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The role of the Callan–Witten anomaly density as a Chern–Simons term in Skyrme model*

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Abstract

We consider axially symmetric solutions of the $U(1)$ gauged Skyrme model supplemented with a Callan–Witten (CW) anomaly density term. The main properties of the solutions are studied, several specific features introduced by the presence of the CW term being identified. We find that the solitons possess a nonzero angular momentum proportional to the electric charge, which in addition to the usual Coulomb part, acquires an extra (topological) contribution from the CW term. Specifically, it is shown that the slope of mass/energy M *vs*. electric charge Q_e and angular momentum J can be both positive *and* negative. Furthermore, it is shown that the gauged Skyrmion persists even when the quartic (Skyrme) kinetic term disappears.

Keywords: Gauged Skyrme, anomaly, soliton, electric charge, spin

(Some figures may appear in colour only in the online journal)

* We dedicate this work to the memory of our colleague Valery Rubakov with whom we discussed this topic extensively.
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1. Introduction

Solutions of the classical Skyrme model [1, 2] were proposed as the solitons describing the dynamics of nucleons a long time ago. Later, topological aspects of the model were highlighted in [3, 4], followed by the influential work of [5]. Subsequently, a detailed study of the $U(1)$ gauged Skyrme model was presented in [6].

More recently, $SO(2)$ gauged Skyrmions[5] in $3+1$ dimensions were studied quantitatively in [7, 8] (see also [9]). These solitons carry both electric charge Q_e [7] and angular momentum J [8], and are characterised by the asymptotic value of the electric component A_0 of the Maxwell potential A_μ. It was observed that the mass/energy M of the Abelian gauged Skyrmion increases monotonically with Q_e and J.

Subsequently the effect of the Chern–Simons (CS) density on gauged Skyrmions on Abelian gauged Skyrme scalars in $2+1$ dimensions [10, 11], and in $4+1$ dimensions [12, 13] was studied. In these particular examples, the solutions are characterised by the asymptotic value of the magnetic component A_i as well as that of A_0 as seen in [7, 8]. It turns out that in the presence of the CS density in the Lagrangian, the (M, Q_e) and (M, J) curves exhibit both positive and negative slopes, in contrast to the monotonic positive slopes observed in [7, 8] in the absence of a Chern–Simon term.

Our aim in the present work is to test whether the $U(1)$ gauged Skyrmions in $3+1$ dimensions exhibit the new pattern of negative (as well as positive) slopes of (M, Q_e) and (M, J) curves when the Lagrangian is augmented with a CS–like term, namely the Anomaly related term employed in [6], whose construction is given in [5] on global anomalies. Our results show that the answer to this question is affirmative!

In section 2 we introduce the model, in section 3 we define the global charges in this model, in section 4 the system is subjected to axial symmetry and in section 5 the numerical construction is presented in detail. We have found it convenient to use a constraint compliant parametrisation of the Skyrme scalar in sections 3 and 4, while in 5 the Lagrange–multiplier method is used in the general parametrisation. The constraint compliant parametrisation employed here is presented in appendix A, and in appendix B an alternative such parametrisation is given.

Conventions. Throughout the paper, Greek alphabet letters $\alpha, \beta, \ldots$ label spacetime coordinates, running from 0 to 3 (with $x^0 = t$); early Latin letters, $a, b, \ldots$ label the internal indices of the scalar field multiplet. As standard, we use Einstein's summation convention, but to alleviate notation, no distinction is made between covariant and contravariant *internal* indices.

The background of the theory is Minkowski spacetime (with the signature $(+---)$), where the spatial $\mathbb{R}^3$ is written first in terms of Cartesian coordinate, $\mathrm{d}s_3^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2$, (with $x_3 = z$). The transformation $x_1 = \rho\cos\varphi$ $x_2 = \rho\sin\varphi$ leads to metric of flat space in cylindrical coordinates, $\mathrm{d}s_3^2 = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\varphi^2 + \mathrm{d}z^2$, where $0 \leqslant \rho < \infty$, $-\infty < z < \infty$ and $0 \leqslant \varphi < 2\pi$. Finally, we shall use also the flat space metric in spherical coordinates, $\mathrm{d}s_3^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2)$, obtained via the coordinate transformation $\rho = r\sin\theta$, $z = r\cos\theta$ (with $0 \leqslant r < \infty$ and $0 \leqslant \theta \leqslant \pi$).

