta}_{0.5}$, $\eta^{\theta}_{min} = 0.7\eta^{\theta}_{0.5}$, and $\eta^{\theta}_{max} = 1.3\eta^{\theta}_{0.5}$, then the discontinuity gap is quite mild but additionally, for each fixed $\eta \in (\eta_{min}, \eta_{max})$, we have $P^{\theta}_{D}(\eta) = \Phi\left(\frac{\eta-\eta^{\theta}_{0.5}}{0.1\eta^{\theta}_{0.5}}\right) = \Phi\left(\frac{\alpha^{\theta}-1}{0.1}\right)$ with $\eta = \alpha^{\theta}\eta^{\theta}_{0.5}$, from where we conclude that if $\eta^{\theta_1}_{0.5} \ge \eta^{\theta_2}_{0.5}$ then $P^{\theta_1}_{D}(\eta) \le P^{\theta_2}_{D}(\eta)$, because Φ is nondecreasing, so the best inspection method turn-out to be always the one with smaller $\eta^{\theta}_{0.5}$ value. A natural way to improve the detectability function of an inspection method θ is to consider the right handside expression in (2),where p_1, p_2 are polynomials of degree n verifying $p_1(0) = 0$, $p_1(\eta^{\theta}_{min}) = \Phi\left(\frac{\eta^{\theta}_{min}-\eta^{\theta}_{0.5}}{\sigma^{\theta}}\right)$, $p_2(\eta^{\theta}_{max}) = \Phi\left(\frac{\eta^{\theta}_{max}-\eta^{\theta}_{0.5}}{\sigma^{\theta}}\right)$, $p_2(1) = 1$. The simplest choice for p_1, p_2 is when they are linear polynomials and there is an even symmetry with respect to $\eta^{\theta}_{0.5}$, i.e. $n = 1$ and $\eta^{\theta}_{max} = 1 - \eta^{\theta}_{min}$. Another choice for p_1, p_2 is to assume they are cubic splines, so the resulting polynomials will agree in monotonicity and concavity at the boundary points of the middle part of P^{θ}_{D}, which is generate by the standard normal cumulative distribution function. Notice also that, using equation (1), we may see the detectability function of θ as a function of time by $\mathcal{P}^{\theta}_{D}(t) = P^{\theta}_{D}\left(\frac{2\,V_{corr}}{D_0}(t - T_{icorr})\right)$ for $t \ge T_{icorr}$.

§2.2 *Inspection Sequences and their Damage Detection before Failure.* From the set of inspection methods $\Theta = \{A, B, C, \ldots\}$, we define the set $\mathcal{S}_{N_S} = \{A, B, C, AA, AB, \ldots\}$ of all possible combinations of elements of Θ up to N_S elements. For any sequence $\rho \in \mathcal{S}$, we define $|\rho|$ as the number of methods in the sequence, and ρ_i as the method which lies at the position $i \in \{1, \ldots, |\rho|\}$, e.g. $\rho = ABC$, $|\rho| = 3$, and $\rho_2 = B$. The detection in time of corrosion damage is quite meaningful from the viewpoint of durability as well as life-cycle cost. The probability of damage detection before failure P^{ρ}_{DBF} of an inspection sequence $\rho \in \mathcal{S}$ is a function which mathematically may be described as

$$P^{\rho}_{DBF}(t_1, \ldots, t_{|\rho|}) \quad = \quad \sum_{j=1}^{|\rho|} \left(\prod_{i=1}^{j} \mathcal{P}^{\rho_j}_{D}(t_j)\,(1 - \mathcal{P}^{\rho_{i-1}}_{D}(t_{i-1}))\, \left(1 - \int_{-\infty}^{t_i} \psi_{T_f}(\tau) d\tau\right) \right), \tag{3}$$

where, for notation simplicity, we assume $\mathcal{P}^{\rho_0}_{D}(t) \equiv 0$ and $t_0 = 0$.

