Generalized Vietoris' Number Sequences from Real and Hypercomplex Points of View

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Abstract. We revisit a special rational number sequence, introduced by L. Vietoris in 1958 in the study of the positivity of some trigonometric sums and used in other contexts by several authors. The aim of the present paper is to embrace and explore real and hypercomplex analytical methods to obtain generalizations of that rational number sequence, where Jacobi polynomials and generalized Appell polynomials are involved.

Introduction

The famous paper of L. Vietoris [13] contains a result about positivity of certain trigonometric sums where a special sequence of rational numbers plays a crucial role. In Askey's version [2] this result is the following:

Theorem 1 (L. Vietoris)

$$\sum_{k=1}^{n} a_k \sin k\theta > 0, \quad 0 < \theta < \pi, \quad \textit{and} \quad \sum_{k=0}^{n} a_k \cos k\theta > 0, \quad 0 \leq \theta < \pi,$$

with

$$a_{2k} = a_{2k+1} = \frac{(\frac{1}{2})_k}{k!}, \qquad k = 0, 1, \ldots,$$

where $(.)_k$ *is the Pochhammer symbol.*

The sequence $(a_k)_{k\geq 0}$ is explicitly given by

$$1, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{3}{8}, \frac{5}{16}, \frac{5}{16}, \frac{35}{128}, \frac{35}{128}, \frac{63}{256}, \frac{63}{256}, \frac{231}{1024}, \frac{231}{1024}, \ldots. \tag{1}$$

Notice that the coefficients in the *sine* sum in Theorem 1 do not include the repetition of the first term 1, i.e, can be considered as terms of the index-shifted sequence $(c_k)_{k\geq 0}$, where $c_k = a_{k+1},\ k \geq 0$, or, explicitly

$$1, \frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{3}{8}, \frac{5}{16}, \frac{5}{16}, \frac{35}{128}, \frac{35}{128}, \frac{63}{256}, \frac{63}{256}, \frac{231}{1024}, \frac{231}{1024}, \ldots. \tag{2}$$

In this paper we focus our attention on this second sequence slightly different from the sequence (1) that Vietoris considered. The main reason is that the sequence (2) appears naturaly as coefficients of generalized Appell polynomials in the context of Clifford analysis.

The paper has two parts: in the first one, we generalize the sequence (2) by using only real analytic methods and, in the second part, we explain how those generalized sequences appear in the hypercomplex framework. Finally, we combine results in order to obtain more features associated with the constructed generalized Vietoris sequences.

A n-parameter generalization of the Vietoris sequence $(c_k)_{k\geq 0}$

The key point for a n-parameter generalization of the sequence $(c_k)_{k\geq 0}$ is the Taylor series of the rational function $g(t;\gamma;\delta) := \dfrac{1}{(1-t)^{\gamma}(1+t)^{\delta}}$ for particular values of γ and δ. This approach is motivated by Askey [2] that observed that the sequence (1) is related to the orthogonal expansion of the function g in terms of Jacobi polynomials $P_k^{(1/2,1/2)}(t)$, for $t = cos\,\theta$, $0 < \theta < \pi$, $\gamma = \frac{3}{4}$ and $\delta = \frac{1}{4}$.

A different expansion of g having the Vietoris numbers (2) as coefficients can be establish with the help of the ordinary (power series) generating function for the Vietoris sequence (2) that was achieved in [7]:

Theorem 2

$$F(t) := \frac{\sqrt{1+t}-\sqrt{1-t}}{t\sqrt{1-t}} = \sum_{k=0}^{\infty} c_k t^k,\ \ t \in]-1,0[\cup]0,1[. \tag{3}$$

By observing that $(tF(t))' = g(t;3/2;1/2)$, from (3) we obtain

$$g(t;3/2;1/2) = \sum_{k=0}^{\infty}(k+1)c_k t^k. \tag{4}$$

Notice that in this development of the function g, the numbers γ and δ are related by $\gamma+\delta = 2$ and $\gamma-\delta = 1$. The main goal of this section is to generalize the sequence $(c_k)_{k\geq 0}$ by generalizing (4) with the introduction of a parameter $n \in \mathbb{N}$ such that g is written in terms of those generalized $c_k(n)$, having in mind that $c_k(2) = c_k$[1]. In this way, it seems natural to consider in the expansion of g, γ and δ such that $\gamma+\delta = n$ and keeping $\gamma-\delta = 1$, i.e. consider $\gamma = \frac{n+1}{2}$ and $\delta = \frac{n-1}{2}$.

Theorem 3 *For $n \in \mathbb{N}$, we have*

$$\frac{1}{(1-t)^{\frac{n+1}{2}}(1+t)^{\frac{n-1}{2}}} = \sum_{k=0}^{\infty} \frac{(n)_k}{k!}\,\frac{(\frac{1}{2})_{\lfloor\frac{k+1}{2}\rfloor}}{(\frac{n}{2})_{\lfloor\frac{k+1}{2}\rfloor}} t^k,\ \ |t| < 1,$$

where $\lfloor . \rfloor$ stands for the floor function.