[5] The term Skyrmion here is used for the soliton of a $O(D+1)$ sigma model on $\mathbb{R}^D$ in $D+1$ dimensional space-time.

2. The model

The model we study is that considered in [6] described by the Lagrangian

$$L = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + L_{\text{Skyrme}} + \kappa\, \Omega_{\text{CW}}, \tag{2.1}$$

$$L_{\text{Skyrme}} = -\frac{1}{2} \text{Tr} D_\mu U D^\mu U^{-1} + \frac{1}{4} \text{Tr} \left[U^{-1} D_\mu U, U^{-1} D_\nu U \right]^2 + V\left[U, U^{-1} \right] \tag{2.2}$$

in which the Skyrme scalar is parametrised by the $SU(2)$ group element U and $V[U,U^{-1}]$ is a potential term[6].

It is convenient to express the Ω_{CW} term as

$$\Omega_{\text{CW}} = \Omega_{\text{CW}}^{(1)} + \Omega_{\text{CW}}^{(2)} \tag{2.3}$$

$$\equiv A_\tau\, W_{(1)}^{\tau} + A_\tau F_{\mu\nu}\, W_{(2)}^{\tau\mu\nu} \tag{2.4}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and

$$W_{(1)}^{\tau} = \varepsilon^{\tau\lambda\mu\nu}\, \text{Tr} Q \left[\left(U^{-1} \partial_\mu U \right) \left(U^{-1} \partial_\nu U \right) \left(U^{-1} \partial_\lambda U \right) + \left(U \partial_\mu U^{-1} \right) \left(U \partial_\nu U^{-1} \right) \left(U \partial_\lambda U^{-1} \right) \right], \tag{2.5}$$

$$W_{(2)}^{\tau\mu\nu} = i \varepsilon^{\tau\lambda\mu\nu} \left\{ \text{Tr} \left[Q^2 \left(U^{-1} \partial_\lambda U + U \partial_\lambda U^{-1} \right) \right] + \frac{1}{2} \text{Tr} \left[\left(Q \partial_\lambda U Q U^{-1} - Q \partial_\lambda U^{-1} Q U \right) \right] \right\}. \tag{2.6}$$

The covariant derivative in (2.1) is defined by[7]

$$D_\mu U = \partial_\mu U + i A_\mu [Q, U] \ , \quad \text{with} \quad Q = \frac{1}{3} \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.7}$$

The alternative to the formulation of the Skyrme model in terms of the $SU(2)$ element U is the formulation in terms of the Skyrme scalar $\phi^a = (\phi^I, \phi^4)$, $I = 1,2,3$ of the $O(4)$ sigma model satisfying the constraint

$$|\phi^a|^2 = 1 \qquad \Leftrightarrow \qquad U^\dagger U = \mathbb{1}\,,$$

expressed through the relation(s)

$$U = \phi^4\, \mathbb{1} + i \phi^I \sigma_I\,, \quad U^{-1} = \phi^4\, \mathbb{1} - i \phi^I \sigma_I\,, \quad I = 1,2,3\,, \tag{2.8}$$

with the $O(4)$ Skyrme scalar $\phi^a = (\phi^I, \phi^4)$.

[6] This potential term is not essential since the Derrick scaling requirement is satisfied in its absence. It is included for convenience in the numerical construction by supplying exponential decay.

[7] Note that the definition in [6] is $D_\mu U = \partial_\mu U + i e A_\mu [Q, U]$ with e the gauge coupling constant, which, in order to simplify various relations, we set to one. Also, this is the value considered in section 5 dealing with the numerical results.

Reassigning the index a as $a = (\alpha, A)$, with $\alpha = 1, 2$ and $A = 3, 4$, the following useful relations can be stated

$$[Q, U] = -\varepsilon_{\alpha\beta}\,\phi^\alpha\,\sigma^\beta\,, \tag{2.9}$$

$$\left[Q, U^{-1}\right] = \varepsilon_{\alpha\beta}\,\phi^\alpha\,\sigma^\beta\,, \tag{2.10}$$

of which (2.9) then leads to the $SO(2)$ gauging prescription

$$D_\mu\phi^\alpha = \partial_\mu\phi^\alpha + A_\mu\,(\varepsilon\phi)^\alpha\ , \quad \alpha = 1, 2 \tag{2.11}$$

$$D_\mu\phi^A = \partial_\mu\phi^A\ , \qquad\qquad A = 3, 4\,, \tag{2.12}$$

for the $O(4)$ Skyrme scalar $\phi^a = (\phi^\alpha, \phi^A)$ employed in [14][8] which through (2.8), is precisely equivalent to the definition (2.7).