§2.3 *Optimal Times for Inspection.* For each inspection sequence $\rho \in \mathcal{S}_{N_S}$, the aim is to compute the moments of time $\bar{t}_i \in [0, T]$, with $\bar{t}_1 < \bar{t}_2 < \cdots < \bar{t}_{|\rho|}$, where the inspection method ρ_i will be applied in order to maximize the total probability of the inspection sequence to detect corrosion before failure, hence we may write

$$PD^\rho \equiv P^\rho_{DBF}(\bar{t}_1, \ldots, \bar{t}_{|\rho|}) = \max_{(t_1,\ldots,t_{|\rho|}) \in [0,T]^{|\rho|}} P^\rho_{DBF}(t_1, \ldots, t_{|\rho|}) \quad s.t. \quad t_i - t_{i-1} \geq m_T \quad \text{and} \quad \sum_{i=1}^{|\rho|} t_i - t_{i-1} \leq T, \tag{4}$$

where $t_0 = 0$ and $m_T \in (0, T/N_S]$ is fixed as the minimum time admissible between consecutive inspections. The continuity of P^θ_D by using the right handside definition (2) is now relevant, together with $\psi_{T_f} \in C([0, T])$ and $[0, T]^{|\rho|}$ being a compact set, to ensure that the maximum in (4) is attained. Also notice that the optimal times $\bar{t}_i$ may not be unique.

§3. **Multi-objective Optimisation via Frontier Analysis (Step 3)** To overcome some of the limitations and further extend some techniques in the literature, we additionally apply benchmarking techniques to compare the (technical) efficiency of the inspection sequences, as the stochastic frontier analysis (SFA), and an improvement of the data envelopment analysis (DEA), i.e. the multidirectional efficiency analysis (MEA).

§3.1 *Cost of Inspection Sequences.* The cost to determine an structure intervention is directly related to the detectability of the inspection methods applied and the number of interventions performed during the service life of the structure. For a given inspection sequence $\rho \in \mathcal{S}_{N_S}$, the cost of the sequence at the moment $(t_1, \ldots, t_{|\rho|}) \in [0, T]^{|\rho|}$ is given by $C^\rho(t_1, \ldots, t_{|\rho|}) = C_0 \sum_{i=1}^{|\rho|} \frac{\alpha^{\rho_i}_{insp}}{(1+r)^{t_i}} (1 - \eta^{\rho_i}_{min})^{20}$ where $\alpha^\theta_{insp} \in (0, 1)$ is the percentage cost of the method $\theta \in \Theta$ as a fraction of the initial construction cost $C_0 > 0$, and $r \in \mathbb{R}$ is the annual discount rate, applied to obtain the present value of money for future investments.

§3.2 *Optimal Times for Detection versus Inspection Costs and Other Costs.* The optimal times for an inspection sequence ρ may be the time T^ρ is are a trade-off between the maximum probability of detection P^ρ_{DBF} and a minimum of cost C^ρ. The problem may be modelled as a vetor value constraint maximisation by $O_\rho(\bar{t}_1, \ldots, \bar{t}_{|\rho|}) = \max_{(t_1,\ldots,t_{|\rho|}) \in [0,T]^{|\rho|}} \left(P^\rho_{DBF}(t_1, \ldots, t_{|\rho|}), -C^\rho(t_1, \ldots, t_{|\rho|})\right)$, under the restrictions in (4). The above optimisation problem is usually solve by gradient free optimisation methods as genetic algorithms or non-dominated sorting in genetic algorithms (NSGA-III). The algorithms provide a Pareto optimal set of solutions which are optimum trade-offs between the two-objectives. Nevertheless, this approach do not consider resources needed for the implementation of the inspections or administrative indirect costs, neither give a relative orderer ranking which allows to determine which are the best sequences. In general, there are several other costs associated with the realisation of an inspection that many times can be associated to other variables which correlate well with such costs, which a priori are difficult to estimate. Here, as a demonstrative scenario, we consider two additional variables that we want to optimize: (i) N^ρ the number of different inspection methods in the sequence ρ (to minimise), and (ii) W^ρ the window of inspections of the sequence ρ (to maximise), which is defined as $W^\rho = T^\rho_{|\rho|} - T^\rho_1$, i.e. the elapse time between the optimal time for the first inspection up to the optimal time of the last inspection.

