Universidade de Aveiro
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Marques da Costa

Conceito de s-numbers na Análise Quaterniónica

Concept of s-numbers in Quaternionic Analysis

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"*Nicht die Neugierde, nicht die Eitelkeit, nicht die Betrachtung der Nützlichkeit, nicht die Pflicht und Gewissenhaftigkeit, sondern ein unauslöschlicher, unglücklicher Durst, der sich auf keinen Vergleich einläßt, führt uns zur Wahrheit.*"

— Georg Hegel

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Dissertação apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Mestre em Matemática e Aplicações, realizada sob a orientação científica do Doutor Uwe Kähler, professor catedrático do Departamento de Matemática da Universidade de Aveiro, e do Doutor Alberto Debernardi Pinos, investigador da Universitat Autònoma de Barcelona.
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o júri / the jury

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agradecimentos / acknowledgements

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Palavras Chave Análise Funcional Quaterniónica, Teoria de Operadores Quaterniónicos, Ideais de Operadores, Teoria Espectral Quaterniónica, s-numbers.

Resumo

O objetivo da tese apresentada é dividido em duas partes. Em primeiro lugar, o conceito de s-numbers na análise quaterniónica é discutido. Vários exemplos de funções de s-numbers são naturalmente extendidos. A noção de números nucleares também é introduzida e a unicidade dos s-números em espaços de Hilbert quaterniônicos é estabelecida.
Em segundo lugar, focamo-nos na teoria de ideais de operadores que atuam em algebras de operadores quaterniónicos. São fornecidos exemplos, como classes de Schatten sobre os quaterniões e discutem-se as relações entre diferentes ideais.
Para isso, adaptamos a abordagem de A. Pietsch ao contexto dos quaterniões, seguindo as ideias de F.Colombo e I.Sabadini.

Keywords

Quaternionic Functional Analysis, Quaternionic Operator Theory, Operator Ideals, Quaternionic Spectral Theory, s-numbers.

Abstract

The presented thesis' objective is twofold. Firstly, it explores the concept of s-numbers in quaternionic analysis and extends several examples of s-number functions. The notion of nuclear numbers is also introduced, and the uniqueness of s-numbers over quaternionic Hilbert spaces is established.
Secondly, it studies the theory of operator ideals of operators acting on quaternionic algebras. We provide examples, such as Schatten classes over the quaternions, and discuss the relationships between different ideals.
To this end we adapt A. Pietsch's approach to the quaternionic framework, following the ideas of F.Colombo and I.Sabadini.

Contents

CHAPTER 1

Introduction

The interest in quaternionic analysis can be attributed to the works of G. Birkhoff and J. von Neumann. Specifically, in their paper on the logic of quantum mechanics [6], they demonstrated that the Schrödinger equation can be formulated in both complex and quaternionic settings, giving rise to the theory of quaternionic quantum mechanics (QQM). Inspired by this, there has been a tendency to generalize classical theories of analysis to the quaternionic context. In this regard, our objective is to extend the concept of s-numbers and explore their implications for the theory of operator ideals over the algebra of the quaternions.

The classic s-numbers theory originates from to the works of E. Schmidt [45], where the concept of singular numbers of integral operators between Hilbert function spaces was introduced. Afterwards, in [43][44], J. von Neumann and R. Schatten extended this concept to the setting of compact operators between Hilbert spaces. Hereby, the n-th s-number, previously referred to as n-th singular value, was defined to be the n-th eigenvalue of $|S| = \sqrt{S^*S}$. More precisely, they defined the non-increasing sequence of those eigenvalues counted according to their algebraic multiplicities, $\lambda_n(|S|)$. These led to the introduction of Schatten classes

$$\mathfrak{S}_p(H) = \{T \in K(H) : \|\lambda_n(T)|\ell_p\| < \infty\}.$$

An axiomatic approach to the theory of s-numbers was introduced by A. Pietsch, enabling the extension of s-numbers to general Banach spaces. We utilize this axiomatization to further expand the theory into the quaternionic framework. Unlike in the Hilbert space setting, the s-numbers are not unique in a Banach space.

As mentioned previously, it is currently well established that s-functions are unique over complex Hilbert spaces. By employing the proposed axiomatizations along with the works of F. Colombo and I. Sabadini on quaternionic Spectral Theory, we are able to extend the uniqueness of s-numbers to the quaternionic setting.

Following the same reasoning behind the construction of Schatten classes, this approach allowed to extend the results of Neumann/Schatten to a general Banach Space. Consequently, various classes of compact operators were obtained, each of which classify the behaviour of certain compact operators with respect to a s-number function. By following this methodology, we derive the quaternionic counterpart of Schatten classes for compact operators acting between quaternionic Banach spaces, thereby establishing a classification for quaternionic compact operators.

Furthermore, A. Pietsch has also introduced an axiomatic approach to the theory of operator ideals. We extend these proposed axioms to the quaternionic setting and, similar to the classical theory, the aforementioned quaternionic Schatten classes give rise to operator ideals.

Hence, the objective of this work is twofold. Firstly, we aim to extend A. Pietsch's axiomatic approach to s-number theory to the quaternionic framework and explore its implications. Secondly, we generalize the theory of operator ideals in the sense of A. Pietsch to the quaternionic setting.

CHAPTER 2

Preliminaries

This section aims to present the quaternionic analogues of basic tools in functional analysis. Hereby, we begin by discussing the algebraic properties of quaternions and their consequences. Next, linearity and boundedness of operators on a quaternionic setting is discussed followed by required notions of duality. We introduce notation for sequence spaces and extend well-known results to the quaternionic case. We conclude the discussion of quaternionic functional analysis by discussion the quaternionic analogues of Hahn-Banach theorem and its consequences, which are crucial for future developments. Afterwards the notions of extensions and liftings of quaternionic Banach spaces is presented. We close the chapter with some considerations on quaternionic spectral theory, namely the construction of a projection valued spectral measure which will be an essential tool throughout the sequel.

2.1 The algebra of quaternions

For a comprehensive exposition on the algebra of quaternions, we refer to [15]. The following presentation is largely based on that source.

Definition 2.1.1. The quaternions, denoted by $\mathbb{H}$, are the real algebra generated by $\{1, i, j, k\}$ satisfying the following properties:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

Thus, an element $q \in \mathbb{H}$ can be written as $q = x_0 + x_1 i + x_2 j + x_3 k$, where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and i, j, k satisfy the above relations. Moreover, we define the real part, imaginary part, and norm of $q \in \mathbb{H}$ as:

$$\text{Re } q = x_0, \qquad \text{Im } q = x_1 i + x_2 j + x_3 k, \qquad \|q\| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

The conjugate of a quaternion is defined as $\overline{q} = \text{Re } q - \text{Im } q$, satisfying $\|q\| = \sqrt{q\overline{q}} = \sqrt{\overline{q}q}$. Consequently, the inverse of any nonzero element q is given by $q^{-1} = \frac{\overline{q}}{\|q\|^2}$. Thus, $\mathbb{H}$ is a skew field. Finally, we denote the unit sphere of purely imaginary quaternions a s

$$\mathbb{S} := \{q = x_1 i + x_2 j + x_3 k : \|q\|^2 = 1\}.$$

2.1.1 Some remarks on quaternionic matrices

As mentioned earlier, $\mathbb{H}$ is a skew field, more formally known as a division algebra. Furthermore, if commutativity is assumed then we obtain a commutative division algebra, i.e., a field. Therefore, the key distinction between real/complex structures and quaternionic structures lies solely in the absence of commutativity. Consequently, many arguments and concepts used in real or complex vector spaces that do not rely on commutativity still hold in the quaternionic algebra. For instance, considering quaternionic matrices A and B, the same arguments as the ones used in real or complex algebra can be employed to show the following inequalities:

$$\operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B) \quad \text{and} \quad \operatorname{rank}(AB) \leq \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}. \tag{2.1}$$

Although seemingly inoffensive, these modifications have profound implications. Namely, to guarantee the existence of eigenvalues one resorts to topological arguments, more precisely using homotopies, as seen in [48]. Moreover, as explained in [2], the classical notion of determinant is no longer well defined in quaternionic matrices. Several alternatives have been proposed, each of which with its own limitations. The most suitable for our purposes is the so-called *Study determinant*, Sdet. We briefly outline its construction. Any $n \times n$ quaternionic matrix M can be expressed as the sum of two complex matrices, i.e. $M = A + jB$[1]. This allows us to construct an homomorphism

$$\psi(M) = \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix}.$$

One then defines

$$\operatorname{Sdet} M = \det_{\mathbb{C}} \psi(M), \tag{2.2}$$

where the subscript notation indicates the underlying structure over which the determinant is calculated. It is important to note that, unlike the classic determinant, multilinearity in rows and columns is lost according this definition. Nevertheless, several other properties that hold in the classic theory still hold:

Proposition 2.1.2. *[2, pp. 9–10] For any $\mathbb{H}$-matrices A and B with suitable dimensions,*

1. *Sdet$(A) = 0$ if and only if A is singular;*
2. *Sdet$(AB) =$Sdet(A)Sdet(B);*
3. *If A' is obtained from A by adding a left-multiple of a row to another row or a right-multiple of a column to another column, then Sdet$(A') =$Sdet(A).*

Since, a priori, multiplying with a quaternion from the left and from the right does not yield to the same value we need to distinguish when the action of the quaternionic structure is taken from the left or from the right. As we will see, this phenomenon leads to the distinction between "one-sided definitions", where it suffices to consider the action of $\mathbb{H}$ just from one

[1] Recall that $ji = k$.

side (usually the right), and "two-sided definitions", where there is no significant difference from where the action of $\mathbb{H}$ is taken.

Nevertheless, mathematicians are still able to discover analogues of classic theories, such as the one presented in the following section.

2.2 Quaternionic Analysis

2.2.1 Preliminaries on quaternionic functional analysis

As mentioned previously we must have a clear distinction between left and right operations. This is done by considering two-sided vector spaces. A two-sided vector space is an abelian (additive) group V equipped with both a left and right action by the structure $\mathbb{K}$, satisfying the associativity condition $(xv)y = x(vy)$ for $x,\ y \in \mathbb{K}$ and $v \in V$. It is called a right vector space if it forms a vector space when considering the operation from the right. Moreover, it is said two sided if it is both a left and right vector space. Additionally, a right Banach space is a complete normed right vector space, while a two-sided Banach space denotes a left and right Banach space. As a shortcut we will often refer to a right Banach space space over $\mathbb{H}$ as a right $\mathbb{H}$-Banach space.

Linear and bounded operators

Let us turn our attention to operators. Consider a right vector space V over $\mathbb{H}$. An operator $T : V \to V$ is a right linear operator if it satisfies the following properties for all $s \in \mathbb{H}$ and $u, v \in V$:

$$T(u + v) = T(u) + T(v), \qquad T(us) = T(u)s.$$

The powers T^n are defined by the relations $T^0 = Id$ and $T^n = TT^{n-1}$. In particular, $(sId)(v) = (Ids)(v) = sv$. Left linear operators and two sided linear operators are defined in a similar manner. It is worth noting that whether T is a right and left linear operator there holds $aT = Ta$ for $a \in \mathbb{R}$.

To discuss the boundedness of operators, we introduce some additional notation. Let us V be a right $\mathbb{H}$-Banach space with norm $\|\cdot\|$. We denote $B^R(V)$ as the two-sided vector space of all right linear bounded operators on V, and $B^L(V)$ as the two-sided vector space of all left linear bounded operators on V. In analogy with the classic theory, it can be shown that $B^R(V)$ and $B^L(V)$ are Banach spaces when equipped with their natural norms [2]:

$$\|T\| := \sup_{v \in V} \frac{\|T(x)\|}{\|x\|}.$$

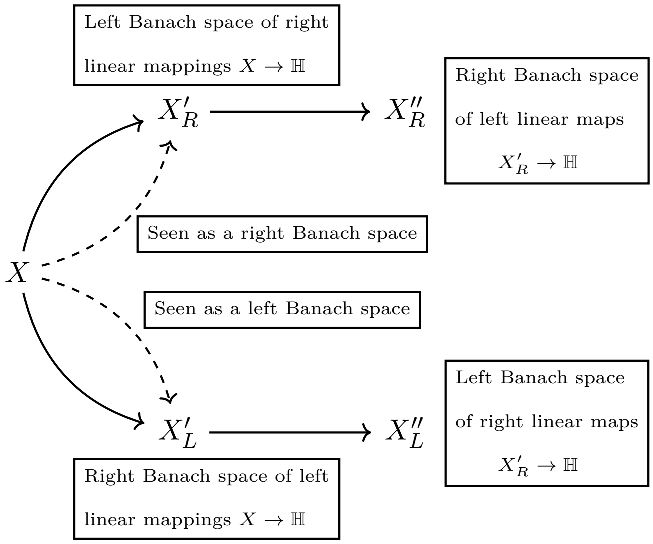
Lastly, when we do not differentiate between left or right linear bounded operators on V we simply use the symbol $B(V)$. Clearly, these concepts can be extended to $B(V, F)$, the set of linear and bounded operators acting between the $\mathbb{H}$-vector spaces V and F.

[2] The definition of a norm is analogous, however homogeneity of the norm is required only from the right or from the left, depending on the nature of the corresponding Banach space.

Duality

Consider a quaternionic right Banach space X_R. Its dual space, denoted as X'_R, is the quaternionic left Banach space consisting of all bounded right linear mappings from X_R to $\mathbb{H}$. Analogously, we define X'_L for a quaternionic left Banach space X_L. Thus, in the case of a two-sided quaternionic Banach space X, we distinguish between two dual spaces: the right dual X'_R, which is the dual space of X when regarded as a right Banach space, and the left dual X'_L, which is the dual space of X when regarded as a left Banach space. This notion allows us to define the bidual spaces of X, denoted as X''_R and X''_L.

If X is a quaternionic right Banach space its bidual will be the quaternionic right Banach space of all bounded left linear mappings from X'_R to $\mathbb{H}$. Similarly, if X is a quaternionic left Banach space its bidual will be the quaternionic left Banach space of all bounded right linear mappings from X'_L to $\mathbb{H}$. Moreover, if X is two-sided then we analogously have to distinguish between the left and right bidual.



In either case, the evaluation map K_X can be defined as follows:

$$K_X : X'_R \to \mathbb{K}, \quad \varphi \mapsto \langle \varphi, x \rangle \quad \text{or} \quad K_X : X'_L \to \mathbb{K}, \quad \varphi \mapsto \langle x, \varphi \rangle.$$

It defines a right (left) functional on X'_R (X'_L). It is a direct consequence of the definition of the operator norm that K_X is an isometry. Moreover, the symbol X' will be used to denote the dual space whenever there is no need to specify whether it is the left or right Banach space[3].

Once the dual spaces are suitably defined, the next step is to define the dual operator T' of an operator T.

Definition 2.2.1. Let X_R and Y_R be quaternionic right Banach spaces, and let $T \in B^R(X_R, Y_R)$. The dual operator of T, denoted as T', is a left linear operator that maps Y'_R to X'_R. Its action is defined as follows: for $\varphi \in Y'_R$, if $T'(\varphi) = \psi$, then $\psi(x) = \varphi(T(x))$.

Observe that, according to this definition, it follows that $T' \in B^L(Y'_R, X'_R)$.

In the case of Hilbert spaces, the dual operator coincides with is the adjoint operator, denoted by T^*.

[3]However, it is important to keep in mind that when we say $T \in B(X)$ and $\varphi \in X'$, it is implicit that the side on which the operations are performed on T and φ must match in order to correctly proceed with computations.

Sequence spaces

Let X be a right $\mathbb{H}$-Banach space. For an index set I, a family $x = (x_i)_{i\in I}$, for which $x_i \in X$, is said to be *absolutely p-summable* whenever $(\|x_i\|)_{i\in I} \in \ell_p(I)$, for $1 \le p < \infty$. The set of these families is denoted by $[\ell_p(I)]$. We define the norm as

$$\|x|\ell_p\| = \left(\sum_{i\in I} \|x_i\|^p\right)^{\frac{1}{p}}.$$

We reserve the notations $[\ell_p]$ and $\ell_p^n(I)$ for $[\ell_p(\mathbb{N})]$ and $\prod_{i=1}^n \ell_p(I)$, respectively. In analogy with to the classic theory of ℓ_p spaces, these spaces form right Banach spaces. Moreover, a family $x = (x_i)_{i\in I}$ is said to be weakly p-summable if $(\langle \varphi, x_i\rangle)_{i\in I} \in \ell_p(I)$ for every $\varphi \in X'$. The set of these families is denoted by $[w_p(I)]$. We define the norm

$$\|x|w_p(I)\| := \sup\{\|(\langle \varphi, x_i\rangle)_{i\in I}|\ell_p\| : \varphi \in B_{X'}\}$$

Similar notation shortcuts are used for these norms, and the resulting spaces are right Banach spaces as well.

The following results, which we extend to the quaternionic setting, can be found in [8].

Lemma 2.2.2. *Let $1 \le p, q < \infty$. For each operator $A \in B^R(\ell_p^n, \ell_q^n)$, and for all $x = (\xi_1, \dots, \xi_n) \in \ell_p^n$ with $\|x\|_p \le 1$, the following inequality holds:*

$$\left(\sum_k \left(\sum_i |\xi_i \langle Ae_i, e_k\rangle|^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} \le c(p)\|A\|,$$

where e_k denotes the k-th unitary sequence, i.e., $e_k = \delta_{ik}$.

Proof. By hypothesis on x we have

$$\sum_k \left|\sum_i \xi_i \langle Ae_i, e_k\rangle\right|^q = \sum_k |\langle Ax, e_k\rangle|^q = \|Ax\|_q^q \le \|A\|^q$$

which implies that for all $v = (\epsilon_1, \dots \epsilon_n)$ for with $\epsilon_i = \pm 1$, we have $\sum_k |\sum_i \epsilon_i \xi_i \langle Ae_i, e_k\rangle|^q \le \|A\|^q$. Since there are 2^n such vectors v, we can apply the arithmetic mean inequality to obtain

$$2^{-n} \sum_v \sum_k \left|\sum_i \epsilon_i \xi_i \langle Ae_i, e_k\rangle\right|^q \le \|A\|^q.$$

The desired claim follows from the Littlewood-Khintchin's inequality [4]. In particular, it allows us to write

$$\sum_k \left(\sum_i |\xi_i \langle Ae_i, e_k\rangle|^2\right)^{\frac{q}{2}} \le c(p) 2^{-n} \sum_k \sum_v \left|\sum_i \epsilon_i \xi_i \langle Ae_i, e_k\rangle\right|^q. \qquad \square$$

[4] cf. [26]. For $v = (\epsilon_i)$ with 2^n elements that are either 1 or -1, there exists a positive constant $c(p)$ such that $c(p) 2^{-\frac{n}{p}} \left(\sum_v \left|\sum_i \epsilon_i \xi_i\right|^p\right)^{\frac{1}{p}} \ge \|x|\ell_2\|$, for all $x = (\xi_1, \dots, \xi_n)$.

Lemma 2.2.3. *Let $A \in B^R(\ell_p^n, \ell_q^n)$. Then if $1 \le q \le 2 \le p \le \infty$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p} \ge \frac{1}{2}$ there holds*

$$\left(\sum_k \|Ae_k\|_2^r\right)^{\frac{1}{r}} \le c(p,q)\,\|A\|\,.$$

Proof. The case $r = 2$, with $p = \infty$ and $q = 2$ follows immediately from Lemma 2.2.2 by taking the sequence $(\xi_i) = (1, 1, \dots)$. Indeed

$$\left(\sum_i \|Ae_i\|_2^2\right)^{\frac{1}{2}} = \left(\sum_k \sum_i |\left\langle Ae_i, e_k\right\rangle|^2\right)^{\frac{1}{2}} \le c_2\,\|A\|\,.$$

Therefore, we can assume $1 \le q < 2$ and $r = 2$. Now, let $A \in B^R(\ell_p^n, \ell_q^n)$, and consider an arbitrary diagonal operator $D(\xi_i) = \lambda_i \xi_i$ from $B(\ell_\infty^n, \ell_p^n)$. We can define the operator $B = AD$. Then, by the above, there exists a positive constant c such that

$$\begin{aligned}\left(\sum_i \|\lambda_i Ae_i\|_2^q\right)^{\frac{1}{q}} &= \left(\sum_i \|Be_i\|_2^q\right)^{\frac{1}{q}} \\ &\le c\left\|B : \ell_\infty^n \to \ell_q^n\right\| \le c\left\|D : \ell_\infty^n \to \ell_p^n\right\|\left\|A : \ell_p^n \to \ell_q^n\right\| \\ &\le c\left(\sum_i |\lambda_i|^p\right)^{\frac{1}{p}}\left\|A : \ell_p^n \to \ell_q^n\right\|\end{aligned}$$

In particular, if we choose $\lambda_i = \|Ae_i\|_2^{\frac{2}{p}}$, we obtain:

$$\left(\sum_i \|\lambda_i Ae_i\|_2^q\right)^{\frac{1}{q}} \le c\left(\sum_i \|\lambda_i Ae_i\|_2^2\right)^{\frac{1}{p}}\left\|A : \ell_p^n \to \ell_q^n\right\|,$$

since

$$\left(\sum_i \|\lambda_i Ae_i\|_2^q\right)^{\frac{1}{q}} = \left(\sum_i \|Ae_i\|_2^2\right)^{\frac{1}{q}} \quad \text{and} \quad \left(\sum_i |\lambda_i|^p\right)^{\frac{1}{p}} = \left(\sum_i \|Ae_i\|_2^2\right)^{\frac{1}{p}}.$$

Therefore, for $\frac{1}{2} = \frac{1}{q} - \frac{1}{p}$, we have:

$$\left(\sum_i \|Ae_i\|_2^2\right)^{\frac{1}{2}} \le c\left\|A : \ell_p^n \to \ell_q^n\right\|.$$

For the case when $q < 2$, let $s = \frac{2-q}{2-p}$. Then,

$$\begin{aligned}\sum_k \|A'e_k\|_2^r &= \sum_k \sum_i |\left\langle e_i, A'e_k\right\rangle|^2\,\|A'e_k\|_2^{r-2} \\ &= \sum_k \sum_i |\left\langle Ae_i, e_k\right\rangle|^2\,\|A'e_k\|_2^{r-2} \\ &= \sum_k \sum_i |\left\langle Ae_i, e_k\right\rangle|^{\frac{2}{s}}\,\|A'e_k\|_2^{r-2}\,|\left\langle Ae_i, e_k\right\rangle|^{\frac{2}{s'}}.\end{aligned}$$

Applying Hölder's inequality leads to

$$\sum_k \|A'e_k\|_2^r \le \sum_i \left(\sum_k |\langle Ae_i, e_k\rangle|^2 \|A'e_k\|_2^{q-2}\right)^{\frac{1}{s}} \left(\sum_k |\langle Ae_i, e_k\rangle|^2\right)^{\frac{1}{s'}}$$
$$= \sum_i \left(\sum_k |\langle Ae_i, e_k\rangle|^2 \|A'e_k\|_2^{q-2}\right)^{\frac{1}{s}} \|Ae_i\|_2^{\frac{2}{s'}}.$$

Let $l = \frac{rs}{q}$. The same reasoning can be applied to obtain

$$\sum_k \|A'e_k\|_2^r \le \left(\sum_i \left(\sum_k |\langle Ae_i, e_k\rangle|^2 \|A'e_k\|_2^{q-2}\right)^{\frac{r}{q}}\right)^{\frac{1}{l}} \left(\sum_i \|Ae_i\|_2^r\right)^{\frac{1}{l'}}.$$

To further estimate the right-hand side consider the expression

$$\sum_i |\langle Ae_i, e_k\rangle|^2 \|A'e_k\|_2^{q-2} |\xi_i|^q = \sum_i |\langle Ae_i, e_k\rangle|^{2-q} \|A'e_k\|_2^{q-2} |\langle Ae_i, e_k\rangle \xi_i|^q.$$

Once again, applying Hölder's inequality with exponent $\frac{2}{2-q}$ yields

$$\sum_i |\langle Ae_i, e_k\rangle|^2 \|A'e_k\|_2^{q-2} |\xi_i|^q \le \left(\sum_i |\langle Ae_i, e_k\rangle|^2 \|A'e_k\|_2^{-2}\right)^{\frac{2-q}{2}} \left(\sum_i |\langle Ae_i, e_k\rangle \xi_i|^2\right)^{\frac{q}{2}}$$
$$= \left(\sum_i |\langle Ae_i, e_k\rangle \xi_i|^2\right)^{\frac{q}{2}}.$$

From Lemma 2.2.2 now follows, when $\sum |\xi_i|^p \le 1$

$$\sum_i |\langle Ae_i, e_k\rangle|^2 \|A'e_k\|_2^{q-2} |\xi_i|^q \le c(q)^q \|A\|^q.$$

The reverse Hölder inequality with $\frac{1}{r} + \frac{1}{p} = \frac{1}{q}$ gives

$$\left(\sum_i \left(\sum_k |\langle Ae_i, e_k\rangle|^2 \|A'e_k\|_2^{q-2}\right)^{\frac{r}{q}}\right)^{\frac{1}{r}} \le c(q) \|A\|$$

and thus finally

$$\sum_k \|A'e_k\|_2^r \le c(p')^{\frac{r}{l}} \|A\|^{\frac{r}{l}} \left(\sum_i \|Ae_i\|_2^r\right)^{\frac{1}{s'}}.$$

Applying these inequalities to the dual operator of A' yields

$$\sum_k \|Ae_k\|_2^r \le c(p')^{\frac{r}{t}} \|A\|^{\frac{r}{t}} \left(\sum_i \|A'e_i\|_2^r\right)^{\frac{1}{t'}}$$

with $t = \frac{r(2-p')}{p'(2-r)}$. Combining both estimates gives us the desired claim:

$$\left(\sum_k \|Ae_k\|_2^r\right)^{\frac{1}{r}} \le c(p')^{\frac{l't'}{t(l't'-1)}} c(q)^{\frac{l'}{l(l't'-1)}} \|A\|. \qquad \square$$

Quaternionic Hahn-Banach theorem

Next, we present a natural extension of the Hahn-Banach theorem to the quaternionic setting. More details can be found in [15, pp. 146–147].

Lemma 2.2.4. *Let V_0 be a right subspace of a right vector space on $\mathbb{H}$. Let ρ denote a seminorm on V, and let $\phi \in V_0'$ such that $|\langle \varphi, v\rangle| \leq \rho(v)$ for all $v \in V_0$. Then it is possible to extend Φ to $\phi \in V'$ for which the same inequality holds.*

Proof. We can write any functional ϕ as $\psi_1(\phi) + \psi_2(\phi)j$ with $\psi_1(\phi) = \phi_0 + \phi_1 i$ and $\psi_2(\phi) = \phi_2 + \phi_3 i$ which are complex functionals. It is immediate that, for all $v \in V_0$

$$\langle \phi, v\rangle = \langle \psi_1, v\rangle - \langle \psi_1, vj\rangle\, j.$$

As ψ_1 is a $\mathbb{C}-$linear functional, the classic Hahn-Banach can be applied to it to guarantee the existence of an extension, $\tilde{\psi_1}$ to V (as a complex vector space). The desired functional ψ is given by

$$\langle \Phi, v\rangle = \left\langle \tilde{\psi_1}, v\right\rangle - \left\langle \tilde{\psi_1}, vj\right\rangle j. \qquad \square$$

We conclude this topic with a lemma that is a direct consequence of the Hahn-Banach theorem. It will be fundamental in the subsequent discussion of a particular s-number, the isomorphism numbers, which we explore in Section 3.2.4.

Lemma 2.2.5. *[40, p. 203] Let $T \in B^R(X, Y)$ be such that rank$(T) \geq n$. Then there exists a Banach space G as well as operators $A \in B^R(G, X)$ and $B \in B^R(Y, G)$ such that $Id = BTA$ and dim$(G) \geq n$.*

The main idea of the proof is the following. By choosing $(x_i)_{i=1}^n \in X$ for which $(Tx_i)_{i=1}^n$ are linearly independent, Hahn-Banach theorem implies that there are $(b_i)_{i=1}^n \in Y'$ with $\langle Tx_i, b_k\rangle = \delta_{ik}$. The claim follows by taking, $G = \ell_2^n$, $A(\xi_i) = \sum_{i=1}^n \xi_i x_i$ for $\xi_i \in \ell_2^n$ and $By := (\langle y, b_i\rangle)$ for $y \in Y$.

Maps, Injections & Surjections

Let Y and X be sets, and let Y be a subset of Y_0 and X_0 be a closed subset of X. We denote the natural injection, also called embedding, from Y to Y_0 and the canonical surjection from X to the quotient space X/X_0 as follows:

$$J_{Y_0}^Y : Y \to Y_0, \quad \text{and} \quad Q_{X_0}^X : X \to {}^X/_{X_0}.$$

Furthermore, we use B_X to represent the closed unit ball of the set X.

The following definitions and corresponding results, which we adapt to the quaternionic setting, can be found [39, pp. 26–28]. Let $T \in B^R(X, Y)$. We define the *injection modulus* as

$$j(T) := \sup\{\tau \geq 0 : \|Tx\| \geq \tau \|x\|,\, \forall x \in X\}, \quad \text{and} \quad j(0) = 0.$$

An operator is called an *injection* if $j(T) > 0$. Moreover, if $\|T\| = j(T) = 1$, then T is said to be a *metric injection.* Analogously we define the *surjection modulus*

$$q(T) := \sup\{\tau \geq 0 : T(B_X) \supseteq \tau B_Y\}, \quad \text{and} \quad q(0) = 0.$$

An operator is called a *surjection* if $q(T) > 0$. If $\|T\| = q(T) = 1$ we say that T is a *metric surjection.* Note that in both cases of metric injection and metric surjection, we require the operator to be an isometry. The following results immediately follow from the definition.

Proposition 2.2.6. *The operator $T \in B(X, Y)$ is an injection if an only if it admits a factorization $T = JT_0$, where T_0 is an isomorphism and J denotes the embedding from $Y_0 := ran(T)$ into Y. In this case, $j(T) = \left\|T_0^{-1}\right\|^{-1}$.*

Proposition 2.2.7. *The operator $T \in B(X, Y)$ is a surjection if and only if it admits a factorization $T = T_0Q$, where T_0 is an isomorphism and Q denotes the canonical surjection from X onto $X_0 := X/ker(T)$. In this case, $q(T) = \left\|T_0^{-1}\right\|^{-1}$.*

The following result is of significant importance for the sequel. It enables us to significantly simplify our work in subsection 4.1.

Proposition 2.2.8. *Let $T \in B^R(X, Y)$. Then $q(T') = j(T)$ and $j(T') = q(T)$.*

Proof. We divide the proof into two steps. In the first step, we demonstrate that the surjection modulus does not depend on whether T is a closed operator or not. We utilize this in the second step to establish the desired equalities. The proof is addapted from [39, p. 26].

Step 1: Define $\overline{q}(T) = \sup \tau \geq 0 : \overline{T(B_X)} \supseteq \tau B_Y$. It is evident that $q(T) \leq \overline{q}(T)$. Without loss of generality, let us assume $\overline{q}(T) > 0$. Consider $y \in B_Y$, $0 < \epsilon < 1$, and set $\tau = (1-\epsilon)\overline{q}(T)$ and $y_1 = y$. Now, we construct a family of elements in X that will establish the remaining inequality. Inductively, choose $x_1, x_2, \cdots \in X$ such that

$$\left\|y_k - \tau^{-1}Tx_k\right\| \leq \epsilon^k \text{ and } \|x_k\| \leq \|y_k\|,$$

where $y_k = y - \sum_{i=1}^{k-1} \tau^{-1}Tx_i$. It is clear that $\|x_1\| \leq \|y_1\| \leq 1$, and

$$\|x_k\| \leq \|y_k\| = \left\|y_{k-1} - \tau^{-1}Tx_{k-1}\right\| \leq \epsilon^{k-1} \text{ for } k \geq 2.$$

Therefore, $\sum_{k=1}^{\infty} \|x_k\| \leq \frac{1}{1-\epsilon}$. Let $x = \sum_{k=1}^{\infty} x_k$. It follows that

$$y = \sum_{i=1}^{\infty} \tau^{-1}Tx_k = \tau^{-1}Tx,$$

and since $\|x\| \leq (1-\epsilon)^{-1}$, we have effectively shown that $T(B_X) \supseteq (1-\epsilon)\tau T(B_Y)$, which immediately implies

$$q(T) \geq (1-\epsilon)^2\overline{q}(T).$$

Therefore, $q(T) = \overline{q}(T)$.

Step 2: Consider $0 < \tau < j(T)$.

By definition, $\|Tx\| \geq \tau \|x\|$ for all $x \in X$. For every $a \in B_{X'}$, the equation

$$\langle y, b_0 \rangle = \left\langle T^{-1}y, a \right\rangle,$$

defines a functional b_0 on $\operatorname{ran}(T)$ with $\|b_0\| \leq \tau^{-1}$. Choose an extension $b \in Y'$ such that $\|b\| \leq \tau^{-1}$. Consequently, we have $\varphi = T'b$. Therefore, $T'(U_{Y'}) \supseteq \tau B_{X'}$, and hence $q(T') \geq j(T)$.
Considering $0 < \tau < q(T')$, for each $x \in X$, choose $\varphi \in B_{X'}$ such that $|\langle x, \varphi \rangle| = \|x\|$. Since $T'(U_{Y'}) \supseteq \tau B_{X'}$, we can find $b \in U_{Y'}$ with $T'b = \tau\varphi$. Consequently, we have

$$\|Tx\| \geq |\langle Tx, b \rangle| = |\langle x, T'b \rangle| = \langle x, \tau a \rangle = \tau \|x\|.$$

This implies that $j(T) \geq q(T')$, and thus $j(T) = q(T')$. Similarly, one can prove that $j(T') \geq q(T)$.
Finally, let us consider $0 < \tau < j(T')$. If there were to exist $y \in B_Y$ such that $\tau y \notin \overline{T(B_X)}$, then by the separation theorem, we could, in particular, find a functional $b \in F'$ such that $|\langle \tau y, b \rangle| > 1$ and $|\langle Tx, b \rangle| \leq 1$ for all $x \in B_X$. This would imply that

$$\|T'b\| = \sup\{|\langle Tx, b \rangle| : x \in U_E\} \leq 1 < |\langle \tau y, b \rangle| \leq \tau \|b\|,$$

which is a contradiction. Thus, $\tau B_Y \subseteq \overline{T(B_X)}$. By the first step, we conclude that $q(T) \geq j(T')$. $\square$

Corollary 2.2.9. *An operator $T \in B^R(X, Y)$ is a (metric) injection/surjection if and only if $T' \in B^R(Y', X')$ is a (metric) injection/surjection.*

Liftings & Extensions

This section is directly adapted from [39, pp. 33–34] as it is a purely topological definition.

$$\begin{tikzcd} X_0 \arrow[r, "S_0"] \arrow[d, "J"'] & Y \\ X \arrow[ur, "S"'] & \end{tikzcd}$$

A right $\mathbb{H}$-Banach space Y possesses the *extension property* if for every injection $J \in B^R(X_0, X)$ and every operator $S_0 \in B^R(X_0, Y)$ there exists an extension $S \in B^R(X, Y)$. The *metric extension property* means that for every metric injection $J \in B^R(X_0, X)$ and every operator $S_0 \in B^R(X_0, Y)$, we can find $S \in B^R(X, Y)$ such that $S_0 = SJ$ and $\|S\| = \|S_0\|$. We will denote $X^{inj} = \ell_\infty(B_{X'})$. Clearly J_X is a metric injection from X into X^{inj}. In this way every Banach space can be identified with a subspace of some Banach space having the metric extension property, as the following Lemma states.

Before proceeding, we require the notion of *complemented space*. A subspace M of X is called complemented if there exists a subspace N of X such that $X = M \bigoplus N$.

Lemma 2.2.10. *A Banach space has the extension property if and only if it is isomorphic to a complemented subspace of some Banach space $\ell_\infty(I)$.*

A Banach space Y has the *lifting property* if for each surjection $Q \in B^R(Y, Y_0)$ and all operators $S_0 \in B^R(X, Y_0)$ there is a lifting $S \in B^R(X, Y)$. *The metric lifting property* means that, given $\epsilon > 0$, for each metric surjection $Q \in B^R(Y, Y_0)$ and every operator $S_0 \in B^R(X, Y_0)$, we can find $S \in B^R(X, Y)$ such that $S_0 = QS$ and $\|S\| = (1+\epsilon)\,\|S_0\|$. We will denote $X^{sur} := \ell_1(U_X)$ and $Q_X(\xi_x) = \sum_{B_X} \xi_x x$ for $(\xi_x) \in \ell_1(B_X)$, which is metric surjection from X^{sur} onto X. Thus, each Banach space can be identified with a quotient space of some Banach space having the metric lifting property.

$$\begin{tikzcd} X \arrow[r, "S_0"] \arrow[d, "S"'] & Y_0 \\ Y \arrow[ur, "Q"'] & \end{tikzcd}$$

Lemma 2.2.11. *Let I be any index set. Then $\ell_1(I)$ has the metric lifting property.*

From Lemma 2.2.6 and Lemma 2.2.7, it follows that a Banach space Y possesses the extension property if, for every operator S_0 mapping a subspace M of an arbitrary Banach space X into Y, there exists an extension S from X into Y such that $\|S\| = \|S_0\|$ $(S = S_0 J_M^X)$. Analogously, a Banach space X possesses the lifting property if, for every operator S_0 mapping X into a quotient space Y/N of an arbitrary Banach space Y, and for every $\epsilon > 0$, there exists a lifting S from X into Y such that $|S| \leq (1+\epsilon)|S_0|$ $(Q_N^F S = S_0)$.

