On the hyperbolicity of the Krein space numerical range

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ABSTRACT
In this paper a necessary and sufficient condition for hyperbolicity of the indefinite numerical range is established. As a consequence, an indefinite version of Brown-Spitkovsky theorem stating the ellipticity of the numerical range of certain tridiagonal matrices is revisited. This result leads to necessary and sufficient conditions for hyperbolicity of indefinite numerical ranges of new classes of tridiagonal matrices.

KEYWORDS
indefinite numerical range, classical numerical range, J-Hermitian matrix; Hyperbolical Range Theorem

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1. Introduction

Let M_n stand for the associative algebra of $n \times n$ complex matrices and I_n be its identity matrix.

We consider the complex vector space $\mathbb{C}^n$ with the Krein space structure induced by the *indefinite inner product* $[x, y] = y^* J\, x$, for $x, y \in \mathbf{C}^n$, where $J = I_r \oplus -I_{n-r}$ and $0 < r < n$. The *indefinite numerical range (INR)* of $A \in M_n$ is denoted and defined by

$$W^J(A) = \left\{ \frac{[Au, u]}{[u, u]} : \, u \in \mathbb{C}^n, \, [u, u] \neq 0 \right\}.$$

This set is the union of the two sets

$$W^J_+(A) = \left\{ \frac{[Au, u]}{[u, u]} : \, u \in \mathbb{C}^n, \, [u, u] > 0 \right\}$$

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and

$$W_-^J(A) = \left\{ \frac{[Au,u]}{[u,u]} : u \in \mathbb{C}^n,\ [u,u] < 0 \right\}.$$

Obviously, $W_-^J(A) = -W_+^{-J}(A)$ and $W^J(A)$ is a singleton if and only if A is a scalar matrix [1]. If $J = I_n$, then $W^J(A) = W_+^J(A)$ reduces to the well-known (classical) *numerical range* $W(A)$. It is well-known that $W(A)$ contains $\sigma(A)$, the spectrum of A, and it is a convex set, as asserted by the Toeplitz-Hausdorff Theorem [2,3]. The set $W^J(A)$ is *pseudo-convex*, that is, for any pair of distinct given points $x, y \in W^J(A)$, either $W^J(A)$ contains the closed line segment $\{tx + (1-t)y : \ 0 \leq t \leq 1\}$ or $W^J(A)$ contains the half-lines $\{tx + (1-t)y : \ t \leq 0 \text{ or } t \geq 1\}$.

A *supporting line* of a convex set $S \subset \mathbb{C}$ is a line containing a boundary point of S and defining two half-planes, such that one of them does not contain S. The supporting lines of $W^J(A)$ are by definition the supporting lines of the convex sets $W_+^J(A)$ and $W_-^J(A)$. A boundary point of $W_\pm^J(A)$ belonging to more than one of its supporting lines is called a *corner* of $W_\pm^J(A)$. The corners of $W^J(A)$ are closely related to the eigenvalues of A. In fact, if $z_0 \in W_\pm^J(A)$ is a corner of the set $W^J(A)$, then z_0 is an eigenvalue of A with associated eigenvector x, such that $[x,x] = \pm 1$ [4].

The *adjoint* of the matrix $A \in M_n$ in this Krein space structure is the matrix $A^\# = JA^*J$ and A is called *J-Hermitian* if $A = A^\#$. Considering

$$\Re^J(A) = \frac{A + A^\#}{2} \qquad \text{and} \qquad \Im^J(A) = \frac{A - A^\#}{2i},$$

we have $A = \Re^J(A) + i\,\Im^J(A)$. We easily see that the orthogonal projections of $W^J(A)$ into the real and imaginary axes are $W^J(\Re^J(A))$ and $W^J(\Im^J(A))$, respectively.

For $A \in M_n$ and each angle $\theta \in \mathbb{R}$, let $H_\theta(A) = \Re^J(\mathrm{e}^{-i\theta} A)$. The characteristic polynomial of the matrix $H_\theta(A)$, given by

$$p_\theta(z) = \det(\Re^J(A)\cos\theta + \Im^J(A)\sin\theta - zI_n),$$

plays an important role in the study of $W^J(A)$. Having this in mind, this polynomial will be called *INR generating polynomial.* To a matrix $A \in M_n$, through the equation

$$\det(\Re^J(A)u + \Im^J(A)v + wI_n) = 0,$$

it is associated a *class n* algebraic curve in homogeneous line coordinates, called the *boundary generating curve* of $W^J(A)$. The supporting lines of $W^J(A)$ are generating elements of this curve. Its real part, denoted by $C^J(A)$, generates the set $W^J(A)$ as its *pseudo-convex hull*, which is obtained in the following way: for any two points x, y in the boundary generating curve, take the line segment defined by them if $[x,x][y,y] > 0$ and the two rays $\{tx + (1-t)y : t \leq 0 \text{ or } t \geq 1\}$ if $[x,x][y,y] < 0$ (see e.g. [5] for more details).

The *Hyperbolical Range Theorem* [5] characterizes $W^J(A)$ in the case $n = 2$ and states the following: for $J = \mathrm{diag}(1,-1)$, the set $W^J(A)$ is bounded by a non-degenerate hyperbola with foci at λ_1, λ_2, the eigenvalues of A, transverse and non-transverse axes of length

$$\Big(\mathrm{Tr}(A^{\#}A) - 2\Re(\lambda_1\bar{\lambda}_2)\Big)^{\frac{1}{2}} \qquad \text{and} \qquad \Big(|\lambda_1|^2 + |\lambda_2|^2 - \mathrm{Tr}(A^{\#}A)\Big)^{\frac{1}{2}},$$

respectively, if and only if $2\Re(\bar{\lambda}_1\lambda_2) < \mathrm{Tr}(A^{\#}A) < |\lambda_1|^2 + |\lambda_2|^2$. For the degenerate cases, $W^J(A)$ may be a singleton, a line, a line except a point or except an open line segment, the whole complex plane or the complex plane except a line.

Hyperbolical shape of $W^J(A)$ persists in certain cases independently of the size of A. Our main goal is to investigate classes of matrices with an hyperbolical indefinite numerical range. The counterpart of the Hyperbolical Range Theorem in the context of the classical numerical range is the Elliptical Range Theorem. Several papers have been published, identifying classes of matrices with $W(A)$ being an elliptical disc (for details we refer to [6–9] and references therein). So, the present study also deserves attention.

The remaining of this note is organized as follows. In Section 2, a criterium for (no degenerate) hyperbolicity of $W^J(A)$ is stated, being the degeneracy into two half-lines also studied. Corollaries for matrices of order 2 useful in subsequent discussions are presented. In Section 3, we revisit an indefinite version of the well known elliptical range theorem due to Brown and Spitkowsky. This result leads to the characterization of new classes of tridiagonal matrices with hyperbolical INR. Illustrative examples of the obtained results are presented. In Section 4, final notes are given.

2. Criterium for hyperbolicity of $W^J(A)$

For $A \in M_n$, we will be repeatedly using the following properties:

(i) $W^J(\alpha A + \beta I_n) = \alpha W^J(A) + \beta$ for any $\alpha, \beta \in \mathbb{C}$;

(ii) $W^J(A)$ is *J-unitarily invariant*: $W^J(U^{\#}AU) = W^J(A)$ for any $U \in M_n$ *J-unitary*, that is, satisfying $U^{\#}U = I_n$.

The matrix

$$H_\theta(A) = \Re^J(A)\cos\theta + \Im^J(A)\sin\theta, \qquad \theta \in \mathbb{R}, \tag{1}$$

is J-Hermitian, and so its eigenvalues are real or occur in complex conjugate pairs [10]. It is known that if $H_\theta(A)$ has non-real eigenvalues, then $W^J(H_\theta(A))$ is the whole real line [11, Proposition 2.1]. To avoid this trivial case, we consider the case when all its eigenvalues are real. In fact, we easily see that if $W^J(A)$ has a supporting line in the direction perpendicular to the angle θ, then the eigenvalues of the matrix $H_\theta(A)$ are all real. In this setup, let us define

$$\sigma^J_\pm(H_\theta(A)) := \{\lambda \in \mathbb{R} :\ \exists\, x \in \mathbb{C}^n,\ [x,x] = \pm 1,\ H_\theta(A)x = \lambda x\},$$

such that

$$\sigma^J_+\big(H_\theta(A)\big) = \{\lambda_1(\theta), \ldots, \lambda_r(\theta) : \lambda_1(\theta) \geq \cdots \geq \lambda_r(\theta)\} \tag{2}$$

and

$$\sigma_-^J\big(H_\theta(A)\big) = \{\lambda_{r+1}(\theta), \dots, \lambda_n(\theta) : \lambda_{r+1}(\theta) \geq \cdots \geq \lambda_n(\theta)\}. \tag{3}$$

We will be concerned with the class $\mathcal{J}$ of matrices $H_\theta(A)$ with real eigenvalues, satisfying (2) and (3), that do *not interlace*, that is, either

$$\lambda_r(\theta) > \lambda_{r+1}(\theta) \tag{4}$$

or

$$\lambda_n(\theta) > \lambda_1(\theta). \tag{5}$$

Notice that if the eigenvalues of $H_\theta(A)$ interlace, then $W^J(H_\theta(A))$ is the whole real line. If $W^J(H_\theta(A))$ has a corner point belonging to $W_\pm^J(H_\theta(A))$, then it is an extremum eigenvalue in $\sigma_\pm^J(H_\theta(A))$.