§3.3 *Frontier Analysis via SFA and MEA.* Here, we briefly state the MEA model used in this work, see the original model proposed in [5]. Let $[m]$ denote the set $\{1, ..., m\}$, for some $m \in \mathbb{N}$. From the previous steps, to any given sequence $\rho \in \mathcal{S}_{N_S}$ we may associate $J \in \mathbb{N}$ outputs $y_j(\rho)$, $j \in [J]$ and $I \in \mathbb{N}$ inputs $x_i(\rho)$, $i \in [I]$. Some of the input variables may be discretionary (i.e. their values can be changed) but others may be non-discretionary (i.e. they are fixed). From now on, the discretionary variables are represented by the first indices from 1 to $d \in [1, I]$. So, $x(\rho) \in \mathbb{R}^I$ is the vector of all the inputs and $y(\rho) \in \mathbb{R}^J$ is the vector of all the outputs. DEA/MEA model my change with respect to a chosen set of complementary variables. We consider the variable returns to scale (VRS) model, by defining the set $\Lambda^N = \left\{\lambda \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^N \lambda_n = 1 \wedge \lambda_n \geq 0\right\}$, where N is the number of sequences under study; although other possibilities exists. Then, the MEA score is found by solving the following linear optimisation problems

$$\begin{array}{lllll}
P^\alpha_m(\bar{\rho}): & \min \alpha_m(\bar{\rho}) & s.t. \quad \sum_\rho \lambda_\rho x_m(\rho) \leq \alpha_m(\bar{\rho}), & F(\bar{\rho}) \leq 0,\ i \in [I]\backslash\{m\}, & G(\bar{\rho}) \leq 0,\ l \in [J], \\
P^\beta_j(\bar{\rho}): & \max \beta_j(\bar{\rho}) & s.t. \quad \sum_\rho \lambda_\rho y_s(\rho) \geq \beta_j(\bar{\rho}), s \in [J], & F(\bar{\rho}) \leq 0,\ i \in [I], & G(\bar{\rho}) \leq 0, l \in [J]\backslash\{j\}, \\
P^\gamma(\alpha^*, \beta^*, \bar{\rho}): & \max \gamma(\bar{\rho}) & s.t \quad F(\bar{\rho}) \leq -\gamma(\bar{\rho})(x_i(\bar{\rho}) - \alpha^*_i(\bar{\rho})), i \in [I], & F(\bar{\rho}) \leq 0, i \in [I] \setminus \{m\}, & G(\bar{\rho}) \geq \gamma(\bar{\rho})(\beta^*_l(\bar{\rho}) - y_l(\bar{\rho})), l \in [J].
\end{array}$$

where $F(\bar{\rho}) = -x_i(\bar{\rho}) + \sum_\rho \lambda_\rho x_i(\rho)$, $G(\bar{\rho}) = -y_l(\bar{\rho}) + \sum_\rho \lambda_\rho y_l(\rho)$, $\lambda \in \Lambda^n$, $\alpha^*_m(\bar{\rho})$, $\beta^*_j(\bar{\rho})$ and $\gamma^*_m(\bar{\rho})$ are the optimal solutions to the problems $P^\alpha_m(\bar{\rho})$, $P^\beta_j(\bar{\rho})$ and $P^\gamma(\alpha^*, \beta^*, \bar{\rho})$, respectively. The MEA score of ρ can be obtained by

$$MEA(\rho) = \left(\frac{1}{\gamma^*(\rho)} - \frac{1}{D} \sum_{i=1}^{D} \frac{x_i(n) - \alpha^*_i(\rho)}{x_i(\rho)}\right)\left(\frac{1}{\gamma^*(\rho)} + \frac{1}{J} \sum_{j=1}^{J} \frac{\beta^*_j(\rho) - y_j(\rho)}{y_j(\rho)}\right)^{-1} \in [0, 1]. \tag{5}$$

A simple and raw rule to decide which are the inputs versus outputs is to consider as inputs, the variables to minimise, and as outputs, the variables to maximize; or consider its complementary values, e.g. for the cost we use the variable $CC^{\rho} = -C^{\rho} + \max_{\rho} C^{\rho}$ instead. Therefore, the best inspection sequences are the ones with higher MEA score.