2.3 Preliminaries on quaternionic Spectral Theory

Although the theory that we will presented is relatively recent, efforts to extend the spectral theorem for normal operators in quaternionic Hilbert Spaces dates to around 1930. Among several papers, O. Teichmüller in [46] was the first to present a spectral theorem for normal operators, even in the absence of a clear notion of a spectrum. In his work, a representation of a normal operator T, was obtained as follows:

$$T = \int_{\mathbb{R}} \int_{\mathbb{R}_0^+} (\lambda' + J_0 \lambda'') dQ_{\lambda''} dP_{\lambda'},$$

where J_0 satisfies $J_0 J_0^* = Id_{\text{ran}B}$ and $J_0^* = -J_0$, and Q and P are projection-valued measures.

However, it was not until [12] that a satisfactory notion of spectrum in the quaternionic setting was developed, which then allowed for the extension of classical theories of functional analysis to the quaternionic framework, specifically spectral theory.

Therefore, among others, the objective of this section is to introduce the quaternionic analogue of a projection-valued spectral measure. This concept follows the same principles as in classical spectral theory, where one starts with the polar representation of an operator and then introduces the notion of a functional calculus. Subsequently, Riesz's representation theorem is utilized to effectively construct the spectral measure. Hence, firstly we need to establish the aforementioned tools in the quaternionic space. While many aspects of the theory resemble their complex counterparts, the quaternionic setting requires additional techniques, namely the Teichmüller decomposition, first introduced in [46]. Moreover, the notion of spectrum in the quaternionic setting, the S-spectrum, first introduced in [12], is required.

The results presented here are taken from [14] and [15] to which we refer for the corresponding proofs.

Quaternionic Hilbert spaces & the Teichmüller decomposition

A right Hilbert space, H, is a right Banach space for which there is a map $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{H}$ that satisfies the following properties:

1. $\langle x\alpha + y\beta, z \rangle = \langle x, z \rangle \alpha + \langle y, z \rangle \beta$;
2. $\langle x, y \rangle = \overline{\langle y, x \rangle}$;
3. $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$.

Furthermore the norm considered is the one induced by the inner product, i.e. $\|x\| = \sqrt{\langle x, x \rangle}$.

To establish the existence of a polar representation in the quaternionic setting, we begin by proving the existence and well definiteness of the square root of a bounded positive operator. In what follows we call an operator A a *positive semidefinite operator* if for every $x \in H$, $\langle Ax, x \rangle \geq 0$.

Theorem 2.3.1. *[14, pp. 198–199] Every positive semidefinite operator $A \in B(H)$ has a unique positive square root $A^{\frac{1}{2}}$ that satisfies $(A^{\frac{1}{2}})^2 = A$. Moreover, every operator $B \in B(H)$ that commutes with A also commutes with its square root.*

As in the classic case, the existence of and uniqueness of the square root is the first step to prove the existence of the so called *polar decomposition* of an operator.

Theorem 2.3.2. *[14, pp. 199–201] Every operator $T \in B(H)$ admits a unique factorization*

$$T = UP \tag{2.3}$$

into the product of a positive operator P and a partial isometry U on ran(P)[5]. The operator P is furthermore given by $P := (T'T)^{\frac{1}{2}}$, and ran(U) =ran(T).

As a consequence of the existence of a polar decomposition we have what is referred to as *Teichmüller decomposition.* This is the first instance where the construction of spectral measures deviates from the classical construction. Here and thereafter, for an operator $T \in B(H)$, we will reserve to the notation $|T| = (T'T)^{\frac{1}{2}}$.

Lemma 2.3.3. *[14, p. 201] Let $T \in B(H)$. Then there exists a triple (A, J, B) of mutually commuting operators in $B(H)$ all of which commute with T such that*

$$T = A + JB \tag{2.4}$$

where, A is self adjoint, B is a positive operator and J is an antiselfadjoint partial isometry operator which is a partial isometry on ker$(T - T')^{\perp}$. The operators A and B are given by

$$A = \frac{1}{2}(T + T'), \quad B = \frac{1}{2}|T - T'| \tag{2.5}$$

and J is the partial symmetry that appears in the polar decomposition of the operator $\frac{1}{2}|T - T'|$. Finally, the adjoint of $T' = A - JB$, and every operator in $B(H)$ commutes with T and T' if and only if it commutes with A, B and J.

[5]Recall that U is a partial isometry when $\|Ux\| = \|x\|$ for every $x \in ran(P)$ and $Ux = 0$ for every $x \in ran(P)^{\perp}$)

The S-spectrum

The second instance, and arguably the most intriguing, where the quaternionic spectral theory deviates from the classical theory is the notion of spectrum. The following considerations can be found in more detail in [11]. If one attempts to directly extend the definition of spectrum by considering right eigenvalues of an operator T, an unsatisfactory result is obtained. It can be observed that if $\lambda \in \mathbb{H}$ is a right eigenvalue of a right operator acting between right Banach spaces, then so is $s^{-1}\lambda s$ for all $s \in \mathbb{H}$. This would imply that any operator would have an infinite number of eigenvalues, even if it has a finite rank. One possible solution would be to consider equivalence classes:

$$\text{Two right eigenvalues } \mu \text{ and } \lambda \text{ are equivalent if and only if } \mu = s^{-1}\lambda s \text{ for some } s \in \mathbb{H}. \tag{2.6}$$

In this case, we achieve the desired analogy to the classical case: a rank-n operator will have n eigenvalues [48]. This restores the desired analogy to the classical case, where a rank-n operator has n eigenvalues. However, this solution introduces another problem: the well-established idea that the trace of an operator is the sum of its eigenvalues no longer holds, which is a desired property. Thus, the correct notion of quaternionic spectrum is required.

The appropriate definition of the spectrum of a linear and bounded operator in the quaternionic setting, known as the *S-spectrum*, can be found in [14, p. 57]. The S-spectrum provides a suitable extension of the classical spectrum to the quaternionic framework and will be utilized throughout this presentation. More precisely, the S-spectrum is defined as follows:

$$\sigma_S(T) = \{s \in \mathbb{H} : \underbrace{T^2 - 2Re(s)T + |s|^2 Id}_{:=\mathcal{Q}_s(T)} \text{ is not invertible}\}$$

A particularly important subset of the S-spectrum is the so called *residual S-spectrum*. It is defined as follows

$$\sigma_{RS}(T) = \{s \in \mathbb{H} : \ker(\mathcal{Q}_s(T)) = \{0\},\ \overline{\operatorname{ran}(\mathcal{Q}_s(T))} \neq H\}.$$

In accordance with the classic theory, the following result shows that some desired of the classic notion of spectrum are still preserved when considering the S-spectrum.

Lemma 2.3.4. *[14, p. 193] Let T be a right linear, self-adjoint and bounded operator acting between right $\mathbb{H}$-Hilbert spaces. Then $\sigma_S(T) \subseteq \mathbb{R}$ and $\sigma_{RS}(T) = \{\emptyset\}$. Additionally, if T is a positive operator, $\sigma_S(T) \subseteq [0, \infty)$.*

Slice functions

Before proceeding with the construction of the spectral measure for bounded normal quaternionic operators, we need to introduce some technical results and definitions. We will refer to Ω as an *axially symmetric set* if for every $q \in U$ there holds $\{Re(q) + j|Im(q)| : j \in \mathbb{S}\} \subset U$. Thus, an axially symmetric set is one that contains the sphere of radius $|Im(q)|$ centered at $Re(q)$ for any $q \in U$. A function $f : U \to \mathbb{H}$ is called a left slice function if it is of the form

$$f(q) = f_0(u, v) + jf_1(u, v), \qquad q = u + jv \in U,$$

where f_0 and f_1 are two functions $f_0, f_1 : \Omega \to \mathbb{H}$ satisfying the conditions:

$$f_0(u, -v) = f_0(u, v), \qquad f_1(u, -v) = -f_1(u, v).$$

If, in addition, both f_0 and f_1 satisfy the Cauchy-Riemann equations, we refer to f as a *left slice hyperholomorphic* function. Furthermore, if f is either a left or right slice function and the functions f_0 and f_1 are real-valued, it will be referred to as an *intrinsic* function. The set of left slice functions is denoted by $\mathcal{SF}_L(U)$, the set of left slice hyperholomorphic functions on U by $\mathcal{SH}_L(U)$, the set of intrinsic slice functions by $\mathcal{NF}(U)$, and the set of slice hyperholomorphic functions on U by $\mathcal{N}(U)$. Some immediate properties of this functions follow.

Theorem 2.3.5. *[14, p. 14] Consider an axially symmetric $U \subseteq \mathbb{H}$. Then*

1. *If $f \in \mathcal{NF}(U)$ and $g \in \mathcal{SF}_L(U)$, then $fg \in \mathcal{SF}_L(U)$. If $f \in \mathcal{SF}_R(U)$ and $g \in \mathcal{NF}(U)$, then $fg \in \mathcal{SF}_R(U)$.*
2. *If $f \in \mathcal{N}(U)$ and $g \in \mathcal{SH}_L(U)$, then $fg \in \mathcal{SH}_L(U)$. If $f \in \mathcal{SH}_R(U)$ and $g \in \mathcal{N}(U)$, then $fg \in \mathcal{SH}_R(U)$.*
3. *If $g \in \mathcal{NF}(U)$ and $f \in \mathcal{SF}_L(g(U))$, then $f \circ g \in \mathcal{SF}_L(U)$. If $g \in \mathcal{NF}(U)$ and $f \in \mathcal{SF}_R(g(U))$, then $f \circ g \in \mathcal{SF}_R(U)$.*
4. *If $g \in \mathcal{N}(U)$ and $f \in \mathcal{SH}_L(g(U))$, then $f \circ g \in \mathcal{SH}_L(U)$. If $g \in \mathcal{N}(U)$ and $f \in \mathcal{SH}_R(g(U))$, then $f \circ g \in \mathcal{SH}_R(U)$.*

The set of left, right and intrinsic slice functions on Ω that are continuous will be denoted by $\mathcal{SC}_L(\Omega)$, $\mathcal{SC}_R(\Omega)$ and $\mathcal{SC}(\Omega)$, respectively. Clearly, for a compact axially symmetric set $\Omega \subset \mathbb{H}$, the set $C(\Omega, \mathbb{H})$ of all continuous quaternion-valued functions on Ω forms a two-sided quaternionic Banach space when endowed with the pointwise multiplications $(af)(q) = af(q)$ and $(fa)(q) = f(q)a$, along with the supremum norm.

It follows from the so called *structure formula*[6] that the uniform limit of a sequence of continuous left, right, or intrinsic slice functions is again a continuous left, right, or intrinsic slice function on Ω. Hence, the set $\mathcal{SC}_L(\Omega)$ is a closed quaternionic right linear subspace of $\mathbb{H}$ and therefore a quaternionic right Banach space. Analogously, $\mathcal{SC}_R(\Omega)$ is a quaternionic left Banach space. On the other hand, $\mathcal{SC}(\Omega)$ is only a closed $\mathbb{R}$-linear subspace of $C(\Omega, \mathbb{H})$, and so it is only a real Banach space.

Furthermore, the previous theorem implies that $\mathcal{NF}(U)$ is closed under pointwise multiplication, and the pointwise product of two intrinsic slice functions is commutative. Thus, we can conclude that $\mathcal{SC}(\Omega)$ is a commutative real Banach algebra.

This result is of major significance. Indeed, starting from a set of quaternionic functions, we have constructed a commutative real Banach algebra. This, as seen in [19], is sufficient for establishing a spectral theory. However, to fully materialize this theory, we need to develop the appropriate quaternionic tools.

[6] The structure formula states that if $U \subseteq \mathbb{H}$ is an axially symmetric and $i \in \mathbb{S}$ then $f : U \to \mathbb{H}$ is a left slice function on U if and only if for every $q = u + jv \in U$ there holds

$$f(q) = \frac{1}{2}\left(f(\overline{z}) + f(z)\right) + \frac{1}{2} ji \left(f(\overline{z}) - f(z)\right)$$

with $z = u + iv$

As mentioned previously, in order to construct a projection-valued spectral measure, it is necessary to establish the quaternionic counterpart of the Riesz representation theorem on the quaternionic Hilbert space. The proof follows the same structure as the classical case.

Theorem 2.3.6. *[1, p. 450] Let H be a right $\mathbb{H}$-Hilbert space with quaternionic inner product $\langle \cdot, \cdot \rangle$. Let φ be a continuous right linear functional on $\mathcal{H}$. Then there exists a unique $y_\varphi \in \mathcal{H}$ such that*

$$\varphi(x) = \langle x, y_\varphi \rangle, \qquad x \in \mathcal{H}.$$

We will also utilize the classic Riesz representation theorem for continuous real-valued functions.

Theorem 2.3.7. *Let X be a compact Hausdorff space. For every $\psi \in (C(X, \mathbb{R}))'$ there exists a unique Borel measure μ on X such that*

$$\psi(f) = \int_X f(t) d\mu(t), \qquad \text{for each } f \in C(X, \mathbb{R}).$$

Additionally, if ψ is positive, then so is μ.

A crucial step in constructing spectral measures is the Stone-Weierstrass Theorem. In practice, the theory is only developed for polynomials, and then density arguments are employed. Therefore, we begin by extending this classical result to the quaternionic setting. However, we will observe that density only holds for continuous intrinsic slice functions, unlike in the classical theory where density can be claimed for any continuous function (over a compact set). Therefore, this marks the first instance where an additional assumption is required in our theory. We will no longer be able to work solely with continuous functions; rather, we require intrinsic slice functions.

Theorem 2.3.8. *[14, p. 207] Consider the multi-index $\ell = (\ell_1, \ell_2)$ and coefficients $a_\ell \in \mathbb{R}$. Then every polynomial P in q of the form*

$$P(q) = \sum_{0 \leq |\ell| \leq n} a_\ell q^{\ell_1} \overline{q}^{\ell_2} \tag{2.7}$$

is a continuous intrinsic slice function on $\mathbb{H}$. Moreover, for every compact axially symmetric set $\Omega \subset \mathbb{H}$ the set of this polynomials is dense in $\mathcal{SC}(\Omega)$.

As mentioned earlier, we now encounter the first deviation from the classical construction. By utilizing the Teichmüller decomposition (Lemma 2.3.3), this theorem enables us to establish the continuous functional calculus. To begin, we define $P(T)$ for a normal operator T and any polynomial of the form (2.7) in the natural manner. More precisely, for a normal operator $T \in B(H)$ and every polynomial of the form (2.7), we define the operator as follows:

$$P(T) := \sum_{0 \leq |\ell| \leq n} a_\ell T^{\ell_1} (T^*)^{\ell_2}. \tag{2.8}$$

The subsequent requirement is the spectral theorem.

Theorem 2.3.9. *[14, p. 208] Let $T \in B(H)$ be a normal operator. For every polynomial of the form (2.7), $P(q)$, with real coefficients, the operator $P(T)$ is a normal operator that commutes with T and T^*, and*

$$\sigma_S(P(T)) = P((\sigma_S)(T)).$$

This in particular implies that $\|P(T)\| = \max_{s \in \sigma_S(T)} |P(s)|$.

With this theorem established, we can now introduce the so-called functional calculus.

Theorem 2.3.10. *[14, p. 211] Consider a quaternionic right $\mathbb{H}$- Hilbert space. If $T \in B(\mathcal{H})$, then there exists a unique continuous homomorphism of real unital *-algebras*

$$\mathcal{SC}(\sigma_S(T)) \ni f \xmapsto{\Psi_T} \Psi_T(T) := f(T) \in B(\mathcal{H}).$$

Moreover,

1. *Ψ_T is an isometry, since $\|f(T)\| = \max_{s \in \sigma_S(T)} |f(s)|$;*
2. *Every operator $f(T)$ is normal and it commutes with T and T^* as well as with the operators A, B and J appearing in the Teichmüller decomposition of T;*
3. *The spectral mapping property $\sigma_S(f(T)) = f(\sigma_S(T))$ holds, and for every function $g \in \mathcal{SC}(\sigma_S(f(T)))$ we have $g(f(T)) = (g \circ f)(T)$.*

Let $\Omega^+ = \{(u, v) \in \mathbb{R} \times \mathbb{R}_0^+ : u + \mathbb{S}v \subset \Omega\}$ for an axially symmetric set $\Omega \subset \mathbb{H}$. Equivalent to the definition of a left slice function f is the existence of functions F_0 and F_1 defined on Ω^+, where $F_1(u, v) = 0$ if $v = 0$, such that

$$f(q) = F_0(u, v) + jF_1(u, v), \qquad q = u + jv \in \Omega,\ v \geq 0.$$

One clearly has a similar result for a right slice function by requiring

$$f(q) = F_0(u, v) + F_1(u, v)j, \qquad q = u + jv \in \Omega,\ v \geq 0.$$

This equivalence implies that for a function to be intrinsic, the functions F_0 and F_1 must be real-valued. The following lemma demonstrates that the approximation using polynomials extends to the component functions. More precisely,

Lemma 2.3.11. *[14, pp. 213–214] Consider a compact, axially symmetric set $K \subset \mathbb{H}$. Let $f = F_0 + jF_1$ in $\mathcal{SC}(K)$, and let $P_n(q) = \sum_{0 \leq |\ell| \leq n} a_{n,\ell} q^{\ell_1} \overline{q}^{\ell_2}$ be a sequence of polynomials of the form (2.7) that converges uniformly to f on K. Then P_n is of the form*

$$P_n(q) = Q_n(u, v) + jvR_n(u, v), \quad q = u + jv$$

where Q_n and R_n are real polynomials such that, as $n \to \infty$, $Q_n(u, v) \to F_0(u, v)$ and $vR_n(u, v) \to F_1(u, v)$ uniformly on K.

Consequently, writing $u = \frac{1}{2}(q + \overline{q})$ and $v = (-j)\frac{1}{2}(q - \overline{q})$ we observe that both $Q_n(u, v)$ and $R_n(u, v)$ are polynomials with real coefficients in q and $\overline{q}$, meaning that they are also of the form (2.7). Additionally, Lemma 2.3.11 allows us to conclude the compatibility of the functional calculus with the component functions F_0 and F_1. Specifically,

Theorem 2.3.12. *[14, pp. 214–216] Consider the Teichmüller decomposition of the operator T, $T = A + JB \in B(\mathcal{H})$. Moreover suppose that T is a normal operator and let $f = F_0 + jF_1 \in \mathcal{SC}(\sigma_S(T))$. Then*

$$f(T) = F_0(T) + JF_1(T).$$

Furthermore, the operators $F_0(T)$ and $F_1(T)$ can be expressed as functions of the operators A and B in terms of the continuous functional calculus for n-tuples of commuting self-adjoint operators as $F_0(T) = F_0(A, B)$ and $F_1(T) = F_1(A, B)$.

Remark: The last statement, in particular, implies independence from the choice of operator J.

Construction of the spectral measure

The following construction and the subsequent results can be found in [14, pp. 234–241]. Let T be a normal operator in $B(H)$ and let j be a fixed imaginary unit. Here and thereafter, we will use the notation $\Omega_j^+ = \sigma_S(T) \cap \mathbb{C}_j^+$. As noted previously, every $f_j \in C(\Omega_j^+, \mathbb{R})$ is the restriction $f_j = f|_{\Omega_j^+}$ of a real-valued continuous slice function f defined on $\sigma_S(T)$. The set of real-valued slice functions on $\sigma_S(T)$ is denoted by $\mathcal{SC}(\sigma_S(T), \mathbb{R})$. In the following, we will not distinguish between the function f_j and the function f, unless it leads to confusion.

For each $x \in H$, we define the map

$$\ell_x(g) = \langle g(T)x, x\rangle, \qquad g \in C(\Omega_j^+, \mathbb{R}) \simeq \mathcal{SC}(\sigma_S(T), \mathbb{R}),$$

where $g(T)$ is the operator obtained using the continuous functional calculus introduced in Theorem 2.3.12. Since T is a bounded operator, its S-spectrum is a compact and non-empty set. Therefore, ℓ_x is a bounded linear functional on $C(\Omega_j^+, \mathbb{R})$ that takes real values. Moreover, it is a positive functional. Indeed, consider a continuous nonnegative function h on Ω_j^+. We can then define the function $g(u, v) = \sqrt{h(u, v)}$, and thus we have $g \in C(\Omega_j^+, \mathbb{R})$ such that $g(T) = g(T)^*$. Therefore,

$$\ell_x(h) = \langle h(T)x, x\rangle = \langle g(T)x, g(T)x\rangle = \|g(T)x\|^2 \geq 0.$$

As a real-valued functional on $C(\Omega_j^+, \mathbb{R})$, the classic Riesz's representation Theorem 2.3.7 guarantees the existence of a uniquely determined positive-valued measure μ_x on the Borel sets $\mathfrak{B}(\Omega_j^+)$ such that

$$\ell_x(g) = \int_{\Omega_j^+} g(p)\, d\mu_x(p), \qquad g \in C(\Omega_j^+, \mathbb{R}).$$

Since

$$\begin{aligned} 4\,\langle g(T)x, y\rangle = &\langle g(T)(x+y), x+y\rangle - \langle g(T)(x-y), x-y\rangle \\ &+ i\,\langle g(T)(x+yi), x+yi\rangle - i\,\langle g(T)(x-yi), x-yi\rangle \\ &+ i\,\langle g(T)(x+yj), x+yj\rangle\, k - i\,\langle g(T)(x-yj), x-yj\rangle\, k \\ &+ \langle g(T)(x+yk), x+yk\rangle\, k - \langle g(T)(x-yk), x-yk\rangle\, k, \end{aligned}$$

it follows that, for each $g \in C(\Omega_j^+, \mathbb{R})$ and for every $x, y \in H$, there is a unique $\mathbb{H}$-valued measure $\mu_{x,y}$ for which

$$\langle g(T)x, y\rangle = \int_{\Omega_j^+} g(p)\, d\mu_{x,y}(p),$$

where $4\mu_{x,y} = \mu_{x+y} - \mu_{x-y} + i\mu_{x+yi} - i\mu_{x-yi} + i\mu_{x-yj}k - i\mu_{x+yj}k + \mu_{x+yk}k - \mu_{x-yk}k$. In analogy with the classic construction we have the following properties:

Lemma 2.3.13. *[14, p. 235] Let $x, y, z \in H$ and $\alpha, \beta \in \mathbb{H}$. The $\mathbb{H}$-valued measures $\mu_{x,y}$ satisfy:*

1. $\mu_{x\alpha+y\beta,z} = \mu_{x,z}\alpha + \mu_{y,z}\beta$;
2. $\mu_{x,y\alpha+z\beta} = \overline{\alpha}\mu_{x,y} + \overline{\beta}\mu_{x,z}$;
3. $|\mu_{x,y}(\Omega_j^+)| \leq \|x\|\,\|y\|$;
4. $\overline{\mu_{x,y}} = \mu_{y,x}$.

The first and third relations imply that for every fixed $y \in H$ and every $\sigma \in \mathfrak{B}(\Omega_j^+)$, the mapping $\Phi_y(x) = \mu_{x,y}(\sigma)$ is a continuous right linear functional on H. Moreover, the second equality shows that $\Phi_{y\alpha}(x) = \overline{\alpha}\Phi_y(x)$ for $\alpha \in \mathbb{H}$. Following the classic reasoning, Theorem 2.3.6 states that there exists a unique $w \in H$ corresponding to every $x \in H$ for which $\Phi_y(x) = \langle x, w\rangle$, i.e., $\mu_{x,y}(\sigma) = \langle x, w\rangle$. This means that for some operator $E(\sigma) \in B(H)$ there holds $E(\sigma)y = w$. Thus, we have constructed an operator such that

$$\mu_{x,y}(\sigma) = \langle x, E(\sigma)y\rangle, \quad \sigma \in \mathfrak{B}(\Omega_j^+). \tag{2.9}$$

The fourth equality of the previous lemma guarantees that, for all $\sigma \in \mathfrak{B}(\Omega_j^+)$, $E(\sigma)$ is self adjoint. Therefore,

$$\mu_{x,y}(\sigma) = \langle E(\sigma)x, y\rangle\,, \quad \sigma \in \mathfrak{B}(\Omega_j^+).$$

By construction, E is also countably additive, i.e., for every sequence of pairwise disjoint sets $(\sigma_n)_n$ in $\mathfrak{B}(\Omega_j^+)$,

$$E\left(\cup_{n=0}^{\infty}\sigma_n\right) = \sum_{n=0}^{\infty} E(\sigma_n),$$

where the limit is taken with respect to the strong operator topology. Therefore, we conclude that, for a normal operator $T \in B(H)$, if $g \in C(\Omega_j^+, \mathbb{R}) \simeq \mathcal{SC}(\Omega, \mathbb{R})$

$$\langle g(T)x, y\rangle = \int_{\Omega_j^+} g(p)\, d\left\langle E(p)x, y\right\rangle, \qquad \forall x, y \in H. \tag{2.10}$$

Moreover, if $f = f_0 + jf_1 \in SC_j(\Omega_j^+) \simeq SC(\Omega)$ then

$$\langle f(T)x, y\rangle = \int_{\Omega_j^+} f_0(p)\, d\left\langle E(p)x, y\right\rangle + \int_{\Omega_j^+} f_1(p)\, d\left\langle JE(p)x, y\right\rangle, \qquad \forall x, y \in H. \tag{2.11}$$

Here, in both cases, E is the spectral measure on $\mathfrak{B}(\Omega_j^+)$ defined above. We will use the simplified notations

$$g(T) = \int_{\Omega_j^+} g(p)\, E(dp) \text{ and } f(T) = \int_{\Omega_j^+} f_0(p)\, E(dp) + \int_{\Omega_j^+} f_1(p)\, JE(dp)$$

when referring to (2.10) and (2.11), respectively. In particular, for a normal $T \in B(H)$ we have the representation

$$T = \int_{\Omega_j^+} Re(p)\, E(dp) + \int_{\Omega_j^+} Im(p)\, JE(dp) \tag{2.12}$$

Remark: We can notice the consistency of this construction with the classical theory by observing how (2.11) collapses to the classical case. As we have already seen, even in the quaternionic setting, the spectrum of a positive operator is positive, i.e., $\Omega_j^+ = \sigma(T) \subseteq [0, \infty)$. Additionally, the Teichmüller decomposition of a positive operator $T \in B(H)$ implies that $J \equiv 0$. This means that, in the case of a positive operator, (2.11) can be written as:

$$T = \int_{\sigma(T)} p\, E(dp). \tag{2.13}$$

Therefore, for positive operators, the results from the classical theory can be directly applied.

CHAPTER 3

Quaternionic s-number Theory

This chapter introduces the theory of quaternionic s-numbers. To achieve this, we adopt an axiomatic approach inspired by A. Pietsch. We demonstrate that these axioms lead to a unique s-number function when considered over $\mathbb{H}$-Hilbert spaces. Furthermore, we expand to the quaternionic Banach space framework certain classical examples of s-numbers and introduce the concept of nuclear numbers. Finally, we conclude the chapter by exploring the relationships between various types of s-numbers.

3.1 Quaternionic s-numbers

3.1.1 Axiomatization of quaternionic s-number theory

In the theory of s-numbers, an operator T is associated with various types of scalar sequences $s_n(T)$. The objective, among others, is to classify the operator based on the behavior of such sequences. The axiomatic framework we employ was first introduced in [40, p. 202]. Since they are given in terms of norms of operators they are well defined in a quaternionic setting. Therefore, there is no need to modify the proposed axioms to our framework.

Thus, let us consider a right linear bounded operator T, acting between right quaternionic Banach Spaces X and Y. A mapping $s : T \to (s_n(T))_n$, which associates a sequence of non-negative numbers to T is referred to as an *s-number function* if it satisfies the following conditions:

A1. $\|T\| \geq s_1(T) \geq s_2(T) \geq \cdots \geq 0$;
A2. $\forall n \in \mathbb{N}$, $s_n(S+T) \leq s_n(S) + \|T\|$ for $X \xrightarrow{S,T} Y$;
A3. $\forall n \in \mathbb{N}$, $s_n(BTA) \leq \|B\| \, s_n(T) \, \|A\|$ for $X_0 \xrightarrow{A} X \xrightarrow{T} Y \xrightarrow{B} Y_0$;
A4. If $dim(X) \geq n$, $s_n(Id_X) = 1$;
A5. If $\operatorname{rank}(T) < n$ then $s_n(T) = 0$.

For a fixed $n \in \mathbb{N}$ we call $s_n(T)$ the n-th s-number of T. As observed in [40, p. 203] it immediately follows from **A2** that $s_n(S) \leq s_n(T) + \|S - T\|$, which implies that

$$|s_n(S) - s_n(T)| \leq |s_n(T) + \|S-T\| - s_n(T)| = \|S - T\| \, .$$

This shows that the s-number functions are continuous functions in the operator topology. Furthermore, the converse of **A5** also holds. If we assume that $s_n(T) = 0$ and that $\operatorname{rank}(T) \geq n$ then by **A4**, $\|Id\| = 1$. From Lemma 2.2.5 we have $Id = BTA$ with $B \in L(\ell^2, E)$ and $A \in L(F, \ell^2)$, which implies that $1 = \|Id\| = \|BTA\| \leq \|B\| \, s_n(T) \, \|A\| = 0$ by **A3**, contradicting our assumption.

The following definitions are taken from [40, p. 206] and [40, p. 208] respectively. An s-number function is *injective* if for any $Y_0 \subset Y$ there holds that $s_n(J_{Y_0}^Y T) = s_n(T)$ for all right linear and bounded T. Moreover, it is said *surjective* if for $X_0 \subset X$ there holds that $s_n(TQ_{X_0}^X) = s_n(T)$ for all right linear and bounded T [1]. Also, the notions of multiplicity and additivity of s-numbers are introduced in [38, p. 327].

A2*. If, for all $m, n \in \mathbb{N}$ and for all operators S T, there holds

$$s_{n+m-1}(S+T) \leq s_m(S) + s_n(T),$$

then we say that the s-numbers are *additive*;

A3*. If, for all $m, n \in \mathbb{N}$ and for all operators S T, there holds

$$s_{n+m-1}(ST) \leq s_m(S) s_n(T),$$

then we say that the s-numbers are *multiplicative.*

In [40, p. 211] the concept of the *dual s-number function* is introduced. For each s-number function we define the dual s-number function, s', via $s'_n(T) = s_n(T')$ for all $T \in B^R(X,Y)$. Note that if we consider a right operator T acting between right Banach spaces, its dual will be a right operator acting between left Banach spaces. Consequently, the proposed axioms do not encompass this case. Therefore, when the notion of a dual s-number is required, it is necessary to additionally consider the axioms for a two-sided structure. This will be the only instance where we require a slight modification to the proposed axioms.

Following [39, p. 152], an s-function, s, is called *symmetric* if, for each $n \in \mathbb{N}$, $s_n(T) \geq s_n(T')$, for all $T \in B^R(X,Y)$. If equality is achieved, it is called *completely symmetric.* Recall that K_Y denotes the canonical evaluation map (2.2.1). An s-function is said *regular* if, for for each $n \in \mathbb{N}$, $s_n(T) = s_n(K_Y T)$, for all $T \in B^R(X,Y)$.

Theorem 3.1.1. *An s-function is completely symmetric if and only if it is regular and symmetric.*

Proof. Clearly a completely symmetric s-number function is symmetric. Since $\|K_Y\| = 1$ $s_n(K_Y S) \leq \|K_Y\| \, s_n(S) = s_n(S)$ and since

$$s_n(S) = s_n(S') = s_n(S'(K_Y)' K_{Y'}) \leq s_n(S'(K_Y)') \, \|K_{Y'}\| = s_n((K_Y S)') = s_n(K_Y S)$$

we have $s_n(S) = s_n(K_Y S)$ which establishes the regularity. Conversely, for a symmetric regular s-number function, to prove that it is completely symmetric it remains to show that

[1] In other words, injectivity means that we have an independence of the codomain of T, while surjectivity means that we have independence of the domain of T.

$s_n(S) \leq s_n(S')$. Indeed $s_n(S) = s_n(K_Y S)$ by regularity. The canonical mapping K_Y is such that

$$s_n(K_Y S) = s_n(S'' K_X) \leq s_n(S'') \|K_X\| = s_n(S'').$$

The symmetry assumption yields the boundedness by $s_n(S')$. $\square$

As seen in [39, p. 154], by defining $s_n(T) = a_n(J_Y T Q_X)$ for $T \in B^R(X,Y)$, where a_n denotes the n-th approximation number (defined later in subsection 3.2.1), we can establish the following existence theorem.

Theorem 3.1.2. *There exists an s-function which is injective, surjective and completely symmetric.*

3.1.2 The uniqueness of the s-number function on quaternionic Hilbert spaces

It has been wellknown for some time that s-numbers in the case of Hilbert spaces had various equivalent definitions, as observed in [22] (these equivalences played a role in establishing some of the axioms defined in subsection 3.1.1). Later, in [40, pp. 203–204], it was demonstrated that the proposed axioms are consistent with this fact. We aim to solidify this idea by showing that also in the quaternionic framework, the s-number function is unique, when considered over $\mathbb{H}$-Hilbert Spaces.

To see this we require some technical results. Consider $T \in B^R(H)$, and let E be the spectral measure associated with the positive operator $|T|$. In what follows, we denote

$$\sigma_n = \inf_{\sigma \geq 0} \{\dim(E(\sigma, \infty)) < n\}.$$

The main objective of this section is to show that $s_n(T) = \sigma_n$. The following lemma simplifies this task, as it shows that it suffices to show that $s_n(|T|) = \sigma_n$.