For simplicity, let $\Omega_A = (\eta, \xi)$ be an interval, with greatest possible diameter, of angles θ, such that $H_\theta(A) \in \mathcal{J}$. For $\theta \in \Omega_A$, the maximal eigenvalue in $\sigma_-^J(H_\theta(A))$ and the minimal eigenvalue in $\sigma_+^J\big(H_\theta(A)\big)$ if (4) holds are denoted by $\lambda_L\big(H_\theta(A)\big)$ and $\lambda_R\big(H_\theta(A)\big)$, respectively; analougously, the maximal eigenvalue in $\sigma_+^J(H_\theta(A))$ and the minimal eigenvalue in $\sigma_-^J\big(H_\theta(A)\big)$ if (5) holds are denoted by $\lambda_L\big(H_\theta(A)\big)$ and $\lambda_R\big(H_\theta(A)\big)$, respectively. Without loss of generality, we may assume that (4) holds.

Chien et al. [6, Theorem 1] presented a criterium for the classical numerical range of a matrix to be an elliptical disc. In the context of Krein spaces, we give a criterium for hyperbolicity of the boundary of $W^J(A)$.

Theorem 2.1. *Let $\widetilde{a}, \widetilde{b} > 0$, $J = I_r \oplus -I_{n-r}$, $0 < r < n$, and $A \in M_n$. The set $W^J(A)$ is bounded by the non-degenerate hyperbola centered at the origin, with horizontal transverse and vertical non-transverse semi-axes of lengths $\widetilde{a}$ and $\widetilde{b}$, respectively, if and only if*

$$\lambda_R\big(H_\theta(A)\big) = \big(\widetilde{a}^2 - \widetilde{c}^2 \sin^2\theta\big)^{\frac{1}{2}} \quad \textit{and} \quad \lambda_L\big(H_\theta(A)\big) = -\big(\widetilde{a}^2 - \widetilde{c}^2 \sin^2\theta\big)^{\frac{1}{2}}, \tag{6}$$

where $\widetilde{c}^2 = \widetilde{a}^2 + \widetilde{b}^2$, for all $\theta \in \Omega_A = (-\theta_0, \theta_0)$ with $\theta_0 = \arctan(\widetilde{a}/\widetilde{b})$.

Proof. ($\Leftarrow$) By hypothesis, for $\theta \in \Omega_A$, we have $H_\theta(A) \in \mathcal{J}$, that is, the eigenvalues $\lambda_1(\theta), \dots, \lambda_n(\theta)$ of the J-Hermitian matrix $H_\theta(A)$ are all real, satisfying (2) and (3), and do not interlace. Without loss of generality, we may suppose that (4) holds, namely

$$\lambda_R\big(H_\theta(A)\big) \in \sigma_+^J(H_\theta(A)) \qquad \text{and} \qquad \lambda_L\big(H_\theta(A)\big) \in \sigma_-^J(H_\theta(A)),$$

so that

$$\lambda_n(\theta) \leq \cdots \leq \lambda_{r+1}(\theta) = \lambda_L\big(H_\theta(A)\big) < 0 < \lambda_R\big(H_\theta(A)\big) = \lambda_r(\theta) \leq \cdots \leq \lambda_1(\theta).$$

In this case, the equation of the supporting line of $W_+^J(A)$ perpendicular to the direction θ is

$$x\cos\theta + y\sin\theta = \lambda_r(\theta), \tag{7}$$

This is a supporting line of $W_+^J(\mathrm{e}^{-i\theta}A)$, because $W_+^J(A) = \mathrm{e}^{i\theta}\, W_+^J(\mathrm{e}^{-i\theta}A)$. The envelope of this family of supporting lines, with θ ranging over Ω_A, gives the boundary of $W_+^J(A)$. To compute this envelope, note that it has parametric equations given by

$$\left\{ \begin{array}{rcl} x & = & \lambda_r(\theta)\,\cos\theta - \lambda_r'(\theta)\,\sin\theta \\ y & = & \lambda_r(\theta)\,\sin\theta + \lambda_r'(\theta)\,\cos\theta \end{array} \right. .$$

Hence,

$$x = \frac{\widetilde{a}^2\,(1+\cos 2\theta)^{\frac{1}{2}}}{\left(2\widetilde{a}^2 - \widetilde{c}^2(1-\cos 2\theta)\right)^{\frac{1}{2}}} \qquad \text{and} \qquad y^2 = \frac{(\widetilde{a}^2-\widetilde{c}^2)^2\,(1-\cos 2\theta)}{2\widetilde{a}^2 - \widetilde{c}^2(1-\cos 2\theta)}. \tag{8}$$

For $\theta \in \Omega_A$, it is clear that

$$2\widetilde{a}^2 - \widetilde{c}^2(1-\cos 2\theta) = 2(\widetilde{a}^2 - \widetilde{c}^2\sin^2\theta) = 2\lambda_r^2(\theta) > 0.$$

Solving the right hand side equation in (8), with respect to $\cos 2\theta$, we obtain

$$\cos 2\theta = \frac{\left(\widetilde{a}^2-\widetilde{c}^2\right)^2 + \left(\widetilde{c}^2 - 2\widetilde{a}^2\right)y^2}{\left(\widetilde{a}^2-\widetilde{c}^2\right)^2 + \widetilde{c}^2 y^2}. \tag{9}$$

We remark that $\left(\widetilde{a}^2-\widetilde{c}^2\right)^2 + \widetilde{c}^2y^2 \neq 0$, since $\widetilde{c}^2 = \widetilde{a}^2 + \widetilde{b}^2$ and $\widetilde{a}, \widetilde{b} > 0$, by hypothesis. Replacing (9) into the left hand side equation of (8), yields

$$x = \widetilde{a}\left(\frac{y^2}{\widetilde{c}^2 - \widetilde{a}^2} + 1\right)^{\frac{1}{2}},$$

the right branch of an hyperbola, representing the boundary of the set $W_+^J(A)$. Analogously, we conclude that $W_-^J(A)$ is bounded by the left branch of the hyperbola, that is, in this case, $W_-^J(A) = -W_+^J(A)$. Hence, the boundary of $W^J(A)$ is defined by

$$\frac{x^2}{\widetilde{a}^2} - \frac{y^2}{\widetilde{c}^2 - \widetilde{a}^2} = 1$$

and the result follows.

($\Rightarrow$) Let $W^J(A)$ be bounded by the hyperbola centered at the origin, with horizontal transverse semi-axis of length $\widetilde{a}$ and vertical non-transverse semi-axis of length $\widetilde{b}$, that is, defined by the equation:

$$\frac{x^2}{\widetilde{a}^2} - \frac{y^2}{\widetilde{b}^2} = 1. \tag{10}$$

Consider a supporting line of $W^J(A)$, which is perpendicular to the direction of argument θ and tangent to one of the branches of this hyperbola.

If the eigenvalues of $H_\theta(A)$ are not all real, then $W^J(H_\theta(A))$ is the whole real line [11, Proposition 2.1] and we would have a contradiction. We conclude that $H_\theta(A)$ has only real eigenvalues. If the eigenvalues of $H_\theta(A)$ are all real and interlace, then by the pseudo-convexity of the J-numerical range, the set $W^J(H_\theta(A))$ is the whole real line and we would have a contradiction too. Hence, we may conclude that $H_\theta(A) \in \mathcal{J}$.

