Starting with the Differential: Representation of Monogenic Functions by Polynomials of Non-monogenic Variables

H. R. Malonek[1,a)], I. Cação[1], M. I. Falcão[2] and G. Tomaz[1,3]

[1]*Center for Research and Development in Mathematics and Applications (CIDMA)*
Department of Mathematics, University of Aveiro, 3810–193 Aveiro, Portugal

[2]*Centre of Mathematics, University of Minho, 4710-057 Braga, Portugal*

[3] *Research Unit for Inland Development, Polytechnic of Guarda,*
6300–559 Guarda, Portugal

[a)]Corresponding author: hrmalon@ua.pt

Abstract. This paper deals with different power series expansions of generalized holomorphic (monogenic) functions in the setting of Clifford Analysis. Our main concern are generalized Appell polynomials as a special class of monogenic polynomials which have been introduced in 2006 by two of the authors using several monogenic hypercomplex variables. We clarify the reasons why a particular pair of non-monogenic variables allows to obtain a power series expansion by those generalized Appell polynomials. The approach is based on the differential of a function. Some other monogenic polynomials as well as applications are mentioned.

Introduction

Higher-dimensional analysis relying on non-commutative Clifford algebra tools is frequently called *Clifford Analysis*. Besides work of Weierstrass at the end of the 19th century about functions of two complex variables or others about functions of quaternions, the time for such methods matured only at the end of the second decade of the 20th century. The incubation time ended mainly due to research done by R. Fueter [14] from the 30ies on. Almost during 20 years he developed with his pupils a theory of quaternionic functions of a quaternion variable as the theory of general-ized Cauchy-Riemann or Dirac differential equations. Naturally, solutions of those systems have been considered as (hypercomplex) *regular or generalized holomorphic functions.* The book [6] favored the equivalent name *monogenic functions* that we also use. Two decades after Fueter, a great part of Fueter's work was renewed or actualized by following Riemann's approach through partial differential equations, a long time thought as the only one reasonable. Nevertheless, it led very quickly to a generalized function theory as refinement of Harmonic Analysis. But this type of function theory heavily relies on representation theoretic and algebraic tools, and much less on instruments from classical complex function theory. For a long time applications, mainly based on polynomials, to number theory (the main motivation for Fueter himself) or methods related to combinatorial questions were almost neglected. This drawback substantially restricted the class of functions useful for a treatment in more analytically oriented research. The main reason for such a situation was that *powers of one hypercomplex variable* (and corresponding polynomials or power series) *do not belong to the set of monogenic functions* in the sense of Fueter. In the 90ies the papers [16, 17, 18] contributed to a radical change of this perspective, describing the same class of regular functions by using several hypercomplex variables (Fueter variables) as a new starting point for a hypercomplex function theory. The papers also clarified the fact that differentiability as property of local linear approximation and derivability (the existence of a hypercomplex derivative) are, contrary to the complex case $n = 1$, dual and have to be considered for $n > 1$ in hypercomplex dimension one resp. in co-dimension one of $\mathbb{R}^{n+1}$ [20]. More about this approach and its applications the reader can find in the recent paper [2].

Starting with the differential we follow now the approach in [16, 17, 18] and consider monogenic functions and their power series representation in general as well as in terms of generalized Appell polynomials [11]. They can be found in 3D-quasi-conformal mappings [19], new analytic methods in elasticity [4, 5], or the construction of generalized Hermite, Laguerre [7], Chebyshev [8], as well as Bernoulli, Euler and other polynomials [1].

Basic Notations

For an independent reading, let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^n$ with the multiplication rules $e_k e_l + e_l e_k = -2\delta_{kl},\ k, l = 1, \cdots, n$, where δ_{kl} is the Kronecker symbol. The set $\{e_A : A \subseteq \{1, \cdots, n\}\}$ with

$$e_A = e_{h_1} e_{h_2} \cdots e_{h_r},\ 1 \le h_1 < \cdots < h_r \le n,\ e_\emptyset = e_0 = 1,$$