Lemma 3.1.3. *Let $T \in B^R(H)$, then $s_n(T) = s_n(|T|)$.*

Proof. From the polar representation of T, as given in Theorem 2.3.2, we know that there is a partial isometry U such that $T = U|T|$ and $T' = U'T$. The third axiom implies that

$$s_n(T) \leq \|U\|\, s_n(|T|) = \|U\|\, s_n((T'T)^{\frac{1}{2}}) = \|U\|\, s_n((U'TT)^{\frac{1}{2}}) \leq \|U\|\, \|U'\|^{\frac{1}{2}}\, s_n(T) = s_n(T)$$

and thus $s_n(T) = s_n(|T|)$. $\square$

Lemma 3.1.4. *$rank\Big(\int_{\sigma_n+\epsilon}^{\infty} pE(dp)\Big) < n$.*

Proof. By definition of σ_n it follows

$$\begin{aligned} n > \operatorname{rank}(E(\sigma_n+\epsilon,\infty)) &= \operatorname{rank}\left(\int_{\sigma_n+\epsilon}^{\infty} E(dp)\right) = \operatorname{rank}\left(\int_{\sigma_n+\epsilon}^{\infty} pp^{-1}E(dp)\right) \\ &= \operatorname{rank}\left(\int_{\sigma_n+\epsilon}^{\infty} pE(dp) \int_{\sigma_n+\epsilon}^{\infty} p^{-1}E(dp)\right) \\ &= \operatorname{rank}\left(\int_{\sigma_n+\epsilon}^{\infty} pE(dp)\right) \operatorname{rank}\left(\int_{\sigma_n+\epsilon}^{\infty} p^{-1}E(dp)\right), \end{aligned}$$

which implies rank $\Big(\int_{\sigma_n+\epsilon}^{\infty} pE(dp)\Big) < n$. $\square$

Lemma 3.1.5. *For any $\epsilon > 0$ there holds that*

$$E(\sigma_n - \epsilon, \infty) = \left(\int_0^\infty pE(dp)\right)\left(\int_{\sigma_n-\epsilon}^\infty p^{-1}E(dp)\right)$$

Proof.

$$\begin{aligned}
\int_0^\infty pE(dp)\int_{\sigma_n-\epsilon}^\infty p^{-1}E(dp) &= \left(\int_0^{\sigma_n-\epsilon} pE(dp) + \int_{\sigma_n-\epsilon}^\infty pE(dp)\right)\left(\int_{\sigma_n-\epsilon}^\infty p^{-1}E(dp)\right) \\
&= \int_0^{\sigma_n-\epsilon} pE(dp)\int_{\sigma_n-\epsilon}^\infty p^{-1}E(dp) + \int_{\sigma_n-\epsilon}^\infty pE(dp)\int_{\sigma_n-\epsilon}^\infty p^{-1}E(dp) \\
&= \int p\chi_{(0,\sigma_n-\epsilon)}E(dp)\int p^{-1}\chi_{(\sigma_n-\epsilon,\infty)}E(dp) + \int_{\sigma_n-\epsilon}^\infty pp^{-1}E(dp) \\
&= \int \underbrace{p\chi_{(0,\sigma_n-\epsilon)}p^{-1}\chi_{(\sigma_n-\epsilon,\infty)}}_{=0}E(dp) + \int_{\sigma_n-\epsilon}^\infty E(dp) \\
&= E(\sigma_n - \epsilon, \infty).
\end{aligned}$$

□

Theorem 3.1.6. *For $S \in B^R(H)$ there holds $s_n(T) = \sigma_n$.*

Proof. Considering an $\epsilon > 0$, then we can write

$$|S| = \int_0^\infty pE(dp) = \int_0^{\sigma_n+\epsilon} pE(dp) + \int_{\sigma_n+\epsilon}^\infty pE(dp).$$

As an application of **A5.**, it follows from Lemma 3.1.4 that

$$s_n(|S|) \leq \left\|\int_0^{\sigma_n+\epsilon} pE(dp)\right\| + s_n\left(\int_{\sigma_n+\epsilon}^\infty pE(dp)\right) \leq \sigma_n + \epsilon$$

because

$$\left\|\int_0^{\sigma_n+\epsilon} pE(dp)\right\| \leq \|Id\|_\infty \|E(0, \sigma_n + \epsilon)\| \leq \|P\|(\sigma_n + \epsilon) = \sigma_n + \epsilon.$$

Now we consider $0 < \epsilon < \sigma_n$. By Lemma 3.1.5

$$P(\sigma_n - \epsilon, \infty) = \left(\int_0^\infty pE(dp)\right)\left(\int_{\sigma_n-\epsilon}^\infty p^{-1}E(dp)\right)$$

By definition of σ_n, $\operatorname{rank}(E(\sigma_n - \epsilon, \infty)) \geq n$. Thus,

$$\begin{aligned}
1 &= s_n(P(\sigma_n - \epsilon, \infty)) \leq s_n(|S|)\left\|\int_{\sigma_n-\epsilon}^\infty p^{-1}E(dp)\right\| \leq s_n(|S|)\sup_{p\in(\sigma_n-\epsilon,\infty)} p^{-1} \\
&= s_n(|S|)(\sigma_n - \epsilon)^{-1}.
\end{aligned}$$

Therefore $\sigma_n - \epsilon \leq s_n(|S|) \leq \sigma_n + \epsilon$ for all $\epsilon > 0$. □

As a consequence, according to the "classic" definition of s-numbers,

Corollary 3.1.7. *Consider a compact right operator acting between right $\mathbb{H}$-Hilbert Spaces, S. Then $s_n(S)$ is the n-th eigenvalue of $|S|$.*

The uniqueness of s-numbers on quaternionic Hilbert spaces is a significant result that will be useful in the sequel. Moreover, s-numbers on quaternionic Hilbert spaces possess the properties of additivity and multiplicativity. This can be proven using ideas inspired by [17, pp. 764–765]. Instead of working directly with general s-numbers, we will work with eigenvalues. More precisely, we consider the sequences of eigenvalues (ordered by algebraic multiplicity) of A^*A, B^*B, $(AB)^*(AB)$, and $(A+B)^*(A+B)$, denoted by (λ_i), (κ_i), (μ_i), and (σ_i) respectively. Since A and B are right linear quaternionic operators acting on a right $\mathbb{H}$-Hilbert space, all the operators considered are self-adjoint, and thus their eigenvalues are real numbers, as seen in Lemma 2.3.4. We aim to show the following inequalities for any non-negative integers m and n:

$$\mu_{m+n+1} \leq \lambda_{m+1}\kappa_{n+1} \tag{3.1}$$

$$\sqrt{\sigma_{m+n+1}} \leq \sqrt{\lambda_{m+1}} + \sqrt{\kappa_{n+1}}. \tag{3.2}$$

To prove these inequalities, let $\{x_s\}$, $\{y_i\}$ be two orthonormal basis such that $AA^*x_i = \lambda_i x_i$, $B^*By_i = \kappa_i y_i$ for $i \in \{1, 2, \dots\}$. Consider the polar representation of AB as WZ, where Z is the non-negative square root of $(AB)^*(AB)$ and W is a partial isometry. For any $z \in Z$, we have by the Cauchy-Schwartz inequality

$$(Zz, z)^2 = (W^*ABz, z)^2 \leq (AA^*Wz, Wz)(B^*Bz, z).$$

By choosing z to be such that $\|z\| = 1$, $(z, W^*x_i) = 0$ for $1 \leq i \leq m$ and $(z, y_j) = 0$ for $1 \leq j \leq n$, we obtain:

$$(AA^*Wz, Wz) \leq \|Wz\|^2 \lambda_{m+1} \leq \lambda_{m+1},$$
$$(B^*Bz, z) \leq \kappa_{n+1}.$$

Therefore, $(Zz, z)^2 \leq \lambda_{m+1}\kappa_{n+1}$. By applying the min-max principle on Z, we obtain (3.1). The proof of (3.2) follows a similar argument.

Hence, on a quaternionic Hilbert space, s-numbers are not only unique but also possess the properties of additivity and multiplicativity. This contrasts with the case of general Banach spaces. More precisely, only on Hilbert spaces can one use Theorem 2.3.7, so that the operator E in (2.9) is well defined. Therefore, a priori, one should not expect the uniqueness claim to hold in a general Banach.

This is indeed the case. In fact, there are several examples of s-number functions when considered over Banach spaces. Moreover, having in mind that s-numbers induce operator ideals in Hilbert spaces (the Schatten classes), it can be expected that a similar phenomenon occurs in the case of a general Banach Space. Furthermore, the loss of uniqueness leads to the existence of multiple examples of operator ideals in Banach spaces, under certain assumptions, as we will see in chapter 4.

3.2 Examples of s-numbers

Throughout this section, unless otherwise stated, X and Y are assumed to be right $\mathbb{H}$-Banach spaces. As we will observe, a key aspect that s-numbers provide is the independence of the choice of basis. This implies that any properties of operators that we obtain via s-numbers are intrinsic and not dependent on any particular representation. For instance, Theorems 3.2.4 and 3.2.6 can be used as a “measure of compactness".

As mentioned previously, in the context of a general Banach space, there exist multiple examples of s-number functions. Many of the s-number functions studied thus far are inspired by geometric phenomena and/or qualitative quantities of Banach spaces.

The following table provides a brief overview of some s-number functions and the corresponding qualitative answer that they provide:

S-numbers	Geometric interpretation
Approximation numbers	Measures the distance from an operator to the set of linear and bounded operators of a fixed rank.
Isomorphism numbers	Measure how much is an operator isomorphic to the identity.
Gelfand numbers	From [16] for uniformly convex and uniformly smooth Banach spaces, if operator has trivial kernel and dense range then it measures the best approximation of the image of the unit ball by linear subspaces of a fixed codimension.
Kolmogorov numbers	Measures the best possible approximation of the image of the unit ball by linear subspaces of a fixed dimension.

In general, in the remainder of this section the reference will indicate where the reader can find the original result, in the classic setting.

3.2.1 Approximation numbers

Introduced in [40, p. 204], for $T \in B^R(X,Y)$, the *n-th approximation number* is defined by

$$a_n(T) := \inf\{\|T - A\| : A \in B^R(X,Y), \operatorname{rank}(A) < n\}.$$

The basis independence is attained by taking the infimum over all finite-rank operators A in $B^R(X,Y)$.

Theorem 3.2.1. *The map app* : $T \to a_n(T)$ *is an additive and multiplicative s-number function. Moreover, it is the largest s-number function.*

Proof. We prove the first two claims together. Consider $S, T \in B^R(X,Y)$.

A1. By definition if $\operatorname{rank}(A) < 1$ then $A = 0$, therefore $\|T\| \geq s_1(T)$. Moreover, since

$$\{A \in B^R(X,Y) : \operatorname{rank}(A) < n\} \subseteq \{A \in B^R(X,Y) : \operatorname{rank}(A) < n+1\}$$

it follows that $a_n(T) \geq a_{n+1}(T)$.

A2*. Consider $m, n \in \mathbb{N}$ and let $X \xrightarrow{S,T} Y$. For any matrix A with rank $n+m$ we can always find two matrices A_n and A_m whose rank is smaller then n and m, respectively, such that $A_m + A_m = A$, upon a proper extension by zeros. Having this we can conclude

$$
\begin{aligned}
a_{n+m-1}(T+S) =& \inf\{\|S+T-A\| : A \in B^R(X,Y), \operatorname{rank}(A) \leq n+m\} \\
=& \inf\{\|S+T-A_n-A_m\| : A_n, A_m \in B^R(X,Y), \\
& \quad \operatorname{rank}(A_n) \leq n \text{ and } \operatorname{rank}(A_m) \leq m\} \\
\leq& \inf\{\|S-A_n\| : A_n \in B^R(X,Y), \operatorname{rank}(A_n) \leq n\} \\
& \quad + \inf\{\|T-A_m\| : A_m \in B^R(X,Y), \operatorname{rank}(A_m) \leq m\} \\
=& a_n(S) + a_m(T).
\end{aligned}
$$

A3*. [39, p. 152] Fix any $m, n \in \mathbb{N}$ and let $T \in B^R(X,Y)$ and $S \in B^R(Y,Z)$. Given $\epsilon > 0$, we can construct operators $B \in B^R(X,Y)$ and $A \in B^R(Y,Z)$ such that

$$
\begin{aligned}
&\|S-A\| \leq (1+\epsilon)a_m(S) \text{ and } \operatorname{rank}(A) < m, \\
&\|T-B\| \leq (1+\epsilon)a_n(T) \text{ and } \operatorname{rank}(B) < n.
\end{aligned}
$$

As mentioned in (2.1), there holds that

$$
\begin{aligned}
\operatorname{rank}(AT+(S-A)B) &\leq \min\{\operatorname{rank}(A), \operatorname{rank}(T)\} + \min\{\operatorname{rank}(S-A), \operatorname{rank}(B)\} \\
&\leq \operatorname{rank}(A) + \operatorname{rank}(B) < m+n-1
\end{aligned}
$$

which implies that

$$
\begin{aligned}
a_{m+n-1}(ST) &\leq \|ST-(AT+(S-A)B)\| = \|(S-A)(T-B)\| \\
&\leq \|S-A\| \, \|T-B\| \leq (1+\epsilon)^2 a_m(S) a_n(T).
\end{aligned}
$$

A4. The proof follows the same lines as the ones presented in [40, p. 204]. Consider $dim(X) \geq n$ and suppose that $s_n(Id_X) < 1$. Then there exists $A \in B^R(X)$ with $\operatorname{rank}(A) < n$ for which $\|Id_X - A\| < 1$. Consequently, $A = Id_X - (Id_X - A)$ is an invertible operator[2,3]. Therefore, $\operatorname{rank}(A) = \dim(X) \geq n$, which is a contradiction. Analogously, if we assume that $s_n(Id_X) > 1$ then we can argue similarly with the operator $(Id_X - A)^{-1}$. Therefore $s_n(Id_X) = 1$.

A5. In this case we can take $A = T$ and conclude that $\|T - A\| = 0$.

[2]This follows from the standard Neumann series argument. Actually, this result holds on a general unital Banach algebra: Let x be an element of a such an algebra such that $\|x\| < 1$. Since it is a Banach algebra, we see that the partial sums form a Cauchy sequence:

$$
\left\| \sum_{n=l}^{m} x^n \right\| \leq \sum_{n=l}^{m} \|x\|^n \to 0
$$

as $l, m \to \infty$. By completeness, the series $\sum_{n=0}^{\infty} x^n$ converges to some element y. For any $m \in \mathbb{N}$ we have $(1-x)\sum_{n=0}^{m} x^n = 1 - x^{m+1}$ and furthermore, $\left\|x^{m+1}\right\| \leq \|x\|^{m+1}$ meaning that $\lim\limits_{m\to\infty} x^{m+1} = 0$. Thus, by taking the limit, we get $(1-x)\sum_{n=0}^{\infty} x^n = 1$ (We can exchange the limit with the multiplication by $(1-x)$, since the multiplication in Banach algebras is continuous). Arguing by multiplying from the right yields to the same conclusion and as such $y = (1-x)^{-1}$.

[3]The set of linear and bounded quaternionic operators even form a non-commutative unital C^*-Algebra, where the involution is the quaternionic conjugation

With this we proved that a_n is an additive and multiplicative s-number function. It remains to prove that it is the largest. As seen in [40, p. 204] for each s-number function, s_n, one has, for all $A \in B^R(X,Y)$ for which $\mathrm{rank}(A) < n$, that

$$s_n(S) = s_n(A + S - A) \leq s_n(A) + \|S - A\| = \|S - A\|.$$

In particular it follows that, $s_n(S) \leq a_n(S)$. □

It follows from definition of the dual operator that $a_n(T') \leq a_n(T)$, i.e. approximation numbers are symmetric. Indeed it suffices to observe that, for $x, \varphi \neq 0$,

$$\frac{\|(T' - A')(\varphi)\|_{X'}}{\|\varphi\|_{Y'}} \leq \frac{\|((T' - A')(\varphi))(x)\|_X}{\|x\|_X \|\varphi\|_{Y'}} = \frac{\|\varphi((T - A)(x))\|_X}{\|x\|_X \|\varphi\|_{Y'}} \leq \frac{\|(T - A)(x)\|_X}{\|x\|_X},$$

to deduce $a_n(T') \leq a_n(T)$.

The following theorem, in the classic setting, is due to Hutton in [25, p. 278]. It in particular, shows that stronger assumptions on T allow us to achieve complete symmetry. Recall that a right compact operator is a right linear operator that maps bounded sets into relatively compact sets.

Theorem 3.2.2. *If a right compact operator $T \in K(X,Y)$, then the approximation numbers are completely symmetric.*

Proof. We shall only consider $T \in B^R(X_R, Y_R)$ for quaternionic right Banach Spaces, as the left linear case is analogous. Then $T'' \in B^R(X''_R, Y''_R)$, where X''_R and Y''_R are quaternionic right Banach Spaces of left linear maps from X'_R to $\mathbb{H}$.

Since T is compact, $T''(X''_R) \subseteq K_Y(Y)$ and $T''(B_{X''})$ is a totally bounded set[4]. For each index i, choose $(x''_{i,n})_{n=1}^{N(i)} \subset B_{X''}$ such that $(T''(x''_{i,n}))_{n=1}^{N(i)}$ is an ϵ_i-net[5] of $T''(B_{X''})$. By being totally bounded we can further assume that $(\epsilon_i)_i \searrow 0$ monotonically. In what follows, we denote

$$G_i := \mathrm{span}\left\{(T''(x''_{i,n}))_{n=1}^{N(i)}\right\} \subseteq Y''.$$

By the Principle of local reflexivity[6] there is a one-to-one operator $\varphi_i : G_i \to Y$ such that

$$\|\varphi_i\| \left\|\varphi_i^{-1}\right\| \leq 1 + \epsilon_i \text{ and } \varphi\Big|_{G_i \cap J(Y)} = Id.$$

Moreover, we define

$$\hat{\varphi} : \cup_{i=1}^{\infty} G_i \to F, \quad \text{such that } \hat{\varphi}(x) = \varphi_i(x), \text{ if } x \in G_i.$$

Since for each i, $G_i \subset K_Y(Y)$, $\hat{\varphi}$ is well defined. Define $G := \overline{\cup_{i=1}^{\infty} G_i}$ and denote the extension of $\hat{\varphi}$ to G by $\tilde{\varphi}$.

[4]For every $\epsilon > 0$, there exists a finite collection of open balls of radius ϵ centered at a point of $T''(B_{X''})$ and whose union contains $T''(B_{X''})$.

[5]Recall that a subset A of X is a ϵ-net if for all $x \in X$, $dist(x, A) < \epsilon$.

[6]Introduced in [34, p. 332], from which we generalize to the quaternionic setting, it states that for a quaternionic right Banach space X_R and for $U \subset X''_R$ of finite dimension, if $\epsilon > 0$, then there exists a one-to-one operator $T : U \to X_R$ with $T(x) = x$ for all $x \in U \cap X_R$ and $\|T\| \left\|T^{-1}\right\| < 1 + \epsilon$.

Let $A : X'' \to Y''$ be such that $\text{rank}(A) < k$ and $\|T'' - A\| < a_k(T'') + \epsilon$. Once again, from principle of local reflexivity we have a bijection $\psi : A(X'') \to Y''$ such that

$$\|\psi\| \left\|\psi^{-1}\right\| < 1 + \epsilon \text{ and } \psi\Big|_{A(X'')\cap K_Y(Y)} = Id.$$

Let $E = G \cup A(X'')$ and define

$$\phi : E \to Y, x \mapsto \begin{cases} \tilde{\varphi}(x) & \text{if } x \in G, \\ \psi(x) & \text{if } x \in A(E'') \cap G \end{cases}.$$

Observe that, if $x \in A(X'') \cap G \subset G \subset K_Y(Y)$ then $K_Y\psi(x) = x = K_Y\tilde{\varphi}(x)$. This implies that $\psi(x) = \tilde{\varphi}(x)$ for $x \in A(X'')\cap G$ meaning that ϕ is indeed is well-defined. Finally, consider the operator $\phi A K_X : X \to Y$. Clearly, $\text{rank}(\phi A K_X) < k$ and for $x \in B_X$ we have

$$\|Tx - \phi A K_X x\|_Y = \|K_Y(Tx - \phi A K_X)x\|_{Y''} = \|T'' K_X x - K_Y \phi A K_X x\|_{Y''}.$$

By construction, $T''K_X x \in G$, thus it follows that $K_Y\phi T''K_X x = T''K_X x$ and therefore,

$$\|T''K_X x - K_Y\phi A K_X x\|_{Y''} = \|K_Y\phi T''K_X x - K_Y\phi A K_X x\|_{Y''} < \|K_Y\| \, \|\phi\| \, (\alpha_k(T'') + \epsilon).$$

That is, $\alpha_k(T) \le \alpha_k(T'')$ for each k. □

In the same paper it is shown that one can not extend this result to bounded operators. Indeed, a counterexample is given:

$$a_2(Id : \ell_1 \to c_0) = 1, \qquad a_2(Id : \ell_1 \to \ell_\infty) = \frac{1}{2}.$$

3.2.2 Gelfand Numbers

The Gelfand widths, introduced in [47], are used to measure the width of a given bounded set using sets of a specific codimension. More precisely, for a compact subset K of X, the Gelfand width, $c_n(K)$, is defined as follows:

$$c_n(K) = \inf_{L} \inf_{x \in K\cap L} \|x\|_X,$$

the infimum being taken over all $L \subset X$, of codimension n. As we mentioned in the beginning of this section although Gelfand widths and Gelfand numbers, which will be defined below, are conceptually similar, they only coincide under certain assumptions.

We follow the definition provided in [40, p. 206]. For $T \in B^R(X, Y)$, the n-th *Gelfand number* is defined as follows:

$$c_n(T) := \inf\{\left\|TJ_M^X\right\| : \text{cod}(M) < n\}$$

where the infimum is taken over all subspaces of X that have codimension n. Therefore, it is basis independent.

Theorem 3.2.3. *The map $gelf : T \to c_n(T)$ defines an additive and multiplicative s-number function. Moreover, it is the largest s-numbers that are injective.*

Proof. We prove the first two claims together.

A1. First, if $\operatorname{cod}(M) = 0$ then the natural injection its the inclusion function $M \ni x \to x \in X$ hence $\left\| J_M^X \right\| = 1$. Therefore $s_1(T) = \inf \left\| T J_M^X \right\| \le \|T\| \inf \left\| J_M^X \right\| = \|T\|$. The claim $s_n(T) \ge s_{n+1}(T)$, follows from

$$\{M : \operatorname{cod}(M) < n\} \subseteq \{M : \operatorname{cod}(M) < n+1\}.$$

A2*. As a consequence of preceding proof it follows that

$$\begin{aligned} s_{n+m-1}(S+T) =& \inf \left\{ \left\| (S+T) J_M^X \right\| : \operatorname{codim}(M < n+m-1) \right\} \\ \le& \inf\{ \left\| S J_M^X \right\| : \operatorname{codim}(M) < n+m-1 \} \\ &+ \inf\{ \left\| T J_M^X \right\| : \operatorname{codim}(M) < n+m-1 \} \\ \le& \inf\{ \left\| S J_M^X \right\| : \operatorname{codim}(M) < n \} \\ &+ \inf\{ \left\| T J_M^X \right\| : \operatorname{codim}(M) < m \} \\ =& s_n(S) + s_m(T). \end{aligned}$$

A3*. We adapt the proof that can be found in [39, p. 149]. Consider $T \in B^R(X,Y)$ and $S \in B^R(Y,Z)$. Fix $m, n \in \mathbb{N}$. Given $\epsilon > 0$, we can find subspaces E and F of X and Y, respectively, such that

$$\begin{aligned} \left\| T J_E^X \right\| &\le (1+\epsilon) s_n(T) \text{ and } \operatorname{codim}(E) < n \\ \left\| S J_F^Y \right\| &\le (1+\epsilon) s_m(S) \text{ and } \operatorname{codim}(F) < m \end{aligned}$$

Define $M := E \cap T^{-1}(F)$. Then it follows from

$$\operatorname{codim}(M) \le \operatorname{codim}(E) + \operatorname{codim}(F) < m+n-1$$

that

$$c_{m+n-1}(ST) \le \left\| ST J_M^X \right\| \le \left\| S J_F^Y \right\| \left\| T J_E^X \right\| \le (1+\epsilon)^2 s_m(T) s_n(S).$$

A4. If $\dim(X) \ge n$ then $\left\| Id_X J_M^X \right\| = \left\| J_M^X \right\| = 1$.

A5. If $\operatorname{rank}(T) < n$ then $\operatorname{codim}(\ker(T)) < n$. Since $M \subset X$ is chosen such that $\operatorname{codim}(M) < n$ then it follows that $J_M^X(M) \subseteq \ker(T)$ thus $\left\| T J_M^X \right\| = \|0\| = 0$.

Finally, consider s to be any injective s-number function. Then for all $S \in B(E,F)$[7]

$$s_n(S) = s_n(J_F S) \le a_n(J_F S) = c_n(S). \qquad \square$$

As a preliminary illustration of the classification of operators achievable through s-numbers, we now present the following result, which is due to H.E. Lacey in [32]. As it is a purely topological argument (recall that $\mathbb{H}$ can be identified with $\mathbb{R}^4$), it consists in similar arguments as those presented in [37, p. 91]. Nevertheless, for the sake of completeness, we provide the proof here.

[7]Recall that $J_Y : Y \hookrightarrow Y^{inj}$. Therefore, if $T : X \to Y$ then $J_Y T \in B(X, Y^{inj})$ where Y^{inj} has the extension property. Then Theorem 3.3.3 allows us to conclude that $a_n(J_Y T) = c_n(J_Y T)$

Theorem 3.2.4. *A right operator T is compact if and only if $(c_n(T))_n \to 0$.*

Proof. Consider a right compact operator T acting between right Banach spaces X and Y. Then, for each $\epsilon > 0$, there are $y_1, \ldots, y_n \in Y$ such that the set

$$\{y_i + \epsilon B_Y : 1 \leq i \leq n\} \quad \text{is a finite cover of } T(B_X).$$

For $i = 1, \ldots n$, consider b_i for which $|\langle y_i, b_i\rangle| = \|y_i\|$ and set

$$M := \{x \in E : \langle Tx, b_j\rangle = 0 \text{ for } i = 1, \ldots n\}.$$

If $x \in M \cap B_X$ then we can take y_j such that $\|Tx - y_j\| \leq \epsilon$ and since

$$\|y_j\| = |\langle y_j, b_j\rangle| \leq |\langle y_j - Tx, b_j\rangle| + |\langle Tx, b_j\rangle| \leq \epsilon$$

we conclude that

$$\left\|TJ_M^X\right\| \leq \|Tx\| \leq \|Tx - y_j\| + \|y_j\| \leq 2\epsilon.$$

This gives us the desired claim since $c_n(T) \leq \left\|TJ_M^X\right\|$. Conversely, assume that $c_n(T) \to 0$. Then, for each $\epsilon > 0$, there is a finite codimensional M for which $\left\|TJ_M^X\right\| \leq \epsilon$. Let N be such that $X = M \oplus N$ and denote the corresponding projections by P_N and P_M. Set $\delta = \min\left(\frac{\epsilon}{\|T\|}, 1\right)$. Since P_N is of finite rank, there are $x_1, \ldots x_m \in B_X$ such that

$$\{P_N x_i + \delta B_X\}_{i=1}^n \text{ covers } P_N(B_X).$$

Therefore, for $x \in B_X$, we can choose x_i with $\|P_N x - P_N x_i\| \leq \delta$. Hence, it follows that

$$\|P_M x - P_M x_i\| \leq \|x - x_i\| + \|P_N x - P_N x_i\| \leq 2 + \delta \leq 3$$

and so

$$\|Tx - Tx_i\| \leq \|TP_M x - TP_M x_i\| + \|TP_N x - TP_N x_i\| \leq 3\left\|TJ_M^X\right\| + \delta\|T\| \leq 4\epsilon.$$

Since ϵ was arbitrary we have therefore found a finite subcover of $T(B_X)$, more precisely being given by

$$T(B_X) = \bigcup_{i=1}^n \{Tx_i + 4\epsilon B_Y\}.$$ □

3.2.3 Kolmogorov numbers

The concept shares similarities with Gelfand widths, but in this instance, the approach is based on dimensionality rather than codimensionality. Despite being a minor modification, it is enough to ensure the equivalence between the Kolmogorov number (of the identity operator) and the Kolmogorov width, (this will be clarified below). The following diagram provides a visual representation of the computation of the first Kolmogorov width of a two-dimensional set.

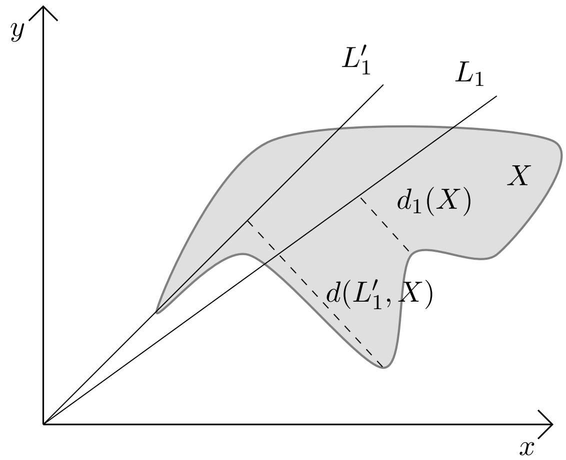
The Kolmogorov n-width of the subset X of Y is given by

$$d_n(X) = \inf_{L_n} d(L_n, X)$$

L_n being an n-dimensional subset of Y, where

$$d(L_n, X) = \sup_{y \in X} \inf_{l \in L_n} \|y - l\| .$$

For instance, the Kolmogorov n-width of a circle will be its radius while the n-width of an ellipse will be it semi-minor axis.



The classic Kolmogorov numbers were introduced in [40, p. 208]. We extend this definition for $T \in B^R(X, Y)$, the n-th *Kolmogorov number* is defined as follows:

$$d_n(T) := \inf\{\left\|Q_N^Y T\right\| : \dim(N) < n\}$$

where the infimum is taken over all n-dimensional subspaces of Y, N. Therefore it is basis independent.

Theorem 3.2.5. *The map kolm* : $T \to d_n(T)$ *is an additive and multiplicative s-number function. Moreover, they are the largest s-numbers that are surjective.*

Proof. We prove the first two claims together

A1. First, if $\dim(N) = 0$ then the natural surjection its the identity map, thus $\left\|Q_N^Y\right\| = 1$. Therefore $s_1(T) = \inf \left\|Q_N^Y T\right\| \leq \|T\| \inf \left\|Q_N^Y\right\| = \|T\|$. Analogously to the cases presented so far, the remainder follows from

$$\{N : \dim(N) < n\} \subseteq \{N : \dim(N) < n + 1\}.$$

A2*.

$$\begin{aligned}
d_{n+m-1}(S+T) =& \inf\{\left\|Q_N^Y (S+T)\right\| : \dim(N) < n+m-1\} \\
\leq& \inf\{\left\|Q_N^Y S\right\| : \dim(N) < n+m-1\} \\
&+ \inf\{\left\|Q_N^Y T\right\| : \dim(N) < n+m-1\} \\
=& d_{n+m-1}(S) + d_{n+m-1}(T) \leq d_n(S) + d_m(T)
\end{aligned}$$

A3*. It is a consequence of the multiplicativity of Gelfand numbers and Theorem 3.3.5.

A4. Consider Y with $\dim(Y) \geq n$. Since we require F with $\dim(F) < n$ then $Y \neq F$. As such $d_n(Id) = \left\|Q_F^Y Id_Y\right\| = \left\|Q_F^Y\right\| = 1$[8] .

[8]This follows from

$$Q_F^Y(B_X) = B_{Y/F}.$$

A5. If $rank(T) < n$ then we can take $N = Y$ and thus Q_N^Y as the zero operator which will imply that $\left\|Q_N^Y T\right\| = 0$.

Finally, consider any surjective s-number function, s. Then, as a consequence of Corollary 3.3.4, for all $T \in B^R(X,Y)$ we have that $s_n(T) = s_n(Q_F^Y T) \leq a_n(Q_F^Y T) = d_n(T)$. □

As a consequence of Theorems 3.2.4 and 3.3.5 we conclude that also Kolmogorov numbers describe compact operators.

Theorem 3.2.6. *A right operator T is compact if and only if $(d_n(T))_n \to 0$.*

3.2.4 Hilbert Numbers & Isomorphism numbers

Motivation behind isomorphism numbers

Consider X and Y as two real, n-dimensional Banach spaces. The notion of the Banach-Mazur distance was introduced to quantify how well two Banach spaces are isomorphic. It is defined as follows:

$$d(X,Y) = \inf\left\{\|T\|\left\|T^{-1}\right\| : X \xrightarrow{T} Y \text{ is an isomorphism}\right\}. \tag{3.3}$$

Let $X/_{\sim}$ denote the set of all equivalence classes of real n-dimensional normed spaces, where the equivalence relation is given by the existence of an isometry, i.e., $X \sim Y$ if and only if there exists an isometry from X to Y. It can be shown that the space $X/_{\sim}$ equipped with $\log d$ forms a metric space, which is known as the *Minkowski compactum.*

In what follows we will use the notations $M_n = \left(X/_{\sim}, \log d\right)$ and $\rho(X,Y) = \log d(X,Y)$. The lemma presented below, also known as *Kadets-Snobar Theorem*, which we will later utilize, is a classical result concerning the geometry of M_n. The classic proof can be found in [27] which can be directly extended to our case.

Lemma 3.2.7. *For any $X \in M_n$ there exists a projection $P : X \to \ell_2^{(n)}$ such that $X = P(M_n)$ and $\|P\| \leq \sqrt{n}$.*

Isomorphism numbers

A slight variation of the metric defined in (3.3) led to the concept of isomorphism numbers, which were first presented in [40, p. 205]. For any operator $T \in B^R(X,Y)$, the isomorphism numbers are defined as follows: if $\text{rank}(T) < n$, we set $i_n(T) = 0$. Otherwise, we define it as follows:

$$i_n(T) = \sup\{\|A\|^{-1}\|B\|^{-1}\},$$

where the supremum is taken over all possible combinations of operators A and B as given by Lemma 2.2.5. Also in [40, p. 205] it is observed that if $T \in B^R(X,Y)$, $A \in B^R(G,X)$ and

---

Indeed, on the one hand, $Q_F^Y(B_X) \subseteq B_{Y/F}$. On the other hand, let $y \in Y$ for which $\epsilon := 1 - \text{dist}(y,F) > 0$. Then there is a $y_0 \in F$ with $\|y - y_0\| < \text{dist}(y,F) + \epsilon = 1$ and hence $Q_F^Y(y) = Q_F^Y(y - y_0) \in Q_F^Y(B_X)$. As such, $\left\|Q_F^Y\right\| \leq 1$. Finally, if $\left\|Q_F^Y\right\| < 1$ it would therefore follow

$$Q_F^Y(B_X) \subseteq \|Q_F^Y\| B_{Y/F} \subsetneq B_{Y/F}.$$

$B \in B^R(Y,G)$ are such that $I_G = BTA$ and $\dim(G) \geq n$, then for each s-number function there holds

$$1 = s_n(I_G) \leq \|B\| \, s_n(T) \, \|A\|$$

which implies that $i_n(S) \leq s_n(S)$, meaning that the isomorphism numbers are the smallest s-numbers.

Hilbert numbers

Consider a right $\mathbb{H}$-Hilbert Space, H. The classic case is studied in [3]. For $T \in B^R(X,Y)$, the n-th *Hilbert number* is defined as follows:

$$h_n(T) = \sup\{s_n(BTA) : A \in B(H,X),\ B \in B(Y,H) \text{ such that } \|A\|, \|B\| \leq 1\}.$$

Recall that, since the operator BTA is acting between Hilbert Spaces, it does not really matter the choice of which s-number we take, and because of it, it is immediate that it leads to an s-numbers function. Moreover, from the additivity and multiplicativity of s-numbers on Hilbert spaces we have the following

Theorem 3.2.8. *The map $hilb : T \to h_n(T)$ defines an additive and multiplicative s-number function.*

Observe that, according to this definition, in particular we have have, for $A \in B(H,X)$, $B \in B(Y,H)$ such that $\|A\|, \|B\| \leq 1$

$$h_n(T) \leq i_n(BTA) \leq \|B\| \, i_n(T) \, \|A\| \leq i_n(T).$$

But since, the isomorphism numbers are the smallest s-numbers this implies that these are the same concepts. Therefore, despite appearing initially distinct and sometimes studied in separate, the isomorphism numbers are, in fact, equivalent to the Hilbert numbers.

Theorem 3.2.9. *[3, p. 184] The Hilbert numbers are completely symmetric.*

Proof. Consider $\epsilon > 0$. Then, by Lemma 2.2.5, there exists a $\rho > 0$ and operators $B \in B^R(H,Y')$, $A \in B^R(X',H)$ such that $\|B\|, \|A\| \leq 1$ for which

$$h_n(T') \leq (1+\epsilon)\rho \text{ and } \rho I_n = AT'B.$$

From the principle of local reflexivity there are $C \in B^R(Y,H)$ and $X \in B^R(H,X)$ such that $\|X\| \leq (1+\epsilon)\|A\|$ for which

$$B = C' \text{ and } Aa = X'a \text{ for } a \in Im(T'B)$$

Since $\rho I_n = AT'B = X'T'C' = (CTX)'$ it follows that $h_n(T')(1+\epsilon)^{-2} \leq h_n(T)$.

The converse inequality follows from the uniqueness of s-numbers between Hilbert spaces, indeed, considering the operators B and A from the definition of Hilbert numbers we observe that $BTA \in B(H)$ and thus $h_n(T) = s_n(BTA) \leq \|B\| \, s_n(T) \, \|A\| \leq s_n(T)$ for any s-number function s_n. Now observe that we can define $l_n(T) := h_n(T')$ which will be an s-number function, and as such $h_n(T) \leq l_n(T)$[9]. Therefore, $h_n(T) \leq h_n(T')$. □

[9]since Hilbert numbers are the smallest s-numbers (because they coincide with isormorphism numbers)

3.2.5 Weyl numbers

The classic Weyl numbers were introduced in [41, p. 149]. For an operator $T \in B^R(X,Y)$, the n-th Weyl number is given by

$$x_n(T) = \sup a_n(TA),$$

where the supremum is taken over all contractions $A \in B^R(\ell_2, X)$. In this way we attain basis independence.

Theorem 3.2.10. *The map weyl* $: T \to x_n(T)$ *is an additive, multiplicative and injective s-number function.*

Proof. The fact that it is an s-number function and the additivity follows immediately from definition. To prove that it is multiplicative consider $T \in B^R(X,Y)$ and $S \in B^R(Y,Z)$ and fix any m and n. For $X \in B^R(\ell_2, F)$ and $\epsilon > 0$ consider $B, A \in B^R(\ell_2, F)$ to be such that $\text{rank}(B) < n$ and $\text{rank}(A) < m$ and

$$\|TX - B\| \leq (1+\epsilon)a_n(TX), \quad \|S(TX-B) - A\| \leq (1+\epsilon)a_n(S(TX-B)).$$

Since $\text{rank}(A+SB) \leq \text{rank}(A)+\text{rank}(B) < m+n-1$ it follows that

$$\begin{aligned} a_{n+m-1}(STX) &\leq \|STX - SB - A\| \leq (1+\epsilon)a_m(A(TX-B)) \\ &\leq (1+\epsilon)x_m(S)\,\|TX - B\| \leq (1+\epsilon)^2 x_m(S)a_n(TX) \\ &\leq (1+\epsilon)^2 x_m(S)x_n(T)\,\|X\|\,. \end{aligned}$$

In particular it follows $x_{n+m-1}(ST) \leq x_n(T)x_m(S)$.

It remains to prove injectivity. Consider a metric injection $J_{Y_0}^Y$. Since Hilbert spaces have the metric extension property, then, by Theorem 3.3.1 and Corollary 3.3.4 it follows that

$$a_n(J_{Y_0}^Y TA) = c_n(J_{Y_0}^Y TA) = c_n(TA) = a_n(TA)$$

for any $A \in B^R(\ell_2, X)$. In particular, $x_n(J_{Y_0}^Y T) = x_n(T)$. $\square$

Next, we present properties regarding Weyl numbers that will prove useful throughout this thesis. We will require the notion of *absolutely (p,q)-summable operators*, $\mathfrak{P}_{p,q}$, which, for the moment, can be thought as operators that map p-summable sequences into weakly q-summable sequences. A precise definition of this class of operators is given in Definition 4.1.4.

The next lemmas are an adaption of [37, pp. 123–125] to the quaternionic setting. Note that in these lemmas we have to use the appropriate tools of the quaternionic setting, namely, the Study determinant as we have explained in (2.2) but also the quaternionic analogue of the Schmidt representation that we discuss later on in (4.1).

Lemma 3.2.11. *Let H and K denote right Hilbert Spaces and consider $T \in B^R(H,K)$.Then*

$$|Sdet(\langle Tx_i, y_j\rangle)| \leq \prod_{k=1}^{n} a_k(T),$$

for all orthonormal families $(x_1, \ldots, x_n)$ and $(y_1, \ldots y_n)$.