Without loss of generality, suppose that $W_+^J(A)$ is bounded by the right branch of the hyperbola (10). The tangent lines to this branch of the hyperbola include $x = \widetilde{a}$ and those with non-zero slope $m_\theta = \tan\theta'$, $\theta' = \pi/2 + \theta$, that is,

$$y = m_\theta\, x + d_\theta \tag{11}$$

for $\theta \in \Omega_A = (-\theta_0, \theta_0)$, with $\theta_0 = \arctan(\widetilde{a}/\widetilde{b})$. By the tangency condition, we have $\widetilde{a}^2 m_\theta^2 - \widetilde{b}^2 = d_\theta^2 > 0$. Then

$$d_\theta = \pm\big(\widetilde{a}^2 m_\theta^2 - \widetilde{b}^2\big)^{\frac{1}{2}}$$

and $|m_\theta| > \widetilde{b}/\widetilde{a}$. The point of the right branch of the hyperbola where the tangent line has the slope m_θ is given by

$$\big(x_\theta, y_\theta\big) = \left(\frac{\widetilde{a}^2|m_\theta|}{|d_\theta|}, -\frac{\widetilde{b}^2}{d_\theta}\right). \tag{12}$$

Since $W_+^J(H_\theta(A))$ is a closed real half-line, its extreme point is a corner of this set, consequently, it is the minimal eigenvalue in $\sigma_+^J\big(H_\theta(A)\big)$, which we denoted by $\lambda_R\big(H_\theta(A)\big)$. Clearly, $\lambda_R\big(H_\theta(A)\big) = \widetilde{a}$ if $\theta = 0$ and the distance of the origin to the tangent line (11), passing through the point (12), is given by

$$\frac{|d_\theta|}{\big(m_\theta^2+1\big)^{\frac{1}{2}}} = \left(\frac{\widetilde{a}^2 m_\theta^2 - \widetilde{b}^2}{m_\theta^2+1}\right)^{\frac{1}{2}} = \Big(\widetilde{a}^2 - (\widetilde{a}^2+\widetilde{b}^2)\cos^2\theta'\Big)^{\frac{1}{2}} = \big(\widetilde{a}^2 - \widetilde{c}^2\sin^2\theta\big)^{\frac{1}{2}},$$

that is, $\lambda_R\big(H_\theta(A)\big) = \big(\widetilde{a}^2 - \widetilde{c}^2\sin^2\theta\big)^{\frac{1}{2}}$ for all $\theta \in \Omega_A$. In this case, $W_-^J(A) = -W_+^J(A)$ is bounded by the left branch of the hyperbola and the result follows. $\square$

The variant of Theorem 2.1 corresponding to the case $\widetilde{a} > 0$ and $\widetilde{b} = 0$ is as follows.

Corollary 2.2. *Let $\widetilde{a} > 0$, $J = I_r \oplus -I_{n-r}$, $0 < r < n$, and $A \in M_n$. We have $W^J(A) = (-\infty, -\widetilde{a}\,] \cup [\,\widetilde{a}, +\infty)$ if and only if*

$$\lambda_R\big(H_\theta(A)\big) = \widetilde{a}\cos\theta \qquad \textit{and} \qquad \lambda_L\big(H_\theta(A)\big) = -\widetilde{a}\cos\theta$$

for all $\theta \in \Omega_A = \big(-\frac{\pi}{2}, \frac{\pi}{2}\big)$.

Proof. ($\Leftarrow$) Analogously to the proof of Theorem 2.1, we may supppose that (4) holds. Then the envelope of the family of supporting lines (7) of $W_+^J(A)$, with θ ranging over $\Omega_A = \big(-\frac{\pi}{2}, \frac{\pi}{2}\big)$ is given by $x = \widetilde{a}$ and $y = 0$. Since $W_-^J(A) = -W_+^J(A)$, the result follows using the fact that $W^J(A)$ is a pseudo-convex set.

($\Rightarrow$) This implication is obvious. $\square$

Now, let $J = \mathrm{diag}(1,-1)$. For any $A = (a_{ij}) \in M_2$, we have

$$A = \mathrm{e}^{i\alpha} \begin{bmatrix} a & b \\ c & -a \end{bmatrix} + \frac{1}{2}\,\mathrm{Tr}(A)\, I_2,$$

considering

$$a = \frac{1}{2}\big|\mathrm{Tr}(JA)\big|, \qquad b = a_{12}\mathrm{e}^{-i\alpha}, \qquad c = a_{21}\mathrm{e}^{-i\alpha}, \qquad \alpha = \arg \mathrm{Tr}(JA). \tag{13}$$

By property (i), $W^J(A)$ may be obtained from the J-numerical range of a matrix of type

$$\tilde{A} = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}, \qquad a \geq 0, \quad b, c \in \mathbb{C}. \tag{14}$$

We set

$$M = 2|a^2+bc| - 2a^2 + |b|^2 + |c|^2, \quad N = 2|a^2+bc| + 2a^2 - |b|^2 - |c|^2. \tag{15}$$

We start with a technical proposition on the eigenvalues and eigenvectors of $H_\theta(A_\phi)$ for $A_\phi = \mathrm{e}^{-i\phi}\tilde{A}$ and $\phi = \frac{1}{2}\arg\big(a^2+bc\big)$, useful in future discussions. Throughout, we may assume that (14) is nonsingular (cf. Remark 1).

Proposition 2.3. *Let $J = \mathrm{diag}(1,-1)$, A be of type (14), $\phi = \frac{1}{2}\arg\big(a^2+bc\big)$ and $A_\phi = \mathrm{e}^{-i\phi}A$. Let M, N be defined as in (15). For each $\theta \in \mathbb{R}$, the eigenvalues of $H_\theta(A_\phi)$ are given by*

$$\lambda_\pm(\theta) = \pm\frac{1}{2}\left(N - (M+N)\sin^2\theta\right)^{\frac{1}{2}}. \tag{16}$$

If $N > 0$, then $\lambda_\pm(\theta)$ are simple eigenvalues, with associated eigenvectors $u_\pm(\theta)$, satisfying

$$[u_-(\theta), u_-(\theta)][u_+(\theta), u_+(\theta)] < 0$$

and $H_\theta(A_\phi) \in \mathcal{J}$, for all $\theta \in (-\theta_0, \theta_0)$ with

$$\theta_0 = \begin{cases} \arctan(N/M)^{\frac{1}{2}}, & M > 0 \\ \pi/2, & M = 0 \end{cases}.$$

If $N \leq 0$, then $H_\theta(A_\phi) \notin \mathcal{J}$, for all $\theta \in \mathbb{R}$.

Proof. Consider $\phi = \frac{1}{2}\arg\big(a^2+bc\big)$ and the rotated matrix $A_\phi = \mathrm{e}^{-i\phi}A$. Let $\theta \in \mathbb{R}$. Then $H_\theta(A_\phi)$ is given by

$$\Re^J\left(\mathrm{e}^{-i(\theta+\phi)}A\right) = \begin{bmatrix} a\cos(\theta+\phi) & \frac{1}{2}\left(b\mathrm{e}^{-i(\theta+\phi)} - \overline{c}\mathrm{e}^{i(\theta+\phi)}\right) \\ \frac{1}{2}\left(c\mathrm{e}^{-i(\theta+\phi)} - \overline{b}\mathrm{e}^{i(\theta+\phi)}\right) & -a\cos(\theta+\phi) \end{bmatrix}.$$

The INR generating polynomial of A_ϕ is equal to

$$p_\theta(\lambda) = \lambda^2 - \frac{2a^2\cos(2(\theta+\phi)) + bc\,\mathrm{e}^{-2i(\theta+\phi)} + \overline{bc}\,\mathrm{e}^{2i(\theta+\phi)} + 2a^2 - |b|^2 - |c|^2}{4}.$$

Then the zeros of the characteristic polynomial of $H_\theta(A_\phi)$ are obtained as

$$\lambda_\pm(\theta) \;=\; \pm\frac{1}{2}\left(4a^2\cos^2(\theta+\phi) - |b|^2 - |c|^2 + 2\,\Re(bc\,\mathrm{e}^{-2i(\theta+\phi)})\right)^{\frac{1}{2}}. \qquad (17)$$

We have

$$\begin{aligned}
&2a^2\cos(2(\theta+\phi)) + bc\,\mathrm{e}^{-2i(\theta+\phi)} + \overline{bc}\,\mathrm{e}^{2i(\theta+\phi)}\\
&= \left(2a^2 + bc + \overline{bc}\right)\cos(2(\theta+\phi)) - i\left(bc - \overline{bc}\right)\sin(2(\theta+\phi))\\
&= 2\,\Re\left(a^2+bc\right)\cos(2(\theta+\phi)) + 2\,\Im\left(a^2+bc\right)\sin(2(\theta+\phi))\\
&= 2\left|a^2+bc\right|\cos(2\phi)\cos(2(\theta+\phi)) + 2\left|a^2+bc\right|\sin(2\phi)\sin(2(\theta+\phi))\\
&= 2\left|a^2+bc\right|\cos(2\theta),
\end{aligned}$$

using trivial trigonometric transformations. Then (17) is equivalent to

$$\lambda_\pm(\theta) \;=\; \pm\frac{1}{2}\left(2|a^2+bc|\cos(2\theta) + 2a^2 - |b|^2 - |c|^2\right)^{\frac{1}{2}}$$

and (16) holds, because

$$\begin{aligned}
2|a^2+bc|\cos(2\theta) + 2a^2 - |b|^2 - |c|^2 &= N - 4|a^2+bc|\sin^2\theta\\
&= N - (M+N)\sin^2\theta.
\end{aligned}$$

If $N > 0$, then

$$g(\theta) = N - (M+N)\sin^2\theta$$

is a continuous even function, which has a maximum given by N attained at $\theta = 0$ and a minimum given by $-M$ attained at $\pi/2$. Then there exists an interval $(-\theta_0, \theta_0)$, such that $g(\theta) > 0$ for $\theta \in (-\theta_0, \theta_0)$ and $g(\pm\theta_0) = 0$. In fact, for $M > 0$, we have $g(\theta_0) = 0$ if and only if

$$\sin^2\theta_0 = N/(M+N) \quad\Leftrightarrow\quad \tan^2\theta_0 = N/M \quad\Leftrightarrow\quad \theta_0 = \arctan(N/M)^{\frac{1}{2}}.$$

Obviously, if $M = 0$, then $\theta_0 = \pi/2$. Hence, for θ in this interval $(-\theta_0, \theta_0)$, the eigenvalues of $H_\theta(A_\phi)$ are real and distinct.