forms a basis of the 2^n-dimensional Clifford algebra $C\ell_{0,n}$ over $\mathbb{R}$. Let $x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$. We consider functions of the form $f(x) = \sum_A f_A(x) e_A$, where $f_A(x)$ are real valued, i.e. $C\ell_{0,n}$-valued functions f defined in some open subset $\Omega \subset \mathbb{R}^{n+1}$. In particular, the embedding of $\mathbb{R}^{n+1}$ in the 2^n-dimensional Clifford algebra $C\ell_{0,n}$ over $\mathbb{R}$ with $\mathcal{A}_n \subset C\ell_{0,n}$ is realized by considering $\mathcal{A}_n \cong \mathbb{R}^{n+1}$ with elements of the form $x = x_0 + e_1 x_1 + \cdots + e_n x_n = x_0 + \underline{x}$. Then $x \in \mathcal{A}_n$ is called a paravector. The *generalized Cauchy-Riemann operator* in $\mathbb{R}^{n+1},\ n \ge 1$, is defined by

$$\overline{\partial} := \tfrac{1}{2}(\partial_0 + \partial_{\underline{x}}), \quad \partial_0 := \frac{\partial}{\partial x_0}, \quad \partial_{\underline{x}} := e_1 \frac{\partial}{\partial x_1} + \cdots + e_n \frac{\partial}{\partial x_n}. \tag{1}$$

C^1-functions f satisfying the equation $\overline{\partial} f = 0$ (resp. $f\overline{\partial} = 0$) are called *left monogenic* (resp. *right monogenic*). This is true if f is *hypercomplex differentiable* in Ω in the sense of [16, 17], i.e. it has a uniquely defined areolar derivative f' in the sense of Pompeiu in each point of Ω . The hypercomplex (areolar) derivative f' of a monogenic function can be obtained as $f' = \partial f = \frac{1}{2}(\partial_0 - \partial_{\underline{x}}) f$ where $\partial := \frac{1}{2}(\partial_0 - \partial_{\underline{x}})$ is just the *conjugate generalized Cauchy-Riemann operator.* The guaranteed existence of the hypercomplex derivative for a monogenic function is vital for the definition of an Appell polynomial sequence. Moreover, like in complex function theory, the (hypercomplex) differentiability of a given function f is equivalent to f being monogenic. Derivating the function $f(x) = x$ we get $\overline{\partial} x = x\overline{\partial} = \frac{1}{2}(\partial_0 x_0 + \partial_{\underline{x}} \underline{x}) = \frac{1}{2}(1 - n)$. This shows that for $n \ge 2$ *the paravector x is not monogenic.* Furthermore, the application of the conjugate operator results in $\partial x = x\partial = \frac{1}{2}(\partial_0 x_0 - \partial_{\underline{x}} \underline{x}) = \frac{1}{2}(1 + n)$. Both formulae together, indicate that only the particular complex case $n = 1$ as specification of the general hypercomplex case gives the desired result, i.e. the variable x itself is a monogenic function with its derivative equal to 1. In other words, the identity function does not belong to the set of monogenic functions if $n \ge 2$.

The Differential of a Monogenic Function of Several Monogenic Variables

Let us introduce the $n + 1$ hypercomplex variables

$$z_k := x_k - e_k x_0; \quad k = 1, \ldots, n, \quad \bar{z} := x_0 - e_1 x_1 - \cdots - e_n x_n. \tag{2}$$

This choice defines - easy to verify by using (1) - n monogenic variables z_k and one non-monogenic variable $\bar{z} = \bar{x}$. It extends the complex case with the usually applied $z = x + iy$ as holomorphic variable resp. $\bar{z} = x - iy$ as non-holomorphic variable. In fact, the use of $z_1 = -iz = y - ix$ instead of z corresponds only to a change of the real resp. imaginary axis of the complex plane. Considering now $f : \mathbb{R}^{n+1} \longrightarrow \mathcal{A}_n$ and the corresponding hypercomplex differentials dz_k and $d\bar{z}$, then the usual differential of $f \in C^1(\mathbb{R}^{n+1}, \Omega)$, i.e. $df = \frac{\partial f}{\partial x_0} dx_0 + \frac{\partial f}{\partial x_1} dx_1 \cdots + \frac{\partial f}{\partial x_n} dx_n$ is transformed into the hypercomplex differential of a right monogenic function f, which because of $f\overline{\partial} = 0$ reduces to

$$df = \left(\frac{2}{n+1} f\overline{\partial}\right) d\bar{z} + \left(\frac{\partial f}{\partial x_1} + \frac{2}{n+1} f\overline{\partial}\right) dz_1 + \cdots + \left(\frac{\partial f}{\partial x_n} + \frac{2}{n+1} f\overline{\partial}\right) dz_n = \frac{\partial f}{\partial x_1} dz_1 + \cdots + \frac{\partial f}{\partial x_n} dz_n. \tag{3}$$