Proof. Let us denote the standard basis of ℓ_2 by (e_i). Consider $X \in B^R(\ell_2, H)$ and $Y \in B^L(\ell_2, K)$ defined by

$$Xx := \sum_{k=1}^{n} \langle x \rangle_k x, e_k \quad \text{and} \quad Yy := \sum_{k=1}^{n} \langle y, e_k\rangle y_k.$$

Moreover, take the Schimdt representation

$$Y'TX(x) = \sum_{k=1}^{\infty} Vv_n \sigma_n \langle u_n, x\rangle,$$

where (u_n) and (v_n) are extended orthonormal sequences in ℓ_2 and $\sigma_n = a_n(Y'TX)$. Therefore, since $\|Y'\|, \|X\| \leq 1$ it follows that $a_n(Y'TX) \leq a_n(T)$. Thus, if $h > n$ we have $\sigma_n = 0$, hence

$$\langle Tx_i, y_j\rangle = \langle Y'TXe_i, e_j\rangle = \sum_{h=1}^{n} a_h(Y'TX) \langle e_i, u_h\rangle \langle Vv_h, e_j\rangle.$$

This implies that

$$Sdet(\langle Tx_i, y_j\rangle) = Sdet(\langle e_i, u_h\rangle) \prod_{h=1}^{n} a_h(Y'TX) Sdet(\langle Vv_h, e_j\rangle).$$

It follows by Hadamard's inequality[10] that $|Sdet(\langle e_i, u_h\rangle)|, |Sdet(\langle Vv_h, e_j\rangle)| \leq 1$ and therefore

$$|Sdet(\langle Tx_i, y_j\rangle)| \leq \prod_{h=1}^{n} a_h(Y'TX) \leq \prod_{h=1}^{n} a_h(T),$$

since $\|Y'\|, \|X\| \leq 1$. □

A particular case of the above mentioned class of operators are the so called *right absolutely 2-summable operators*, $\mathfrak{B}_{2,2}^R$ (which we will write $\mathfrak{B}_2^R$).

Lemma 3.2.12. *Consider $T \in B^R(E,F)$ and $S \in \mathfrak{B}_2^R(F,G)$. Then*

$$\left(\prod_{k=1}^{2n-1} x_k(ST)\right)^{\frac{1}{2n-1}} \leq e^{\frac{\left\|S|\mathfrak{B}_2^R\right\|^2}{2n-1}} \left(\prod_{k=1}^{n} x_k(T)\right)^{\frac{1}{n}}.$$

[10] Hadamard's inequality states that, for a matrix $M = (\mu_{ij})$ there holds

$$|det(M)| \leq \prod_{i=1}^{n} \left(\sum_{j=1}^{n} |\mu_{ij}|^2\right)^{\frac{1}{2}}.$$

The same technique can be applied to show a similiar result for the Study determinant.

Proof. Since Weyl numbers are multiplicative we have

$$x_{2k-1}(ST) \le x_k(S)x_k(T) \text{ and } x_{2k}(ST) \le x_k(S)x_{k+1}(T).$$

Separating the product between odd and even indices, it follows from the above inequalities that

$$\prod_{k=1}^{2n-1} x_k(ST) = \prod_{k=1}^{n} x_{2k-1}(ST) \prod_{k=1}^{n-1} x_{2k}(ST) \le \prod_{k=1}^{n} x_k(S) \prod_{k=1}^{n-1} x_k(S) \prod_{k=1}^{n} x_k(T) \prod_{k=1}^{n-1} x_{k+1}(T).$$

From Lemma 4.1.5 we know that $\sqrt{k} x_k(S) \le \left\| S|\mathfrak{B}_2^R \right\|$. Thus

$$\prod_{k=1}^{n} x_k(S) \prod_{k=1}^{n-1} x_k(S) \le \sqrt{\frac{\|S|\mathfrak{B}_2^R\|^{2n}}{n!} \frac{\|S|\mathfrak{B}_2^R\|^{2(n-1)}}{(n-1)!}} \le e^{\|S|\mathfrak{B}_2^R\|^2}$$

the latter inequality follows by induction. It follows by the first axiom of a s-number function that $(\prod_{k=1}^{n} x_k(T))^{\frac{1}{n}} \le \|T\|$. Therefore,

$$\prod_{k=1}^{n} x_k(T) \prod_{k=1}^{n-1} x_{k+1}(T) = \|T\|^{-1} \left(\prod_{k=1}^{n} x_k(T) \right)^2 \le \left(\prod_{k=1}^{n} x_k(T) \right)^{\frac{2n-1}{n}}.$$

The desired inequality follows as a combination of both of these inequalities. □

3.2.6 Chang numbers

First introduced in [38, p. 330], for an operator $T \in B^R(X,Y)$, the Chang numbers are the natural dual counterpart of Weyl numbers as we will see in Theorem 3.3.5. They are defined as follows,

$$y_n(T) = \sup a_n(BT),$$

where the supremum is taken over all contractions $B \in B^R(Y, \ell_2)$. In this way we attain basis independence.

Theorem 3.2.13. *The map chang* $: T \to y_n(T)$ *is an additive, multiplicative and surjective s-number function.*

Proof. All the claims easily follow from the fact that the approximation numbers are an additive and multiplicative s-number function. As a consequence of Theorem 3.3.3 and the surjectivity of Kolmogorov numbers, it follows

$$d_n(BTQ_{X_0}^X) = d_n(BT) = a_n(BT)$$

for any $B \in B^R(Y, \ell_2)$. In particular, it we have $y_n(TQ_{X_0}^X) = y_n(T)$. □

3.2.7 Nuclear numbers

The notion of *nuclear operator* goes back to the works of A. Grothendieck in [23]. In a concise manner, these are the operators acting between Banach Spaces for which the classic notion of the trace of an operator holds true. Hereby, a nuclear operator is linear and bounded operator $T : X \to Y$ such that there are $(x_i') \in X'$ and $(y_i) \in Y$ for which

$$T = \sum_{i=1}^{\infty} x_i' \otimes y_i \qquad \text{and} \qquad \sum_{i=1}^{\infty} \|x_i'\| \, \|y_i\| < \infty.$$

To the former term we refer to *nuclear representation of the operator* T. We equip it with the norm

$$\|T\| = \inf \sum_{i=1}^{\infty} \|x_i'\| \, \|y_i\| \,,$$

where the infimum is taken over all nuclear representations of T.

These concept was latter generalized by Pietsch in [39, p. 243] to *(r,p,q)-nuclear operators* as follows: for a sequence $(\sigma_i) \in \ell_r$, $(x_i) \in w(q')$ and $(y_i) \in w(p')$ a linear and bounded operator $T : X \to Y$ is said (r,p,q)-nuclear operator if

$$T = \sum_{i=1}^{\infty} \sigma_i x_i \otimes y_i \qquad \text{and} \qquad \sum_{i=1}^{\infty} \|\sigma_i|\ell_r\| \, \|x_i'|w(q')\| \, \|y_i|w(p')\| < \infty. \tag{3.4}$$

The set of these operators shall be denoted by $\mathfrak{N}_{r,p,q}(X,Y)$. Moreover, the former term in (3.4) is referred to as *(r,p,q)-nuclear representation of the operator* T. We equip it with the norm

$$\|T|\mathfrak{N}_{r,p,q}\| := \inf \sum_{i=1}^{\infty} \|\sigma_i|\ell_r\| \, \|x_i'|w(q')\| \, \|y_i|w(p')\| \,,$$

where the infimum is taken over all (r,p,q)-nuclear representations of T. As noted in [39, p. 382], for an Hilbert space H, $\mathfrak{N}_{1,1,2}(H,H) = \mathfrak{S}_1(H)$. Therefore, Grothendieck-Lidskii trace formula[11] appears as a special case for nuclear operators. Moreover, since $\mathfrak{N}_{\frac{2}{3},1,1} \subseteq \mathfrak{N}_{1,1,2}$ the result also holds for $(\frac{2}{3},1,1)$. However, for $\frac{2}{3} < r \leq 1$ one cannot expect a trace formula to hold. Indeed, counter examples have already been built, cf. [39, p. 138].

These considerations, together with the role that s-number theory play in extending the notion of trace of an operator to a Banach Space, as we will see in 4.1.2, led us to introduce the concept of *nuclear numbers*.

Consider the sequence of (σ_n), (x_n) and (y_n) for which $\sum_{i=1}^{\infty} \|\sigma_n|\ell(r)\| \, \|x_n|w(q')\| \, \|y_n|w(p')\| = \left\|T|R_{(r,p,q)}\right\|$. Then we define the n-nuclear number of the operator T as

$$n_n(T) = \|\sigma_n^*|\ell(r)\| \, \|x_n^*|w(q')\| \, \|y_n^*|w(p')\| \,.$$

where the * notation represent the decreasing rearrangement of the corresponding sequence.

Theorem 3.2.14. *Let $T \in B^R(X,Y)$. Then nuclear : $T \to (n_n(T))_n$ is an additive s-number function.*

[11] The Grothendieck-Lidskii trace formula states that, for a given operator T there holds $Tr(T) = \sum_{i=1}^{\infty} \lambda_i(T)$, $(\lambda_i(T))$ being the sequence of eigenvalues of T.

Proof. **A1.** Follows directly by construction.

A2*. If the (r,p,q)-nuclear reprsentation of $S+T$ is given by $\sum_{i=1}^{\infty} \sigma_i x_i \otimes y_i$ then $\sigma_i = \lambda_i + \mu_i$ where $S = \sum_{i=1}^{\infty} \mu_i x_i \otimes y_i$ and $T = \sum_{i=1}^{\infty} \lambda_i x_i \otimes y_i$. It then follows that

$$\begin{aligned} n_{n+m-1}(S+T) &= \|\sigma^*_{n+m-1}|\ell_r\| \, \|x^*_{n+m-1}|w(q')\| \, \|y^*_{n+m-1}|w(p')\| \\ &\leq \|\lambda^*_{n+m-1}|\ell_r\| \, \|x^*_{n+m-1}|w(q')\| \, \|y^*_{n+m-1}|w(p')\| \\ &\quad + \|\mu^*_{n+m-1}|\ell_r\| \, \|x^*_{n+m-1}|w(q')\| \, \|y^*_{n+m-1}|w(p')\| \\ &= s_{n+m-1}(S) + s_{n+m-1}(T) \leq s_n(S) + s_m(T). \end{aligned}$$

A3. Suppose that the (r,p,q)-nuclear representation of T is given by $\sum_{i=1}^{\infty} \sigma_i x_i \otimes y_i$. Then the (r,p,q)-nuclear representation of BTA is given by $\sum_{i=1}^{\infty} \sigma_i A' x_i \otimes B y_i$. Therefore

$$\begin{aligned} n_n(BTA) &= \|\sigma^*_i|\ell_r\| \, \|A'x^*_i|w(q')\| \, \|By^*_i|w(p')\| \\ &\leq \|\sigma^*_i|\ell_r\| \, \|A\| \, \|x^*_i|w(q')\| \, \|B\| \, \|y^*_i|w(p')\| = \|B\| \, n_n(T) \, \|A\| \end{aligned}$$

A4. Since $Id = 1 \otimes 1$ then for every n we have $n_n(Id) = 1$.

A5. Without loss of generality let $rank(T) = n-1$. Then $T = \sum_{i=1}^{n-1} \sigma_i x_i \otimes y_i$. We can technically extend this sum to n since after the n-1 term we are just adding zeros. It immediately follows that $s_n(T) = 0$.

□

3.3 Relations between s-numbers

Theorem 3.3.1. *[37, p. 90] For a right $\mathbb{H}$-Hilbert space H and a right $\mathbb{H}$-Banach space Y, if $T \in B^R(H,Y)$, then $c_n(T) = a_n(T)$. On the other hand if $T \in B^R(X,H)$, for a right $\mathbb{H}$-Banach space X, there holds $d_n(T) = a_n(T)$.*

Proof. We only show the first claim as the second one follows the same lines. Indeed, it remains to show that $a_n(T) \leq c_n(T)$ for any n. Consider a n-codimensional subspace of H, M. Let $P \in B(H)$ denote the orthogonal projection from H onto M and set $L = T(Id - P)$. As $M \subseteq \ker(L)$ it follows that rank(L) =codim(ker(L)) ≤codim(M) $< n$. Therefore,

$$a_n(T) \leq \|T - L\| = \|TP\| = \left\|TJ^H_M\right\|.$$

Since M was arbitrary the claim follows. □

Consequently, for a right $\mathbb{H}$-Hilbert space, H, and a right $\mathbb{H}$-Banach space Y, from the injectivity of Gelfand numbers, if $T \in B^R(H,Y)$ there holds that

$$x_n(T) = a_n(TX) = c_n(TX) = c_n(T) = a_n(T).$$

Analogously from the surjectivity of Kolmogorov numbers we can write $y_n(T) = a_n(T)$.

Corollary 3.3.2. *[37, p. 94] Let X and Y be right linear $\mathbb{H}$-Banach spaces. For all $T \in B^R(X,Y)$, $x_n(T) \leq c_n(T)$. Analogously, $y_n(T) \leq d_n(T)$.*

Proof. Once again we only show the first claim. Consider a right linear $\mathbb{H}$-Hilbert space, H. From Theorem 3.3.1 it follows that, for $X \in B^R(H, X)$, such that $\|X\| \leq 1$

$$x_n(T) \leq a_n(TX) = c_n(TX) \leq c_n(T) \|X\| \leq c_n(T). \qquad \square$$

Next we present a generalization of Theorem 3.3.1.

Theorem 3.3.3. *[40, p. 205] Let X and Y be right $\mathbb{H}$-Banach Spaces. If $T \in B^R(X, Y)$ and additionally Y has the metric extension property then $c_n(T) = a_n(T)$. Moreover, if additionally X has the metric lifting property $d_n(T) = a_n(T)$.*

Proof. Assume that Y has the extension property. Take $S \in B^R(X, Y)$. Since a_n is the largest s-number it suffices to show that $a_n(S) \leq c_n(S)$. Considering $\epsilon > 0$ we can choose a subspace M of X such that

$$\left\| SJ_M^X \right\| \leq c_n(S) + \epsilon$$

for which $\operatorname{codim}(M) < n$. As Y has the extension property, there is an extension $T \in B^R(X, Y)$ of SJ_M^X for which $\|T\| = \left\| SJ_M^X \right\|$. Set $A = S - T$. By definition, for all $x \in M$, $Ax = 0$, therefore $\dim(A) < n$. Hence,

$$a_n(S) \leq \|S - A\| = \|T\| = \left\| SJ_M^X \right\| \leq c_n(S) + \epsilon.$$

Now assume that X has the lifting property. Take $S \in B^R(X, Y)$. Again it suffices to show that $a_n(S) \leq d_n(S)$. Considering $\epsilon > 0$ we can choose a subspace N of F such that

$$\left\| Q_N^Y S \right\| \leq d_n(S) + \epsilon$$

for which $\dim(N) < n$. As X has the lifting property, there is a lifting $T \in B^R(X, Y)$ of $Q_N^Y S$ for which $\|T\| = (1 + \epsilon) \left\| Q_N^Y S \right\|$. Set $A = S - T$. By definition, for all $x \in X$, $Ax \in N$, therefore $\dim(A) < n$. Therefore

$$a_n(S) \leq \|S - A\| = \|T\| = (1 + \epsilon) \left\| Q_N^Y S \right\| \leq (1 + \epsilon) d_n(S). \qquad \square$$

Corollary 3.3.4. *[38, p. 329] Given X and Y, we may choose a metric surjection Q from some $\ell^1(I)$ onto X and a metric injection J from Y into some $\ell^\infty(I)$ in such that $c_n(T) = a_n(JT)$ and $d_n(T) = a_n(TQ)$.*

Theorem 3.3.5. *[40, pp. 211–212] Consider a right linear and bounded operator, acting between right $\mathbb{H}$-Banach spaces, $T : X \to Y$. Then $c_n(T) = d_n(T')$ and $d_n(T) \geq c_n(T')$. Additionally, if the operator is compact, then $d_n(T) = c_n(T')$.*

Proof. Since J_Y is a metric injection it follows from Corollary 2.2.9 that J_Y' is a metric surjection. Thus, by the surjectivity of Kolmogorov numbers and the symmetry of approximation numbers, we have

$$d_n(T') = d_n(T' J_Y') \leq a_n(T' J_Y') \leq a_n(J_Y T) = c_n(T).$$

Analogously from Corollary 2.2.9, Q'_X is a metric injection which allows us to show that $c_n(T') \leq d_n(T)$. The regularity of Gelfand numbers allows to conclude that

$$c_n(T) = c_n(K_Y T) = c_n(T'' K_X) \leq c_n(T'') \leq d_n(T').$$

Now assume that the operator is compact. By definition of X^{sur} and by Lemma 2.2.10 it follows that $(X^{sur})'$ has the metric extension property. Now, Theorem 3.2.2 gives the complete symmetry of approximation numbers and with Theorem 3.3.3 we have

$$d_n(T) = a_n(SQ_X) = a_n(Q'_X T') = c_n(Q'_X T') \leq c_n(T'). \qquad \square$$

Theorem 3.3.6. *[38, p. 329] Consider a right linear and bounded operator acting between right $\mathbb{H}$-Banach Spaces, T. Then $x_n(T') = y_n(T)$ and $y_n(T') = x_n(T)$.*

Proof. Consider $E : \ell_2 \to X$ such that $x_n(T) = a_n(TX)$. Then, since $\|X'\| \leq \|X\|$, it follows from Theorem 3.3.1 that

$$x_n(T) = a_n(TX) = c_n(TX) = d_n(X'T') = a_n(X'T') = y_n(T').$$

On the other hand

$$x_n(T') = d_n(T'Y') = c_n(YT) = a_n(YT) = y_n(T). \qquad \square$$

The following lemma will be useful.

Lemma 3.3.7. *Let X be a right $\mathbb{H}$-Hilbert space and Y be a right $\mathbb{H}$-Banach space. If $T \in B^R(X, Y)$ and if $a_{2n-1}(T) > 0$ then for every $\epsilon > 0$ there exists an orthonormal family $(x_1, \dots x_n) \in X$ and a family $(b_1, \dots, b_n) \in Y'$ such that $\|b_k\| = 1$,*

$$a_{2k-1}(T) \leq (1 + \epsilon) |\langle Tx_k, b_k \rangle|$$

for $k = 1, \dots, n$ and $\langle Tx_i, b_j \rangle = 0$ whenever $1 \leq j < i \leq n$.

Proof. The required families can be constructed by induction. If $x_l, \dots x_{n-1} \in X$ and $b_l, \dots b_{n-1} \in Y'$ have already been found, then we define the subspace

$$M_n := \{x \in X : \langle x, x_k \rangle = 0 \text{ and } \langle Tx, b_k \rangle = 0 \text{ for } k = 1, \dots, n-1\}.$$

Since $\operatorname{codim}(M_n) < 2n - 1$, it follows from Theorem 3.3.1 that $a_{2n-1}(T) = c_{2n-1}(T) \leq \left\| T J_{M_n}^X \right\|$. Hence there exists $x_n \in M_n$ such that $a_{2n-1}(T) \leq (1 + \epsilon) \|Tx_n\|$ and $\|x_n\| = 1$. Next we choose $b_n \in Y'$ with $|\langle Tx_n, b_n \rangle| = \|Tx_n\|$ and $\|b_n\| = 1$. $\square$

The following diagram indicates the "trivial" order of relation, in the sense that, the arrows point from the larger s-numbers to the smaller ones:

$$\begin{tikzcd}
 & c_n(T) \arrow[r] & x_n(T) \arrow[dr] & \\
a_n(T) \arrow[ur] \arrow[dr] & & & h_n(T) \\
 & d_n(T) \arrow[r] & y_n(T) \arrow[ur] &
\end{tikzcd}$$

The converse inequalities aren't so ideal. Indeed, they depend on constants that depend on the order of the s-number, as the next Lemma shows.

Lemma 3.3.8. *[37, pp. 115–117] For any $T \in B^R(X,Y)$ there holds*

1.

$$a_n(T) \leq \sqrt{2n}c_n(T) \text{ and } a_n(T) \leq \sqrt{2n}d_n(T);$$

2.

$$c_{2n-1}(T) \leq 2e\sqrt{n}\left(\prod_{k=1}^{n} x_k(T)\right)^{\frac{1}{n}} \text{ and } d_{2n-1}(T) \leq 2e\sqrt{n}\left(\prod_{k=1}^{n} y_k(T)\right)^{\frac{1}{n}};$$

3.

$$x_{2n-1}(T) \leq \sqrt{n}\left(\prod_{k=1}^{n} h_k(T)\right)^{\frac{1}{n}} \text{ and } y_{2n-1}(T) \leq \sqrt{n}\left(\prod_{k=1}^{n} h_k(T)\right)^{\frac{1}{n}}.$$

Proof. 1. Consider $S \in B^R(X,Y)$. For $\epsilon > 0$ we choose a subspace N of Y such that

$$\left\|Q_N^Y S\right\| \leq d_n(S) + \epsilon$$

and $\dim(N) < n$. Then by Lemma 3.2.7 there exists a projection $P \in B^R(X,Y)$ for which $N = P(Y)$ and $\|P\| \leq (n-1)^{\frac{1}{2}}$. Let $J(y+N) := y - Py$. Then $J \in B^R\left({}^{Y}/_{N}, Y\right)$. Moreover,

$$\|J\| \leq \|I_Y - P\| \leq 1 + \sqrt{n-1} \leq \sqrt{2n}.$$

From $S - PS = (I_Y - P)S = JQ_N^Y S$ we obtain

$$a_n(S) \leq \|S - PS\| \leq \|J\| \left\|Q_N^Y S\right\| \leq \sqrt{2n}(d_n(S) + \epsilon).$$

The other inequality is proved analogously.

2. Consider $\epsilon > 0$ and inductively choose $x_1, x_2, \ldots, x_{2n-1} \in E$ and $b_1, b_2, \ldots, b_{2n-1} \in F'$ such that $\|x_i\| \leq 1$, $\|b_j\| \leq 1$, $\langle Sx_i, b_j\rangle = 0$ for $i > j$ and $c_k(S) \leq |\langle Sx_k, b_k\rangle|$. Set

$$M_n = \{x \in E : \langle Sx, b_j\rangle = 0 \text{ for } j < n\}.$$

As M_n is a codimension $n-1$ subspace of E we find $x_n \in M_n$ such that $\|x_n\| \leq 1$ and

$$(1+\epsilon)\|Sx_n\| \geq \left\|SJ_{M_n}^E\right\| \geq c_n(S).$$

Moreover, we know that there is $b_n \in F'$ with $\|b_n\| \leq 1$ and $|\langle Sx_n, b_n\rangle| = \|Sx_n\|$. Thus the required (x_i) and (b_j) do exist. Now, define $X_n \in B^R(\ell_2^n, E)$ and $B_n \in B^R(F, \ell_2^n)$ via

$$X_n(\xi_n) := \sum_{i=1}^n \xi_i x_i \text{ and } B_n y := (\langle y, b_j\rangle).$$

It is clear by choice of x_n that $\|X_n\| \leq \sqrt{n}$ and $\|B_n|\mathfrak{B}_2\| \leq \sqrt{n}$. By construction, the matrix $S_n = B_n S X_n$ has upper triangular form it follows from Lemmas 3.2.11 and 3.2.12, from the choice of (x_n) and (b_n), and by definition of S_n that

$$\begin{aligned}
\frac{1}{1+\epsilon} c_{2n-1}(T) &\leq \left(\prod_{k=1}^{2n-1} |\langle Tx_k, b_k\rangle|\right)^{\frac{1}{2n-1}} = |Sdet(\langle B_{2n-1}TX_{2n-1}e_i, e_j\rangle)|^{\frac{1}{2n-1}} \\
&\leq \left(\prod_{k=1}^{2n-1} x_k(B_{2n-1}TX_{2n-1}))\right)^{\frac{1}{2n-1}} \leq e^{\frac{\|B_{2n-1}|\mathfrak{B}_2\|^2}{2n-1}} \left(\prod_{k=1}^{n} x_k(TX_{2n-1})\right)^{\frac{1}{n}} \\
&\leq 2e\sqrt{n} \left(\prod_{k=1}^{n} x_k(T)\right)^{\frac{1}{n}}.
\end{aligned}$$

Now take $\epsilon \to 0$. The remaining inequality is proved analogously.

3. Let $E \in L(\ell_2, X)$. Consider $u_1, \ldots u_n \in \ell_2$ and $b_1, \ldots, b_n \in Y'$ obtained as an application of Lemma 3.3.7 to the operator $TE \in L(\ell_2, Y)$, for a given $\epsilon > 0$. Define $U_n \in L(\ell_2)$ and $B_n \in L(Y, \ell_2)$ by

$$U_n := \sum_{k=1}^n e_k \otimes u_k \text{ and } B_n := \sum_{k=1}^n b_k \otimes e_k.$$

Then $\|U\| = 1$ and $\|B_n\| \leq \sqrt{n}$. Since the matrix $(\langle TEu_i, b_j\rangle = ((B_nTEU_ne_i, e_j))$ has upper triangular form, it follows from 3.2.11 that

$$\begin{aligned}
\frac{1}{1+\epsilon} a_{2n-1}(TE) &\leq \left(\prod_{k=1}^{n} |\langle TEu_k, b_k\rangle|\right)^{\frac{1}{n}} = (Sdet((B_nTEU_ne_i, e_j)))^{\frac{1}{n}} \\
&\leq \left(\prod_{k=1}^{n} a_k(B_nTEU_n)\right)^{\frac{1}{n}} \leq \sqrt{n} \left(\prod_{k=1}^{n} h_k(T)\right)^{\frac{1}{n}} \|E\|
\end{aligned}$$

Hence

$$\frac{1}{1+\epsilon} x_{2n-1}(T) \leq \sqrt{n} \left(\prod_{k=1}^{n} h_k(T)\right)^{\frac{1}{n}}. \qquad \square$$

CHAPTER 4

Quaternionic Operator Ideals Theory

This chapter focuses on ideals of quaternionic operators, utilizing an adapted axiomatic approach by A. Pietsch to the quaternionic framework. We examine examples, discuss the quaternionic analog of Schatten classes, and briefly explore specific components of operator ideals. The chapter concludes with considerations of the diagonal limit order.

4.1 Quaternionic ideals on Banach spaces

4.1.1 Basic concepts of ideal theory on Banach spaces

Following the intuition of the ring theoretic perspective an axiomatic approach to the operator ideal theory is introduced in [39, p. 45]. We extend these considerations to the quaternionic framework as follows

Definition 4.1.1. Consider right $\mathbb{H}$-Banach spaces X and Y. We call $A^R(X,Y) \subseteq B^R(X,Y)$ a right $\mathbb{H}$-operator ideal if

1. For a 1-dimensional Banach space K, there holds $Id_K \in A^R(X,Y)$;
2. for all $S,T \in A^R(X,Y)$, $S+T \in A^R(X,Y)$;
3. If $B^R \in B^R(X_0,X)$, $T \in A^R(X,Y)$ and $M \in B^R(Y,Y_0)$ then $MTL \in A^R(X_0,Y_0)$

Therefore, $A^R(X,Y)$ is a left $\mathbb{H}$-vector space. Moreover we shall denote

$$A^R = \bigcup_{X,Y} A^R(X,Y).$$

Analogously one defines the left ideal A^L and considering the two-sided counterpart one can define the two-sided ideal A.

Remark: We emphasize that, unlike the classic theory, where the terms “right” and “left” refer to the side from which the ideal “absorbs” operations, whether an ideal is right or left refers to the side on which the normed space structure is defined. Additionally, note that the third axiom of the operator ideal theory implies that we only consider two-sided ideals in the sense of the classic theory. Therefore, in this context and thereafter, the ideals "absorb" operations from both sides, and the reference to the side of the ideal will indicate the side from which the action of the structure is considered. Therefore, we can write Calkin's Theorem as follows

Theorem 4.1.2. *Let H be a separable right $\mathbb{H}$-Hilbert Space. For any right operator ideal I either $I = B^R$ or $I \subset K^R$, the set of compact right operators.*

The proof of this theorem follows the same approach as presented in [7, p. 841]. However, as previously mentioned, the classic concept of eigenvalues is not suitable for the quaternionic setting. Therefore, we need to address this issue by using equivalence classes, as discussed in (2.6). Specifically, if we denote the representative of the equivalence class $[\lambda_j]$ as λ_j for $j \geq 1$, we can still obtain the following representation, of a given operator T:

$$H = \bigoplus_{i=0}^{n} E([\lambda_i]) \text{ and consequently } T = \sum_{[\lambda_j] \in \sigma_S(T)} \lambda_j E(\lambda_j),$$

where $E([\lambda_0]) = \ker(T)$ and $E([\lambda_j]) = \ker[\mathcal{Q}_{\lambda_j}]$. Therefore, by working with representatives of each eigenspace, we can overcome the issues associated with the traditional notion of eigenvalues.

The implications of Theorem 4.1.2 are quite interesting: any ideal of operators is either the set of linear and bounded operators or a subset of the compact operators. In the sequel, we will specifically focus on subsets of compact operators.

Before proceeding further, it is worth noting that the first axiom can also be expressed in terms of tensor products. Consider a right $\mathbb{H}$-Banach space, X. Its dual, X', is a left space. Hence, we can define the following mapping:

$$X \ni x \overset{a\otimes y}{\longmapsto} y \langle a, x \rangle \in Y.$$

From the following diagram

$$\begin{tikzcd} X \arrow[r, "a \otimes y"] \arrow[d, "a \otimes 1"'] & Y \\ K \arrow[r, "Id_K"'] & K \arrow[u, "1 \otimes y"'] \end{tikzcd}$$

we can conclude that $Id_K \in A^R(X,Y)$ if and only if $a \otimes y \in A^R(X,Y)$ for all $a \in X'$ and $y \in Y$. This is because $a \otimes 1 : X \to \mathbb{K}$ and $1 \otimes y : \mathbb{K} \to Y$, allowing us to write $a \otimes y = 1 \otimes y Id_K a \otimes 1$. Furthermore, the third axiom implies $Id_K \in A^R(X,Y)$. We will use these interchangeably, i.e., to prove that the first axiom of the operator ideal theory is satisfied, one can prove that $Id_K \in A$ holds, where K is a one-dimensional Banach space, or that $a \otimes y \in A^R(X,Y)$ holds for all $a \in X'$ and $y \in F$.

In this context, we can introduce a quasi-norm as follows:

Definition 4.1.3. Consider a right operator ideal A^R. A map $\alpha : A^R \to \mathbb{R}^+$ is called a quasi-norm in A^R if

1. For a 1-dimensional right $\mathbb{H}$-Banach space K, there holds $\alpha(Id_K) = 1$ or , equivalently, $\alpha(a \otimes y) = \|a\| \, \|y\|$ for all $a \in X'$ and $y \in F$;
2. There exists a constant $c \geq 1$ for which $\alpha(S + T) \leq c(\alpha(S) + \alpha(T))$ holds true for $S, T \in A^R(X, Y)$;
3. $\alpha(MTL) \leq \|M\| \, \alpha(T) \, \|L\|$ for $L \in B^R(X_0, X)$, $T \in A^R(X, Y)$ and $M \in B^R(Y, Y_0)$.

We refer to $[A^R, \alpha]$ as a quasi-normed operator ideal. If the constant in the second condition is 1, then we call it a normed operator ideal. Finally, if every component of $A^R(X, Y)$ is complete with respect to α, we call $[A^R, \alpha]$ a quasi-Banach operator ideal. This definition can be easily extended to left and two-sided $\mathbb{H}$-operator ideals.

For a right quasi-normed ideal A^R [38, p. 342], we define

- the *dual Ideal*, $A^{dual} = \left\{T : T' \in A^R\right\}$. It turns into a left quasi-normed operator ideal by taking $\left\|T|A^{dual}\right\| = \left\|T'|A^R\right\|$;
- the *injective ideal*, $A^{inj} = \left\{T : TJ \in A^R(X_0, Y) \text{ for a metric injection } J : X_0 \to X\right\}$. It turns to a right quasi-normed operator ideal by taking $\|T|A^{inj}\| = \left\|TJ|A^R\right\|$;
- the *surjective ideal*, $A^{surj} = \left\{T : QT \in A^R(X, Y_0) \text{ for a metric surjection } Q : Y \to Y_0\right\}$. It turns to a right quasi-normed operator ideal by taking $\|T|A^{surj}\| = \left\|QT|A^R\right\|$.

An ideal A is said injective if $A = A^{inj}$, and surjective if $A = A^{sur}$. These definitions can be extended to quasi-normed operator ideals by, respectively, imposing the additional requirements:

$$\left\|\cdot|A^{inj}\right\| = \|\cdot|A\| \text{ and } \left\|\cdot|A^{surj}\right\| = \|\cdot|A\| \, .$$

A word on the Schmidt representation

In this paragraph, we briefly present the Schmidt representation in the context of quaternionic spaces. Although some properties may be lost, they are not essential for our purposes. Consider separable right $\mathbb{H}$-Hilbert spaces X and Y, and let T be a compact self-adjoint operator. From Lemma 2.3.4, T has a real S-spectrum. Thus, we obtain the Schmidt representation by using two orthonormal families, $(x_i) \in X$ and $(y_i) \in Y$. Then, the *Schmidt representation of* T is given by

$$T = \sum_{i \in I} \sigma_i x_i' \otimes y_i,$$

where $(\sigma_i) \to 0$. The Schmidt representation of an operator can be illustrated using the following diagram,

$$\begin{tikzcd}
X \arrow[r, "T"] \arrow[d, "A'"] & Y \arrow[d, "B'"'] \\
\ell_2(I) \arrow[r, "D"'] \arrow[u, dashed, bend left, "A"] & \ell_2(I) \arrow[u, dashed, bend right, "B"']
\end{tikzcd}$$

where,

$$A((\xi_n)) = \sum_{n=1}^{\infty} x_n \xi_n, \quad B((\xi_n)) = \sum_{n=1}^{\infty} y_n \xi_n.$$

The decomposition $T = BDA'$ well be referred to as the *Schmidt decomposition* of T. With this notation it follows that $D = B'TA$. If T is not a self-adjoint operator, we can address this issue by considering its polar decomposition.

However, in the quaternionic setting, the polar decomposition is not unique. Specifically, there exists a partial isometry V such that $T = V|T|$. Since $|T|$ is a positive operator, the classical Schmidt representation can still be applied, resulting in

$$T = \sum_{i \in I} \sigma_i x_i' \otimes V y_i. \tag{4.1}$$

The Schmidt representation is a fundamental concept that will be extensively used in the sequel. This is because, if we consider the Schmidt decomposition of the operator T as $A'DB$, the third axiom of operator ideal theory implies that if the diagonal matrix D belongs to a given ideal with ℓ_2 components, then the operator T also belongs to that ideal, with the corresponding Hilbert spaces X and Y as depicted in the diagram above. This is due to the boundedness of the operators A' and B. Consequently, the Schmidt decomposition not only allows us to simplify the study of an operator to its associated diagonal operator, but it also reduces the spaces under consideration to the sequence space ℓ_2.

4.1.2 Examples of Operator Ideals

Absolutely summable operators

A survey on the theory of absolutely summable operators can be found in [37, pp. 41–57], which serves as the basis for our generalization to the quaternionic setting.

Definition 4.1.4. Assume $1 \leq s \leq r \leq \infty$. An operator $L \in B^R(X, Y)$ is called *absolutely (r,s)-summable* if there exists $c \geq 0$ such that

$$\|\|(Tx_i)_i\| \,|\ell_r(I)\| \leq c \,\|(x_i)_i|w_s\|$$

holds for every finite family $x_1, \ldots, x_n \in X$. The set of these operators shall be denoted as $\mathfrak{B}^R_{r,s}(X, Y)$ and we equip with the (quasi-) norm $\left\|T|\mathfrak{B}^R_{r,s}\right\| := \inf c$. We shall denote $\mathfrak{B}^R_{r,r}$ by $\mathfrak{B}^R_r$ and refer to these operators as *absolutely r-summable* operators. This definition can be naturally extended to left and to two-sided linear and bounded operators.[1].

The Weyl numbers are interestingly related to the (r,s)-norm of an operator.

Lemma 4.1.5. *[37, p. 98] Consider* $2 \leq r < \infty$. *Then*

$$n^{\frac{1}{r}} x_n(T) \leq \left\|T|\mathfrak{B}^R_{r,2}\right\|, \qquad \forall\, T \in \mathfrak{B}^R_{r,2}(E, F).$$

Proof. Let $T \in \mathfrak{B}^R_{r,2}(H, F)$. Consider $(x_i)_{i=1}^k \in H$ such that $a_k(T) \leq (1+\epsilon)\|Tx_k\|$ and $\|x_k|w_2\| = 1$ [2]. Moreover, by definition of absolute $(r,2)$-summability it follows

$$\|a_k(T)|\ell_r\| \leq (1+\epsilon)\|T(x_k)|\ell_r\| \leq (1+\epsilon)\left\|T(x_k)|\mathfrak{B}^R_{r,2}\right\| \|x_k|w_2\|.$$

[1] However, one needs to be careful with the concept of weak norm, since, if one considers a two sided structure, then there are two dual spaces to consider.