Consider $\theta \in (-\theta_0, \theta_0)$. If $H_\theta(A_\phi)$ becomes a non-zero diagonal matrix, then $(1,0), (0,1)$ are eigenvectors of $H_\theta(A_\phi)$ associated to the real eigenvalues

$$\lambda_\pm(\theta) \;=\; \pm a\,\cos(\theta+\phi) \;\neq\; 0.$$

If $H_\theta(A_\phi)$ is not a diagonal matrix, then it has eigenvectors associated to the eigenvalues $\lambda_\pm(\theta)$ given by

$$u_\pm(\theta) = \Big(a\cos(\theta+\phi) + \lambda_\pm(\theta), \big(c\mathrm{e}^{-i(\theta+\phi)} - \bar{b}\mathrm{e}^{i(\theta+\phi)}\big)/2\Big)$$

and $\lambda_\pm(\theta)$ differ from the main diagonal entries of $H_\theta(A_\phi)$. Hence,

$$0 \;<\; \lambda_+(\theta) \;\leq\; \frac{1}{2}\left(4a^2\cos^2(\theta+\phi) - (|b|-|c|)^2\right)^{\frac{1}{2}} \;\leq\; a\,|\cos(\theta+\phi)|$$

and one of the two inequalities in the above chain of inequalities must be strict. Under this hypothesis, we have

$$\big(\lambda_+(\theta) - a\cos(\theta+\phi)\big)\big(\lambda_+(\theta) + a\cos(\theta+\phi)\big) < 0$$

and the corresponding eigenvectors satisfy

$$\begin{aligned}
[u_+(\theta), u_+(\theta)] \;&=\; \big(\lambda_+(\theta) + a\cos(\theta+\phi)\big)^2 - \frac{|b|^2+|c|^2}{4} + \frac{1}{2}\Re\left(bc\,\mathrm{e}^{-2i(\theta+\phi)}\right)\\
&=\; 2\lambda_+(\theta)\big(\lambda_+(\theta) + a\cos(\theta+\phi)\big) \neq 0.
\end{aligned}$$

Analogously, $[u_-(\theta), u_-(\theta)] \neq 0$. We easily see that $[u_-(\theta), u_-(\theta)]$ and $[u_+(\theta), u_+(\theta)]$ have opposite signs. Therefore, one of the eigenvalues $\lambda_\pm(\theta)$ belongs to $\sigma_+^J(H_\theta(A_\phi))$ and the other belongs to $\sigma_-^J(H_\theta(A_\phi))$, that is, we conclude that $H_\theta(A_\phi) \in \mathcal{J}$.

The previous cases imply that $a \neq 0$ and $\theta \neq \frac{\pi}{2} - \phi + k\pi$, $k \in \mathbb{Z}$.

If $N \leq 0$, then $\lambda_\pm(\theta) \in i\mathbb{R}$ and so $H_\theta(A_\phi) \notin \mathcal{J}$, for any $\theta \in \mathbb{R}$. □

Remark 1. If $A \in M_2$ is non-zero singular of type (14), then $a^2 = |bc|$, which yields

$$N = -(|b|-|c|)^2 = -M < 0.$$

Indeed, the case $N = 0$ would give $\sigma\big(H_\theta(A)\big) = \{0\}$ for all $\theta \in \mathbb{R}$, that is, $A = O$. From $N = -M < 0$ and (16), we get $\sigma\big(H_\theta(A)\big) \subset i\,\mathbb{R}\setminus\{0\}$. By [11, Proposition 2.1], we have $W^J\big(H_\theta(A)\big) = \mathbb{R}$ for all $\theta \in \mathbb{R}$. In this case, $W^J(A) = \mathbb{C}$.

Now, the non-degenerate case of the Hyperbolical Range Theorem for nonsingular matrices of type (14) is easily derived.

Theorem 2.4. *Let $J = \mathrm{diag}(1,-1)$, $A \in M_2$ be of the form* (14) *and let M, N be defined as in* (15)*. The set $W^J(A)$ is bounded by the non-degenerate hyperbola centered at the origin, with foci at the eigenvalues of A,*

$$\pm\left(a^2+bc\right)^{\frac{1}{2}}, \tag{18}$$

whose lengths of the transverse and the non-transverse axes are $N^{\frac{1}{2}}$ and $M^{\frac{1}{2}}$, respectively, if and only if $M > 0$ and $N > 0$.

Proof. ($\Leftarrow$) Suppose that $M > 0$ and $N > 0$. Thus, the eigenvalues of A in (18) are not equal. Let $\phi = \frac{1}{2}\arg\left(a^2+bc\right)$ and consider the rotated matrix $A_\phi = \mathrm{e}^{-i\phi}A$. Since $\sigma(A_\phi) = \mathrm{e}^{-i\phi}\sigma(A)$, the eigenvalues of A_ϕ are real and simple, given by $\pm|a^2+bc|^{\frac{1}{2}}$.

By Proposition 2.3, the eigenvalues of $H_\theta(A_\phi)$ are given by (16) and $H_\theta(A_\phi) \in \mathcal{J}$, for all $\theta \in (-\theta_0, \theta_0)$, with $\theta_0 = \arctan(N/M)^{\frac{1}{2}}$. Clearly, $H_{\pm\theta_0}(A_\phi) \notin \mathcal{J}$ holds. Then $\Omega_{A_\phi} = (-\theta_0, \theta_0)$.

Let $\widetilde{a}^2 = N/4$ and $\widetilde{b}^2 = M/4$. The eigenvalues of $H_\theta(A_\phi)$ are given

$$\lambda_\pm(\theta) \;=\; \pm\left(\widetilde{a}^2 - \widetilde{c}^2 \sin^2\theta\right)^{\frac{1}{2}},$$

with $\widetilde{c}^2 = \widetilde{a}^2 + \widetilde{b}^2$, fo all $\theta \in \Omega_{A_\phi}$. Applying Theorem 2.1, we conclude that $W^J(A_\phi)$ is bounded by the hyperbola, centered at the origin, with transverse and non-transverse axes on the x-axis and y-axis, respectively, of lengths $2\widetilde{a} = N^{\frac{1}{2}}$ and $2\widetilde{b} = M^{\frac{1}{2}}$. In this case, the foci of the hyperbola are the eigenvalues of A_ϕ, which are on the real axis. Finally, the boundary of $W^J(A)$ is a rotated hyperbola, centered at the origin, since $W^J(A) = \mathrm{e}^{i\phi}\, W^J(A_\phi)$ and the result easily follows.

($\Rightarrow$) Suppose that $W^J(A)$ is bounded by an hyperbola centered at the origin, with foci at the eigenvalues of A, whose lengths of the transverse and the non-transverse axes are, respectively, $N^{\frac{1}{2}}$ and $M^{\frac{1}{2}}$. Then we must have $M > 0$ and $N > 0$. $\square$

For M, N defined as in (15), given the matrix A in (14), we remark that

$$M \;\geq\; 2|a^2 + bc| - 2(a^2 - |bc|) \;\geq\; 2\big|a^2 - |bc|\big| - 2(a^2 - |bc|) \;\geq\; 0$$

and we find that

$$\begin{aligned} M = 0 \quad &\Leftrightarrow \quad |b| = |c| \;\wedge\; \arg(bc) = -\pi \;\wedge\; a^2 \geq |bc| \\ &\Leftrightarrow \quad a \geq |b| \;\wedge\; c = -\overline{b}. \end{aligned} \tag{19}$$

Therefore, if $M = 0$, then $A^{\#} = A$, that is, A is J-Hermitian, A has real spectrum and

$$N \;=\; 2\big|a^2 - |b|^2\big| + 2(a^2 - |b|^2) \;=\; 4(a^2 - |b|^2) \geq 0. \tag{20}$$

From Corollary 2.2, we readily characterize the degenerate case of $W^J(A)$ as a disjoint union of two closed real half-lines, in terms of the entries of A.