Remark. The final form of the differential shows that the particular choice (2) implies that a monogenic function depends only on n- monogenic variables corresponding to co-dimension 1 of $\mathbb{R}^{n+1}$, but not on $\bar{z}$. In this sense they behave like holomorphic functions of one complex variable z not depending on $\bar{z}$. Like complex holomorphic functions they are infinitely differentiable and admit locally their expansion in Taylor series. The differential (3) implies, analogously to the ordinary real multivariate calculus and its conventions, the following result (cf. [18]).

Proposition 1 *The Taylor series expansion of a (right) monogenic function f in terms of n hypercomplex variables around the origin and ordered by powers of the same homogeneous degree k has the form*

$$f(z_1, \ldots, z_n) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\partial}{\partial x_1} z_1 + \frac{\partial}{\partial x_2} z_2 + \cdots + \frac{\partial}{\partial x_m} z_m \right)^k f(0) = \sum_{|\nu|=0} \frac{1}{\nu!} \frac{\partial^{|\nu|} f(0)}{\partial x_1{}^{\nu_1} \cdots \partial x_n{}^{\nu_n}}\ z_1^{\nu_1} \times \cdots \times z_n^{\nu_n}. \tag{4}$$

with the usual multi-index $\nu = (\nu_1, \ldots, \nu_n)$; $|\nu| = \nu_1 + \cdots + \nu_n$; $\nu! = \nu_1! \cdots \nu_n!$. The symbol " $\times$ " is based on an additional convention about an n-nary symmetric product symbolized by " $\times$ " of the hypercomplex variables so that for any multi-index ν all functions $z_1^{\nu_1} \times \cdots \times z_n^{\nu_n}$ are separable and monogenic.

The Differential of a Monogenic Function of Several Non-monogenic Variables

Let $J_i(e_i) := -e_i$, $J_i(e_k) := e_k$, for $k \neq i$, $J_0 = \mathrm{id}$, be linear mappings $J_k : C\ell_{0,n} \longrightarrow C\ell_{0,n}, k = 0, 1, \ldots, n$, then a set of $n+1$ non-monogenic hypercomplex variables can be defined by

$$\tilde{z}_k := J_k(J_{k-1}(\cdots(J_0(z)))) = J_{k,k-1,\ldots,0}(z), \quad k = 0, 1, \ldots, n. \tag{5}$$

Any real linear mapping $\mathcal{L}$ from $\mathcal{A}$ to $\mathcal{A}$ (such as the differential df) may be represented by a linear combination $\mathcal{L}(z) = \sum_k \tilde{c}_k \tilde{z}_k$, (cf. [15]). This means that formally and after a very tedious determination of the gradient

$$\nabla_f := (\partial_{\tilde{z}_0}, \cdots, \partial_{\tilde{z}_n}) \tag{6}$$

corresponding to the chosen set of variables, one would get in analogy to (4)

$$f(\tilde{z}_0, \tilde{z}_1, \ldots, \tilde{z}_n) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\partial}{\partial \tilde{z}_0} \tilde{z}_0 + \frac{\partial}{\partial \tilde{z}_1} \tilde{z}_1 + \cdots + \frac{\partial}{\tilde{z}_n} \tilde{z}_n \right)^k f(0). \tag{7}$$

But as we can see, the Taylor series expansion (7) is *not of some reduced hypercomplex form* since it is an expansion with respect to $n+1$ *non- monogenic variables* and at the same time with *non-commutative partial derivatives*. Nevertheless, we can find in (5) a pair of two non-monogenic variables with the algebraic advantage of being commutative, namely $\tilde{z}_0 = x$ and $\tilde{z}_n = \bar{x}$. Corresponding operations in the differential $df = \frac{\partial f}{\partial x_0} dx_0 + \frac{\partial f}{\partial x_1} dx_1 \cdots + \frac{\partial f}{\partial x_n} dx_n$, taking into account that $f\overline{\partial} = 0$, lead to the following result.