Proof The result follows from the Cauchy product of the series $\frac{1}{(1-y)^m} = \sum_{k=0}^{\infty}\frac{(m)_k}{k!}y^k,\ \ m \geq 0;\ \ |t| < 1$, for the cases $m = \frac{n+1}{2}$, $y = t$ and $m = \frac{n-1}{2}$, $y = -t$, respectively, and from the properties of the Pochhammer symbol. For details, see [5].

Definition 1 (cf. [5]) For each $n \in \mathbb{N}$, the n-parameter generalized Vietoris sequence $(c_k(n))_{k\geq 0}$ is defined by

$$c_k(n) = \frac{(\frac{1}{2})_{\lfloor\frac{k+1}{2}\rfloor}}{(\frac{n}{2})_{\lfloor\frac{k+1}{2}\rfloor}},\ \ \ k = 0,1,2,\ldots. \tag{5}$$

Notice that all the terms of this sequence, apart from the first, are pairwise repeated, i.e. $c_{2m}(n) = c_{2m-1}(n)$, $m = 1,2,\ldots$ and we get the sequence (2) for the particular case $n = 2$.

Methods of holonomic differential equations can be successfully used to obtain an exponential generating function of the n-parameter generalized Vietoris sequence defined above.

Theorem 4 *For each $n \in \mathbb{N}$, if*

$$F(t;n) = \sum_{k=0}^{\infty} c_k(n)\,\frac{t^k}{k!}$$

is an exponential generating function of the sequence $(c_k(n))_{k\geq 0}$, then $F(t,n)$ is the solution of the second order holonomic differential equation

$$tF''(t) + nF'(t) - (1+t)F(t) = 0$$

[1] Notice that, requiring $\gamma+\delta = 1$ and $\gamma-\delta = 1$ gives $g(t;1;0) = \frac{1}{1-t} = \sum_{k=0}^{\infty} t^k$ and $c_k(1) = 1$, for all $k \in \mathbb{N}_0$.

or, equivalently, of the Sturm-Liouville equation

$$(t^n F')' - t^{n-1}(1+t)F = 0$$

with the initial conditions

$$F(0) = 1, F'(0) = \frac{1}{n}.$$

For the proof of this theorem, we refer to [4]. The solution of the considered Sturm-Liouville initial value problem in terms of confluent hypergeometric functions is known (see [8]). More precisely,

$$F(t;n) = e^{-t} M\Big(\frac{n+1}{2}, n, 2t\Big),$$

where $M(a,b,z)$ is the Kummer's confluent hypergeometric function ${}_1F_1(a;b;z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!}$. It is well known that when $b-2a$ is a nonnegative integer (which is the case here), the Kummer function can be expressed in terms of modified Bessel functions of the first kind I_α. This means that F can be written as

$$F(t;n) = \Gamma\Big(\frac{n}{2}\Big)\Big(\frac{t}{2}\Big)^{1-\frac{n}{2}}\Big(I_{\frac{n}{2}-1}(t) + I_{\frac{n}{2}}(t)\Big). \tag{6}$$

The n-parameter generalized Vietoris sequence $(c_k)_{k\geq 0}$ from the hypercomplex point of view

We start by recalling some basic knowledge about Clifford analysis. Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^n$ endowed with the non-commutative product $e_i e_j + e_j e_i = -2\delta_{ij}$, $i, j = 1, 2, \ldots, n$, where δ_{ij} is the Kronecker symbol. A basis for the associative 2^n−dimensional Clifford algebra $C\ell_{0,n}$ over $\mathbb{R}$ is the set $\{e_A : A \subseteq \{1, \ldots, n\}\}$ formed by $e_A = e_{h_1} e_{h_2} \ldots e_{h_r}$, $1 \leq h_1 < \ldots < h_r \leq n$, $e_\emptyset = e_0 = 1$. In general, the vector space $\mathbb{R}^{n+1}$ is embedded in $C\ell_{0,n}$ by identifying $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ with $x = x_0 + \sum_{k=1}^n e_k x_k \in \mathcal{A}_n := \mathrm{span}_{\mathbb{R}}\{1, e_1, \ldots, e_n\} \subset C\ell_{0,n}$. The element $\underline{x} := \sum_{k=1}^n e_k x_k$ is called vector and $x = x_0 + \underline{x}$ is a paravector. The conjugate of x is $\bar{x} = x_0 - \underline{x}$ and its norm is given by $|x| = (x\bar{x})^{1/2} = (\bar{x}x)^{1/2} = \Big(\sum_{k=0}^n x_k^2\Big)^{1/2}$. We consider $C\ell_{0,n}$−valued functions defined in an open subset $\Omega \subseteq \mathbb{R}^{n+1} \cong \mathcal{A}_n$, i.e. functions of the form $f(z) = \sum_A f_A(z) e_A$ where $f_A(z)$ are real valued functions. The generalized Cauchy-Riemann operator in $\mathbb{R}^{n+1}$ is defined by $\overline{\partial} := \frac{1}{2}(\partial_0 + \partial_{\underline{x}})$, with $\partial_0 := \frac{\partial}{\partial x_0}$, and $\partial_{\underline{x}} := \sum_{k=1}^n e_k \frac{\partial}{\partial x_k}$. Its conjugate $\partial := \frac{1}{2}(\partial_0 - \partial_{\underline{x}})$ is also the hypercomplex derivative operator for *monogenic* (or *hyperholomorphic*) functions, i.e. solutions of the generalized Cauchy-Riemann system $\overline{\partial} f = 0$ ($f\overline{\partial} = 0$). The hypercomplex differentiability as generalization of complex differentiability has to be understood in the following way (see [11]): a function f defined in an open domain $\Omega \subseteq \mathbb{R}^{n+1}$ is hypercomplex differentiable if there exists in each point of Ω a uniquely defined areolar derivative f'. Then f is real differentiable and $f' := \partial f$. Furthermore, f is hypercomplex differentiable in Ω if and only if f is monogenic Ω (cf. [11]). For a deeper study of monogenic function theory we refer [3].