Corollary 2.5. *Under the conditions of Theorem 2.4 the set $W^J(A)$ is the disjoint union of two closed real half-lines with endpoints at the eigenvalues of A if and only if $M = 0$ and $N > 0$, equivalently, $a > |b|$ and $c = -\overline{b}$.*

Proof. ($\Leftarrow$) By (19) and (20), $M = 0$ and $N > 0$ if and only if $a > |b|$ and $c = -\overline{b}$. In this case, A has real eigenvalues $\pm\,\widetilde{a}$ with $\widetilde{a} = (a^2 - |b|^2)^{\frac{1}{2}} \neq 0$. By Proposition 2.3, with $\phi = 0$, we have $H_\theta(A) \in \mathcal{J}$, for $\theta \in (-\pi/2, \pi/2)$, with eigenvalues $\lambda_\pm(\theta) = \pm\,\widetilde{a}\cos\theta$. By Corollary 2.2, this is equivalent to

$$W^J(A) = (-\infty, -\widetilde{a}] \cup [\widetilde{a}, +\infty).$$

($\Rightarrow$) Suppose $W^J(A)$ is the disjoint union of two closed real half-lines, being the endpoints, as corners of this set, the eigenvalues $\pm\,\widetilde{a}$ of A, with $\widetilde{a} = (a^2 + bc)^{\frac{1}{2}} \neq 0$.

By Corollary 2.2, we have

$$\lambda_R\big(H_\theta(A)\big) = \widetilde{a}^2 \cos\theta \qquad \text{and} \qquad \lambda_L\big(H_\theta(A)\big) = -\widetilde{a}^2 \cos\theta \tag{21}$$

for $\theta \in (-\pi/2, \pi/2)$. From Proposition 2.3, the eigenvalues of $H_\theta(A)$ can be written as

$$\pm\frac{1}{2}\left(N\cos^2\theta \; - M\sin^2\theta\right)^{\frac{1}{2}}. \tag{22}$$

In this case, (21) and (22) yield $N = 4\widetilde{a}^2 > 0$ and $M = 0$. □

To end this section, the Hyperbolical Range Theorem in terms of the invariants of the matrix is easily obtained.

Corollary 2.6. *Let* $J = \mathrm{diag}(1, -1)$ *and* $A \in M_2$. *The set* $W^J(A)$ *is bounded by a non-degenerate hyperbola, centered at* $\frac{1}{2}\mathrm{Tr}(A)$, *with foci at the eigenvalues* λ_1, λ_2 *of* A, *transverse and non-transverse axes of lengths*

$$\Big(\mathrm{Tr}(A^{\#}A) - 2\Re(\lambda_1\bar{\lambda}_2)\Big)^{\frac{1}{2}} \qquad \textit{and} \qquad \Big(|\lambda_1|^2 + |\lambda_2|^2 - \mathrm{Tr}(A^{\#}A)\Big)^{\frac{1}{2}},$$

respectively, if and only if

$$2\Re(\lambda_1\bar{\lambda}_2) < \mathrm{Tr}(A^{\#}A) < |\lambda_1|^2 + |\lambda_2|^2. \tag{23}$$

Proof. If $A = (a_{ij}) \in M_2$, then

$$W^J(A) = \mathrm{e}^{i\alpha}\, W^J(B) + \frac{1}{2}\mathrm{Tr}(A), \tag{24}$$

where B is the matrix in (14), considering a, b, c defined in (13). This matrix B has eigenvalues $\lambda_1' = -\left(a^2 + bc\right)^{\frac{1}{2}}$ and $\lambda_2' = \left(a^2 + bc\right)^{\frac{1}{2}}$, such that

$$2\Re(\lambda_1'\bar{\lambda}_2') = -2|a^2 + bc|, \qquad |\lambda_1'|^2 + |\lambda_2'|^2 = 2|a^2 + bc|, \tag{25}$$

$$\mathrm{Tr}(B^{\#}B) = 2a^2 - |b|^2 - |c|^2. \tag{26}$$

Let λ_1, λ_2 be the eigenvalues of A. Hence,

$$M \;=\; |\lambda_1|^2 + |\lambda_2|^2 - \mathrm{Tr}(A^{\#}A) \qquad \text{and} \qquad N \;=\; \mathrm{Tr}(A^{\#}A) - 2\Re(\lambda_1\bar{\lambda}_2).$$

It is clear that

$$2\Re(\lambda_1\bar{\lambda}_2), \qquad |\lambda_1|^2 + |\lambda_2|^2 \qquad \text{and} \qquad \mathrm{Tr}(A^{\#}A)$$

are obtained by adding $\frac{1}{2}|\mathrm{Tr}(A)|^2$ to the respective expressions, concerning B, in (25) and (26). Then

$$M \;=\; 2|a^2 + bc| - 2a^2 + |b|^2 + |c|^2, \qquad N \;=\; 2a^2 - |b|^2 - |c|^2 + 2|a^2 + bc|$$

and we may conclude that $M > 0$ and $N > 0$ if and only if (23) holds. By Theorem 2.4 and (24) the result easily follows. $\square$

From Corollary 2.5 and following the proof of the previous corollary we have the next result.

Corollary 2.7. *Under the hypothesis of Corollary 2.6, $W^J(A)$ is the disjoint union of two closed half-lines, with endpoints at the eigenvalues λ_1, λ_2 of A, if and only if*

$$2\Re(\lambda_1\bar{\lambda}_2) < \mathrm{Tr}(A^{\#}A) = |\lambda_1|^2 + |\lambda_2|^2.$$

3. Indefinite version of Brown-Spitkowski Theorem

A matrix $A = (a_{ij})$ is tridiagonal if $a_{ij} = 0$ for $|i-j| > 1$. In this section, we consider tridiagonal matrices with biperiodic main diagonal, that is, $a_{ii} = a_1$ if i is odd and $a_{ii} = a_2$ if i is even, with $a_1, a_2 \in \mathbb{C}$.

By property (i), considering $J = \mathrm{diag}(1, -1, 1, -1, \ldots) \in M_n$ and

$$a = \frac{1}{n}|\mathrm{Tr}(JA)|, \qquad b_j = a_{j,j+1}\mathrm{e}^{-i\alpha}, \qquad c_j = a_{j+1,j}\mathrm{e}^{-i\alpha}, \qquad j = 1, \ldots, n-1,$$

for $\alpha = \arg \mathrm{Tr}(JA)$, analogously to the case $n = 2$, we may focus our study on the following class of tridiagonal matrices with biperiodic real main diagonal:

$$T(n; \mathbf{c}, \mathbf{a}, \mathbf{b}) = \begin{bmatrix} a & b_1 & 0 & 0 & 0 & \ldots \\ c_1 & -a & b_2 & 0 & 0 & \ldots \\ 0 & c_2 & a & b_3 & 0 & \ldots \\ 0 & 0 & c_3 & -a & b_4 & \ldots \\ 0 & 0 & 0 & c_4 & a & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \in M_n, \tag{27}$$

where

$$\mathbf{a} = (a, -a, a, -a, \ldots), \qquad \mathbf{b} = (b_1, b_2, b_3, \ldots), \qquad \mathbf{c} = (c_1, c_2, c_3, \ldots),$$

with $a \geq 0$. For convenience, we also consider the bidiagonal matrix

$$X(\mathbf{b}) = \begin{bmatrix} b_1 & 0 & 0 & \ldots \\ b_2 & b_3 & 0 & \ldots \\ 0 & b_4 & b_5 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{28}$$

We recall [13, Lemma 2.1] that the J-numerical range of an $n \times n$ tridiagonal matrix is invariant under the interchange of the $(j, j+1)$ and $(j+1, j)$ entries for any $j = 1, \ldots, n-1$.

We revisit the following theorem partially obtained in [13] and we give a simpler proof.