NUMERICAL SIMULATION AND CONCLUSIONS

A combined Matlab and R package was developed to do the numerical analysis of the methodology presented. As a demonstrative example, we present here the results for $T = 80$ years, $C_0 = 1000$ cost units, $r = 0.055$ and $\eta_{cr} = 0.25$. We also assume that the same inspection method cannot be used consecutively. The probability density function for T_f was obtained from a Monte Carlo simulation method, where the randomness of the parameters are assumed as is shown in Table 1, a sample size of 50000 and subdividing the select sampling region into 1000 even spaced regions. The figure below shows the plots of the inspection methods detectability, see (2), for the values in Table 2.

TABLE 1. Parameters for generating T_f.

Variable(s)	Units	Value(s) or Distribution
D_0	cm	$LogN$, $\mu = 2.5$, $\sigma = 0.020$
T_{icorr}	years	$LogN$, $\mu = 3.84$, $\sigma = 1.200$
V_{corr}	cm/year	$LogN$, $\mu = 0.0065$, $\sigma = 0.0015$

TABLE 2. Parameters of the inspection methods.

Inspection Method	$\eta_{0.5}$	σ	η_{max}	α_{insp}
A	0.15	0.035	0.95	0.003
B	0.18	0.030	1.00	0.004
C	0.05	0.005	0.55	0.003
D	0.08	0.03	0.75	0.004

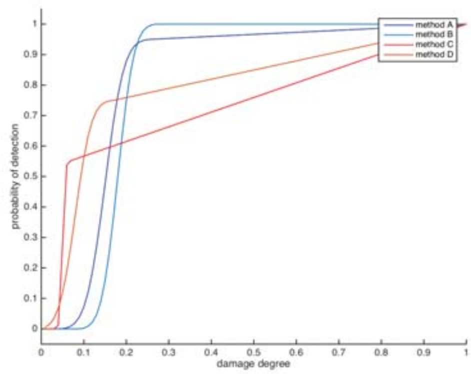

TABLE 3. SFA and MEA results.

InspSeq	t_1	t_2	t_3	N^{ρ}	W^{ρ}	PD^{ρ}	C^{ρ}	SFA(ρ)	MEA(ρ)
B	45.77	0.000	0.000	1	0.000	0.812	0.052 (1)	0.859	1.000 (1)
A	43.425	0.000	0.000	1	0.000	0.862	0.117 (2)	0.903	1.000 (1)
CB	39.728	45.770	0.000	2	6.042	0.926	0.228	0.919	1.000 (1)
CBC	39.733	45.146	46.646	2	6.913	0.966	0.350	0.953	1.000 (1)
DBC	38.502	45.147	46.647	3	8.145 (1)	0.979	0.796	0.943	1.000 (1)
DBD	38.502	44.717	46.217	2	7.715 (3)	0.984	1.088	0.967	1.000 (1)
DB	38.489	45.770	0.000	2	7.281	0.953	0.674	0.939	0.500 (2)
CDB	37.899	39.399	45.769	3	7.870 (2)	0.981	0.838	0.947	0.401 (3)
DAB	38.493	43.425	45.770	3	7.277	0.994 (2)	0.791	0.961	0.342
ADA	41.492	42.992	44.492	2	3.000	0.994 (2)	0.728	0.999 (2)	0.162
ABA	42.881	44.381	45.881	2	3.000	0.995 (1)	0.279	1.000 (1)	0.150
BAB	43.447	44.947	46.447	2	3.000	0.994 (2)	0.217	0.998 (3)	0.148
BA	43.985	45.485	0.000	2	1.500	0.967	0.162 (3)	0.979	0.129
ADB	42.59	44.090	45.771	3	3.182	0.993 (3)	0.635	0.979	0.098

In Table 3, the first three relative ranks for each variable is presented in brackets. It's clear that our approach is broader in scope and the best sequences obtained from the standard methods are suboptimal when taken in consideration further variables. Also SFA seems to be a not appropriate frontier analysis tool in the scope of inspection planning for reinforced concrete structures. Several other conclusions and results (e.g. monotonicity ranking behaviour, error estimations and sensitivity analysis) may be derived from our approach but, for space reasons, such will be discussed in another publication.

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