Proposition 2 *The differential of a monogenic function $f = f(x, \bar{x})$ has the form*

$$df = \partial_x f dx + \partial_{\bar{x}} f d\bar{x},$$

with the corresponding hypercomplex gradient $\nabla_f = (\partial_x, \partial_{\bar{x}}) = \left(\frac{1}{2}(\partial_0 - \frac{1}{n}\partial_{\underline{x}}), \frac{1}{2}(\partial_0 + \frac{1}{n}\partial_{\underline{x}})\right)$. Moreover, the Taylor series expansion of a paravector valued function f as function of the paravector x and its conjugate $\bar{x}$ is given by

$$f(x, \bar{x}) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\partial}{\partial x} x + \frac{\partial}{\partial \bar{x}} \bar{x} \right)^k f(0) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{s=0}^{k} \binom{k}{s} \frac{\partial^k f(0)}{\partial x^{k-s} \partial \bar{x}^s} \, x^{k-s} \bar{x}^s = \sum_{k=0}^{\infty} \sum_{s=0}^{k} \frac{1}{(k-s)!s!} \frac{\partial^k f(0)}{\partial x^{k-s} \partial \bar{x}^s} \, x^{k-s} \bar{x}^s. \tag{8}$$

Since $\partial_{\underline{x}} \underline{x} = -n$ it is easy to see that $\partial_x x = \partial_{\bar{x}} \bar{x} = 1$ and $\partial_x \bar{x} = \partial_{\bar{x}} x = 0$. Since $\partial_0 x_0 = 1$ and $\partial_0 \underline{x} = 0$, the action of the hypercomplex gradient on the paravector-functions x and $\bar{x}$ implies $(\partial_x, \partial_{\bar{x}})x = 1$ and $(\partial_x, \partial_{\bar{x}})\bar{x} = 1$. It is evident that these relations are the essential tools for a direct differential calculus adapted to monogenic functions $f = f(x, \bar{x})$ given in terms of the hypercomplex mutual conjugated variables x and $\bar{x}$ which for $n \geq 2$, contrary to the complex case, are mutual independent. Like in the case of two independent real variables the gradient plays the role of the derivative. Indeed, it is easy to verify that this is in agreement with the hypercomplex derivative obtained simply by $f' = \partial_0 f$. But the last expression in formula (8) is nothing else than the expansion of the given function f in a series of polynomials of the form

$$\mathcal{P}_k^n(x, \bar{x}) = \sum_{s=0}^{k} T_s^k(n) \, x^{k-s} \, \bar{x}^s, \quad k = 1, 2, \ldots. \tag{9}$$

Such polynomials with

$$T_s^k = \binom{k}{s} \frac{\left(\frac{n+1}{2}\right)_{k-s} \left(\frac{n-1}{2}\right)_s}{(n)_k} \tag{10}$$

form a generalized Appell sequence (cf. [3, 19]) of hypercomplex monogenic polynomials associated to the hypercomplex derivative ∂. With $a = 1, b = \frac{n+1}{2}, b' = \frac{n-1}{2}, c = n, u = x$, and $v = \bar{x}$, the corresponding function $F = F(x, \bar{x})$ in (8) is recognized as Appell's function F_1 of two variables (see [21], § 139)

$$F_1(a; b, b'; c; x, \bar{x}) = \sum_{r,t=0}^{\infty} \frac{(a)_{r+t} (b)_t (b')_r u^t v^r}{t! r! (c)_{r+t}} = \sum_{k=0}^{\infty} \sum_{s=0}^{k} \binom{k}{s} \frac{\left(\frac{n+1}{2}\right)_{k-s} \left(\frac{n-1}{2}\right)_s}{(n)_k} \, x^{k-s} \bar{x}^s = \sum_{k=0}^{\infty} \mathcal{P}_k^n(x, \bar{x}). \tag{11}$$

Acknowledgment

This work was supported by Portuguese funds through the CMAT - Research Centre of Mathematics of University of Minho - and through the CIDMA-Center of Research and Development in Mathematics and Applications (University of Aveiro) and the Portuguese Foundation for Science and Technology ("FCT - Fundação para a Ciência e Tecnologia"), within projects UIDB/00013/2020, UIDP/00013/2020, UIDB/04106/2020 and UIDP/04106/2020.

REFERENCES

[1] L. Aceto, H. Malonek, and G.Tomaz , Matrix approach to hypercomplex Appell polynomials. *Appl. Numer. Math.* 116 (2017), 2–9.