Theorem 3.1. *Let $J = \operatorname{diag}(1, -1, 1, -1, \ldots) \in M_n$, $n \geq 2$, and T be a tridiagonal matrix of type $T(n; \mathbf{c}, \mathbf{a}, \mathbf{b})$ with $\mathbf{a} = (a, -a, a, -a, \ldots) \in \mathbb{R}^n$, $\mathbf{b}, \mathbf{c} \in \mathbb{C}^{n-1}$, such that $\mathbf{c} = \kappa \overline{\mathbf{b}}$ for some $\kappa \in \mathbb{C}$. Let $s_1 \geq \ldots \geq s_{\lfloor \frac{n}{2} \rfloor}$ be the singular values and r be the rank of the bidiagonal matrix $X(\mathbf{b})$. For each $j = 1, \ldots, r$, consider*

$$M_j = 2|a^2 + \kappa s_j^2| - 2a^2 + (1 + |\kappa|^2) s_j^2, \tag{29}$$

$$N_j = 2|a^2 + \kappa s_j^2| + 2a^2 - (1 + |\kappa|^2) s_j^2, \tag{30}$$

and the hyperbola $\mathcal{H}_j$, centered at the origin, with foci at

$$f_j^{\pm} = \pm \left(a^2 + \kappa s_j^2\right)^{\frac{1}{2}},$$

transverse and non-transverse axes of lengths $N_j^{\frac{1}{2}}$ and $M_j^{\frac{1}{2}}$, respectively. The boundary generating curves of $W^J(T)$ are the non-degenerate nested hyperbolas $\mathcal{H}_1, \ldots, \mathcal{H}_r$ (the points $a, -a$ if $r < \lfloor \frac{n}{2} \rfloor$) and the point a, when n is odd, if and only if $M_1 > 0$ and $N_1 > 0$, that is,

$$\left| a^2 - \frac{1 + |\kappa|^2}{2} s_1^2 \right| < |a^2 + \kappa s_1^2|. \tag{31}$$

In this case, $W^J(T)$ is bounded by the non-degenerate hyperbola $\mathcal{H}_1$.

Proof. Firstly, we prove the result in the case $n = 2m$, $m \in \mathbb{N}$. Without loss of generality, by [13, Lemma 2.1], we may consider T of the form $T(n; \widetilde{\mathbf{c}}, \mathbf{a}, \widetilde{\mathbf{b}})$, with

$$\mathbf{a} = (a, -a, a, -a, \ldots), \qquad \widetilde{\mathbf{b}} = (b_1, \kappa \overline{b_2}, b_3, \ldots), \qquad \widetilde{\mathbf{c}} = (\kappa \overline{b_1}, b_2, \kappa \overline{b_3}, \ldots),$$

for some $\kappa \in \mathbb{C}$. Let $P_\pi \in M_n$ be the permutation matrix associated with the permutation $\pi \in S_n$ defined in the following way:

$$\pi(i) = 2i - 1, \quad 1 \leq i \leq m, \qquad \text{and} \qquad \pi(m + i) = 2i, \quad 1 \leq i \leq m.$$

Let $\widetilde{J} = I_m \oplus -I_m$. By easy computations, we have $P_\pi J P_\pi^T = \widetilde{J}$ and

$$\widetilde{A} = P_\pi T P_\pi^T = \begin{bmatrix} a & 0 & 0 & \ldots & b_1 & 0 & 0 & \ldots \\ 0 & a & 0 & \ldots & b_2 & b_3 & 0 & \ldots \\ 0 & 0 & a & \ldots & 0 & b_4 & b_5 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\ \kappa \overline{b_1} & \kappa \overline{b_2} & 0 & \ldots & -a & 0 & 0 & \ldots \\ 0 & \kappa \overline{b_3} & \kappa \overline{b_4} & \ldots & 0 & -a & 0 & \ldots \\ 0 & 0 & \kappa \overline{b_5} & \ldots & 0 & 0 & -a & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} aI_m & X \\ \kappa X^* & -aI_m \end{bmatrix},$$

where $X = X(\mathbf{b}) \in M_m$, with $\mathbf{b} = (b_1, b_2, b_3, \ldots) \in \mathbb{C}^{2m-1}$, is the bidiagonal matrix defined in (28). It can be easily seen that

$$W^J(T) = W^{P_\pi J P_\pi^T}(P_\pi T P_\pi^T) = W^{\widetilde{J}}(\widetilde{A}).$$

Hence, we focus our proof in the study of $W^{\widetilde{J}}(\widetilde{A})$. By the singular value decomposition, there exist matrices U, V in the unitary group of degree m, such that the block X of $\widetilde{A}$ satisfies $X = U\,D\,V^*$, where $D = \operatorname{diag}(s_1, s_2, \ldots, s_m)$ and $s_1 \geq s_2 \geq \cdots \geq s_m$ are the singular values of X. It is easy to see that the matrix $Z = U \oplus V$ is $\widetilde{J}$-unitary. Moreover, for $Z^{\#} = \widetilde{J} Z^* \widetilde{J}$, we get

$$Z^{\#} \widetilde{A}\, Z = (U^* \oplus V^*) \begin{bmatrix} aI_m & X \\ \kappa X^* & -aI_m \end{bmatrix} (U \oplus V) = \begin{bmatrix} aI_m & U^* X V \\ \kappa\, V^* X^* U & -aI_m \end{bmatrix}.$$

Using the unitary invariance of the $\widetilde{J}$-numerical range, we have

$$W^{\widetilde{J}}(\widetilde{A}) = W^{\widetilde{J}}(Z^{\#} \widetilde{A} Z) = W^{\widetilde{J}}(\widetilde{B}),$$

being

$$\widetilde{B} = \begin{bmatrix} aI_m & D \\ \kappa D & -aI_m \end{bmatrix}$$

permutationally similar to

$$\widetilde{B}_1 \oplus \cdots \oplus \widetilde{B}_m \;=\; \begin{bmatrix} a & s_1 \\ \kappa s_1 & -a \end{bmatrix} \oplus \ldots \oplus \begin{bmatrix} a & s_m \\ \kappa s_m & -a \end{bmatrix},$$

and, under the same permutation, $\widetilde{J}$ is permutationally similar to

$$J_1 \oplus \cdots \oplus J_1 \;=\; \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \oplus \ldots \oplus \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

that is, to J. Hence, $W^J(T) = W^{\widetilde{J}}(\widetilde{B})$ is the pseudo-convex hull of $W^{J_1}(\widetilde{B}_j)$ for $j = 1, \ldots, m$.

Let $\gamma = \arg(\kappa)$. For each $\theta \in \mathbb{R}$, the eigenvalues $\lambda_{\pm,j}(\theta)$ of the matrix $H_\theta(\widetilde{B}_j)$ are such that

$$\lambda_{\pm,j}^2(\theta) = a^2 \cos^2\theta - \frac{1}{4} s_j^2 \left(1 + |\kappa|^2 - 2|\kappa| \cos(\gamma - 2\theta)\right).$$

By the Hyperbolical Range Theorem (cf. Theorem 2.4), the set $W^{J_1}(\widetilde{B}_1)$ is bounded by a non-degenerate hyperbola, namely $\mathcal{H}_1$, if and only if $M_1 > 0$ and $N_1 > 0$. In this case, $a \neq 0$, $\kappa \neq -1$ and we clearly have

$$1 - 2|\kappa| \cos(\gamma - 2\theta) + |\kappa|^2 \geq (1 - |\kappa|)^2 > 0,$$

so we may conclude that

$$0 < \lambda_{\pm,1}^2(\theta) \leq \lambda_{\pm,2}^2(\theta) \leq \cdots \leq \lambda_{\pm,m}^2(\theta),$$

for $\theta \in (\phi_1 - \theta_1, \phi_1 + \theta_1)$ with $\theta_1 = \arctan(N_1/M_1)^{\frac{1}{2}}$ and $\phi_j = \frac{1}{2}\arg(a^2 + ks_j^2)$. For $j = 1, \dots, r$, we remark that $s_j > 0$ implies

$$-M_j = -4\lambda_{\pm,j}^2\left(\frac{\pi}{2} + \phi_j\right) < 0 < 4\lambda_{\pm,j}^2\left(\phi_j\right) = N_j,$$

whenever $M_1 > 0$ and $N_1 > 0$. We conclude that we have a collection of r non-degenerate nested hyperbolas, namely $\mathcal{H}_1, \dots, \mathcal{H}_r$, some possibly coincident, all centered at the origin, defining the boundary generating curves of the sets $W^{J_1}(\widetilde{B}_j)$ for $j = 1, \dots, r$. If $r < m$, then $s_{r+1} = 0$ and $\widetilde{B}_j = \widetilde{B}_{r+1}$, $j > r$, is a real diagonal matrix and the main diagonal entries $a, -a$ of T, which clearly belong to $W^J(T)$, are the corners and form the boundary generating curve of $W^{J_1}(\widetilde{B}_j)$, $j > r$.

We may conclude that the outer hyperbola $\mathcal{H}_1$ defines the hyperbolical boundary of $W^J(T)$ if and only if $N_1 > 0$ and $M_1 > 0$. Clearly, we have $M_1 > 0$ and $N_1 > 0$ if and only (31) holds.