Since the variable x and its powers x^k are, in general, not monogenic, the usual way of constructing polynomials is not valid in this class of functions. In order to obtain an higher dimensional analogue of the holomorphic powers z^k, $z \in \mathbb{C}$, the concept of Appell's power-like polynomials (cf. [1]) was generalized to the hypercomplex setting as homogeneous monogenic polynomials $\mathcal{F}_k$ of degree k such that

$$\partial \mathcal{F}_k(x) = k\,\mathcal{F}_{k-1}(x),\ k = 1, 2, ... \text{ and } \mathcal{F}_0(x) = const., \quad x \in \mathcal{A}_n.$$

The construction of the simplest case of generalized monogenic Appell polynomials by choosing $\mathcal{F}_0(x) = 1$, for all $x \in \mathcal{A}_n$, lead to the homogeneous polynomials

$$P_k(x) = \sum_{s=0}^{k} \binom{k}{s} c_s(n) x_0^{k-s} \underline{x}^s,$$

that include the real powers x_0^k, when $\underline{x} = 0$ and the holomorphic powers $z = (x_0 + x_1\, e_1)^k$ when $n = 1$ (with the usual identification of e_1 with the imaginary unit)[2]. The coefficients $c_s(n)$, $s = 0, 1, \ldots, k$, are precisely the terms (5) of the generalized Vietoris sequence $(c_k(n))_{k\geq 0}$ (for details, see [6], [9] and [10]).

[2]Notice that for the case $n = 1$, we are dealing with polynomials from $\mathbb{R}^2 \cong \mathbb{C}$ to $\mathbb{R}^2 \cong \mathbb{C}$ and, in this case, $c_k(1) = 1$, for all $k \in \mathbb{N}_0$.

As a first application of Appell polynomials in the framework of hypercomplex function theory, a monogenic exponential function was studied in [10]:

$$\mathrm{Exp}_n\,(x) = \mathrm{Exp}_n\,(x_0 + \omega\,|\underline{x}|) := \sum_{k=0}^{\infty} \frac{P_k(x)}{k!} = e^{x_0}\Gamma\left(\frac{n}{2}\right)\left(\frac{2}{|\underline{x}|}\right)^{\frac{n}{2}-1}\left(J_{\frac{n}{2}-1}(|\underline{x}|) + \omega J_{\frac{n}{2}}(|\underline{x}|)\right),$$

where J_α are Bessel functions of the first kind and $\omega := \dfrac{\underline{x}}{|\underline{x}|}$ is such that $\omega^2 = -1$. Applying different methods, Laville et al. (cf. [12]) constructed an equivalent monogenic exponential function, but represented in terms of the modified Bessel functions:

$$\mathcal{E}_n\,(x) := e^{x_0}\mathcal{E}_n(\underline{x}),$$

where

$$\mathcal{E}_n(\underline{x}) = \Gamma\left(\frac{n}{2}\right)\left(\frac{\underline{x}}{2}\right)^{1-\frac{n}{2}}\left(I_{\frac{n}{2}-1}(\underline{x}) + I_{\frac{n}{2}}(\underline{x})\right). \tag{7}$$

Notice that the paravector valued function (7) is formally identical with the real valued exponential generating function (6) of the n-parameter generalized Vietoris sequence $(c_k(n))_{k\geq 0}$.

The functions (6) and (7) are related to hyperbolic trigonometric functions, for n odd and to Bessel functions of integer order, when n is even.

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