[2] D. Alpay, I. L. Paiva, and D. Struppa , A general setting for functions of Fueter variables: differentiability, rational functions, Fock module and related topics, *Israel Journal of Mathematics*, 236, (2020), 207–246.

[3] P. Appell, Sur une classe de polynômes, *Ann. Sci. École Norm. Sup.* 9, (1880) no.2, 119–144.

[4] S. Bock and K. Gürlebeck, *Math. Methods Appl. Sci.,* 33 (2010), no. 4, 394–411.

[5] Bock S. On Monogenic Series Expansions with Applications to Linear Elasticity. *Adv. Appl. Clifford Algebras* 24, (2014), 931 - 943.

[6] F. Brackx, R.Delanghe, and F. Sommen, *Clifford analysis* . Pitman, Boston-London-Melbourne; 1982.

[7] I. Cação, M. I. Falcão, and H. R. Malonek, Laguerre derivative and monogenic Laguerre polynomials: An operational approach. *Math. Comput. Model.* 2011;53(5-6): 1084–1094.

[8] I. Cação and H. R. Malonek, On an hypercomplex generalization of Gould-Hopper and related Chebyshev polynomials. In: D. Murgante et al (eds) *Computational Science and Its Applications - ICCSA 2011, LN in Computer Science*, Springer-Verlag Berlin Heidelberg, 6784, (2011, 316–326.

[9] I. Cação, M. I. Falcão, and H. R. Malonek, Hypercomplex Polynomials, Vietoris' Rational Numbers and a Related Integer Numbers Sequence. *Complex Anal. Oper. Theory.* 11,(2017) no. 5, 1059–1076.

[10] I. Cação, M. I. Falcão, and H. R. Malonek, On generalized Vietoris' number sequences. *Discret. App. Math.* 296, (2019), 77–85.

[11] M.I. Falcão, J. Cruz, and H.R. Malonek, Remarks on the generation of monogenic functions, In: K. Gürlebeck and C. Könke, (eds.), *Proc. of the 17-th Inter. Conf. on the Application of Computer Science and Mathematics in Architecture and Civil Engineering, Bauhaus-University Weimar,* ISSN 1611- 4086, 2006.

[12] M. I. Falcão and H. R. Malonek, Generalized exponentials through Appell sets in $\mathbb{R}^{n+1}$ and Bessel functions.*AIP Conf. Proc.,* edited by T. E. Simos, G. Psihoyios, and C. Tsitouras, 936, (2007), 738–741.

[13] M. I. Falcão and H. R. Malonek, A note on a one-parameter family of non-symmetric number triangles. *Opuscula Mathematica.* 32 (2012) no.4,661–673.

[14] R. Fueter, Analytische Funktionen einer Quaternionenvariablen. *Comment. Math. Helv.* 4, (1932), 9–20.

[15] B. Goldschmidt, A theorem about the representation of linear combinations in Clifford algebras. *Beiträge Algebra Geom.* 13 (1982), 21-24.

[16] K. Gürlebeck, H. R. Malonek, A hypercomplex derivative of monogenic functions in $\mathbb{R}^{n+1}$ and its applications. *Complex Variables Theory Appl.* 39, (1999),199–228.

[17] H. Malonek, A new hypercomplex structure of the Euclidean space $\mathbb{R}^{m+1}$ and the concept of hypercomplex differentiability, *Complex Variables Theory Appl.*, 14, (1990), no. 4, 25–33

[18] H. Malonek, Power series representation for monogenic functions in $\mathbb{R}^{n+1}$ based on a permutational product. *Complex Variables, Theory Appl.* 15, (1990) no. 3, 181-191.

[19] H. R. Malonek and M. I. Falcão, 3D-mappings using monogenic functions, In: Th. E. Simos et al. (eds.) *ICNAAM-2006 Conference Proceedings,* Wiley-VCH, Weinheim, (2006), 615–619.

[20] H. R. Malonek, I. Cação, M. I. Falcão, and G. Tomaz, Harmonic Analysis and Hypercomplex Function Theory in Co-dimension One. In: A. Karapetyants, V. Kravchenko, E. Liflyand (eds) *Modern Methods in Operator Theory and Harmonic Analysis. OTHA 2018.* Springer, Cham. Springer Proceedings in Mathematics & Statistics. 291, (2019), 93–115.

[21] E.D. Rainville, *Special Functions*, Macmillan, New York, 1960.