In the case $n = 2m + 1$, $m \in \mathbb{N}$, let $P_\tau \in M_n$ be the permutation matrix associated with the permutation $\tau \in S_n$ defined in the following way:

$$\tau(i) = 2i - 1, \quad 1 \le i \le m+1, \qquad \text{and} \qquad \tau(m+1+i) = 2i, \quad 1 \le i \le m.$$

Now, let $\check{J} = I_{m+1} \oplus -I_m$. By easy computations, we have $P_\tau J P_\tau^T = \check{J}$ and

$$\check{A} = P_\tau T P_\tau^T = \begin{bmatrix} aI_{m+1} & X \\ \kappa X^* & -aI_m \end{bmatrix},$$

for $X = X(\mathbf{b}) \in M_{(m+1)\times m}$, with $\mathbf{b} = (b_1, b_2, b_3, \dots) \in \mathbb{C}^{2m}$, a bidiagonal matrix defined as in (28). By the singular value decomposition, there exist unitary matrices $U \in M_{m+1}$ and $V \in M_m$, such that the block X of $\check{A}$ is given by $X = U\,\check{D}\,V^*$, where $\check{D} \in M_{(m+1)\times m}$ contains the diagonal matrix of the singular values $s_1 \ge s_2 \ge \cdots \ge s_m$ of X and has the last row of zeros. As above, we may conclude that $\check{A}$ and $\check{J}$ are permutationallly similar to

$$\widetilde{B}_1 \oplus \cdots \oplus \widetilde{B}_m \oplus \left[a\right] \qquad \text{and} \qquad J_1 \oplus \cdots \oplus J_1 \oplus \left[1\right],$$

respectively. The proof follows now analogous steps as the even case, also noting now that $a \in W_+^{J_1}(\widetilde{B}_1)$. Thus, the boundary of $W^J(T)$ is the hyperbola $\mathcal{H}_1$ if and only (31) holds. $\square$

Condition (31) for hyperbolicity of $W^J(T)$ does not hold when $\kappa = -1$, in this case, the tridiagonal matrix T defined as in Theorem 3.1 becomes J-Hermitian and $W^J(T)$, as the union of two rays, is characterized below.

Corollary 3.2. *Under the conditions of Theorem 3.1, we have*

$$W^J(T) = \left(-\infty, -\widetilde{a}\right] \cup \left[\widetilde{a}, +\infty\right), \qquad \widetilde{a} > 0, \tag{32}$$

if and only if $M_1 = 0$ and $N_1 > 0$, equivalently, $|a| > s_1$ and either $\kappa = -1$ or $s_1 = 0$.

Proof. Analogousgly to the proof of Theorem 3.1, $W^J(T)$ is the pseudo-convex hull of $W^{J_1}(\widetilde{B}_j)$, $j = 1, \ldots, \lfloor \frac{n}{2} \rfloor$, where $J_1 = \mathrm{diag}(1, -1)$ and

$$\widetilde{B}_j = \begin{bmatrix} a & s_j \\ \kappa s_j & -a \end{bmatrix},$$

with $s_1 \geq \cdots \geq s_{\lfloor \frac{n}{2} \rfloor}$ the singular values of the bidiagonal matrix $X(\mathbf{b})$.

($\Leftarrow$) By hypothesis, $M_1 = 0$ and $N_1 > 0$, equivalently, $|a| > s_1$ and either $\kappa = -1$ or $s_1 = 0$. For $j = 1, \ldots, \lfloor \frac{n}{2} \rfloor$, considering $\widetilde{a}_j = (a^2 - s_j^2)^{\frac{1}{2}}$, from $|a| > s_1$, we have $0 < \widetilde{a}_1 \leq \widetilde{a}_j$ and, using Corollary 2.5, we get

$$W^{J_1}(\widetilde{B}_1) \supseteq W^{J_1}(\widetilde{B}_j) = \big(-\infty, -\widetilde{a}_j\big] \cup \big[\widetilde{a}_j, +\infty\big) \supseteq \{a\}.$$

Thus, $W^J(T) = W^{J_1}(\widetilde{B}_1)$ and (32) holds with $\widetilde{a} = \widetilde{a}_1$.

($\Rightarrow$) Suppose $W^J(T)$ is the disjoint union of two closed real half-lines. In particular, we have

$$W^{J_1}(\widetilde{B}_1) \subseteq W^J(T)$$

and $\widetilde{B}_1$ has real spectrum. Thus, $\arg(a^2 + \kappa s_1^2) = 0$ and $H_\theta(\widetilde{B}_1) \in \mathcal{J}$, for all $\theta \in (-\pi/2, \pi/2)$. If $N_1 \leq 0$, by Proposition 2.3, we would have $H_\theta(\widetilde{B}_1) \notin \mathcal{J}$, which is a contradiction. Hence, $N_1 > 0$. If $M_1 > 0$, by Theorem 2.4, $W^J(\widetilde{B}_1)$ would be bounded by a non-degenerate hyperbola, another contradiction. Then $M_1 = 0$ and $N_1 > 0$. □

Corollary 3.3. *Under the conditions of Theorem 3.1, let s be the operator norm of the bidiagonal matrix $X(\mathrm{e}^{i \arg b_1}, \ldots, \mathrm{e}^{i \arg b_{n-1}})$, with*

$$|b_j| + |c_j| = |\alpha| + |\beta| \qquad \text{and} \qquad b_j c_j = \alpha\beta, \qquad j = 1, \ldots, n-1, \tag{33}$$

for some $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. Considering

$$M = 2|a^2 + \alpha\beta s^2| - 2a^2 + (|\alpha|^2 + |\beta|^2)s^2,$$

$$N = 2|a^2 + \alpha\beta s^2| + 2a^2 - (|\alpha|^2 + |\beta|^2)s^2,$$

the set $W^J(T)$ is bounded by the non-degenerate hyperbola, centered at the origin, with foci at

$$\pm \left(a^2 + \alpha\beta s^2\right)^{\frac{1}{2}},$$

transverse and non-transverse axes of lengths $N^{\frac{1}{2}}$ and $M^{\frac{1}{2}}$, respectively, if and only if

$$\left| a^2 - \frac{|\alpha|^2 + |\beta|^2}{2} s^2 \right| < |a^2 + \alpha\beta s^2|.$$

Proof. Under the hypothesis (33), we may conclude that either $|b_j| = |\alpha|$ and $|c_j| = |\beta|$ or $|b_j| = |\beta|$ and $|c_j| = |\alpha|$, for each $j = 1, \ldots, n-1$. Hence, recalling

[13, Lemma 2.1], we may consider T of type $T(n; \mathbf{c}, \mathbf{a}, \mathbf{b})$ with

$$\mathbf{b} = |\alpha|(\mathrm{e}^{i \arg b_1}, \ldots, \mathrm{e}^{i \arg b_{n-1}}), \qquad \mathbf{c} = |\beta|(\mathrm{e}^{i \arg c_1}, \ldots, \mathrm{e}^{i \arg c_{n-1}}).$$

Moreover, from $b_j c_j = \alpha\beta$, we have

$$\arg(c_j) - \arg(\alpha) = \arg(\beta) - \arg(b_j) + 2k\pi, \quad k \in \mathbb{Z},$$

for each $j = 1, \ldots, n-1$. Therefore, $\overline{\alpha}\,\mathbf{c} = \beta\overline{\mathbf{b}}$. Since $\alpha \neq 0$, we have $\mathbf{c} = \kappa\,\overline{\mathbf{b}}$, with $\kappa = \beta/\overline{\alpha}$. The result follows now by Theorem 3.1. Let s_1 be the operator norm of the bidiagonal matrix

$$X(\mathbf{b}) = |\alpha|\, X(\mathrm{e}^{i \arg b_1}, \ldots, \mathrm{e}^{i \arg b_{n-1}}).$$

Clearly, $s_1 = |\alpha| s$ gives $\kappa s_1^2 = \alpha\beta s$ and $(1 + |\kappa|^2)s_1^2 = (|\alpha|^2 + |\beta|^2)s^2$. □

Lemma 3.4. *Let* $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^{n-1}$, $n \geq 2$. *The singular values of the bidiagonal matrix* $X(\mathbf{1})$ *are*

$$s_k = 2\cos\frac{k\pi}{n+1}, \qquad k = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor.$$

Proof. If $n = 2m$, $m \in \mathbb{N}$, then $X = X(\mathbf{1}) \in M_m$. If $n = 2m+1$, $m \in \mathbb{N}$, then $X = X(\mathbf{1}) \in M_{m+1,m}$. Let $p = \left\lfloor \frac{n}{2} \right\rfloor$. We can see that

$$X^*X = T(p; \tilde{\mathbf{1}}, \mathbf{a}, \tilde{\mathbf{1}}), \quad \tilde{\mathbf{1}} = (1, \ldots, 1) \in \mathbb{R}^{p-1}, \quad \mathbf{a} = (2, \ldots, 2, 2-\delta) \in \mathbb{R}^p,$$

where $\delta = 1$ if n is even or $\delta = 0$ if n is odd. If n is even, then X^*X is a 'perturbed' Toeplitz matrix, with eigenvalues given [14, Theorem 1] by

$$\lambda_k = 2 + 2\cos\left(\frac{2k\pi}{2p+1}\right), \qquad k = 1, \ldots, p.$$