[2] Indeed, such family is constructed inductively: if $x_1, \ldots x_{n-1}$ are already known we define

$$M_n = \{x \in H : \langle x, x_k \rangle = 0 \text{ for } k = 1, \ldots, n-1\}.$$

It then follows, from Theorem 3.3.1, that $a_n(T) = c_n(T) \leq \left\|TJ^H_M\right\|$ which in turn implies that there is $x_n \in M_n$ such that $a_n(T) \leq (1+\epsilon)\|Tx_n\|$ and $\|x_n\| = 1$.

Therefore, $\|a_k(T)|\ell_r\| \leq \|T(x_k)|\mathfrak{B}_{r,2}\|$. Thus if $X \in B^R(H,E)$ is such that $\|X\| \leq 1$, then

$$n^{\frac{1}{r}} x_n(T) \leq n^{\frac{1}{r}} a_n(TX) \leq \|a_k(TX)|\ell_r\| \leq \|TX|\mathfrak{B}_{r,2}\| \leq \|T|\mathfrak{B}_{r,2}\| \, \|X\| \leq \|T|\mathfrak{B}_{r,2}\| . \qquad \square$$

Proposition 4.1.6. *[39, p. 41] Let I be any infinite index set. A linear and bounded operator T is absolutely (r,s)-summable if and only if the map*

$$[w_s(I)] \ni (x_i) \xrightarrow{T(I)} (Tx_i) \in [\ell_r(I)]$$

is well defined. Moreover, we have $\left\|T|\mathfrak{B}^R_{r,s}\right\| = \|T(I)\|$.

Proof. Assume that $T \in \mathfrak{B}^R_{r,s}$. On the one hand, it is clear by definition that $\left\|T|\mathfrak{B}^R_{r,s}\right\| \leq \|T(I)\|$. On the other hand, consider $(x_i) \in [w_s(I)]$. For a finite subset of I, N there holds

$$\|(Tx_i)_i|\ell_r(N)\| \leq \left\|T|\mathfrak{B}^R_{r,s}\right\| \|(x_i)_i|w_s(N)\| .$$

As (x_i) is assumed to be weakly s-summable, passing to limit we can replace N with I to obtain $\|(Tx_i)_i|\ell_r(I)\| \leq \left\|T|\mathfrak{B}^R_{r,s}\right\| \|(x_i)_i|w_s(I)\|$. Therefore $\|T(I)\| \leq \left\|T|\mathfrak{B}^R_{r,s}\right\|$.

For the converse, suppose that $T \in B^R(X,Y)$ is not absolutely (r,s)-summable. Then, for $h = 1, 2, \ldots$, we can find $x_{h_1}, \ldots, x_{h_{n(h)}} \in X$ such that

$$\|(Tx_{h_k})_k|\ell_r(\{1,\ldots,n(h)\})\|^r \geq 1 \quad \text{and} \quad \|(x_{h_k})_k|w_s(\{1,\ldots,n(h)\})\|^s \leq 2^{-h}.$$

Set $x_{h_k} = 0$ if $k > n(h)$. Then the subsequence (x_{h_k}) is weakly-s-summable because , for $a \in B_{X'}$,

$$\sum_{h=1}^{\infty}\sum_{k=1}^{\infty} |\langle x_{h_k}, a\rangle|^s \leq \sum_{h=1}^{\infty} 2^{-h} = 1.$$

On the other hand

$$\sum_{h=1}^{\infty}\sum_{k=1}^{\infty} \|Tx_{h_k}\|^r \quad \text{diverges.}$$

Thus (Tx_{h_k}) cannot be absolutely r-summable which means that $T(\mathbb{N}\times\mathbb{N})$ fails to map $[w_s(\mathbb{N}\times\mathbb{N})]$ into $[\ell_r(\mathbb{N}\times\mathbb{N})]$. This finishes the proof because clearly every infinite index set contains a copy of $\mathbb{N}\times\mathbb{N}$ as a subset meaning that we can extend the subsequence (x_{h_k}) to a family (x_i) with $i \in I$ by taking $x_i = 0$ whenever $i \notin \mathbb{N}\times\mathbb{N}$. This means that there exists $(x_i)_{i\in I} \in [w_s(\mathbb{N}\times\mathbb{N})]$ such that $(Tx_i)_{i\in I} \notin [\ell_r(\mathbb{N}\times\mathbb{N})]$. $\square$

This proposition allows us to identify $\mathfrak{B}^R_{r,s}$ with $B^R([w_s],[\ell_r])$ and since the latter is clearly an ideal of operators we have effectively shown that $\mathfrak{B}^R_{r,s}$ is an Operator Ideal.

Theorem 4.1.7. *$\mathfrak{B}^R_{r,s}$ is a right quasi-Banach operator ideal with respect to $\left\|\cdot|\mathfrak{B}^R_{r,s}\right\|$.*

Proof. It is straightforward that $\left\|\cdot|\mathfrak{B}^R_{r,s}\right\|$ is a quasi-norm with respect to the Definition 4.1.3. Thus it remains to show that $\mathfrak{B}^R_{r,s}$ is complete with respect to it.

First note that the inequality

$$\|(Tx_i)_i|\ell_r(I)\| \leq \left\|T|\mathfrak{B}^R_{r,s}\right\| \|(x_i)|w_s(I)\|$$

allows us to write $\|T\| \leq \left\|T|\mathfrak{B}^R_{r,s}\right\|$ by considering $I = \{1\}$.

Consider a Cauchy sequence $(T_k) \in \mathfrak{B}^R_{r,s}$. It has limit in $B^R(X,Y)$ with respect to the operator norm since $\|T_k - T_h\| \leq \left\|T_k - T_h|\mathfrak{B}^R_{r,s}\right\|$. Given $\epsilon > 0$, choose k_0 such that $\left\|T_k - T_h|\mathfrak{B}^R_{r,s}\right\| \leq \epsilon$ for $h > k \geq k_0$. Then we have

$$\|\|((T_h - T_k)x_i)_i\| \,|\ell_r\| \leq \epsilon \,\|(x_i)_i|w_s\|\,.$$

Letting $h \to \infty$ we have $\|\|((T - T_k)x_i)_i\| \,|\ell_r\| \leq \epsilon \,\|(x_i)_i|w_s\|$. Therefore, not only $T - T_k \in \mathfrak{B}^R_{r,s}$ but also $\left\|\|T - T_k\| \,|\mathfrak{B}^R_{r,s}\right\| \leq \epsilon$ for $k \geq k_0$. □

In what follows, we will present properties of this ideal that will be useful. Let X and Y be right Banach spaces. By definition, it follows that if $1 \leq q \leq \infty$:

$$\mathfrak{B}^R_{\infty,q}(X,Y) = B^R(X,Y). \tag{4.2}$$

Furthermore, it is a consequence of the classic relations between the sequence spaces ℓ^p that if $r \leq p$ and $s \geq q$:

$$\mathfrak{B}^R_{r,s}(X,Y) \subset \mathfrak{B}^R_{p,q}(X,Y). \tag{4.3}$$

The classic version of the following theorem can be found in [31, p. 334], from which we directly extend it. We present its proof for completeness.

Theorem 4.1.8. *Let $1 \leq r_0 \leq r_1 < \infty$ and $1 \leq s_0 \leq s_1 < \infty$. Then, if $\frac{1}{r_0} - \frac{1}{r_1} = \frac{1}{s_0} - \frac{1}{s_1}$ we have $\mathfrak{B}^R_{r_0,s_0} \subseteq \mathfrak{B}^R_{r_1,s_1}$. More precisely, there holds*

$$\left\|T|\mathfrak{B}^R_{r_1,s_1}\right\| \leq \left\|T|\mathfrak{B}^R_{r_0,s_0}\right\|.$$

Proof. Denote $\frac{1}{r} = \frac{1}{r_0} - \frac{1}{r_1}$ and $\frac{1}{s} = \frac{1}{s_0} - \frac{1}{s_1}$. By hypothesis we have $r = s$. Let $T \in \mathfrak{B}^R_{r_0,s_0}(X,Y)$, $x_1, \dots x_n \in X$ and $\alpha_1, \dots, \alpha_n \in \mathbb{H}$. Then, as a routine application of Hölder's inequality, we have

$$\|\alpha_i T x_i|\ell_{r_0}\| \leq \left\|T|\mathfrak{B}^R_{r_0,s_0}\right\| \|\alpha_i x_i|w_{s_0}\| \leq \left\|T|\mathfrak{B}^R_{r_0,s_0}\right\| \|x_i|w_{s_1}\| \,\|\alpha_i|\ell_s\|\,.$$

In particular, choosing $\alpha_i = \|Tx_i\|^{\frac{r_1}{r}}$ we can write,

$$\|\alpha_i T x_i|\ell_{r_0}\|^{r_0} = \|\alpha_i|\ell_r\|^r = \|Tx_i|\ell_{r_1}\|^{r_1}$$

and therefore,

$$\|Tx_i|\ell_{r_1}\| \leq \left\|T|\mathfrak{B}^R_{r_0,s_0}\right\| \|x_i|w_{s_1}\|\,,$$

this means that $T \in \mathfrak{B}^R_{r_1,s_1}$ and $\left\|T|\mathfrak{B}^R_{r_1,s_1}\right\| \leq \left\|T|\mathfrak{B}^R_{r_0,s_0}\right\|$. □

For the following theorem, we recall that, for any $1 \leq p \leq \infty$, $\ell^n_p = \prod_{i=1}^n \ell_p$.

Theorem 4.1.9. *If $1 \leq p \leq q \leq \infty$ and $2 \leq q \leq \infty$ then $Id_{p,q} \in \mathfrak{B}^R_{(p,1)}(\ell_p, \ell_q)$.*

Proof. For each k let $e_k = (\delta_{ik})_i$. For $A \in B^R(\ell_\infty^n, \ell_q^n)$, for $1 \le p \le 2 \le q \le \infty$, since $\|\cdot\|_q \le \|\cdot\|_2$, from Lemma 2.2.3

$$\left(\sum_{k=1}^n \|Ae_k\|_q^p\right)^{\frac{1}{p}} \le \|A\| .$$

The case $2 \le p \le q \le \infty$ follows from Lemma 2.2.2, for arbitraty $x = (\xi_1, \dots, \xi_n)$, $\|x\|_\infty \le 1$ we have

$$\begin{aligned}\left(\sum_i |\xi_i|^p \|Ae_i\|_p^p\right)^{\frac{1}{p}} &= \left(\sum_i |\xi_i|^p \sum_k |\langle Ae_i, e_k\rangle|^p\right)^{\frac{1}{p}} = \left(\sum_k \sum_i |\xi_i \langle Ae_i, e_k\rangle|^p\right)^{\frac{1}{p}} \\ &\le \left(\sum_k \left(\sum_i |\xi_i \langle Ae_i, e_k\rangle|^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} \le c(p)\|A\| .\end{aligned}$$

In particular for $(\xi_i) = (1, \dots, 1)$ we have $\left(\sum_i \|Ae_i\|_p^p\right)^{\frac{1}{p}} \le c(p)\|A\|$ and since $\|\cdot\|_q \le \|\cdot\|_p$ we have shown that $\left(\sum_i \|Ae_i\|_q^p\right)^{\frac{1}{p}} \le c(p)\|A\|$. Now we show that the same holds for $A \in B^R(\ell_\infty, \ell_p)$. For this, define $P_n : \ell_p \to \ell_p^n$ and $Q_n : \ell_\infty^n \to \ell_\infty$ via

$$\begin{aligned}&P_n(\xi_1, \dots, \xi_n, \xi_{n+1}, \dots) = (\xi_1, \dots, \xi_n), \qquad \|P_n\| \le 1\\ &Q_n(\xi_1, \dots, \xi_n) = (\xi_1, \dots, \xi_n, 0, \dots), \qquad \|Q_n\| \le 1\\ &B_n = P_n A Q_n.\end{aligned}$$

Clearly, $B \in B^R(\ell_\infty^n \to \ell_p^n)$. Moreover, for $m \le n$

$$\left(\sum_{k=1}^m \|B_n e_k\|_q^p\right)^{\frac{1}{p}} \le c(p)\|B_n\| \le c(p) \le \|A\| .$$

Since $\lim_{n\to\infty} \|B_n e_k\|_q = \|Ae_k\|_q$ for any k we obtain, for all m:

$$\left(\sum_{k=1}^m \|Ae_k\|_q^p\right)^{\frac{1}{p}} \le \|A\| .$$

Now, considering any finite system $x_1, \dots, x_n \in \ell_p$, and defining the operator $A \in B^R(\ell_\infty, \ell_p)$ via $Ax = \sum_{i=1}^n \xi_i x_i$. Then for such an operator it follows that

$$\left(\sum_{i=1}^n \|x_i\|_q^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^n \|Ae_i\|_q^p\right)^{\frac{1}{p}} \le c(p)\|A\|$$

but this clearly means that $Id \in \mathfrak{B}^R_{(p,1)}(\ell_p, \ell_q)$. □

Theorem 4.1.10. *If $1 \le p \le q \le 2$ and $\frac{1}{r} = \frac{1}{p} - \frac{1}{q} + \frac{1}{2}$ then $Id_{p,q} \in \mathfrak{B}^R_{r,1}(\ell_p, \ell_q)$ as well as $Id_{p,q} \in \mathfrak{B}^R_{q,1}(\ell_p, \ell_q)$, for $1 \le q < p$.*

Proof. Take $0 < \theta < 1$ for which $\frac{1}{q} = \frac{1-\theta}{2} + \frac{\theta}{p}$. Then, for $x \in \ell_p$ it follows from Hölder's inequality that

$$\|x\|_q \leq \|x\|_2^{1-\theta} \|x\|_p^{\theta}.$$

Applying this inequality and Hölder's inequality once again, for $\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{2}$, yields

$$\left(\sum_{i=1}^{n} \|x_i\|_q^r\right)^{\frac{1}{r}} \leq \left(\sum_{i=1}^{n} \|x\|_2^{(1-\theta)r} \|x\|_p^{\theta r}\right)^{\frac{1}{r}} \leq \left(\sum_{i=1}^{n} \|x_i\|_2^p\right)^{\frac{1-\theta}{p}} \left(\sum_{i=1}^{n} \|x_i\|_p^2\right)^{\frac{\theta}{2}}.$$

One can directly generalize the results found in [36] to conclude that, under these assumptions, $Id \in \mathfrak{B}_{2,1}^R(\ell_p, \ell_p)$. Moreover, from Theorem 4.1.9 we derive

$$\begin{aligned}
\left(\sum_{i=1}^{n} \|x_i\|_q^r\right)^{\frac{1}{r}} &\leq \left(\sum_{i=1}^{n} \|x_i\|_2^p\right)^{\frac{1-\theta}{p}} \left(\sum_{i=1}^{n} \|x_i\|_p^2\right)^{\frac{\theta}{2}} \\
&\leq \left\|Id|\mathfrak{B}_{p,1}^R\right\| \|x_i|w(p')\|^{1-\theta} \left\|Id|\mathfrak{B}_{2,1}^R\right\|^{\theta} \|x_i|w(p')\|^{\theta} \\
&\leq \left\|Id|\mathfrak{B}_{p,1}^R\right\|^{1-\theta} \left\|Id|\mathfrak{B}_{2,1}^R\right\|^{\theta} \|x_i|w(p')\|
\end{aligned}$$

for $\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{2}$ and $\frac{1}{q} = \frac{1-\theta}{2} + \frac{\theta}{p}$ which yields $\frac{1}{r} = \frac{1}{p} - \frac{1}{q} + \frac{1}{2}$. □

Ideals derived from s-numbers

We extend the concept of Schatten classes to a general Banach space, by considering any s-number function, s.

Definition 4.1.11. Consider right $\mathbb{H}-$Banach spaces X and Y. For $1 \leq p < \infty$ we denote,

$\mathfrak{S}_p^{(s)} = \left\{T \in B^R(X,Y) : \|s_n(T)|\ell_p(I)\| < \infty\right\}$ and $\mathfrak{S}_\infty^{(s)} = \left\{T \in B^R(X,Y) : \lim_{n\to\infty} s_n(T) = 0\right\}$

and equip it with the quasi-norm

$$\left\|T|\mathfrak{S}_p^{(s)}\right\| = \|s_n(T)|\ell_p\|.$$

Observe that, by Theorems 3.2.4 and 3.2.6, $\mathfrak{S}_\infty^{(c)} = \mathfrak{S}_\infty^{(d)} = K^R$.

Theorem 4.1.12. *Let s be an additive s-number function. Then, for any $1 \leq p \leq \infty$, $\mathfrak{S}_p^{(s)}$ is a quasi-Banach operator ideal.*

Proof. We split the proof in three steps. On the first step we show that under the additivity assumption $\mathfrak{S}_p^{(s)}$ is indeed an operator ideal. On the second step we show that $\left\|\cdot|\mathfrak{S}_p^{(s)}\right\|$ is a quasi-norm and finally on the third step we prove the completeness.

Step 1: $\mathfrak{S}_p^{(s)}$ is an operator ideal.

Consider a finite dimensional Banach space K and its identity Id_K. Then $\text{rank}(Id_K) = 1$, as such, by the axioms of s-numbers $s_n(Id_K) = 0$, for any $n \in \mathbb{N}$. This immediately implies that $Id_K \in \mathfrak{S}_p^{(s)}$ for any $1 \leq p \leq \infty$.

For the second axiom of the theory of operator ideals, the assumption of the additivity of s is crucial. Consider $T_1, T_2 \in \mathfrak{S}_p^{(s)}(X,Y)$. If $1 < p < \infty$, for any non negative x_1 and x_2 we have

$(x_1+x_2)^p \leq \alpha(x_1^p + x_2^p)$ with $\alpha = \max\{2^{p-1}, 1\}$. Since s-numbers are non increasing we can write

$$\sum_{n=1}^{\infty} s_n(T_1+T_2)^p \leq 2\sum_{n=1}^{\infty} s_{2n-1}(T_1+T_2)^p \leq 2\sum_{n=1}^{\infty}(s_n(T_1)+s_n(T_2))^p$$
$$\leq 2\alpha \sum_{n=1}^{\infty} s_n(T_1)^p + s_n(T_2)^p \leq \infty.$$

The case $p=\infty$ follows as well from the additivity:

$$\lim_{n\to\infty} s_n(T_1+T_2) = \lim_{n\to\infty} s_{2n-1}(T_1+T_2) \leq \lim_{n\to\infty} s_n(T_1)+s_n(T_2) = 0. \tag{4.4}$$

The third condition is immediate.

Step 2: $\left\|\cdot|\mathfrak{S}_p^{(s)}\right\|$ is a quasi-norm.

For the first condition, observe that for any operator $S \in \mathfrak{S}_p^{(s)}$, by considering $k \in B_X$ and $z \in B_{Y'}$, there holds[3]:

$$|\langle Sk, z\rangle| \leq \|k\otimes 1\| \left\|S|\mathfrak{S}_p^{(s)}\right\| \|1\otimes z\| \leq \left\|S|\mathfrak{S}_p^{(s)}\right\|.$$

This means that $\|S\| \leq \left\|S|\mathfrak{S}_p^{(s)}\right\|$. From the first step $a\otimes y \in \mathfrak{S}_p^{(s)}$ and thus $\|a\|\,\|y\| = \|a\otimes y\| \leq \left\|a\otimes y|\mathfrak{S}_p^{(s)}\right\|$. On the other hand

$$\left\|a\otimes y|\mathfrak{S}_p^{(s)}\right\| \leq \|a\otimes 1\| \left\|Id_k|\mathfrak{S}_p^{(s)}\right\| \|1\otimes y\| = \|a\otimes 1\|\,\|1\otimes y\|,$$

because, since K is a 1 dimensional Banach space, $\left\|Id_K|\mathfrak{S}_p^{(s)}\right\| = s_1(Id_K) = 1$ by or fourth and fifth axioms of the s-number theory. This allows us to conclude that $\left\|a\otimes y|\mathfrak{S}_p^{(s)}\right\| = \|a\otimes 1\|\,\|1\otimes y\|$ as desired.

For the second condition, take $\alpha = 2^{\frac{1}{p}}\max(2^{\frac{1}{p}-1}, 1)$ then

$$\left\|T_1+T_2|\mathfrak{S}_p^{(s)}\right\| = \left(\sum_{n=1}^{\infty} s_n(T_1+T_2)^p\right)^{\frac{1}{p}} \leq \left(2\sum_{n=1}^{\infty} s_{2n-1}(T_1+T_2)^p\right)^{\frac{1}{p}}$$
$$\leq \left(2\sum_{n=1}^{\infty}(s_n(T_1)+s_n(T_2))^p\right)^{\frac{1}{p}}$$
$$\leq \alpha\left(\left(\sum_{n=1}^{\infty} s_n(T_1)^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} s_n(T_2)^p\right)^{\frac{1}{p}}\right)$$
$$= \alpha\left(\left\|T_1|\mathfrak{S}_p^{(s)}\right\| + \left\|T_2|\mathfrak{S}_p^{(s)}\right\|\right).$$

The third condition is immediate from the third axiom of s-numbers.

Step 3: $\mathfrak{S}_p^{(s)}$ is complete. Consider a Cauchy sequence $(T_k) \in \mathfrak{S}_p^{(s)}$. From the second step we know that there exists a limit T with the respect to the operator topology because $\|T_k - T_h\| \leq \left\|T_k - T_h|\mathfrak{S}_p^{(s)}\right\|$. From the continuity of s-number functions it follows that $s_n(T_k - T_h) \to 0$ as $h \to k$, therefore $\left\|T_k - T_h|\mathfrak{S}_p^{(s)}\right\| \to 0$. □

[3]this actually holds for any operator ideal

Moreover, it follows from the definition of injectivity/surjectivity of an operator ideal that, if an s-number function is injective/surjective then so is the ideal derived from it.

In order to give another example we will denote the non-increasing rearrangement of the sequence x by x^*. More precisely, we define $x^*_n = \inf\{\sigma \geq 0 : card(k : |x_k| \geq \sigma) \leq n\}$ which basically means that we sort the elements of (x_n) in such a way that elements of the new sequence (x^*_n) are ordered in non increasing, way.

The Lorentz sequence space $\ell_{r,w}(I)$ consists of all families $x = (\xi_i)$ such that $(n^{\frac{1}{r}-\frac{1}{w}}x^*_n) \in \ell_w$. We equip it with $\|x|\ell_{r,w}(I)\| := \left\|n^{\frac{1}{n}-\frac{1}{w}}x^*_n|\ell_w(I)\right\|$. It is a clear generalization of the ℓ_p sequence spaces, in the sense that $\ell_{p,p}(I) = \ell_p(I)$. Following the same idea as in the previous example, this allows us to introduce operator ideals associated to s-numbers.

Definition 4.1.13. The operator ideal $\mathfrak{L}^{(s)}_{r,w}(X,Y)$ consists of all the operators $S \in B^R(X,Y)$ such that the sequence $(s_n(S))$ belongs to the Lorentz sequence space $\ell_{r,w}$. In this case we have a quasi-norm given by

$$\left\|S|\mathfrak{L}^{(s)}_{r,w}\right\| := \|(s_n(S))|\ell_{r,w}\|\,.$$

We will refer to $\mathfrak{L}^{(s)}_{r,w}$ with respect to which s-number function it is constructed, for example $\mathfrak{L}^{(c)}_{r,w}$ will be defined as the *Ideal of* (r,w)*-Gelfand operators.*

Now we are ready to classify operator ideals in terms of s-numbers. But first let us show that these sets are ideals according to our definition.

Theorem 4.1.14. *Let s be an additive s-number function. Then, for any $0 < p,q \leq \infty$, $\mathfrak{L}^{(s)}_{p,q}$ is a quasi-Banach operator ideal.*

Proof. Again, we split the proof in three steps.

Step 1: $\mathfrak{L}^{(s)}_{p,q}$ is an Operator Ideal.
The first axiom of the operator ideal theory follows again from the fact that, for a 1 dimensional Banach Space, K $s_1(Id_K) = 1$ and $s_n(Id_K) = 0$ for $n > 1$. Indeed consider $a \in X'$ and $y \in Y$, then

$$\|s_n(a \otimes y)|\ell_{r,w}\| \leq \|\|a\otimes 1\|\, s_n(Id_K)\, \|1\otimes y\|\, |\ell_{r,w}\| = \|a\otimes 1\|\, \|1\otimes y\|\,.$$

For the second condition, consider $S,T \in \mathfrak{L}^{(s)}_{p,q}$. Then as observed in [37, p. 76] with $c_0 = \max(2^{\frac{1}{r}}, 2^{\frac{1}{w}})$ then we can write [4]

$$\begin{aligned}\|x|\ell_{r,w}\| &= \left(\sum_{n=1}^{\infty}\left(n^{\frac{1}{r}-\frac{1}{w}}x^*_n\right)^w\right)^{\frac{1}{w}} = \left(\sum_{n=1}^{\infty}\left((2n-1)^{\frac{1}{r}-\frac{1}{w}}x^*_{2n-1}\right)^w + \sum_{n=1}^{\infty}\left((2n)^{\frac{1}{r}-\frac{1}{w}}x^*_{2n}\right)^w\right)^{\frac{1}{w}}\\ &\leq c_0\left(\sum_{n=1}^{\infty}\left(n^{\frac{1}{r}-\frac{1}{w}}x^*_{2n-1}\right)^w\right)^{\frac{1}{w}}.\end{aligned}$$

[4]Since, for $0 < r,w \leq \infty$, if $r \leq w$

$$(2n-1)^{\frac{1}{r}-\frac{1}{w}} \leq (2n)^{\frac{1}{r}-\frac{1}{w}} = 2^{\frac{1}{r}-\frac{1}{w}}n^{\frac{1}{r}-\frac{1}{w}}.$$

On the other hand, if $r \geq w$

$$(2n)^{\frac{1}{r}-\frac{1}{w}} \leq (2n-1)^{\frac{1}{r}-\frac{1}{w}} \leq= n^{\frac{1}{r}-\frac{1}{w}}.$$

This allows us to obtain

$$\begin{aligned}\|s_n(S+T)|\ell_{r,w}\| &\leq c_0 \,\|s_{2n-1}(S+T)|\ell_{r,w}\| = c_0 \left\| n^{\frac{1}{r}-\frac{1}{w}} s_{2n-1}(S+T)|\ell_w \right\| \\ &\leq c_0 \left\| n^{\frac{1}{r}-\frac{1}{w}} s_n(S) + n^{\frac{1}{r}-\frac{1}{w}} s_n(T)|\ell_w \right\| \\ &\leq c(\|s_n(S)|\ell_{r,w}\| + \|s_n(T)|\ell_{r,w}\|)\end{aligned}$$

which shows that $\left\|S+T|L_{r,w}^{(s)}\right\| \leq c\left(\left\|S|L_{r,w}^{(s)}\right\| + \left\|T|L_{r,w}^{(s)}\right\|\right)$. The third requirement is easy to prove.

Step 2: To see that $\left\|\cdot|\mathfrak{L}_{p,q}^{(s)}\right\|$ is a quasi-norm, one can follow the same lines as the ones presented in the proof of $\mathfrak{S}_p^{(s)}$.

Step 3: Consider a Cauchy sequence $(T_k) \in \mathfrak{L}_{r,w}^{(s)}(X,Y)$. Then, since $\|T_h - T_k\| \leq \left\|T_h - T_k|\mathfrak{L}_{r,w}^{(s)}\right\|$ there is a limit $T \in B^R(X,Y)$ with respect to the operator norm. For $\epsilon > 0$, choose k_0 such that $\left\|T_h - T_k|\mathfrak{L}_{r,w}^{(s)}\right\| \leq \epsilon$ for $h,k \geq k_0$. From the continuity of s-numbers we have, by taking $h \to \infty$, $\left\|T - T_k|\mathfrak{L}_{r,w}^{(s)}\right\| \leq \epsilon$ for $k \geq k_0$. □

Quaternionic Schatten classes & s-numbers

Efforts are been made in the direction of defining a basis independent notion of Schatten classes of quaternionic operators, as observed in [13, p. 14]. Following the classic reasoning, we can introduce what is known as Schatten classes with respect to a given s-number function, which will be denoted by $\mathfrak{S}_p^{(s)}$. From the uniqueness of s-numbers on Hilbert spaces obtained in Theorem 3.1.6 we can define a unique Schatten class for each p as follows: For right $\mathbb{H}$-Hilbert spaces X and Y, let $\sigma_n(T) = \inf_\sigma\{\dim(E(\sigma,\infty)) < n\}$ where E is the spectral measure associated to the operator $|T|$, then

$$\mathfrak{S}_p = \left\{T \in B^R(X,Y) : \|\sigma_n(T)|\ell_p\| < \infty\right\}.$$

Moreover, if we consider just right $\mathbb{H}$-Banach spaces we then have several notions of Schatten classes, each of which associated to a specific s-number function. For example one can refer to the approximation-Schatten class of an operator as

$$\mathfrak{S}_p^{(a)} = \left\{T \in B^R(X,Y) : \|a_n(T)|\ell_p\| < \infty\right\}.$$

The independence of the basis of the s-number function, seen so far, yields the desired basis independence. As such we have effectively created several notions of quaternionic Schatten classes, each of which related to the s-number that generates the operator ideal $\mathfrak{S}_p^{(s)}$.

4.2 Specific components of operator ideals

Different operator ideals $\mathfrak{U}$ and $\mathfrak{A}$ might coincide for specific Banach spaces. More precisely, there might exist Banach spaces X and Y for which $\mathfrak{U}(X,Y) = \mathfrak{A}(X,Y)$ holds true. Of particular interest is to determine the Hilbert space component of a given ideal. The Theorem 4.2.3 describes each Hilbert space component of the ideal $\mathfrak{B}_{p,q}^R$. The proof on the complex setting can be found in [31, pp. 334–336], from where we extend this result. As mentioned previously, it suffices to consider the ideal over the sequence space ℓ_2.

We required additional machinary and theorems.

Theorem 4.2.1. *[5, p. 363] Let $2 \leq q < \infty$, there is a constant K', depending only on q, so that if (for each $m, n = 1, 2, \dots$) $A_{m,n}$ is an $m \times n$ random matrix whose entries are independent, mean zero random variables with $|a_{ij}| \leq 1$ for all i, j. Then there holds*

$$\limsup_{\max(m,n)\to\infty} \frac{\|A_{m,n}\|}{\max(m^{\frac{1}{q}}, n^{\frac{1}{2}})} \leq K'.$$

Therefore, one can choose matrix entries (a_{ij}) independently so that

$$P(a_{ij} = 1) = P(a_{ij} = -1) = \frac{1}{2}.$$

It follows that there exists, for each positive integer n, at least one matrix of order $[n^{\frac{q}{2}}] \times n$ with all ± 1 entries and satisfying $\|A\|_{2,q} \leq K\sqrt{n}$, where K is a constant depending only on q.

Lemma 4.2.2. *[4, p. 28] Let $(\lambda_k)_{k=1}^{\infty}$ be a decreasing sequence of non-negative real numbers. Then for $2 < p < \infty$, $2 < q < \infty$, the following conditions are equivalent*

i. $\sum_{k=1}^{\infty} k^{\frac{q}{p}-1}(\lambda_k)^q < \infty$;

ii. $\sum_{t=1}^{\infty} t^{\frac{q}{p}-1-q} \left(\sum_{k=1}^{t} \lambda_k\right)^q < \infty$;

iii. $\sum_{t=1}^{\infty} t^{\frac{q}{p}-1-\frac{q}{2}} \left(\sum_{k=1}^{t} \lambda_k^2\right)^{\frac{q}{2}} < \infty$;

iv. $\sum_{i=1}^{\infty} 2^{tq\left(\frac{1}{p}-\frac{1}{2}\right)} \left(\sum_{k=2^t}^{2^{t+1}-1} \lambda_k^2\right)^{\frac{q}{2}} < \infty$.

Theorem 4.2.3. *Suppose that $1 \leq q \leq p \leq \infty$. Then*

1. *if $\frac{1}{q} - \frac{1}{p} \geq \frac{1}{2}$ or $p = +\infty$, then $\mathfrak{B}_{p,q}^R(\ell_2, \ell_2) = B^R(\ell_2, \ell_2)$;*
2. *if $q \leq 2$ and $\frac{1}{q} - \frac{1}{p} < \frac{1}{2}$, then $\mathfrak{B}_{p,q}^R(\ell_2, \ell_2) = \mathfrak{S}_r(\ell_2, \ell_2)$, where $r = \frac{1}{p} - \frac{1}{q} + \frac{1}{2}$;*
3. *if $2 < q < p < \infty 2$ then $\mathfrak{B}_{p,q}^R(\ell_2, \ell_2) = \mathfrak{S}_{\frac{2p}{q},p}(\ell_2, \ell_2)$;*
4. *if $p = q$ then $\mathfrak{B}_{p,q}^R(\ell_2, \ell_2) = \mathfrak{S}_2(\ell_2, \ell_2)$;*

Proof.

1. It remains to show that $B^R(\ell_2, \ell_2) \subseteq \mathfrak{B}_{p,q}^R(\ell_2, \ell_2)$.

   The case where $p = \infty$ follows for any $1 \leq q \leq \infty$ from (4.2). The case where $p < \infty$ is split in two cases. If $q = 1$ then, from (4.3) we have that

   $$\mathfrak{B}_{2,1}^R(\ell_2, \ell_2) \subset \mathfrak{B}_{p,q}^R(\ell_2, \ell_2).$$

   Which to completes the proof. Indeed, from Theorem 4.1.9 that $Id_{\ell_2} \in \mathfrak{B}_{2,1}^R$. Given that $\mathfrak{B}_{2,1}^R$ is a ideal of operators, then $B^R(\ell_2, \ell_2) \subset \mathfrak{B}_{2,1}^R$.

   Finally, if $q > 1$, follows from Theorem 4.1.8. Indeed if $\frac{1}{2} - \frac{1}{p} = 1 - \frac{1}{q}$ (which is possible because we assume $\frac{1}{q} - \frac{1}{p} \geq \frac{1}{2}$) there holds $\mathfrak{B}_{2,1}^R \subseteq \mathfrak{B}_{p,q}^R$.
2. We start by proving that $\mathfrak{B}_{p,q}^R \supset \mathfrak{S}_r^R$. Consider the Schmidt decomposition of $A \in \mathfrak{S}_r^R$ to be given by $A = UBV$. For $0 \leq s \leq \frac{1}{2}$ define

   $$\ell_2 \ni (\xi_n) \overset{B_s}{\longmapsto} (|\lambda_n|^{\frac{1}{s}} \xi_n) \in \ell_2.$$

By Theorem 4.1.8, since $B_{\frac{1}{2}} \in \mathfrak{B}_{1,1}(\ell_2, \ell_2)$, it follows that $B_{\frac{1}{2}} \in \mathfrak{B}_{q,q}(\ell_2, \ell_2)$. Moreover, the previous item implies that $B_0 \in \mathfrak{B}^R_{\frac{2q}{2-q},q}(\ell_2, \ell_2)$. Therefore, there is a positive constant M such that, for every finite family $x = (x_i) \in \ell_2(I)$, there holds

$$\left\| B_{\frac{1}{2}} x | \ell_q(I) \right\| \leq M \left\| x | w_q(I) \right\|, \qquad \left\| B_0 x | \ell_{\frac{2q}{2-q}}(I) \right\| \leq M \left\| x | w_q(I) \right\|.$$

For each $i \in I$, let $x_i = (\alpha_n^i) \in \ell_2$ and $y_i = (\beta_n^i) \in \ell_2$ such that $\|y_i | \ell_2\| \leq 1$. Then

$$\|Bx_i\| = \left\| B_{\frac{1}{r}} x_i \right\| = \left\langle B_{\frac{1}{r}} x_i, y_i \right\rangle = \sum_{n=1}^{\infty} |\alpha_n^i| |\beta_n^i| |\lambda_n|^{\frac{1}{r}}.$$

If $\frac{1}{p} = \frac{1}{q} + \frac{1}{r} - \frac{1}{2}$, then denoting the Hölder's conjugate of p by p^* (i.e., $p^* = \frac{p}{1-p}$) we have

$$\begin{aligned} \|Bx_i | \ell_p(I)\| &= \sup_{\|\xi_i | \ell_1(I)\| \leq 1} \left\| B_{\frac{1}{r}} x_i |\xi_i|^{\frac{1}{p^*}} | \ell_1(I) \right\| = \sup_{\|\xi_i | \ell_1(I)\| \leq 1} \sum_{i \in I} \left\| B_{\frac{1}{r}} x_i \right\| |\xi_i|^{\frac{1}{p^*}} \\ &= \sup_{\|\xi_i | \ell_1(I)\| \leq 1} \sum_{i \in I} \sum_{n=1}^{\infty} |\alpha_n^i| |\beta_n^i| |\lambda_n|^{\frac{1}{r}} |\xi_i|^{\frac{1}{p^*}}. \end{aligned}$$

Now, for a natural number N, a family $(\xi_i)_{i \in I}$ with $\|\xi_i | \ell_1(I)\| \leq 1$ and $0 \leq s \leq \frac{1}{2}$ define the function

$$f_N(s) = \sum_{i \in I} \underbrace{\sum_{n=1}^{N} |\alpha_n^i| |\beta_n^i| |\lambda_n|^s}_{\leq \|B_s x_i\|} |\xi_i|^{\frac{3}{2} - \frac{1}{q} - s}$$

It follows from Hölder's inequality that

$$\begin{aligned} f_N(0) &\leq \sum_{i \in I} |\xi_i|^{\frac{3}{2} - \frac{1}{q}} \|B_0 x_i\| \leq \left\| B_0 x_i | \ell_{\frac{2q}{2-q}}(I) \right\| \leq M \left\| x_i | w_q(I) \right\|, \\ f_N\left(\frac{1}{2}\right) &\leq \sum_{i \in I} |\xi_i|^{1 - \frac{1}{q}} \left\| B_{\frac{1}{2}} x_i \right\| \leq \left\| B_{\frac{1}{2}} x_i | \ell_q(I) \right\| \leq M \left\| x_i | w_q(I) \right\|. \end{aligned}$$

One can show that functions of this kind are convex and thus $\sup_{a \leq s \leq b} f_N(s) = \max(f_N(a), f_N(b))$. In particular

$$f_N\left(\frac{1}{r}\right) \leq \max\left(f_N(0), f_N\left(\frac{1}{2}\right)\right) \leq M \left\| x_i | w_q(I) \right\|.$$

But

$$\sup_{N \in \mathbb{N}} \sup_{\|\xi_i | \ell_1(I)\| \leq 1} f_N\left(\frac{1}{r}\right) \geq \|Bx_i | \ell_p(I)\|,$$

which implies that $\|Bx_i | \ell_p(I)\| \leq M \|x_i | w_q(I)\|$, i.e. $B \in \mathfrak{B}^R_{p,q}(\ell_2, \ell_2)$.