Here, we will choose to use the definition (2.7) for $D_\mu U$, consistently with the definition $D_\mu\phi^a$ given by (2.11) and (2.12). Further abbreviating our notation for $D_\mu\phi^a = (D_\mu\phi^\alpha, D_\mu\phi^A)$

$$\phi^a_\mu \stackrel{\text{def.}}{=} D_\mu\phi^a\ , \quad \text{and} \quad \phi^{ab}_{\mu\nu} \stackrel{\text{def.}}{=} \phi^a_{[\mu}\phi^b_{\nu]}$$

the Skyrme Lagrangian (2.2) can be written compactly as

$$L_{\text{Skyrme}} = -\frac{1}{2}\,|\phi^a_\mu|^2 + \frac{1}{8}\,|\phi^{ab}_{\mu\nu}|^2 + V[\phi^a] \tag{2.16}$$

where $V[\phi^a]$ is the (pion mass) potential term intended to localise the soliton exponentially.

With this parametrisation of the Skyrme scalar, the stress-energy tensor $T_{\mu\nu}$ can be expressed compactly as

$$T_{\tau\lambda} = -\frac{1}{2}\left(F_{\tau\mu}F_\lambda{}^\mu - \frac{1}{4}\,g_{\tau\lambda}\,|F_{\mu\nu}|^2\right) + \frac{1}{2}\left(\phi^a_\tau\,\phi^a_\lambda - \frac{1}{2}\,g_{\tau\lambda}\,|\phi^a_\mu|^2\right) - \frac{1}{4}\left(\phi^{ab}_{\tau\mu}\,(\phi_\lambda{}^\mu)^{ab} - \frac{1}{4}\,g_{\tau\lambda}\,|\phi^{ab}_{\mu\nu}|^2\right)\,. \tag{2.17}$$

The component T_{00} is the energy density, while the angular momentum density $\mathcal{J}$ in the (x_1, x_2)-plane is

$$\begin{aligned}\mathcal{J} &\stackrel{\text{def.}}{=} x_2\,T_{10} - x_1\,T_{20} \\ &= (\varepsilon x)_{\bar\alpha}\,T_{\bar\alpha 0}\,,\end{aligned} \tag{2.18}$$

[8] In [6], the definition used for the covariant derivative of U is

$$D_\mu U = \partial_\mu U - iA_\mu\,[Q, U] \tag{2.13}$$

in which case the definition of the covariant derivative $D_\mu\phi^a$ given by (2.11) and (2.12), used in [14], changes to

$$D_i\phi^\alpha = \partial_i\phi^\alpha - A_i\,(\varepsilon\phi)^\alpha\ , \quad \alpha = 1, 2 \tag{2.14}$$

$$D_i\phi^A = \partial_i\phi^A, \qquad\qquad A = 3, 4\,. \tag{2.15}$$

in which the spacelike index notation ($\bar{\alpha} = 1, 2$) is used. The quantity $T_{\bar{\alpha}0}$ is

$$T_{\bar{\alpha}0} = \frac{1}{2}\left(F_{\bar{\alpha}\bar{\beta}}F_{0\bar{\beta}} + F_{\bar{\alpha}z}F_{0z}\right) + \frac{1}{2}\phi^a_{\bar{\alpha}}\phi^a_0 + \frac{1}{2}\left(\phi^{ab}_{\bar{\alpha}\bar{\beta}}\phi^{ab}_{0\bar{\beta}} + \phi^{ab}_{\bar{\alpha}z}\phi^{ab}_{0z}\right) \tag{2.19}$$

which can be evaluated only after imposition of axial symmetry.

Having introduced the re-expression of U in terms of the Skyrme scalar $\phi^a = (\phi^I, \phi^4) = (\phi^\alpha, \phi^A)$ with $I = 1, 2, 3$ and ($\alpha = 1, 2$; $A = 3, 4$), one exploits the relation (2.10), and one finds a useful simplification of $W^{\tau\mu\nu}_{(2)}$ in (2.6) by noting