If n is odd, then X^*X becomes a Toeplitz matrix with eigenvalues given by

$$\lambda_k = 2 + 2\cos\left(\frac{k\pi}{p+1}\right), \qquad k = 1, \ldots, p.$$

Thus, we may write

$$\lambda_k = 2 + 2\cos\left(\frac{2k\pi}{n+1}\right) = 4\cos^2\frac{k\pi}{n+1}, \qquad k = 1, \ldots, p,$$

and the singular values of X are as stated. □

Corollary 3.5. *Under the conditions of Theorem 3.1, let* $\mathbf{c} = \mathbf{0}$ *and* s_1 *be the operator norm of the bidiagonal matrix* $X(\mathbf{b})$. *The set* $W^J(T)$ *is bounded by the hyperbola, centered at the origin, with foci at the main diagonal entries* $a, -a$ *of* T

and non-transverse axis of length s_1 *if and only if* $0 < s_1 < 2|a|$. *If, in addition* $\mathbf{b} = (b, b, \ldots, b) \in \mathbb{C}^{n-1} \setminus \{\mathbf{0}\}$, *then* $W^J(T)$ *has hyperbolical boundary if and only if*

$$\cos\frac{\pi}{n+1} < \left|\frac{a}{b}\right|.$$

Proof. Let $\kappa = 0$ in Theorem 3.1. In addition, if b is a non-zero constant vector, then use Lemma 3.4 too. □

Now, we consider the hyperbolical shape for the J-numerical range of a tridiagonal J-Toeplitz matrix, that is, a tridiagonal matrix T such that JT is a Toeplitz matrix.

Corollary 3.6. *Let* $J = \operatorname{diag}(1, -1, 1, -1, \ldots) \in M_n$, $n \geq 2$, $a \in \mathbb{R}$, $b, c \in \mathbb{C}$, $b \neq 0$ *and consider*

$$d_{k,n} = 4\cos^2\frac{k\pi}{n+1}, \qquad \Delta_{k,n} = a^2 + b\,c\,d_{k,n}, \qquad k = 1, \ldots, \left\lfloor\frac{n}{2}\right\rfloor.$$

The J*-numerical range of a* J*-Toeplitz matrix* T *of type* $T(n; \mathbf{c}, \mathbf{a}, \mathbf{b})$, *with*

$$\mathbf{a} = (a, -a, a, -a, \ldots), \qquad \mathbf{b} = (b, b, \ldots, b), \qquad \mathbf{c} = (c, c, \ldots, c),$$

is bounded by the non-degenerate hyperbola, with foci at $\pm\Delta_{1,n}^{\frac{1}{2}}$, *transverse and non-transverse axes of lengths*

$$\left(2|\Delta_{1,n}| + 2a^2 - (|b|^2 + |c|^2)d_{1,n}\right)^{\frac{1}{2}}, \qquad \left(2|\Delta_{1,n}| - 2a^2 + (|b|^2 + |c|^2)d_{1,n}\right)^{\frac{1}{2}}$$

respectively, if and only if

$$\left|a^2 - \frac{|b|^2 + |c|^2}{2}d_{1,n}\right| < |\Delta_{1,n}|.$$

Under these conditions, the boundary generating curves of $W^J(T)$ *are* $\lfloor\frac{n}{2}\rfloor$ *nested hyperbolas, with foci at* $\pm\Delta_{k,n}^{\frac{1}{2}}$, *transverse and non-transverse axes of lengths*

$$\left(2|\Delta_{k,n}| + 2a^2 - (|b|^2 + |c|^2)d_{k,n}\right)^{\frac{1}{2}}, \qquad \left(2|\Delta_{k,n}| - 2a^2 + (|b|^2 + |c|^2)d_{k,n}\right)^{\frac{1}{2}}$$

respectively, $k = 1, \ldots, \lfloor\frac{n}{2}\rfloor$, *and the point* a, *when* n *is odd.*

Proof. Under the hypothesis, since $b \neq 0$, then there exists $\kappa = c/\overline{b} \in \mathbb{C}$, such that $\mathbf{c} = \kappa\overline{\mathbf{b}}$. It follows from Lemma 3.4 that the singular values of the bidiagonal matrix $X(\mathbf{b})$ are all non-zero, given by

$$2|b|\cos\frac{k\pi}{n+1}, \qquad k = 1, \ldots, \left\lfloor\frac{n}{2}\right\rfloor.$$

By Theorem 3.1, the result follows. □

The following two examples illustrate Theorem 3.1 and Corollary 3.6.

Example 3.7. Let $J = \mathrm{diag}(1, -1, 1, -1, 1, -1)$ and consider the J-Toeplitz tridiagonal matrix T given by

$$T(6; \mathbf{c}, \mathbf{a}, \mathbf{b}) = \begin{bmatrix} 2 & 3 & 0 & 0 & 0 & 0 \\ 1 & -2 & 3 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix},$$

with

$$\mathbf{a} = (2, -2, 2, -2, 2, -2), \qquad \mathbf{b} = (3, 3, 3, 3, 3)\,, \qquad \mathbf{c} = (1, 1, 1, 1, 1)\,.$$

By Corollary 3.6, the boundary generating curves of $W^J(T)$ are three hyperbolas, centered at the origin, with foci at the points

$$\pm 2\sqrt{1 + 3\cos^2(k\pi/7)}, \qquad k = 1, 2, 3,$$

and with horizontal transverse axes, as shown in Figure 1. The lenghts of the transverse and the non-transverse axes are, respectively, given by

$$4\sin(k\pi/7) \qquad \text{and} \qquad 8\cos(k\pi/7), \qquad k = 1, 2, 3.$$

The set $W^J(T)$ is bounded by the outer hyperbola, with cartesian equation

$$\frac{x^2}{4\sin^2(\pi/7)} - \frac{y^2}{16\cos^2(\pi/7)} = 1,$$

whose foci, lenghts of the transverse and non-transverse axes are given, approximately, by ± 3.70688, 1.73553 and 7.20775, respectively.

Figure 1.

Example 3.8. Let $J = \mathrm{diag}(1, -1, 1, -1, 1, -1, 1, -1)$ and

$$T = \begin{bmatrix} 2 & i & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 2i & 2 & 1+i & 0 & 0 & 0 \\ 0 & 0 & 1+i & -2 & 3i & 0 & 0 \\ 0 & 0 & 0 & 3 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2i & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & i & 2 \end{bmatrix},$$

that is, T is a tridiagonal matrix of type $T(7; \mathbf{c}, \mathbf{a}, \mathbf{b})$ with

$$\mathbf{a} = (2, -2, 2, -2, 2, -2, 2),$$

$$\mathbf{b} = (i, 2, i+1, 3i, 2, 1)\,, \qquad \mathbf{c} = (1, 2i, 1+i, 3, 2i, i)\,.$$

According to Theorem 3.1 with $\kappa = i$ and its proof, we are considering

$$P_\pi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad \text{and} \qquad X = \begin{bmatrix} i & 0 & 0 \\ 2 & i+1 & 0 \\ 0 & 3i & 2 \\ 0 & 0 & 1 \end{bmatrix},$$

in order that

$$\widetilde{A} = P_\pi T P_\pi^T = \begin{bmatrix} 2I_4 & X \\ iX^* & -2I_3 \end{bmatrix} \qquad \text{and} \qquad \widetilde{J} = I_4 \oplus -I_3,$$

in which case $W^J(T) = W^{\widetilde{J}}(\widetilde{A})$. The singular values of X are

$$s_1 = (8+\sqrt{53})^{\frac{1}{2}}, \qquad s_2 = \sqrt{5}, \qquad s_3 = (8-\sqrt{53})^{\frac{1}{2}}.$$

The set $W^J(T)$ is bounded by the hyperbola centered at the origin, with foci at

$$\pm\left(4 + i(8+\sqrt{53})\right)^{\frac{1}{2}},$$

transverse and non-transverse axes given by

$$\sqrt{2}\left(\sqrt{16\sqrt{53}+133} - 4 - \sqrt{53}\right)^{\frac{1}{2}}, \qquad \sqrt{2}\left(\sqrt{16\sqrt{53}+133} + 4 + \sqrt{53}\right)^{\frac{1}{2}},$$

respectively. In this case, the boundary generating curves of $W^J(T)$ are three non-degenerate hyperbolas, with transverse axes not collinear, and the point 2, as shown in Figure 2.

Figure 2.

4. Final notes

We have investigated classe of matrices with hyperbolical indefinite numerical range. We have imposed some restrictions in order to ensure non-degeneracy of these hyperbolas. We have also studied the degeneracy of the indefinite numerical range to two half-lines. Our approach also allows to obtain conclusions, concerning degeneracy of other cases. In view of the proof of Theorem 3.1, this discussion relies on the 2×2 case, already well studied (see e.g. [5,15]).

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