Now let us prove that $\mathfrak{S}^R_r \supset \mathfrak{B}^R_{p,q}$ for $q \leq 2$ and $\frac{1}{r} = \frac{1}{p} - \frac{1}{q} + \frac{1}{2} > 0$. By Theorem 4.1.2 , every (p,q)-absolutely summing operator is compact. By being compact, we know that there is a Schmidt decomposition $A = UBV$, where B is a diagonal operator $x_n \to \lambda_n x_n$ with $\lambda_n \to 0$. Let $I = \{1, 2, \ldots N\}$.

$$\begin{aligned} \|\lambda_n | \ell_r(I)\| &= \sup_{\|\xi_i | \ell_1(I)\| \leq 1} \left\| \lambda_i |\xi_i|^{\frac{1}{r^*}} | \ell_1(I) \right\| \leq \sup_{\|\xi_i | \ell_1(I)\| \leq 1} \left\| \lambda_i |\xi_i|^{\frac{1}{r^*} - \frac{1}{p^*}} | \ell_p(I) \right\| \left\| |\xi_i|^{\frac{1}{p^*}} | \ell_{p^*}(I) \right\| \\ &\leq \sup_{\|\xi_i | \ell_1(I)\| \leq 1} \left\| Be_i |\xi_i|^{\frac{1}{q} - \frac{1}{2}} | \ell_p \right\| \leq \left\| B | \mathfrak{B}^R_{p,q} \right\| \left\| e_i |\xi_i|^{\frac{1}{q} - \frac{1}{2}} | w(q) \right\| \\ &\leq \left\| B | \mathfrak{B}^R_{p,q} \right\| \|e_i | w(2)\| \left\| |\xi_i|^{\frac{1}{q} - \frac{1}{2}} | \ell_{\frac{2q}{q-2}} \right\| \leq \left\| B | \mathfrak{B}^R_{p,q} \right\| < \infty, \end{aligned}$$

because $\frac{1}{r^*} - \frac{1}{p^*} = \frac{1}{q} - \frac{1}{2}$ and because $q\left(\frac{2}{q}\right)^* = \frac{2q}{q-2}$. This means $B \in \mathfrak{S}_r^R$ which suffices to conclude that every operator in $\mathfrak{B}_{p,q}^R$ is also in $\mathfrak{S}_r^R$ since $\mathfrak{B}_{p,q}^R$ is an ideal of operators.

3. Following the same argumentation given in the previous item, it suffices to consider a diagonal operator. Let $(\lambda_n) \in \ell_{\frac{2p}{q},p}$ and $x^{(1)}, \ldots, x^{(n)} \in \ell_2$, then, we ought to show that

$$\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{\infty}|\lambda_k x_k^{(j)}|^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} \leq M \underbrace{\sup_{\|f\|_2\leq 1}\left(\sum_{j=1}^{n}\left|\sum_{k=1}^{\infty} x_k^{(j)} f_k\right|^q\right)^{\frac{1}{q}}}_{:=N},$$

where M depends on λ, p and q. Without loss of generality we may assume that $(|\lambda_k|)_{k=1}^{\infty}$ is a decreasing sequence[5]. Fix $x^{(1)}, \ldots, x^{(n)} \in \ell_2$ and a natural number t. For each j define μ_j, v_j and w_j as follows

$$\mu_j = \sum_{k=1}^{\infty}|\lambda_k x_k^{(j)}|^2 = \sum_{k=1}^{t}|\lambda_k x_k^{(j)}|^2 + \sum_{k=t+1}^{\infty}|\lambda_k x^{(j)_k}|^2 = v_j + w_j.$$

On the one hand, since λ_k is decreasing,

$$w_j \leq \sup_{k>t}|\lambda_k|^2 \sum_{k=t+1}^{\infty}|x_k^{(j)}|^2 \leq |\lambda_{t+1}|^2 \max_{1\leq j\leq n}\sum_{k=1}^{\infty}|x_k^{(j)}|^2 = |\lambda_{t+1}|^2 \max_{1\leq j\leq n}\sup_{\|f\|_2\leq 1}\sum_{t+1}^{\infty}|x_k^{(j)} f_k|^2.$$

By Jensen's inequality the latter term can be bounded by $|\lambda_{n+1}|^2 N^2$. Thus,

$$\max_{1\leq j\leq n} w_j \leq \frac{N^2}{t}\sum_{k=1}^{t}|\lambda_k|^2.$$

On the other hand since

$$\sum_{j=1}^{n} v_j^{\frac{q}{2}} = \sum_{j=1}^{n}\left(\sum_{k=1}^{t}|\lambda_k x_k^{(j)}|^2\right)^{\frac{q}{2}} \leq \left(\sum_{j=1}^{n}\left(\sum_{k=1}^{t}|\lambda_k x_k^{(j)}|^q\right)^{\frac{2}{q}}\right)^{\frac{q}{2}}.$$

Minkowski's inequality in $\ell_{\frac{q}{2}}$ gives us

$$\leq \left(\sum_{k=1}^{t}|\lambda_k|^2\right)^{\frac{q}{2}} \max_{1\leq k\leq t}\sum_{j=1}^{n}|x_k^{(j)}|^q = \left(\sum_{k=1}^{t}|\lambda_k|^2\right)^{\frac{q}{2}} \sup_{\|f\|_1\leq 1}\sum_{j=1}^{n}\left|\sum_{k=1}^{\infty} x_k^{(j)} f_k\right|^q$$

by three applications of Landau's theorem. Analogously, Jensen's inequality allows us to derive the upper bound for the rearrangement of v_j

$$v_j^* \leq \frac{N^2}{j^{\frac{2}{q}}}\sum_{k=1}^{t}|\lambda_k|^2.$$

Therefore

$$v^*_{[t^{\frac{q}{2}}]} + w^*_{[t^{\frac{q}{2}}]} \leq \frac{2N^2}{t}\sum_{k=1}^{t}|\lambda_k|^2$$

[5] Indeed, the inequality above does not depend on the order of λ which can be seen by rearranging the coordinates $x^{(1)}, \ldots, x^{(n)}$

where the bracket notation is used to denote the ceilling number, i.e, $[x]$ is the least positive integer greater then x. By construction, at most $2[t^{\frac{q}{2}}]-2$ values of $\mu_j = v_j + w_j$ can exceed $v^*_{[t^{\frac{q}{2}}]} + w^*_{[t^{\frac{q}{2}}]}$, so that

$$\mu_{2[t^{\frac{q}{2}}]-1} \le \frac{2N^2}{t} \sum_{k=1}^{t} |\lambda_k|^2$$

which holds for any positive integer t. Therefore,

$$\sum_{t=1}^{n} \mu_t^{\frac{p}{2}} = \sum_{t=1}^{n} (\mu^*)_t^{\frac{p}{2}} = \sum_t \sum_{s=2[t^{\frac{q}{2}}]-1}^{2[(t+1)^{\frac{q}{2}}]-2} (\mu_s^*)^{\frac{p}{2}} = C(q) \sum_t t^{\frac{q}{2}-1} \left(\mu^*_{2[t^{\frac{q}{2}}]-1}\right)^{\frac{p}{2}}$$

$$\le 2N^p C(q) \sum_{t=1}^{\infty} t^{\frac{q}{2}-1-\frac{p}{2}} \left(\sum_{k=1}^{t} |\lambda_k|^2\right)^{\frac{p}{2}} \le NC(p,q) \|\lambda\|^p_{\frac{2p}{q},p}$$

By 4.2.2 ($\sum_{t=1}^{\infty} t^{\frac{q}{p}-1-q} (\sum_{k=1}^{\infty} \lambda_k)^q < \infty$ is equivalent to $\sum_{t=1}^{\infty} t^{\frac{q}{p}-1-\frac{q}{2}} (\sum_{k=1}^{\infty} \lambda_k^2)^{\frac{q}{2}} < \infty$). Thus, by taking $M = C(p,q) \|\lambda\|^p_{\frac{2p}{q},p}$ we obtain the desired inequality. Hence we have shown that $2 < q < p < \infty$ $\mathfrak{S}^R_{\frac{2p}{q},p} \subset \mathfrak{B}_{p,q}$.

To see the converse inequality one requires a totally different approach. As observed in [4, p. 27] it suffices to show that there holds if $\sum_{j=1}^{\infty} (\sum_{k=1}^{\infty} |\lambda_k a_{jk}|^2)^{\frac{p}{2}} < \infty$, whenever $A = (a_{ij})$ is a matrix satisfying $\sum_{j=1}^{\infty} |\sum_{k=1}^{\infty} a_{jk} x_k|^q < \infty$ for each $x \in \ell_2$ then $\lambda \in \ell_{\frac{2p}{q},p}$. By changing the order of the columns of A we conclude that the last statement is independent of the order of the terms λ_k. Moreover, considering $a_{i,j} = \delta_{ij}$ it follows that $\lambda \in \ell_p$, so that the decreasing rearrangement of λ, λ^*, must exist. Thus we can assume λ is a decreasing sequence. The proof now is constructive, in the sense that, a matrix A is constructed for which $\lambda \in \ell_{\frac{2p}{q},p}$. It is a consequence of 4.2.1 that we can construct a matrix $A = \sum_{t=1}^{\infty} A^{(t)}$, where each block $A^{(t)}$ is r-orthonormal[6] and such that,

$$\sum_{\|x\|_2 \le 1} \sum_{j=1}^{\infty} \left| \sum_{k=1}^{\infty} a_{jk} x_k \right|^q = \sup_t \left\| A^{(t)} \right\|^q_{2,q} \le r!.$$

Consequently by our assumption on the λ's

$$\infty > \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} |\lambda_k a_{jk}|^2 \right)^{\frac{p}{2}} = \sum_{t=1}^{\infty} \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} |\lambda_k a^{(t)}_{jk}|^2 \right)^{\frac{p}{2}} = \sum_{t=1}^{\infty} 2^{rt} (2^{rt})^{-\frac{p}{2r}} \left(\sum_{k \in K_t} |\lambda_k|^2 \right)^{\frac{p}{2}}$$

$$= \sum_{t=1}^{\infty} 2^{tp\left(\frac{q}{2p}-\frac{1}{2}\right)} \left(\sum_{k \in K_t} |\lambda_k|^2 \right)^{\frac{p}{2}},$$

so that $\lambda \in \ell_{\frac{2p}{q},p}$ by Lemma 4.2.2.

[6] A $m \times n$ matrix $A = (a_{ij})$ is called r-orthogonal if, for $1 \le j_1, \ldots, j_r, k_1, \ldots k_r \le n$ we have

$$\sum_{j=1}^{\infty} \prod_{h=1}^{r} a_{j,j_h} \overline{a_{j,k_h}} = 1,$$

if $\{j_1, \ldots, j_r\} = \{k_1, \ldots, k_r\}$ and zero otherwise. Additionally if $|a_{jk}| = m^{-\frac{r}{2}}$, it is called r-orthogonal.

4. Suppose that $T \in \mathfrak{B}_2^R(\ell_2, \ell_2)$. By definition we have

$$\left(\sum_{i=1}^{k} \|Tx_i\|^2\right)^{\frac{1}{2}} \leq \left\|T|\mathfrak{B}_2^R\right\| \|x_i|w_2\| \,.$$

Consider a complete orthogonal system of ℓ_2, $(e_i)_{i\in I}$. From Bessel's inequality we have $\|x_i|w(2)\|^2 = \sum_{i=1}^k |\langle \varphi, e_i\rangle|^2 \leq \|\varphi\|^2$. Therefore,

$$\left(\sum_{i=1}^{k} \|Te_i\|^2\right)^{\frac{1}{2}} \leq \left\|T|\mathfrak{B}_2^R\right\| \sup_{\|\varphi\|\leq 1} \|\varphi\| \leq \left\|T|\mathfrak{B}_2^R\right\| .$$

Thus, taking the limit $k \to \infty$ implies $\left\|T|\mathfrak{S}_2^R\right\| \leq \left\|T|\mathfrak{B}_2^R\right\|$. Conversely, assume $T \in \mathfrak{S}_2$. Then for any $x \in X$ it follows from Schmidt's decomposition that there are two orthogonal systems (e_n) and $(f_n) = (Ve_n)$ for which

$$Tx = \sum_{n=1}^{\infty} \lambda_n f_n \langle x, e_n\rangle \,, \quad \|Tx\| = \sum_{n=1}^{\infty} |\lambda_n|^2 |\langle x, e_n\rangle|^2, \quad \left\|T|\mathfrak{S}_2^R\right\| = \left(\sum_{n=1}^{\infty} |\lambda_n|^2\right)^{\frac{1}{2}} .$$

Then it follows

$$\left(\sum_{i=1}^{k} \|Tx_i\|^2\right)^{\frac{1}{2}} = \left(\sum_{n=1}^{\infty} |\lambda_n|^2 \sum_{i=1}^{k} |\langle x_i, e_n\rangle|^2\right)^{\frac{1}{2}} \leq \left\|T|\mathfrak{S}_2^R\right\| \sup_{\|\varphi\|\leq 1} \left\{\left(\sum_{i=1}^{k} |\langle \varphi, x_i\rangle|^2\right)^{\frac{1}{2}}\right\}$$

which implies $\left\|T|\mathfrak{B}_2^R\right\| \leq \left\|T|\mathfrak{S}_2^R\right\|$.

$\square$

4.3 The Diagonal Limit Order

In this section we will omit the upper script notation for simplicity. However, one should keep in mind that all the structures in considerations are right structures and that, unless otherwise stated, the operators are right linear operators.

4.3.1 S-numbers of the diagonal operator

We now compute the the s-number for a given diagonal operator. For now we shall consider the right diagonal operator D acting between the right ℓ_p^m and the right ℓ_q^m spaces to be given by

$$\ell_p^m \ni (x_1, x_2, \dots x_m) \overset{D}{\longmapsto} (\sigma_1 x_1, \sigma_2 x_2, \dots, \sigma_m x_m) \in \ell_q^m \qquad \text{with } \sigma_1 \geq \cdots \geq \sigma_m > 0.$$

By definition, as $s_n(D) = 0$ for $n > m$, we might as well just consider $n \leq m$. Before proceeding we will need some technical results, which can be directly extended from the ones found in [40, pp. 213–215].

Lemma 4.3.1. *For each s-number function s_n, if $1 \leq p = q \leq \infty$, then $s_n(D) = \sigma_n$.*

Lemma 4.3.2. *Let $M \subset \ell_\infty^m$ with codim$(M) < n$; then, there exists $e = (\epsilon_1, \dots \epsilon_m) \in M$ with $\|e\|_\infty = 1$ such that the set $K := \{k : |\epsilon_k| < 1\}$ satisfies $|K| < n$.*

Lemma 4.3.3. *Let $0 < q < p < \infty$, $\mu_1, \dots, \mu_{n+1} > 0$, and $|\xi_{n+1}| \le |\xi_k|$ for $k = 1, \dots, n$. Then*

$$\frac{\left(\sum_{k=1}^{n+1} |\xi_k|^q \mu_k\right)^{\frac{1}{q}}}{\left(\sum_{k=1}^{n+1} |\xi_k|^p \mu_k\right)^{\frac{1}{p}}} \ge \frac{\left(\sum_{k=1}^{n} |\xi_k|^q \mu_k\right)^{\frac{1}{q}}}{\left(\sum_{k=1}^{n} |\xi_k|^p \mu_k\right)^{\frac{1}{p}}}.$$

We are now ready to give the first explicit computations of some s-numbers.

Lemma 4.3.4. *For $n = 1, ..., m$, we have*

$$a_n(D : \ell_p^m \to \ell_q^m) = c_n(D : \ell_p^m \to \ell_q^m) = d_n(D : \ell_p^m \to \ell_q^m) = \left(\sum_{k=n}^{m} \sigma_k^r\right)^{\frac{1}{r}},$$

where $1/r := 1/q - 1/p$, whenever $1 \le q \le p \le \infty$.

Proof. As above we set $A(x_1, \dots x_m) := (\sigma_1 x_1, \dots \sigma_{n-1} x_{n-1}, 0, \dots, 0)$. Then,

$$a_n(D) \le \|D - A\| = \sup_{\|x|\ell_p\|=1} \|(0, \dots, 0, \sigma_n x_n, \dots, \sigma_m x_m)|\ell_q\| = \left(\sum_{k=n}^{m} \sigma_k^r\right)^{\frac{1}{r}},$$

as a consequence of Hölder's inequality. On the other hand, consider a n-codimensional subspace $M \subset \ell_p^m$. Set

$$\ell_p^m \ni (x_1, \dots x_m) \overset{B}{\longmapsto} (\sigma_1^{-\frac{r}{p}} x_1, \dots, \sigma_m^{-\frac{r}{p}} x_m) \in \ell_\infty^m.$$

By Lemma 4.3.2, there exists $\mathbf{e} = (\epsilon_1, \dots, \epsilon_m) \in B(M)$ with $\|\mathbf{e}\|_\infty = 1$ such that K=$\{k : |\epsilon_k| < 1\}$ has less than n elements. Set $x := B^{-1}\mathbf{e}$. Then from Lemma 4.3.3 it follows

$$\left\|D J_M^{\ell_p^m}\right\| \ge \frac{\|Dx\|_q}{\|x\|_p} = \frac{\left(\sum_{k=1}^{m} |\epsilon_k|^q \sigma_k^r\right)^{\frac{1}{q}}}{\left(\sum_{k=1}^{m} |\epsilon_k|^p \sigma_k^r\right)^{\frac{1}{p}}} \ge \frac{\left(\sum_{k \notin K} |\epsilon_k|^q \sigma_k^r\right)^{\frac{1}{q}}}{\left(\sum_{k \notin K} |\epsilon_k|^p \sigma_k^r\right)^{\frac{1}{p}}} = \left(\sum_{k \notin K} \sigma_k^r\right)^{\frac{1}{r}} \ge \left(\sum_{k=1}^{m} \sigma_k^r\right)^{\frac{1}{r}}$$

and thus $c_n(D) \ge \left(\sum_{k=1}^{m} \sigma_k^r\right)^{\frac{1}{r}}$. By Theorem 3.3.5 we know $d_n(D) = c_n(D') \ge \left(\sum_{k=1}^{m} \sigma_k^r\right)^{\frac{1}{r}}$. Which proves the claim for $1 \le q < p < \infty$. The case $p = \infty$ follows the same lines but we do not require Lemma 4.3.3. □

Consequently,

$$a_n(D : \ell_p \to \ell_q) = c_n(D : \ell_p \to \ell_q) = d_n(D : \ell_p \to \ell_q) = \left(\sum_{k=n}^{\infty} \sigma_k^r\right)^{\frac{1}{r}},$$

whenever $1 \le q \le p \le \infty$. $1/r := 1/q - 1/p$. Many more results of this flavour are known, for example,

Theorem 4.3.5. *[42, p. 463] If $1 \le p \le q < \infty$, then for the diagonal operator $D : \ell_p \to \ell_q$, we have $d_n(D) = a_n(D) = \lambda$ when D is a Diagonal operator associated with (λ_n) and $\lambda_n \to \lambda$.*

For abbreviation we shall denote the identity operator from l_p^m to l_q^m by $I_{p,q}^m$. As particular case of the above theorem yields to

$$s_n(I_{p,q}^m) = c_n(I_{p,q}^m) = d_n(I_{p,q}^m) = (m - n + 1)^{\frac{1}{q} - \frac{1}{p}}.$$

4.3.2 The diagonal limit order

What is presented here follows the considerations found in [33]. Given positive scalar sequences (α_n) and (β_n), we write $\alpha_n \preceq \beta_n$ if there is a constant $c > 0$ for which $\alpha_n \leq c\beta_n$ for all n. The symbol $\alpha_n \asymp \beta_n$ means that $\alpha_n \preceq \beta_n$ and $\beta_n \preceq \alpha_n$.

Let $a = (\alpha_n) \in \ell_{t,w}$. We denote the corresponding diagonal operator by D_a, this means that $D_a(x_n) = (\alpha_n x_n)$. If $1 \leq p, q \leq \infty$ and $\frac{1}{t} > \frac{1}{q} - \frac{1}{p}$ then it is a consequence of Hölder's inequality that D_a is an operator from ℓ_p into ℓ_q. In the special case where $\alpha_n = n^{-\frac{1}{t}}$ we shall denote the corresponding diagonal operator by D_t. Now fix any s-function s. We aim to classify D_a with respect to a parameter r in the sense of finding the value r for which $D_a \in \mathfrak{L}^{(s)}_{r,w}(\ell_p, \ell_q)$. The following theorem introduces equivalent manners of solving this problem. Before we will require an interpolation formula between Lorentz spaces, whose proof on the real setting can be found in [30]

Theorem 4.3.6. *Let s be any s-function, $0 < r_0 < r_1 \leq \infty$, $0 < w, w_0, w_1 \leq \infty$ and $0 < \theta < 1$. If $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$, then there exists a constant $c > 0$, depending on the s-function and the numbers r, r_0, r_1, w, w_0, w_1 such that*

$$\left\| T|\mathfrak{L}^{(s)}_{r,w} \right\| \leq c \left\| T|\mathfrak{L}^{(s)}_{r_0,w_0} \right\|^{1-\theta} \left\| T|\mathfrak{L}^{(s)}_{r_1,w_1} \right\|^{\theta},$$

for all $T \in B^R(X, Y)$ where X and Y are right $\mathbb{H}$-Banach Spaces.

Theorem 4.3.7. *[33, p. 89] Consider $D_t : \ell_p \to \ell_q$. Let s be any s-function, and let $1 \leq p, q \leq \infty$. Assume that the exponents r and t satisfy $\frac{1}{r} = \frac{\mu}{t} + \nu$ for $0 \leq t_0 < t < t_1 < (\frac{1}{q} - \frac{1}{p})^{-1}$, where the parameters μ and ν may be dependent of p and q. Then the following are equivalent:*

1. *If $0 < w \leq \infty$ and $t_0 < t < t_1$, then*

$$a \in l_{t,w} \text{ implies } D_a \in \mathfrak{L}^{(s)}_{r,w}(\ell_p, \ell_q).$$

2. *If $t_0 < t < t_1$, then for each $n \in \mathbb{N}$*

$$s_n(D_t) \preceq n^{-\frac{1}{r}}.$$

3. *If $0 < w \leq \infty$ and $t_0 < t < t_1$, then for each $m \in \mathbb{N}$*

$$\left\| I^m_{p,q}|\mathfrak{L}^{(s)}_{r,w} \right\| \preceq m^{\frac{1}{t}}.$$

4. *For $t_0 < t < t_1$ and every $n \in \mathbb{N}$*

$$\left\| I^m_{p,q}|\mathfrak{L}^{(s)}_{r,\infty} \right\| \preceq m^{\frac{1}{t}}.$$

Proof.

(1) $\Rightarrow$ (2): It follows from the fact that for each $n \in \mathbb{N}$,

$$n^{\frac{1}{r}} s_n(D_t) \leq \left\| D_t|\mathfrak{L}^{(s)}_{r,\infty} \right\| < \infty.$$

(2) $\Rightarrow$ (3): Let $a \in \ell_{t,\infty}$. Then
$$s_n(D_a) \leq \sup_k |k^{\frac{1}{t}} \alpha_k| s_n(D_t)$$
implies that $D_a \in L^{(s)}_{r,\infty}(\ell_p, \ell_q)$. By the closed graph theorem it follows that there is a constant $c > 0$ such that
$$\left\| D_a | \mathfrak{L}^{(s)}_{r,\infty} \right\| \leq c \, \|a|\ell_{t,\infty}\|$$
for all $a \in \ell_{t,\infty}$. Yielding to
$$\left\| I^m_{p,q} | \mathfrak{L}^{(s)}_{r,\infty} \right\| \leq c m^{\frac{1}{t}}.$$
Finally, with Theorem 4.3.6 we complete this step.

(3) $\Rightarrow$ (4): Trivial.

(4) $\Rightarrow$ (1): Let $t_0 < t < t_1$. We may assume that $\mathfrak{L}^{(s)}_{r,\infty}$ is u-normed [7]. Without loss of generality we might assume that $u \leq t$. Moreover, for $k = 0, 1, \ldots$ we define
$$N_k = \{n \in \mathbb{N} : 2^k \leq n < 2^{k+1}\}.$$
If $a = (\alpha_n) \in \ell_{t,u}$, then it follows
$$\begin{aligned} \|a|\ell_{t,u}\|^u &= \sum_{n=1}^{\infty} \left(n^{\frac{1}{t} - \frac{1}{u}} \alpha_n \right)^u = \sum_{k=1}^{\infty} \sum_{n \in N_k} \left(n^{\frac{1}{t} - \frac{1}{u}} \alpha_n \right)^u \\ &\geq \sum_{k=0}^{\infty} 2^k \left(2^{(k+1)(\frac{1}{t} - \frac{1}{u})} \alpha_{2^{k+1}} \right)^u = \frac{1}{2} \sum_{k=0}^{\infty} \left(2^{\frac{k+1}{t}} \alpha_{2^{k+1}} \right)^u. \end{aligned}$$
Let $D_k : \ell_p \to \ell_q$ be defined by
$$D_k(\xi_n) = \begin{cases} \alpha_n \xi_n \text{ for} & n \in N_k \\ 0 & \text{otherwise} \end{cases}.$$
Then $D_a = \sum_{k=0}^{\infty} D_k$ and since $\|D_k\| = \alpha_{2^k}$ by hypothesis we have
$$\begin{aligned} \left\| D_a | \mathfrak{L}^{(s)}_{r,\infty} \right\| &\leq \left(\sum_{k=0}^{\infty} \left\| D_k | \mathfrak{L}^{(s)}_{r,\infty} \right\|^u \right)^{\frac{1}{u}} \leq c \left(\sum_{k=0}^{\infty} \left(2^{\frac{k}{t}} \alpha_{2^k} \right)^u \right)^{\frac{1}{u}} \\ &\leq c\alpha_0 + c \left(\sum_{k=0}^{\infty} \left(2^{\frac{k+1}{t}} \alpha_{2^{k+1}} \right)^u \right)^{\frac{1}{u}} \leq c\alpha_0 + 2^{\frac{1}{u}} c \, \|a|\ell_{t,u}\| < \infty. \end{aligned}$$
Since this is true for all $t \in (t_0, t_1)$ then (1) follow from Theorem 4.3.6.

□

Given an s-number function s we define the *diagonal limit order* by the value
$$\rho_{diag}(t, p, q|s) := \sup \left\{ p \geq 0 : s_n(D_t : \ell_p \to \ell_q) \preceq \frac{1}{n^p} \right\}.$$

[7] As observe in [39, p. 93] on the classic setting, if A^R is a right quasi normed operator ideal with quasi-norm α, then there exists a u-norm, α_u which is equivalent to α. Here a u-norm stands for a quasi norm that satisfies $\alpha_u(T+S)^p \leq \alpha_u(T)^u + \alpha_u(S)^u$, for any $T, S \in A^R(X,Y)$.

Indeed, for a right linear and bounded operator S such norm is given by $\alpha_u(S) = \inf \|\alpha(S_i)|\ell_u\|$, the infimum being taken over all representation $S = \sum_{i=1}^{\infty} S_i$ with $S_i \in A^R(X,Y)$.

This is how Theorem 4.3.7 enters the picture. In order to compute $\rho_{diag}(t,p,q|s)$ for a given s-number function, we can equivalently, study the asymptotic behaviour of the expression

$$\left\| I^m_{p,q} | \mathfrak{L}^{(s)}_{r,w} \right\|,$$

for the different s-numbers.

Because of the duality relations it is enough to consider the approximation numbers, the Weyl numbers, the Gelfand numbers and the Hilbert numbers. Moreover, following the works of E.D. Gluskin, we can further simplify this task. In section 6.1 we explicitly show that

$$a_n(I^m_{p,q}) \asymp \max\{c_n(I^m_{p,q}), d_n(I^m_{p,q})\},$$

which allow us to restrict ourselves to the study of Gelfand, Weyl and Hilbert numbers.

The Ideal of Gelfand Operators

Also in Section 6.1 we explicitly compute the asymptotic behaviour of Kolmogorov widths. We have shown in Theorem 3.3.5 that the Gelfand numbers are the dual counterpart of the Kolmogorov numbers, hereby we can write

$$c_n(I^m_{p,q}) = d_n(I^m_{q',p'}).$$

Therefore, in conjuction with Lemma 4.3.4, the results obtained in section 6.1 give us the following theorem

Theorem 4.3.8. *1. Let $1 \leq q \leq p \leq \infty$. Then*

$$c_n(I^m_{p,q}) = (m-n+1)^{\frac{1}{q}-\frac{1}{p}} \qquad \textit{for } 1 \leq n \leq m.$$

2. Let $2 \leq p \leq q \leq \infty$. Then

$$c_n(I^m_{p,q}) \asymp \begin{cases} \left(\frac{m-n+1}{m}\right)^{\frac{\frac{1}{p}-\frac{1}{q}}{1-\frac{2}{q}}} & \textit{for } 1 \leq n \leq m - m^{\frac{2}{q}}, \\ m^{\frac{1}{q}-\frac{1}{p}} & \textit{for } m = n = m - m^{\frac{2}{q}}. \end{cases}$$

3. Let $1 < p \leq 2 \leq q \leq \infty$. Then

$$c_n(I^m_{p,q}) \asymp \begin{cases} \left(\frac{m-n+1}{m}\right)^{\frac{1}{2}} & \textit{for } 1 \leq n \leq m^{\frac{2}{p'}}, \\ \left(\frac{m-n+1}{m}\right)^{\frac{1}{2}} m^{\frac{1}{p'}} n^{-\frac{1}{2}} & \textit{for } m^{\frac{2}{p'}} \leq n \leq \frac{m}{m^{\frac{2}{q}-1}+1}, \\ m^{\frac{1}{q}-\frac{1}{p}} & \textit{for } \frac{m}{m^{\frac{2}{q}-1}+1} \leq n \leq m. \end{cases}$$

4. Let $1 \leq p \leq q \leq 2$. Then

$$c_n(I^m_{p,q}) \asymp \begin{cases} 1 & \textit{for } 1 \leq n \leq m^{\frac{2}{p'}}, \\ \left(m^{\frac{1}{p'}} n^{-\frac{1}{2}}\right)^{\frac{\frac{1}{p}-\frac{1}{q}}{\frac{1}{p}-\frac{1}{2}}} & \textit{for } m^{\frac{2}{p'}} \leq n \leq m. \end{cases}$$

As a consequence of Theorem 4.3.7, we have

Lemma 4.3.9. *Let* $1 \leq p, q \leq \infty$ *and* $0 < r < \infty$. *Then*

$$\left\| I^m_{p,q} | L^{(c)}_{r,\infty} \right\| \asymp m^{\frac{1}{t}}$$

with the values of $\frac{1}{t}$ *being:*

1. *Let* $1 \leq q \leq p \leq \infty$. *Then*
$$\frac{1}{t} = \frac{1}{r} - \frac{1}{p} + \frac{1}{q}.$$
2. *Let* $2 \leq p \leq q \leq \infty$. *Then*
$$\frac{1}{t} = \frac{1}{r}$$
3. *Let* $1 < p \leq 2 \leq q \leq \infty$. *Then*
$$\frac{1}{t} = \begin{cases} \frac{1}{r} - \frac{1}{p} + \frac{1}{2} & \text{for } 0 < r \leq 2, \\ \frac{2}{p'r} & \text{for } 2 < r < \infty. \end{cases}$$
4. *Let* $1 \leq p \leq q \leq 2$. *Then*
$$\frac{1}{t} = \begin{cases} \frac{1}{r} - \frac{1}{p} + \frac{1}{q} & \text{for } 0 < r \leq 2\frac{\frac{1}{p}-\frac{1}{2}}{\frac{1}{p}-\frac{1}{q}}, \\ \frac{2}{p'r} & \text{for } 2\frac{\frac{1}{p}-\frac{1}{2}}{\frac{1}{p}-\frac{1}{q}} < r < \infty. \end{cases}$$

Proof. This is an immediate consequence of Theorem 4.3.8. Indeed, since $1 \leq n \leq m$, we have

$$\left\| I^m_{p,q} | L^{(c)}_{r,\infty} \right\| = \sup_n n^{\frac{1}{r}} c_n(I^m_{p,q}) = m^{\frac{1}{r}} \sup_n c_n(I^m_{p,q}).$$

The claim follows from the assymptotic behaviour obtained in Theorem 4.3.8. $\square$

The previous result has a rather interesting interpretation, when writing $\frac{1}{r}$ as a function of t, p and q, in the unit square with the coordinates $\frac{1}{p}$ and $\frac{1}{q}$, acting as x-axis and y-axis, respectively.

Theorem 4.3.10. *If* $1 \leq p, q \leq \infty$, $\frac{1}{t} > \left(\frac{1}{q} - \frac{1}{p}\right)_+$, $0 < w \leq \infty$ *and* $a \in \ell_{t,t}$ *then* $D_a \in L^{(c)}_{r,w}(\ell_p, \ell_q)$, *where* $\frac{1}{r}$ *takes values as as indicated in the following diagrams.*

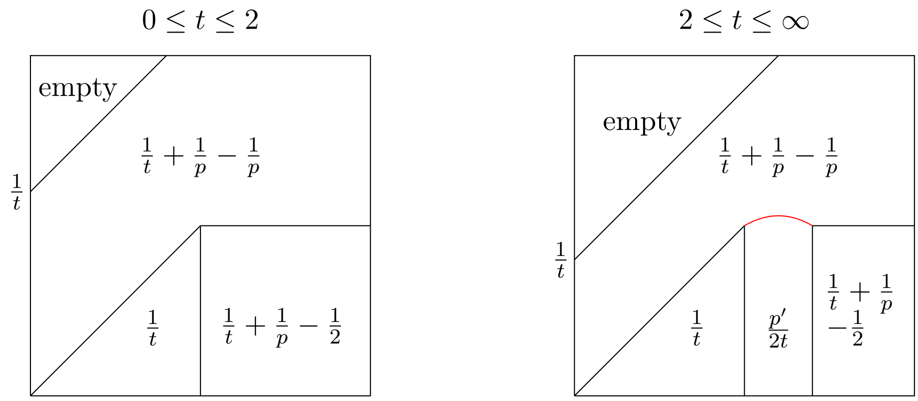
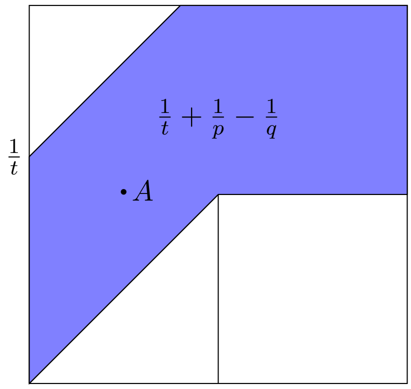


Figure 4.1: Diagonal limit order of the Gelfand operator ideal

The curve in red being given by

$$\frac{1}{t} = \frac{\frac{1}{p} - \frac{1}{q}}{p'\left(\frac{1}{p} - \frac{1}{2}\right)}.$$

To clarify what these diagrams represent, let us focus on the case of $0 \le t \le 2$.



At any point in the blue region we can conclude that, since $0 \le t \le 2$,

$$\frac{1}{2} + \frac{1}{p} - \frac{1}{q} \le r \le \infty$$

then D_a belongs to $\mathfrak{L}^{(c)}_{r,w}(\ell_p, \ell_q)$ for any $0 \le w \le \infty$. For example, the point $A = (1/4, 1/2)$, we have that $D_a \in \mathfrak{L}^{(c)}_{r,w}(\ell_2, \ell_4)$ for any $0 \le w \le \infty$ if $\frac{3}{4} \le r \le \infty$. Thus, the equation presented at each region, dictates the relation that must exist between r, p, q so that $D_a \in \mathfrak{L}^{(c)}_{r,w}(\ell_p, \ell_q)$ for any $0 \le w \le \infty$.

The Ideal of Weyl Operators

The following results were initially obtained in [35]. We present the details in section 6.2.

Theorem 4.3.11. *Let $1 \le p < \max(2, q) \le \infty$ and $1 \le n \le \frac{m}{2}$. Then*

$$x_n(I^m_{p,q}) \asymp \begin{cases} 1 & \textit{for } 2 \le p \le q \le \infty, \\ n^{\frac{1}{q} - \frac{1}{p}} & \textit{for } 1 \le p \le q \le 2, \\ n^{\frac{1}{2} - \frac{1}{p}} & \textit{for } 1 \le p \le 2 \le q \le \infty, \\ m^{\frac{1}{q} - \frac{1}{p}} & \textit{for } 1 \le q \le p \le 2. \end{cases}$$

If $\max(2, q) \le p$, *then*

$$x_n(I^m_{p,q}) \asymp \begin{cases} m^{\frac{1}{q} - \frac{1}{p}} & \textit{for } 1 \le n \le m^{\frac{2}{p}}, \\ m^{\frac{1}{q}} n^{-\frac{1}{2}} & \textit{for } m^{\frac{2}{p}} \le n \le m. \end{cases}$$

From this we deduce the following behaviour of the corresponding quasi-norms.

Lemma 4.3.12. *Let $1 \le p, q \le \infty$ and $0 < r < \infty$. Then*

$$\left\| I^m_{p,q} | \mathfrak{L}^{(x)}_{r,\infty} \right\| \asymp m^{\frac{1}{t}}$$

with the following values of $\frac{1}{t}$: if $1 \le p < \max(2, q) \le \infty$ and $1 \le n \le \frac{m}{2}$, then

$$\frac{1}{t} = \begin{cases} \frac{1}{r} & \textit{for } 2 \le p \le q \le \infty, \\ \left(\frac{1}{r} - \frac{1}{p} + \frac{1}{q}\right)_+ & \textit{for } 1 \le p \le q \le 2, \\ \left(\frac{1}{r} - \frac{1}{p} + \frac{1}{2}\right)_+ & \textit{for } 1 \le p \le 2 \le q \le \infty, \\ \frac{1}{r} - \frac{1}{p} + \frac{1}{q} & \textit{for } 1 \le q \le p \le 2. \end{cases}$$

if $\max(2,q) \leq p$. *Then*

$$\frac{1}{t} = \begin{cases} \frac{1}{r} - \frac{1}{p} + \frac{1}{q} & \text{for } 0 \leq r \leq 2, \\ \frac{2}{pr} - \frac{1}{p} + \frac{1}{q} & \text{for } 2 < r < \infty. \end{cases}$$

And hence we obtain

Theorem 4.3.13. *If* $1 \leq p, q \leq \infty$, $\frac{1}{t} > \left(\frac{1}{q} - \frac{1}{p}\right)_+$, $0 < w \leq \infty$ *and* $a \in \ell_{t,w}$, *then* D_a *belongs to* $\mathfrak{L}^{(x)}_{r,w}(\ell_p, \ell_q)$, *where* $\frac{1}{r}$ *takes values as indicated in the following diagrams, where* $* = \frac{p}{2}\left(\frac{1}{t} + \frac{1}{p} - \frac{1}{q}\right)$ *and* $** =$ *empty.*

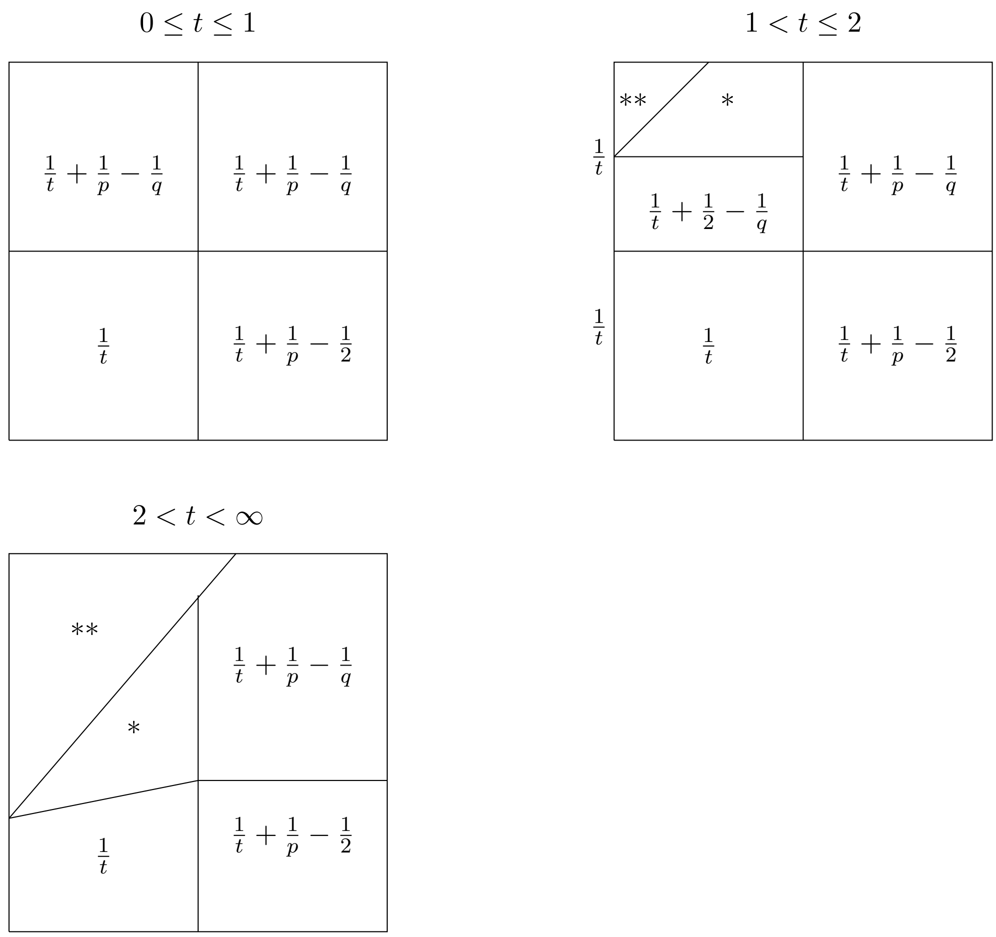


Figure 4.2: Diagonal limit order of the Weyl operator ideal.

The Ideal of Hilbert Operators

In section 6.3 the explicit computations are presented together with their references.

Theorem 4.3.14. *1. Let* $1 \leq p' \leq q \leq 2$. *Then*

$$h_n(I^m_{p,q}) \asymp \begin{cases} m^{\frac{1}{q} - \frac{1}{p}} & \text{for } 1 \leq n \leq m^{\frac{2}{p}}, \\ m^{\frac{1}{q}} n^{-\frac{1}{2}} & \text{for } m^{\frac{2}{p}} \leq n \leq m^{\frac{2}{q'}}, \\ mn^{-1} & \text{for } m^{\frac{2}{q'}} \leq n \leq m. \end{cases}$$

2. Let $2 \le p, q \le \infty$. *Then*

$$h_n(I^m_{p,q}) \asymp \begin{cases} m^{\frac{1}{q}-\frac{1}{p}} & \text{for } 1 \le n \le m^{\frac{2}{p}} \\ m^{\frac{1}{q}} n^{-\frac{1}{2}} & \text{for } m^{\frac{2}{p}} \le n \le m \end{cases}.$$

3. Let $2 \le p' \le q \le \infty$. *Then, for* $1 \le n \le m$.

$$h_n(I^m_{p,q}) \asymp n^{\frac{1}{q}-\frac{1}{p}}.$$

Therefore, we can derive the following asymptotic behaviour

Lemma 4.3.15. *For* $1 \le p, q \le \infty$ *and* $0 < r < \infty$ *we have,*

$$\left\| I^m_{p,q} | \mathfrak{L}^{(h)}_{r,\infty} \right\| \asymp m^{\frac{1}{t}}$$

with the values of t *being given as follows:*

1. Let $1 \le p' \le q \le 2$. *Then*

$$\frac{1}{t} = \begin{cases} \frac{1}{r} & \text{for } 0 < r \le 1 \\ \frac{2}{q'r} + \frac{2}{q} - 1 & \text{for } 1 < r \le 2 \\ \frac{2}{pr} + \frac{1}{q} - \frac{1}{p} & \text{for } 2 < r < \infty \end{cases}.$$

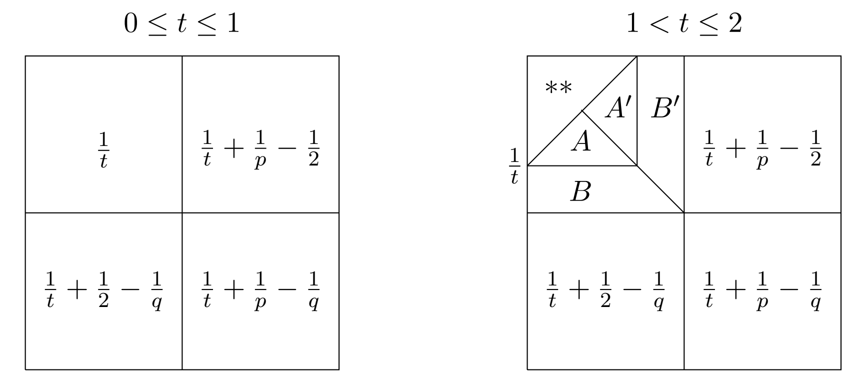
2. Let $2 \le p, q \le \infty$. *Then*

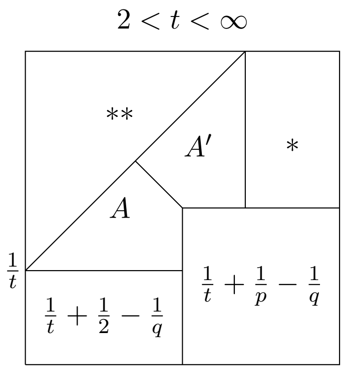
$$\frac{1}{t} = \begin{cases} \frac{1}{r} + \frac{1}{q} - \frac{1}{2} & \text{for } 0 < r \le 2 \\ \left(\frac{2}{pr} + \frac{1}{q} - \frac{1}{p}\right)_+ & \text{for } 2 < r < \infty \end{cases}.$$

3. Let $2 \le p' \le q \le \infty$. *Then, for* $0 < r < \infty$

$$\frac{1}{t} = \left(\frac{1}{r} + \frac{1}{q} - \frac{1}{p}\right)_+ := \max\left\{\frac{1}{r} + \frac{1}{q} - \frac{1}{p}, 0\right\}.$$

Theorem 4.3.16. *If* $1 \le p, q \le \infty$, $\frac{1}{t} > \left(\frac{1}{q} - \frac{1}{p}\right)_+$, $0 < w \le \infty$ *and* $a \in \ell_{t,w}$, *then* D_a *belongs to* $\mathfrak{L}^{(h)}_{r,w}(\ell_p, \ell_q)$, *where* $\frac{1}{r}$ *takes values as indicated in the following diagrams.*





$A = \frac{p}{2}\left(\frac{1}{t} + \frac{1}{p} - \frac{1}{q}\right);$

$A' = \frac{q'}{2}\left(\frac{1}{t} + \frac{1}{p} - \frac{1}{q}\right);$

$B = \frac{q'}{2}\left(\frac{1}{t} + 1 - \frac{2}{q}\right);$

$B' = \frac{p}{2}\left(\frac{1}{t} + \frac{2}{p} - 1\right);$

$* = \frac{1}{t} + \frac{1}{p} - \frac{1}{2};$

$** =$ empty.

Figure 4.3: Diagonal limit order of the Hilbert operator ideal.

CHAPTER 5

Conclusions

Our objective of extending the s-number theory to the quaternionic setting has been successfully achieved. By employing the proposed axiomatic framework, we have obtained several significant results concerning quaternionic Hilbert spaces, Banach spaces, and operators acting on them.

In particular, we have demonstrated the uniqueness of s-number functions over quaternionic Hilbert spaces through the quaternionic functional calculus. This achievement has allowed a natural extension of the concept of Schatten classes to compact operators over quaternionic Hilbert spaces. Additionally, we have derived various examples of s-number functions in quaternionic Banach spaces, thereby expanding the classification of operators acting on these spaces beyond the realm of classical Banach spaces.

Furthermore, we have explored the consequences of s-number theory in the theory of ideals of quaternionic operators. By employing the proposed axiomatic setting, we have identified instances where specific components of operator ideals coincide. Moreover, we have computed the limit order of specific ideals, resulting in a more precise classification of diagonal operators and a more intricate categorization of ideals.

It is important to note that s-number theory in the context of quaternionic analysis is by no means complete; the results in this thesis are only the basis of the theory, and as such, it opens up numerous avenues for further exploration. Some of these potential directions include:

- Trace of quaternionic operators: Pietsch is famous for, not exclusively, generalizing the results on nuclear operators, due to Grothendieck, to a more specific structure: the (r, p, q)-nuclear operators. These allow the investigation of operators acting Banach spaces for which a trace remains well defined. This extension could also lead to the adaptation of Grothendieck-Lidskii formulas for specific (r, p, q)-nuclear operators. The proposed definition of nuclear numbers is motivated by these endeavors.
- Distribution of eigenvalues of quaternionic operators: This particular subject shows great promise. s-numbers, especially Weyl numbers, are closely related to the behavior of eigenvalues of an operator. A lot of research has been produced in this topic. Equipped

with the suitable notion of spectrum in quaternionic analysis, this may yield to successful extensions.

- Quaternionic C^*- algebras: While our focus has been primarily on operator ideals, the realm of linear and bounded operators constitutes a subset of the larger theory of C^*-algebras. The axiomatic approach proposed for operator ideals sheds light on the study of ideals in this algebraic structure. Generalizing the ideas of Gelfand-Shillov-Smirnov to a broader context is another possible line of research.
- Clifford analysis: The algebra of the quaternions is a specific case of the so called Clifford algebras, both sharing associativity and the existence of a unit. In such general settings, new difficulties arise, as for instance, the existence of zero divisors (the quaternions being the "largest" Clifford algebra where these still do not appear). This direction is perhaps the most challenging one, since even in the finite-dimensional case, it is not clear how to describe the right spectrum.

While there is still much work to be done, this thesis serves as evidence that, under the appropriate modifications, important theories within the framework of functional analysis can be developed in the quaternionic setting.

CHAPTER 6

Appendix

6.1 Computations of the asymptotic behaviour of Kolmogorov numbers

We adapt the results obtained in [20] and the references therein to effectively compute the asymptotic behaviour of $I^m_{p,q}$. But first we need several technical results and definitions. We will denote the group of $m \times m$ orthogonal matrices by $\mathcal{O}(m)$ and by $\mathcal{P}$ the Haar measure[1] on $\mathcal{O}(m)$. For $n \leq m$ and $x \in \mathbb{R}^n$ we set $\varphi_n(x) = \|x|\ell_2(N)\|$ where $N = \{1, \dots, n\}$.

M_n will denote the median of φ_n with respect to the normalized Lebesgue measure on the $(m-1)$-sphere, here and thereafter denoted by μ, i.e.,

$$\mu\{x \in \mathbb{S}^{m-1} : \varphi_n(x) \leq M_n\} = \frac{1}{2}.$$

We remark that for any nonzero $x \in \mathbb{R}^m$, the measure $\mathcal{P}$ induces the measure μ under the mapping $U \to \frac{Ux}{\|x\|_2}$. Moreover, it can be verified directly that, there is a positive constant $\mathfrak{g}$ for which $\mathfrak{g}\sqrt{\frac{n}{m}} \leq M_n \leq \sqrt{\frac{2n}{m}}$. The following result will be necessary,

Lemma 6.1.1. *[18, p. 26] For any $\lambda > 0$ there holds*

$$\mu\{x \in S^{m-1} : |\varphi_n(x) - M_n| > \lambda\} \leq 4e^{\frac{-\lambda^2 m}{2}}. \tag{6.1}$$

In particular,

$$\mu\{x \in S^{m-1} : |\varphi_n^2(x) - M_n^2| > (2M_n + \lambda)\lambda\} \leq 4e^{\frac{-\lambda^2 m}{2}}. \tag{6.2}$$

Consider $A \subset \{1, 2, \dots, m\}$. We define $B_p^A := \{x \in \mathbb{R}^m : \|x\|_p \leq 1,\ x_i = 0 \text{ for } i \notin A\}$. $\mathcal{F}_0(N)$ will stand for the set of absolutely convex polyhedra in B_2^m that have at most $2N$ vertices. Moreover, for an absolutely convex subset V, we will say that $V \in \mathcal{F}(N)$ if there exists a $K \in \mathcal{F}_0(N)$ such that $V \subset K$. Furthermore, we define

$$\mathcal{E}_A = \{x \in B_2^A : |x_i| = |A|^{-\frac{1}{2}}, i \in A\},$$

[1] i.e. the positive, left invariant Radon measure

where $|A|$ denotes the cardinality of A. The symbol $\mathcal{E}'_A$ will be used to refer to the minimal ϵ-net[2] for the set B_2^A, while $\Lambda_A(\epsilon)$ will denote the minimal ϵ-net for the set $\sqrt{|A|}B_1^A$ both of which with respect to the metric of ℓ_2^m. As observed in [29] we can relate the cardinality of these sets with cardinality of A. Indeed, there it is shown that $|\Lambda_A(\epsilon)| \leq (\frac{\mathfrak{a}}{\epsilon})^{|A|}$ and that $|\mathcal{E}'_A| \leq 5^{|A|}$. Finally define $P_{n,m}$, the orthoprojector of $\mathbb{R}^m$ onto the subspace generated by the n first basis elements and $Q_{n,m} = I_m - P_{n,m}$. In what follows, E_W will denote the the n-dimensional normed space with unit ball W.

We start with the construction of a specific polyhedra $K_q(\lambda)$. Fix $q \in [2, \infty[$ and let $\lambda \geq 1$. Set

$$k(\lambda) = \left[\lambda^{\frac{2q}{q-2}}\right], \qquad t_k = \left[\lambda^2 k^{\frac{2}{q}}\right] + 1, \quad \text{for } k > k(\lambda), \qquad \text{and } s_k = \left[\frac{k}{t_k}\right].$$

It follows that $t_k \leq k$ and, consequently, $\frac{k}{2t_k} \leq s_k \leq \frac{k}{t_k}\cdot$ Decompose the set $\{1, \ldots, m\}$ into s_k subsets $\sigma_1, \ldots, \sigma_{s_k}$ such that $|\sigma_i| \leq \frac{2m}{s_k}$, $i = 1, \ldots, s_k$. Take $k(\lambda) < k \leq m$, and define

$$\Delta_k := \{A \subset \{1, \ldots, m\} : |A| = t_k \text{ and there exists } i \in \{1, \ldots s_k\} \text{ for which } A \in \sigma_i\}.$$

Moreover, set Δ_0 as the family of all subsets of $\{1, \ldots, m\}$ whose cardinality is $k(\lambda)$. Finally, take

$$\mathcal{E} = \bigcup_{k(\lambda)<k\leq m} \bigcup_{A\in\Delta_k} \mathcal{E}_A, \quad \mathcal{E}' = \bigcup_{A\in\Delta_0} \mathcal{E}'_A$$

and define $K_q(\lambda)$ as the convex hull of the union of these sets, i.e.

$$K_q(\lambda) = conv\{\mathcal{E} \cup \mathcal{E}'\}.$$

The following lemma shows that $K_q(\lambda)$ is indeed an absolutely convex polyhedron.

Lemma 6.1.2. *There exists a constant solely dependent on q, $\mathfrak{c}(q)$ for which*

$$K_q(\lambda) \in \mathcal{F}_0(2e^{\mathfrak{c}(q)}\lambda^2 m^{\frac{2}{q}}).$$

Proof. By the above presented construction, it is clear that $K_q(\lambda) \subset B_2^m$ and also that it suffices to show that both $\mathcal{E}$ and $\mathcal{E}'$ have at most $e^{\lambda^2 m^{\frac{2}{q}}\mathfrak{c}(q)}$ points, for if that is the case then $K_q(\lambda)$ will be a convex subset of B_2^m with at most $2e^{\lambda^2 m^{\frac{2}{q}}\mathfrak{c}(q)}$ points. Set the constant

$$\mathfrak{c}(q) = 3 \sup\left\{x^{-\frac{2}{q}} \ln(8ex) : \ x \geq 1\right\},$$

[2]Recall that a subset A of X is an ϵ-net if $d(x, A) < \epsilon$ for any $x \in X$. It will be refer to minimal if it is minimal in the sense of cardinality.

then we can write[3]

$$\begin{aligned}
|\mathcal{E}| &= \sum_{k(\lambda)<k\leq m}\sum_{A\in\Delta_k}|\mathcal{E}_A| = \sum_{k(\lambda)<k\leq m} 2^{t_k}|\Delta_k| \\
&= \sum_{k(\lambda)<k\leq m}\sum_{i\leq s_k} 2^{t_k}\binom{|\sigma_i|}{t_k} = \sum_{k(\lambda)<k\leq m} s_k 2^{t_k}\binom{\left[\frac{2m}{s_k}\right]}{t_k} \\
&\leq \sum_{k(\lambda)<k\leq m} s_k 2^{t_k}\binom{\left[\frac{2m}{s_k}\right]}{t_k} \leq \sum_{k(\lambda)<k\leq m} s_k 2^{t_k}\left(\frac{e\left[\frac{2m}{s_k}\right]}{t_k}\right)^{t_k} \\
&\leq \sum_{k(\lambda)<k\leq m} s_k 2^{t_k}\left(4\frac{e\,[m]}{k}\right)^{t_k} = \sum_{k(\lambda)<k\leq m} s_k\left(\frac{8em}{k}\right)^{t_k} \\
&= \sum_{k(\lambda)<k\leq m} e^{\lambda^2 k^{\frac{2}{q}}\ln\left(\frac{8em}{k}\right)+\ln\left(\frac{8ems_k}{k}\right)} \leq e^{\lambda^2 m^{\frac{2}{q}}\frac{\mathfrak{c}(q)}{3}}\sum_{k(\lambda)<k\leq m} e^{\ln(8em)} \\
&\leq e^{\lambda^2 m^{\frac{2}{q}}\frac{\mathfrak{c}(q)}{3}} e^{2\ln(8em)} \leq e^{\lambda^2 m^{\frac{2}{q}}\frac{\mathfrak{c}(q)}{3}} e^{\frac{2}{3}\mathfrak{c}(q)m^{\frac{2}{q}}} \leq e^{\lambda^2 m^{\frac{2}{q}}\mathfrak{c}(q)}.
\end{aligned}$$

Moreover, since $0\leq 1-\frac{2}{q}<1$,

$$|\mathcal{E}'| \leq \sum_{A\in\Delta_0}|\mathcal{E}'_A| \leq 5^{k(\lambda)}\binom{m}{k(\lambda)} \leq e^{k(\lambda)\ln\left(\frac{5em}{k(\lambda)}\right)} \leq e^{\mathfrak{c}(q)m^{\frac{2}{q}}k(\lambda)^{1-\frac{2}{q}}} \leq e^{\mathfrak{c}(q)m^{\frac{2}{q}}\lambda^2}. \qquad \square$$

Next we introduce two functions that have useful properties. Consider the function

$$v_{q,\lambda}(x) = \sup\left\{\frac{1}{2}\varphi_{k(\lambda)}(x), \sup_{k(\lambda)<k\leq m}\lambda k^{\frac{1}{q}}x_k^*\right\},$$

and, for $2\leq r<\infty$ and $\mu\geq 1$, the function

$$w_{r,\mu}(x) = \inf_{k\leq m}\left(\varphi_k(x)+\mu\left(\sum_{k<i\leq m}(x_i^*)^r\right)^{\frac{1}{r}}\right), \qquad x\in\mathbb{R}^m.$$

Lemma 6.1.3. *For all $x\in\mathbb{R}^m$ there holds*

$$v_{q,\lambda}(x) \leq \max\{|\langle x,y\rangle| : y\in K_q(\lambda)\}. \tag{6.3}$$

Proof. We start by assuming that

$$v_{q,\lambda}(x) = \frac{1}{2}\left(\sum_{i\leq k(\lambda)}(x_i^*)^2\right)^{\frac{1}{2}} = \frac{1}{2}\sup_{A\in\Delta_0}\left(\sum_{i\in A}|x_i|^2\right)^{\frac{1}{2}}.$$

Then,

$$\frac{1}{2}\sup_{A\in\Delta_0}\left(\sum_{i\in A}|x_i|^2\right)^{\frac{1}{2}} = \sup\left\{|\langle x,y\rangle| : y\in B_2^A\right\} \leq 2\sup\left\{|\langle x,y\rangle| : y\in\mathcal{E}'_A\right\},$$

[3] Recall the inequality $\binom{n}{k}\leq\left(\frac{en}{k}\right)^k$

which means that there is an $y \in \mathcal{E}'_A$ such that $v_q\lambda(x) \leq |\langle x, y\rangle|$. Now assume assume that $v_{q,\lambda}(x) = \lambda k^{\frac{1}{q}} x^*_k$ for some $k \in k(\lambda)$. Then, there exists a subset $B \subset \{1, \dots, m\}$ with k elements for which $v_{q,\lambda}(x) = \lambda k^{\frac{1}{q}} |x_i|$ for $i \in B$. Since

$$\sum_{j \leq s_k} |\sigma_j \cap B| = k,$$

there is a j such that $|\sigma_j \cap B| \leq \frac{k}{s_k} \leq t_k$. Thus, some $A \in \Delta_k$ lies completly in B. Now observe that

$$v_{q,\lambda}(x) \leq \frac{t_k^{\frac{1}{2}}}{\lambda k^{\frac{1}{q}}} v_{q,\lambda}(x) \leq |A|^{-\frac{1}{2}} \sum_{i \in A} |x_i| = \sup\{|\langle x, y\rangle| \, |y \in \mathcal{E}_A\}.$$

And by construction of $K_q(\lambda)$ the claim follows. □

Lemma 6.1.4. *There is a constant solely dependent on q and r, $\mathfrak{d}(q, r)$, for which there holds*

$$w_{r,\mu}(x) \leq \mathfrak{d}(q, r) v_{q,\lambda}(x), \quad \text{for } q > r \text{ and } \lambda = m^{\frac{1}{r} - \frac{1}{q}} \mu. \tag{6.4}$$

Proof. By homogeneity, it can be assumed without loss of generality that $v_{q,\lambda}(x) \leq 1$. In this case

$$\phi_k(x) \leq 2 \text{ and } x^*_k \leq (\lambda k^{\frac{1}{q}})^{-1} \text{ for } k > k(\lambda).$$

Therefore we ought to show that $w_{r,\mu}(x) \leq \mathfrak{d}(q, r)$. Set the constant

$$\mathfrak{d}(q, r) = 2 + \left(1 - \frac{r}{q}\right)^{-\frac{1}{r}}$$

It immediately follows that,

$$w_{r,\mu}(x) \leq 2 + \mu \left(\sum_{k(\lambda) < k \leq m} (\lambda k^{\frac{1}{q}})^{-r} \right)^{\frac{1}{r}} \leq 2 + m^{\frac{1}{q} - \frac{1}{r}} \left(\left(1 - \frac{r}{q}\right)^{-1} m^{1 - \frac{r}{q}} \right)^{\frac{1}{r}} \leq \mathfrak{d}(q, r). \quad \square$$

The last result that we require follows,

Lemma 6.1.5. *Let $1 < p \leq 2$ and $\mu \geq 1$. Then, there are constants $\mathfrak{e}(p)$ and $\mathfrak{f}(p)$ that only depend of p for which,*

$$(B_2^m \cap \mu B_p^m) \in \mathfrak{e}(p) \mathcal{F}(e^{\mathfrak{f}(p)} \mu^2 m^{2-2p}).$$

Proof. To see this, set $\lambda = m^{\frac{1}{p'} - \frac{1}{2p'}} \mu$ and take $\mathfrak{e}(p) = \mathfrak{d}(2p', p')$ and $\mathfrak{f}(p) = \mathfrak{c}(2p')$. By Lemma 6.1.2 it suffices to show

$$(B_2^m \cap \mu B_p^m) \subset \mathfrak{e}(p) K_{2p'}(\lambda).$$

By duality the previous inclusion is equivalent to

$$\sup\{|\langle x, y\rangle| : y \in B_2^m \cap \mu B_p^m\} \leq \mathfrak{e}(p) \sup\{|\langle x, y\rangle| : y \in K_{2p'}(\lambda)\}.$$

By definition, we have

$$\sup\{|\langle x, y\rangle| : y \in B_2^m \cap \mu B_p^m\} \leq w_{r,\mu}(x)$$

and (6.3) gives us

$$v_{q,\lambda}(x) \leq \sup\{|\langle x, y\rangle| : y \in K_q(\lambda)\}.$$

The result is shown because of (6.4). $\square$

Now we are able to prove the desired estimates. To simplify the notation we will use the following:

$$\Phi(m,n,p,q) = \begin{cases} \left(\min\left\{1, m^{\frac{1}{q}} n^{-\frac{1}{2}}\right\}\right)^{\frac{\frac{1}{p}-\frac{1}{q}}{\frac{1}{2}-\frac{1}{q}}}, & 2 \leq p < q \leq \infty, \\ \max\left\{m^{\frac{1}{q}-\frac{1}{p}}, \min\left\{1, m^{\frac{1}{q}} n^{-\frac{1}{2}}\right\} \sqrt{1-\frac{n}{m}}\right\}, & 1 \leq p < 2 < q \leq \infty, \\ \max\left\{m^{\frac{1}{q}-\frac{1}{p}}, \left(\sqrt{1-\frac{n}{m}}\right)^{\frac{\frac{1}{p}-\frac{1}{q}}{\frac{1}{p}-\frac{1}{2}}}\right\}, & 1 \leq p < q \leq 2, \end{cases}$$

Before preceding to the main result of this section we take from [21] a sufficient condition that also holds in our case

Lemma 6.1.6. *If* $d_n(I^m_{2,r}) \preceq m^{\frac{1}{r}} n^{-\frac{1}{2}}$ *then*

$$d_n(I^m_{p,q}) \preceq \Phi(m,n,p,q) \tag{6.5}$$

for $1 \leq p < q \leq r$.

Theorem 6.1.7. *Let* $1 \leq p < q < \infty$ *and* $n < m$. *Then*

$$d_n(I^m_{p,q}) \asymp \Phi(m,n,p,q). \tag{6.6}$$

Proof. In [21] it was proved that $\Phi(m,n,p,q) \preceq d_n(I^m_{p,q})$. Now, we prove the remaining estimate $d_n(I^m_{pq}) \preceq \Phi(m,n,p,q)$. It is immediate, from Lemma 6.1.6, that it suffices to prove $d_n(I^m_{2,r}) \preceq m^{\frac{1}{r}} n^{-\frac{1}{2}}$ for $r < \infty$. We split the proof in two steps. On the first step, we construct a set for which the Gelfand number of the identity mapping that set to ℓ^m_2 is 1. This set, on the second step, will allow us to prove the desired result

Step 1: Consider $\Lambda, W \subset \mathbb{H}^m$, such that

$$N_1 = |\Lambda| < \frac{e^{\frac{\mathfrak{g}^2 n}{8}}}{8}, \qquad W \in \mathcal{F}(N_2), \text{ for } N_2 < \frac{e^{\frac{\tau^2 \mathfrak{g}^2 n}{8}}}{8} \tag{6.7}$$

for some $\tau \in (0, \infty)$. Take

$$V \subset \bigcup_{z \in \Lambda} \left(z + \frac{\|z\|_2}{2+2\tau} W\right) \cup B^m_2$$

and consider $x_1, \ldots x_n \in B^m_2$ for which $W \subset conv\{\pm x_1, \ldots x_{N_2}\}$. Fix $U_0 \in \mathcal{O}(m)$ and consider the operator $T_0 = P_{n,m} U_0$[4]. Then we have

$$\|T_0 : B_W \to \ell^m_2\| < M_n(1+\tau) \text{ and } \|T_0 z\|_2 > \frac{M_n \|z\|_2}{2}, \quad \forall z \in \Lambda.$$

[4] Note that $\operatorname{codim}(\ker T_0) = n$.

Indeed, to see this is the case, one turns to a probabilistic argument. Recall that $\mathcal{P}$ stands for the Haar measure on $\mathcal{O}(m)$. Denoting $T(U) = P_{n,m}U$ for $U \in \mathcal{O}(m)$, it follows by (6.1),

$$\begin{aligned}
&\mathcal{P}\Bigg(\left\{ U \in \mathcal{O}(m) : \frac{\|T(U) : B_W \to \ell_2^m\|}{M_n(1+\tau)} \geq 1 \right\} \cup \left\{ U \in \mathcal{O}(m) : \exists z \in \Lambda : \frac{2\,\|T(U)z\|_2}{M_n\,\|z\|_2} > 1 \right\} \Bigg) \\
=&\mathcal{P}\Bigg(\left\{ U \in \mathcal{O}(m) : \frac{\|T(U) : B_W \to \ell_2^m\|}{M_n(1+\tau)} \geq 1 \right\} \Bigg) + |\Lambda| \mu \Bigg(\left\{ x \in \mathbb{S}^{m-1} : \|T(U)x\|_2 > \frac{M_n}{2} \right\} \Bigg) \\
\leq&\mathcal{P}\left(\{ U \in \mathcal{O}(m) : \exists i \leq N_2 : \|T(U)x_i\|_2 \geq M_n(1+\tau) \} \right) + N_1 4 e^{-\frac{M_n^2 m}{8}} \\
\leq& N_2 \mu \left(\left\{ x \in \mathbb{S}^{m-1} : \|T(U)x\|_2 \geq M_n(1+\tau) \right\} \right) + N_1 4 e^{-\frac{M_n^2 m}{8}} \\
\leq& N_2 4 e^{-\tau^2 \frac{M_n^2 m}{2}} + N_1 4 e^{\frac{M_n^2 m}{8}} < 4 \frac{e^{\frac{\tau^2 \mathfrak{g}^2 n}{8}}}{8} e^{-\tau^2 \frac{M_n^2 m}{2}} + 4 \frac{e^{\frac{\mathfrak{g}^2 n}{8}}}{8} e^{\frac{-M_n^2 m}{8}} \\
<&\frac{1}{2} \left(e^{\tau^2 \left(\frac{\mathfrak{g}^2 n}{8} - \frac{n}{2} \right)} + e^{\frac{1}{8}(\mathfrak{g}^2 n - n)} \right) < 1, \text{ for some } \tau \in (0, \infty).
\end{aligned}$$

Now suppose that the inclusion $\ker T_0 \cap V \subset B_2^m$ is false. In this case there is a $z \in \Lambda$ such that $x \in \ker T_0 \cap \left(z + \frac{\|z\|_2}{2+2\tau} W \right)$. But then

$$0 = \|T_0 x\|_2 \geq \|T_0 z\|_2 - \left(\frac{\|z\|_2}{2+2\tau} \right) \|T_0 : B_W \to \ell_2^m\| > \|z\|_2 \frac{M_n}{2} - \frac{\|z\|_2}{2+2\tau} M_n(1+\tau) = 0.$$

Therefore $\ker T_0 \cap V \subset B_2^m$. Thus,

$$\forall x \in \ker T_0 \cap V \subset B_2^m, \|x\| \leq 1.$$

This means that there are n-codimensional spaces whose intersection with V is contained in the unit ball, therefore, $c_n(Id : V \to \ell_2^m) \leq 1$.

Step 2: Let $p = r'$ and $\delta = \frac{\min\{\mathfrak{c}(p)^{-1}, 1\}}{16}$. We apply the previous step to the set

$$V = \delta l^{\frac{1}{p} - \frac{1}{2}} B_p^m$$

for a still to be choosen l. Here we set $\tau = 1$, $W = \mathfrak{c}(p)^{-1} \left(B_2^m \cap l^{\frac{1}{p} - \frac{1}{2}} B_p^m \right)$, and $\Lambda = \cup \Lambda_A(\delta)$, where the union is over all subsets A of $\{1, \ldots, m\}$ of cardinality l. We start by verifying the necessary inclusion. Take $x \in V$. It is immediate that $\|x\|_p \leq \delta l^{\frac{1}{p} - \frac{1}{2}}$. Then $x_l^* \leq \delta l^{\frac{1}{p} - \frac{1}{2}} l^{-\frac{1}{p}} = \delta l^{-\frac{1}{2}}$. Define a vector

$$u = \begin{cases} u_i = x_i, & |x_i| \leq x_l^* \\ u_i = 0, & \text{otherwise} \end{cases}.$$

It follows that $\|u\|_p \leq \delta l^{\frac{1}{p} - \frac{1}{2}}$ and

$$\|u\|_2 \leq \|u\|_p^{\frac{p}{2}} \|u\|_\infty^{1 - \frac{p}{2}} \leq \left(\delta l^{\frac{1}{p} - \frac{1}{2}} \right)^{\frac{p}{2}} \left(\delta l^{-\frac{1}{2}} \right)^{(1 - \frac{p}{2})} \leq \delta l^{\frac{1}{2} - \frac{p}{4}} l^{\frac{p}{4} - \frac{1}{2}} = \delta.$$

Further, for some A with $|A| = l$

$$v = x - u \in \delta l^{\frac{1}{p} - \frac{1}{2}} B_p^A \subset l^{\frac{1}{2}} B_1^A \subset |A|^{\frac{1}{2}} B_1^A.$$

Therefore, there is a $z \in \Lambda$ such that $\|v-z\|_2 \leq \delta$ and, consequently $\|v+z\|_p \leq \delta l^{\frac{1}{p}-\frac{1}{2}}$. Hence $\|x-z\|_2 \leq 2\delta$ and $\|x-z\|_p \leq 2\delta l^{\frac{1}{p}-\frac{1}{2}}$. Thus $x-z \in \frac{1}{8}W$. If $\|x\|_2 > 1$, then $\|z\|_2 \geq 1-2\delta \geq \frac{1}{2}$. This concludes the proof of the desired inclusion. Now define $n_0 = 4\mathfrak{f}(p)\mathfrak{g}^{-2}\log(8)$ and

$$\mathfrak{h}(p) = \sup\left\{x^{2-\frac{2}{p}}\log\left(\frac{8\mathfrak{a}e}{\delta x}\right) : \; 0 < x \leq 1\right\}.$$

Finally, let $\mathfrak{i}(p) = \min\left\{\frac{\mathfrak{g}^2}{4}\mathfrak{f}(p), \frac{\mathfrak{g}^2}{8}\mathfrak{h}(p)\right\}$. Now we show that the restrictions (6.7) on $|\Lambda|$ and W are satisfied as we take $l = \left[\left(\mathfrak{i}(p)nm^{\frac{2}{p}-2}\right)^{\frac{1}{\frac{2}{p}-1}}\right]$ for $n > n_0$. This will conclude the prove because we then we have

$$d_n(I^m_{2,r}) = c_n(I^m_{p,2}) \leq \delta^{-1} l^{\frac{1}{2}-\frac{1}{p}}$$

which give the desired result when replacing l as mentioned above. By Lemma 6.1.2 we have

$$W \in \mathcal{F}(e^{\mathfrak{f}(p)} l^{\frac{2}{p}-1} m^{2-\frac{2}{p}}).$$

Therefore, the restriction on W follows from the choice of l and the inequality $\mathfrak{i}(p) \leq \frac{\mathfrak{g}^2}{4}\mathfrak{f}(p)$[5]. Moreover,

$$|\Lambda| \leq \binom{m}{l}\left(\frac{\mathfrak{a}}{8}\right)^l \leq \left(\frac{em\mathfrak{a}}{l\delta}\right)^l = e^l \log\left(\frac{\mathfrak{a}em}{l\delta}\right).$$

The choice of l implies that [6]

$$l\log\left(\frac{\mathfrak{a}em}{l\delta}\right) \leq \mathfrak{i}(p)n\left(\frac{l}{m}\right)^{2-2p}\log\left(8\mathfrak{a}\frac{em}{\delta l}\right) - \log 8 \leq \frac{n\mathfrak{g}^2}{8} - \log(8),$$

which is the necessary estimate of $|\Lambda|$. □

Theorem 6.1.8. *Let $1 \leq p < q < \infty$, $(p,q) \neq (1,\infty)$ and $n < m$. Then*

$$d_n(I^m_{pq}) \asymp \Psi(m,n,p,q) = \begin{cases} \Phi(m,n,p,q), & 1 \leq p < q \leq p', \\ \Phi(m,n,q',p'), & \max\{p,p'\} < q \leq \infty. \end{cases}.$$

Proof. For an $m \times m$ orthogonal matrix U define the operator $S(U) = U^*(Q_{n,m} - \kappa_n P_{n,m})U$[7], where $\kappa_n = \frac{M^2_{m-n}}{M^2_n}$.

Step 1: Consider $\theta > 0$ and $x, y \in B^m_2$. Set $\lambda_1 = (2M_{m-n}+\theta)\theta$, $\lambda_2 = \kappa_n(2M_n+\theta)\theta$ and $\lambda = 2(\lambda_1+\lambda_2)$. Then

$$\mathcal{P}\left\{U \in \mathcal{O}(m) : |\langle S(U)x, y\rangle| > \lambda\right\} \leq 16e^{-\frac{\theta^2 m}{2}}.$$

[5] More precisely, one takes $\tau > \sqrt{\frac{8}{\mathfrak{g}^2 n}\ln\left(\frac{n\mathfrak{g}^2\mathfrak{f}}{2}e^{\mathfrak{f}}\right)}$

[6]

$$l \leq \mathfrak{i}(p)n\left(\frac{l}{m}\right)^{2-2p} \Leftrightarrow l^{2p-1} \leq \mathfrak{i}(p)nm^{2p-2} \Leftrightarrow l \leq \left(\mathfrak{i}(p)nm^{2p-2}\right)^{\frac{1}{2p-1}} \Leftrightarrow l \leq \left(\mathfrak{i}(p)nm^{2-2p}\right)^{\frac{1}{1-2p}}$$
$$\Leftrightarrow \ldots$$

[7] Here $Q_{n,m} = I_m - P_{n,m}$

We start be assuming that $x = y \neq 0$. Since $\mathcal{P}$ induces μ under the mapping $U \to \frac{Ux}{\|x\|_2}$, in this case

$$\begin{aligned}
h &= \mathcal{P}\left\{U \in \mathcal{O}(m) : |\,\langle S(U)x, x\rangle\,| > \frac{\lambda}{2}\right\} \\
&\leq \mathcal{P}\left\{U \in \mathcal{O}(m) : |\,\langle S(U)x, x\rangle\,| > \frac{\lambda\,\|x\|_2^2}{2}\right\} \\
&= \mu\left\{z \in \mathcal{S}^{m-1} : |\,\langle(Q_{n,m} - \kappa_n P_{n,m})z, z\rangle\,| > \frac{\lambda}{2}\right\} \\
&= \mu\left\{z \in \mathcal{S}^{m-1} : |(\langle Q_{n,m}z, z\rangle - M_{m-n}^2) - \kappa_n(\langle P_{n,m}z, z\rangle - M_n^2)| > \frac{\lambda}{2}\right\} \\
&\leq \mu\left\{z \in \mathcal{S}^{m-1} : |(\langle Q_{n,m}z, z\rangle - M_{m-n}^2)| > \lambda_1\right\} \\
&+ \mu\left\{z \in \mathcal{S}^{m-1} : \kappa_n|\,\langle P_{n,m}z, z\rangle - M_n^2| > \lambda_2\right\}.
\end{aligned}$$

Observe now that $\langle P_{n,m}z, z\rangle = \phi_n^2(z)$ and that the functions $\langle Q_{n,m}z, z\rangle$ and $\phi_{n-m}^2(z)$ are identically distributed on $\mathcal{S}^{m-1}$. Now applying (6.2) we have $h \leq 8e^{-\frac{\theta^2 m}{2}}$.

To pass to the general case we first note that

$$\langle S(u)x, y\rangle = \left\langle \frac{S(U)(x+y)}{2}, \frac{(x+y)}{2}\right\rangle - \left\langle \frac{S(U)(x-y)}{2}, \frac{(x-y)}{2}\right\rangle$$

Thus,

$$\begin{aligned}
\{U \in \mathcal{O}(m) : |\,\langle S(U)x, y\rangle\,| > \lambda\} \subset &\left\{U \in \mathcal{O}(m) : |\left\langle \frac{S(U)(x+y)}{2}, \frac{(x+y)}{2}\right\rangle| > \frac{\lambda}{2}\right\} \\
&\cup \left\{U \in \mathcal{O}(m) : |\left\langle \frac{S(U)(x-y)}{2}, \frac{(x-y)}{2}\right\rangle| > \frac{\lambda}{2}\right\}.
\end{aligned}$$

Step 2: Suppose that $\theta > 0$, $N > \frac{1}{16}e^{\frac{\theta^2 m}{4}}$ and the sets $V, W \in \mathcal{F}(N)$ are absolutely continuous. Then[8]

$$a_n(V, E_W^*) \leq \mathrm{j}\left(\theta\sqrt{\frac{m-n}{n}} + \frac{\theta^2 m}{n}\right) \tag{6.8}$$

for some constant j.

Consider $x_1, \ldots, x_N, y_1, \ldots, y_N \in B_2^m$ to be such that

$$V \subset conv\{\pm x_1, \ldots x_N\}, \quad W \subset conv\{\pm y_1, \cdots \pm y_N\},$$

which implies that

$$\|S(U)\|_{E_V \to E_W^*} \leq \sup_{i,j \leq N} |\langle S(U)x_i, y_j\rangle|\,.$$

Take λ as in the previous step. Then we have

$$\begin{aligned}
&\mathcal{P}\left\{U \in \mathcal{O}(m) : \|S(U)\|_{E_V \to E_W^*} > \lambda\right\} = \sum_{i,j \leq N} \mathcal{P}\left\{U \in \mathcal{O}(m) : |\langle S(U)x_i, y_j\rangle| > \lambda\right\} \\
&\leq N^2 16e^{-\frac{\theta^2 m}{2}} < 1.
\end{aligned}$$

[8]Here we will perform a slight modification in the notation: in what follows $a_n(X, Y)$ will refer to the approximation number of the identity mapping X to Y.

Thus, there are $U \in \mathcal{O}(m)$ for which $\|S(U)\| \le \lambda$. Since $\operatorname{rank}(I_m - S(U)) = n$, we have $a_n(V, E_W^*) \le \lambda$. Now observe that

$$\lambda \le \theta \left(\frac{8\sqrt{2}}{\mathfrak{g}}\right) \sqrt{\frac{m-n}{n}} + \theta^2 \left(\frac{4}{\mathfrak{g}^2}\right) \frac{m}{n}.$$

Taking $\mathfrak{j} = \max\left\{\frac{8\sqrt{2}}{\mathfrak{g}}, \frac{4}{\mathfrak{g}^2}\right\}$ concludes this step.

Step 3: for any k, m, n, p and q such that $1 \le p \le q \le \infty$ we have

$$a_{kn}\left(B_p^{km}, \ell_q^{km}\right) \le a_n\left(B_p^m, \ell_q^m\right) \tag{6.9}$$

This formula follows at once from the representation of $\mathbb{R}^{km}$ in the form $\mathbb{R}^k \otimes \mathbb{R}^m$ if we use the operator $I_k \otimes T$ for approximating the set B_p^{km}, where T is an extremal operator for $a_n(B_p^m, \ell_q^m)$.

Step 4: By the duality $d_n'(BX, Y) = d_n'(BY^*, X^*)$, it suffices for us to confine ourselves to the case $1 < p < q < p'$. The inequality $d_n'(B, X) \ge d_n(B, X)$ gives the necessary lower estimates. The results found in [28] allows us to exclude the cases $p = 1, q < \infty$ and $p > 1, q < \infty$ from consideration. By Lemma 6.1.5, (6.8) might be applied to the subsets $V = W = \mathfrak{c}(p)^{-1} B_p^m$, with

$$\theta = 2(\log 16 + \mathfrak{f}(p))^{\frac{1}{2}} m^{\frac{1}{2}-\frac{1}{p}}.$$

This gives the necessary estimate for $d_n'(B_p^m, \ell_{p'}^m)$. In particular for $m - n < \frac{m^{2-\frac{2}{p}}}{2}$ we have

$$a_n(B_p^m, \ell_p^m) \le \mathfrak{k}(p, p') m^{\frac{1}{p'}-\frac{1}{p}}.$$

As $q < p'$

$$a_n(B_p^m, \ell_p^m) \le \mathfrak{k}(p, p') m^{\frac{1}{p'}-\frac{1}{p}}.$$

For the case $1 < p < q' \le 2$ we have

$$a_n(B_p^m, \ell_q^m) \le \mathfrak{l}(p) m^{\frac{1}{q}-\frac{1}{p}}$$

for $1 < p < q \le 2$, $m - n \le \frac{m^{2-\frac{2}{p}}}{2}$. Obviously for $1 < p < q \le 2$

$$a_n(B_p^m, \ell_q^m) \le 1 \le (1 - \frac{n}{m})^{-1} \Psi(m, n, p, q). \tag{6.10}$$

Therefore, to conclude the proof it suffices for us to confine ourselves to the case when $1 < p < q \le 2$ and $\frac{m^{2-\frac{2}{p}}}{2} < m - n < 6^{\frac{1}{\frac{2}{p}-2}} m$. Let

$$k = \left[(6(m-n) m^{\frac{2}{p}-2})^{\frac{1}{\frac{2}{p}-1}}\right] + 1, \quad n_1 = \left[\frac{n}{k}\right], \quad m_1 = \left[\frac{m}{k}\right] + 1$$

Under our restrictions on $m - n$ it follows that $k < m - n$ and $m_1 - n_1 < \frac{m_1^{2-\frac{2}{p}}}{2}$. Using (6.10), we find that

$$a_{n_1}(B_p^{m_1}, B\ \ell_p^{m_1}) \le \mathfrak{l}(p) m_1^{\frac{1}{q}-\frac{1}{p}}.$$

Finally, by (6.9) we have

$$\begin{aligned}
a_n(B_p^m, \ell_q^m) &\leq d_{kn_1}(B_p^{km_1}, \ell_q^{km_1}) \leq \mathfrak{l}(p) m_1^{\frac{1}{q}-\frac{1}{p}} \\
&\leq \mathfrak{l}(p) 2^{\frac{1}{q}-\frac{1}{p}} 6(m-n) m^{1-\frac{2}{p}} m^{(\frac{2}{p}-2)(\frac{1}{p}-\frac{1}{q})(\frac{2}{p}-1)} \\
&= \mathfrak{l}(p) 2^{\frac{1}{q}-\frac{1}{p}} 6^{(\frac{1}{p}-\frac{1}{q})(\frac{2}{p}-1)} \Psi(m,n,p,q). \qquad \square
\end{aligned}$$

6.2 Computations of the asymptotic behaviour of Weyl numbers

Here we follow [35].

Theorem 6.2.1. *Let* $1 \leq p \leq q \leq \infty$. *Then*

$$x_n(Id_{p,q}) \asymp \begin{cases} n^{\frac{1}{q}-\frac{1}{p}}, & 1 \leq p \leq q \leq 2 \\ n^{\frac{1}{2}-\frac{1}{p}}, & 1 \leq p \leq 2 \leq q \leq \infty \\ 1 & 2 \leq p \leq q \leq \infty \end{cases}.$$

Proof. Let $2 \leq p \leq q \leq \infty$. From Theorem 4.3.5 $a_n(Id_{p,q}) = 1$ for $1 < p \leq q \leq \infty$ and therefore $x_n(Id_{p,q}) \leq 1$. On the other hand it clearly follows that $\|Id_{p,q}\| = 1$ for $p \leq q$. Therefore in particular, since $q \geq 2$, by the definition of Weyl numbers

$$x_n(Id_{p,q}) \geq a_n(Id_{p,q}) = 1.$$

Now, let $1 \leq p \leq q \leq 2$. From Theorem 4.1.10 we have that $\|Id_{p,q}|\mathfrak{B}_{r,1}\| \leq c$ with $\frac{1}{r} = \frac{1}{p} - \frac{1}{q} + \frac{1}{2}$. From Theorem 4.1.8 it follows that, for $\frac{1}{s} = \frac{1}{r} - \frac{1}{2}$,

$$\|Id_{p,q}|\mathfrak{B}_{s,2}\| \leq \|Id_{p,q}|\mathfrak{B}_{r,1}\| \leq c.$$

Since, for $2 \leq p \leq \infty$, if $T \in \mathfrak{B}_{p,2}$ there holds $(n+1)^{\frac{1}{p}} x_n(T) \leq \|T|\mathfrak{B}_{p,2}\|$[9]. This leads us to the upper bound

$$x_n(Id_{p,q}) \lesssim n^{-\frac{1}{s}} = n^{\frac{1}{q}-\frac{1}{p}}.$$

Analogously, for $1 \leq p \leq 2 \leq \infty$, via $\|Id_{p,q}|\mathfrak{B}_{p,1}\| \leq c$, we obtain $x_n(Id_{p,q}) \lesssim n^{-\frac{1}{s}} = n^{\frac{1}{2}-\frac{1}{p}}$.

To estimate from below recall the estimates obtained for the Gelfand numbers

$$c_n(Id_{2,q}^{2n}) \gtrsim \begin{cases} n^{\frac{1}{q}-\frac{1}{2}}, & q \leq 2, \\ 1 & q \geq 2. \end{cases}$$

Moreover, we have already seen that, for $p \leq 2$, $\left\|Id_{2,p}^{2n}\right\| = (2n)^{\frac{1}{p}-\frac{1}{2}}$. This yields to, due to the axioms of s-numbers and definition of Weyl numbers,

$$c_n(Id_{2,q}^{2n}) \leq (2n)^{\frac{1}{p}-\frac{1}{2}} x_n(Id_{p,q}^{2n}) \leq x_n(Id_{2,q}). \qquad \square$$

[9]More generally, if $X \in B^R(\ell_2, E)$ then for $Y \in B^R(\ell_2, Fx)$ and for a positive ϵ consider an orthogonal family $f_i \in \ell_2$ to be constructed inductively, such that $(1+\epsilon)\|Yf_n\| \geq c_n(Y)$. Then it follows that

$$(n+1)^{\frac{1}{p}} c_n(TX) \leq (1+\epsilon)\|TXf_k|\ell_p(\{0,\dots,n\})\| \leq (1+\epsilon)\|TX|\mathfrak{B}_{p,2}\| \leq (1+\epsilon)\|T|\mathfrak{B}_{p,2}\|\,\|X\|.$$

The claim now follows from Corollary 3.3.2.

It is a clear consequence of this proof that for $2n \leq m$

$$x_n(Id^m_{p,q}) \asymp \begin{cases} n^{\frac{1}{q}-\frac{1}{p}}, & 1 \leq p \leq q \leq 2, \\ n^{\frac{1}{2}-\frac{1}{p}}, & 1 \leq p \leq 2 \leq q \leq \infty, \\ 1 & 2 \leq p \leq q \leq \infty. \end{cases}$$

For the case $p > q$ we require an interpolation theorem, which was first observed in [10, p. 212]. It is a quick consequence of the classic interpolation theorem between ℓ_p spaces.

Lemma 6.2.2. *For* $0 < \theta < 1$ *there holds:*

$$\left\| Id^m_{p,q} | \mathfrak{B}_{r,2} \right\| \leq \left\| Id^m_{p,q_1} | \mathfrak{B}_{r_1,2} \right\|^{\theta} \left\| Id^m_{p,q_2} | \mathfrak{B}_{r_2,2} \right\|^{1-\theta},$$

with $\frac{1}{r} = \frac{\theta}{r_1} + \frac{1-\theta}{r_2}$ *and* $\frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}$.

Theorem 6.2.3. *1.* $\left\| Id^n_{p,q} | \mathfrak{B}_{2,2} \right\| = n^{\frac{1}{q}}$ *for* $1 \leq q \leq 2 \leq p \leq \infty$.

2. $\left\| Id^n_{p,q} | \mathfrak{B}_{r,2} \right\| \leq n^{\frac{1}{r}}$ *for* $2 \leq q < p \leq \infty$ *and* $\frac{1}{r} = \frac{\frac{1}{q}-\frac{1}{p}}{1-\frac{2}{p}}$.

Proof. The first claim is directly adapted from [39, p. 309]. By definition $\mathfrak{B}_{2,2} = \mathfrak{B}_2$. Thus, since $\left\| Id^n_{\infty,2} | \mathfrak{B}_2 \right\| = \sqrt{n}$ and

$$\left\| Id^n_{p,q} | \mathfrak{B}_2 \right\| \leq \left\| Id^n_{p,\infty} \right\| \left\| Id^n_{\infty,2} | \mathfrak{B}_2 \right\| \left\| Id^n_{2,q} \right\| = n^{-\frac{1}{p}} n^{\frac{1}{2}} n^{\frac{1}{q}-\frac{1}{2}} = n^{\frac{1}{q}-\frac{1}{p}},$$

it follows that $\left\| Id^n_{p,q} | \mathfrak{B}_2 \right\| \leq n^{\frac{1}{q}}$, since $1 \leq q \leq 2 \leq p$.

The second claim is a consequence of the previous lemma together with the first claim. Indeed, set $\theta = \frac{\frac{1}{q}-\frac{1}{p}}{\frac{1}{2}-\frac{1}{p}}$, then we can write $\frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{p}$ and $\frac{1}{r} = \frac{\theta}{2}$. Thus, using Lemma 6.2.2 we have

$$\left\| Id^m_{p,q} | \mathfrak{B}_{r,2} \right\| \leq \left\| Id^m_{p,2} | \mathfrak{B}_{2,2} \right\|^{\theta} \left\| Id^m_{p,p} | \mathfrak{B}_{\infty,2} \right\|^{1-\theta} = m^{\frac{\theta}{2}} = m^{\frac{1}{r}}. \qquad \square$$

As a consequence of $\left\| Id^m_{p,q} \right\| = m^{\frac{1}{q}-\frac{1}{p}}$, Theorem 6.2.3 and Lemma 4.1.5 we obtain

Corollary 6.2.4. *For* $1 \leq q < p \leq \infty$*:*

1. $x_n(Id : \ell^m_p \to \ell^m_q) \leq m^{\frac{1}{q}-\frac{1}{p}}$, $1 \leq q < p \leq \infty$;
2. $x_n(Id : \ell^m_p \to \ell^m_q) \lesssim m^{\frac{1}{q}} n^{-\frac{1}{2}}$, $1 \leq q \leq 2 < p \leq \infty$;
3. $x_n(Id : \ell^m_p \to \ell^m_q) \lesssim \left(\frac{m}{n}\right)^{\frac{1}{r}}$, $2 \leq q < p \leq \infty$, $\frac{1}{r} = \frac{\frac{1}{q}-\frac{1}{p}}{1-\frac{2}{p}}$.

Moreover, it follows that if $2n \leq m$, then it follows for $1 \leq q < p \leq 2$

$$x_n(\ell^m_p \to \ell^m_q) \geq x_{\left[\frac{m}{2}\right]}(Id : \ell^m_p \to \ell^m_q) \gtrsim \frac{c_{\left[\frac{m}{2}\right]}(Id : \ell^m_2 \to \ell^m_q)}{\left\| Id : \ell^m_2 \to \ell^m_p \right\|} \gtrsim m^{\frac{1}{q}-\frac{1}{p}}.$$

The following results were taken from [24]. From the complete regularity of Hilbert numbers, it suffices to consider the case $1 \le p' \le q \le \infty$. Let $k \le m$ be natural numbers and let $t_{ij}(\omega)$ with $i = 1, \dots, m$ and $j = 1, \dots, k$ be a system of independent, mean-zero random variables on some probability space (Ω, μ) taking values $+1$ and -1 only. Let $T(\omega)$ be the matrix $(t_{ij}(\omega))$. The following result is found in [5].

Lemma 6.3.1. *Let $2 \le u < \infty$. Then there exists a constant $c = c(u) \ge 1$ such that for all natural numbers k and m with $k \le m$,*

$$\mu\left(\{\omega \in \Omega : \left\|T(\omega) : \ell_2^k \to \ell_u^m\right\| \le c \max(m^{\frac{1}{u}}, k^{\frac{1}{2}})\}\right) > 1 - e^{-2k}.$$

Proof. **Step 1:** Let $(X_j)_{j=1}^n$ be independent, mean-zero random variables with $|X_j| \le 1$ for all j; then for any $\lambda > 0$ and real $b_1, \dots b_n$

$$\mu\left(\sum_{j=1}^n b_j X_j \ge \lambda\right) \le 2e^{\left(-\frac{\lambda^2}{4}\sum_{j=1}^n b_j^2\right)}.$$

To see this first observe that $e^x - x \le e^{x^2}$. Then for any real m

$$E\left[e^{m^* X_j}\right] = E\left[e^{m^* X_j} - m^* X_j\right] \le E\left[e^{(m^*)^2 X_j^2}\right] \le e^{(m^*)^2}.$$

The independence of X_j leads to $E\left[e^{m^* \sum_{j=1}^n b_j X_j}\right] \le e^{(m^*)^2 \sum_{j=1}^n b_j^2}$. A routine application of Chebyshev's inequality yields

$$\mu\left(\sum_{j=1}^n b_j X_j \ge \lambda\right) \le e^{(m^*)^2 \sum_{j=1}^n b_j^2 - \lambda m^*},$$

for any $m^* > 0$. The claim follows by taking $m^* = \frac{\lambda}{2}\sum_{j=1}^n b_j^2$.

Step 2: For each $q \ge 2$, there is a constant C depending only on q, so that if $(X_j)_{j=1}^n$ satisfy the assumptions of the previous step, then

$$E\left[e^{m^*\left|\sum_{j=1}^n b_j X_j\right|^q}\right] \le 1 + Cm^*\left(\sum_{j=1}^n b_j^2\right)^{\frac{q}{2}}$$

for $0 \le m^* \le \left(\sum_{j=1}^n b_j^2\right)^{-\frac{q}{2}} \frac{n^{1-\frac{q}{2}}}{8}$.

To see this, we assume without loss of generality that $\sum_{j=1}^n b_j^2 = 1$. As application of integration by parts

$$\begin{aligned} E\left[e^{m^*\left|\sum_{j=1}^n b_j X_j\right|^q}\right] &= \int_0^\infty e^{m^*\lambda^q} d\mu\left(\left|\sum_{j=1}^n b_j X_j\right| \le \lambda\right) \\ &= 1 + \int_0^\infty q m^* \lambda^{q-1} e^{m^*\lambda^q} \mu\left(\left|\sum_{j=1}^n b_j X_j\right| > \lambda\right) d\lambda. \end{aligned}$$

Since $\left|\sum_{j=1}^n b_j X_j\right| \le \left(\sum_{j=1}^n b_j^2\right)^{\frac{1}{2}} \sqrt{n}$, it follows $\mu\left(\left|\sum_{j=1}^n b_j X_j\right| > \sqrt{n}\right) = 0$. Now, the previous step allows to write

$$\begin{aligned} E\left[e^{m^*\left|\sum_{j=1}^n b_j X_j\right|^q}\right] &\le 1 + 2qm^* \int_0^{\sqrt{n}} \lambda^{q-1} e^{\left(m^*\lambda^q - \frac{\lambda^2}{4}\right)} d\lambda \\ &= 1 + 2qm^* \int_0^{\infty} \lambda^{q-1} e^{\left(m^*\lambda^q - \frac{\lambda^2}{8}\right)} d\lambda, \end{aligned}$$

since $0 \le m^* \le \frac{n^{1-\frac{q}{2}}}{8}$. The claim follows by taking

$$C = 2q \int_0^\infty \lambda^{q-1} e^{\left(-\frac{\lambda^2}{8}\right)} d\lambda = 8^{\frac{q}{2}} \Gamma\left(\frac{q}{2}\right).$$

Step 3: Fix $q \ge 2$ and suppose that t_{ij} are independent random variables satisfying the hypotheses of the first step. Given real x_j we obtain estimates on the probability distribution for the random variable

$$Y = \frac{\sum_{i=1}^m \left|\sum_{j=1}^n x_j t_{ij}\right|}{\left(\sum_{j=1}^n x_j^2\right)^{\frac{1}{2}}}.$$

Indeed, considering the constant C obtained in the previous step, for any $\lambda \in \mathbb{R}$ and positive integers m, n there holds

$$\mu\left(Y \ge Cm + 8\lambda n^{\frac{q}{2}}\right) \le e^{-\lambda n}.$$

For a real m^* we set $K(m^*) = \log E[e^{m^* Y}]$ so that $E[e^{m^* Y - K(m^*)}] = 1$. From Chebyshev's inequality, for any real ν we have

$$\mu(m^* Y \ge K(m^*) + \nu) \le e^{-\nu}.$$

On the other hand, from the second step we have for $0 \le m^* \le \frac{n^{1-\frac{q}{2}}}{8}$,

$$E\left[e^{m^* Y}\right] = \prod_{i=1}^m \left[E\left[e^{m^*\left|\sum_{j=1}^n b_j t_{ij}\right|^q}\right]\right] \le \prod_{i=1}^m (1 + m^* C) \le e^{mCm^*}$$

where $b_j = \frac{x_j}{\left(\sum_{j=1}^n x_j^2\right)^{\frac{1}{2}}}$ so that $K(m^*) \le mCm^*$. Setting $\nu = \lambda n$ and $m^* = \frac{n^{1-\frac{q}{2}}}{8}$, the claim follows.

Step 4: Denote $\|T\|_{2,q} = \sup_{\|x\|_2=1}\left(\|Ax\|_q\right)$. Then, there are constants c_1, c_2 such that for all $\lambda > 0$,

$$\mu\left(\|T\|_{2,q} \ge c_1 (m + \lambda n^{\frac{q}{2}})^{\frac{1}{q}}\right) \le e^{-(\lambda - c_2)n}.$$

Fix $0 < \epsilon < 1$ and denote by N an ϵ-net for the unit n-sphere, $\mathbb{S}$ [10]. Put $c_1 = \frac{(\max(C,8))^{\frac{1}{q}}}{1-\epsilon}$.

[10] by an ϵ-net for the unit n-sphere (with respect to the ℓ_2 norm), $\mathbb{S}$, we mean a finite subset, N, of $\mathbb{S}$ for which $\sup_{x\in\mathbb{S}} \inf_{y\in N} \|x-y\|_2 < \epsilon$. Since

$$\|T\|_{2,q} = \sup_{x\in S}\left(\|Tx\|_q\right) \le \max_{y\in N} \|Tx\|_q + \sup_{x\in S}\min_{y\in N}\left(\|T(x-y)\|_q\right)$$

it follows $\|T\|_{2,q} \le \frac{1}{1-\epsilon} \max_{y\in N} \|Tx\|_q$.

Then

$$\mu\left(\|T\|_{2,q} \geq c_1(m+\lambda n^{\frac{q}{2}})^{\frac{1}{q}}\right) \leq \mu\left(\max_{y\in N}\|Tx\|_q \geq (Cm+8\lambda n^{\frac{q}{2}})^{\frac{1}{q}}\right)$$
$$\leq \sum_{y\in N} e^{-\lambda n} = |N|e^{-\lambda n}$$

by the third step. Clearly one may choose $|N| \leq e^{c_2 n}$ with c_2 t depending only on ϵ.
Step 5: Let $k = c_1^q(m + c_2 n^{\frac{q}{2}})$. As a consequence of Jensen's inequality

$$\begin{aligned}
\left(E\left[\|T\|_{2,q}\right]\right)^q &\leq E\left[\left(\|T\|_{2,q}\right)^q\right] \\
&= \int_0^\infty \mu\left[\left(\|T\|_{2,q}\right)^q \geq x\right]\, dx \\
&\leq k + \int_k^\infty \mu\left[\|T\|_{2,q} \geq x^{\frac{1}{q}}\right]\, dx \\
&\leq k + \int_k^\infty e^{-n^{1-\frac{q}{2}}\left(\frac{x}{c_1^q}-m\right)+c_2 n}\, dx \\
&= c_1^q\left(m + c_2 n^{\frac{q}{2}} + n^{\frac{q}{2}-1}\right)
\end{aligned}$$

from which the desired result immediately follows. □

Note that in the case $u = \infty$ we obviously have $\left\|T(\omega) : \ell_2^k \to \ell_\infty^m\right\| \leq k^{\frac{1}{2}}$ for all ω. We need norm-estimates in different spaces simultaneously. They are given in the following lemma which is an immediate consequence of the previous result.

Lemma 6.3.2. *Let $1 \leq v \leq 2 \leq u \leq \infty$. Then there exists a constant $c = c(u,v) \leq 1$ such that for all $k \leq m$ there is a matrix $T = (t_{ij})_{m,k}$ with $|t_{ij}| = 1$ satisfying*

$$\begin{aligned}
\left\|T(\omega) : \ell_2^k \to \ell_u^m\right\| &\leq c\max(m^{\frac{1}{u}}, k^{\frac{1}{2}}), \\
\left\|T(\omega) : \ell_2^k \to \ell_{v'}^m\right\| &\leq c\max(m^{\frac{1}{v'}}, k^{\frac{1}{2}}), \\
\left\|T(\omega) : \ell_2^k \to \ell_2^m\right\| &\leq cm^{\frac{1}{2}}.
\end{aligned}$$

Lemma 6.3.3. *Let $k \leq m$ and let $T = (t_{ij})_{m,k}$ be a matrix satisfying $|t_{ij}| = 1$ and $\left\|T(\omega) : \ell_2^k \to \ell_2^m\right\| \leq cm^{\frac{1}{2}}$ for some real number $c \leq 1$. Then*

$$a_n(T : \ell_2^k \to \ell_2^m) \geq 2^{-\frac{1}{2}} m^{\frac{1}{2}}$$

for all $n \leq \frac{k}{2c^2}$.

Proof. Consider the composition $\ell_2^k \xrightarrow{T} \ell_2^m \xrightarrow{T^*} \ell_2^k$. It follows that trace$(T^*T) = mk$[11]. Therefore, denoting the eigenvalues of the operator T^*T by $\lambda_j(T^*T)$, by Theorem 3.1.6 we have

$$\begin{aligned}
mk = \sum_{j=1}^k \lambda_j(T^*T) &= \sum_{j=1}^k a_j^2(T) = \sum_{j=1}^{n-1} a_j^2(T) + \sum_{j=n}^k a_j^2(T) \\
&\leq (n-1)\|T\|^2 + (k-n+1)a_n^2(T) \leq (n-1)c^2 m + k a_n^2(T).
\end{aligned}$$

[11] because T^*T is a $km \times km$ matrix with ones in the diagonal

By hypothesis $n \leq \frac{k}{2c^2}$ which allows us to obtain the bound

$$a_n(T) \geq \sqrt{m\left(1-(n-1)\frac{c^2}{k}\right)} \geq \sqrt{\frac{m}{2}}. \qquad \square$$

Proposition 6.3.4. *1. Let $2 \leq v' \leq u \leq \infty$. Then there exists a constant $c_1(u,v) > 0$ such that*

$$h_n(I:\ell_u^m \to \ell_v^m) \geq c_1(u,v)\begin{cases} m^{\frac{1}{v}-\frac{1}{u}} & \text{for } 1 \leq n \leq m^{\frac{2}{u}}, \\ m^{\frac{1}{v}}n^{-\frac{1}{2}} & \text{for } m^{\frac{2}{u}} \leq n \leq m^{\frac{2}{v'}}, \\ mn^{-1} & \text{for } m^{\frac{2}{v'}} \leq n \leq m. \end{cases}$$

2. Let $2 \leq u,v \leq \infty$. Then there exists a constant $c_2(u,v) > 0$ such that

$$h_n(I:\ell_u^m \to \ell_v^m) \geq c_2(u,v)\begin{cases} m^{\frac{1}{v}-\frac{1}{u}} & \text{for } 1 \leq n \leq m^{\frac{2}{u}}, \\ m^{\frac{1}{v}}n^{-\frac{1}{2}} & \text{for } m^{\frac{2}{u}} \leq n \leq m. \end{cases}$$

Proof. (1) Let $c(u,v) \geq 1$ be the constant appearing in Lemma 6.3.2. The case $n \geq \frac{m}{2c^2}$ follows easily from the relation

$$1 = h_n(I:\ell_2^m \to \ell_2^m) \leq h_n(I:\ell_u^m \to \ell_v^m).$$

Let now $n < \frac{m}{2c^2}$. Let T be the matrix as in Lemma 6.3.2. We take $k = [2c^2n]+1$ and consider

$$\ell_2^k \xrightarrow{T} \ell_u^m \xrightarrow{I} \ell_v^m \xrightarrow{T^*} \ell_2^k.$$

By Lemma 6.3.3 we have

$$a_n(T^*IT) = \lambda_n(T^*T) = a_n(T)^2 \geq \frac{m}{2}.$$

It follows that

$$\begin{aligned} h_n(I:\ell_u^m \to \ell_v^m) &\geq \frac{m}{2\left\|T:\ell_2^k \to \ell_u^m\right\|\left\|T:\ell_2^k \to \ell_{v'}^m\right\|} \geq \frac{m}{2c^2\max(m^{\frac{1}{u}},k^{\frac{1}{2}})\max(m^{\frac{1}{v'}},k^{\frac{1}{2}})} \\ &\geq \frac{m}{6c^4\max(m^{\frac{1}{u}},n^{\frac{1}{2}})\max(m^{\frac{1}{v'}},n^{\frac{1}{2}})} \end{aligned}$$

(2) Let $c = c_2(u,2) \geq 1$ be the constant in Lemma 6.3.2. Using the relation

$$h_n(I:\ell_u^m \to \ell_v^m) \geq m^{\frac{1}{v}-\frac{1}{2}}$$

the case $n \geq \frac{m}{2c^2}$ is again easily verified. We put $k = [2c^2n]+1$ and consider

$$\ell_2^k \xrightarrow{T} \ell_u^m \xrightarrow{I} \ell_v^m \xrightarrow{I} \ell_2^m.$$

where T is the matrix in Lemma 6.3.2. This together with Lemma 6.3.3 yields the estimates

$$h_n(I:\ell_u^m \to \ell_v^m) \geq \frac{\sqrt{m}}{\sqrt{2}c\max(m^{\frac{1}{u}},k^{\frac{1}{2}})m^{\frac{1}{2}-\frac{1}{v}}} \geq \frac{m^{\frac{1}{v}}}{\sqrt{6}c^2\max(m^{\frac{1}{u}},n^{\frac{1}{2}})m^{\frac{1}{2}-\frac{1}{v}}}. \qquad \square$$

Together with the estimates obtained in [9] we can write the following Theorem.

Theorem 6.3.5. *Let* $1 \le u' \le v \le \infty$*. Then, for all* $n \le m$*, there holds*

$$h_n(I : \ell_u^m \to \ell_v^m) \asymp \Phi(n, m, u, v),$$

where

$$\Phi(n, m, u, v) = \begin{cases} \min(m^{\frac{1}{v}-\frac{1}{u}}, m^{\frac{1}{v}} n^{-\frac{1}{2}}, mn^{-1}) & \textit{for } 1 \le u' \le v \le 2, \\ \min(m^{\frac{1}{v}-\frac{1}{u}}, m^{\frac{1}{v}} n^{-\frac{1}{2}}) & \textit{for } 2 \le v \le u \le \infty, \\ n^{\frac{1}{v}-\frac{1}{u}} & \textit{for } 2 \le u' \le v \le \infty. \end{cases}$$

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