$$\begin{aligned}
&\mathrm{Tr}\left(Q\partial_\lambda UQU^{-1} - Q\partial_\lambda U^{-1}QU\right)\\
&\quad = \mathrm{Tr}\,Q^2\left(U^{-1}\partial_\lambda U + \partial_\lambda UU^{-1}\right) + \varepsilon_{\alpha\beta}\phi^\alpha \mathrm{Tr}\,\sigma^\beta Q\partial_\mu\left(U + U^{-1}\right)\\
&\quad = \mathrm{Tr}\,Q^2\left(U^{-1}\partial_\lambda U + \partial_\lambda UU^{-1}\right) + \varepsilon_{\alpha\beta}\,\phi^\alpha\,\partial_\lambda\phi^4\,\mathrm{Tr}\,Q\,\sigma^\beta\\
&\quad = \mathrm{Tr}\,Q^2\left(U^{-1}\partial_\lambda U + \partial_\lambda UU^{-1}\right)
\end{aligned} \tag{2.20}$$

since $\mathrm{Tr}\,Q\,\sigma^\beta = 0$, and hence the second term in (2.6) equals $\frac{1}{2}$ times the first term there, *i.e.*,

$$W^{\tau\mu\nu}_{(2)} = \frac{3i}{2}\varepsilon^{\tau\lambda\mu\nu}\,\mathrm{Tr}\left[Q^2\left(U^{-1}\partial_\lambda U + \partial_\lambda UU^{-1}\right)\right], \tag{2.21}$$

which simplifies the expression (2.6). This is of practical importance when expressing the Lagrangian using a parametrisation in which the Skyrme scalar is compliant with the constraint $U^\dagger U = \mathbb{1}$ (or $|\phi^a|^2 = 1$). Working with a constraint compliant parametrisation is very convenient in the definition of the electric charge, and, in verifying that the Euler–Lagrange equations of Ω_{CW} are gauge invariant.

One such parametrisation, designated '$3+1$', is the one employed in the present work and is given in appendix A. An alternative one, designated '$2+2$', is relegated to appendix B. Our preference for the particular parametrisation of appendix A is dictated by technicalities of numerical constructions.

3. Definitions of global charge densities

In this section, the definitions of the densities of global charges (baryon number and electric charge) are presented, in preparation to applying axial symmetry and their evaluation in the next section. Here, the definitions of the gauge deformed baryon number density of the electric charge are given explicitly, while the definition of the angular momentum (2.18) given above in terms of components of the stress-energy tensor (2.17), will be made precise in the following section after imposition of symmetry.

3.1. Gauge deformed baryon number: energy lower bound

The baryon number q of the nucleon is defined to be the winding number of the Skyrme scalar $\phi^a = (\phi^I, \phi^4)$, $I = 1, 2, 3$, seen in the previous section. This is the volume integral of the winding number density

$$\varrho_0 = \frac{1}{3}\varepsilon_{ijk}\varepsilon^{abcd}\,\partial_i\phi^a\partial_j\phi^b\partial_k\phi^c\phi^d \tag{3.22}$$

$$= \varepsilon_{ijk}\varepsilon^{IJK}\left[\partial_i\phi^I\partial_j\phi^J\partial_k\phi^K\phi^4 - 3\,\partial_i\phi^I\partial_j\phi^J\partial_k\phi^4\phi^K\right] \tag{3.23}$$

and with the axially symmetric parametrisation

$$\phi^I = \begin{pmatrix} \sin f\, n^\alpha \\ \cos f \end{pmatrix} \quad \text{with} \quad n^\alpha = \begin{pmatrix} \cos n\varphi \\ \sin n\varphi \end{pmatrix} \tag{3.24}$$

and boundary values $f(0) = \pi, f(\infty) = 0$, the integral of ϱ_0 equals the integer (winding number) n. Most importantly, this number $q = n$ presents a lower bound[9] on the energy of the Skyrmion on $\mathbb{R}^3$.

After $SO(2)$ gauging according to (2.11) and (2.12), the winding number density (3.22) no longer results in a lower bound on the energy of the gauged Skyrmion.

It was proposed in [14] (and references therein) that the energy lower bound (density) replacing the winding number density for a $O(D+1)$ Skyrme scalar on $\mathbb{R}^D$ suffers a deformation caused by the gauging, which must now be both total divergence like the latter, and gauge invariant.

These gauge deformed winding number densities are given by two equivalent definitions, formally expressed as

$$\varrho = \varrho_G + W[F, D\Phi] \tag{3.25}$$

$$= \varrho_0 + \partial_i \Omega_i [A, \Phi] \, . \tag{3.26}$$

The definition (3.26), which consists of the explicitly total divergence term $\partial_i \Omega_i [A, \Phi]$ and the winding number density ϱ_0 which is essentially total divergence, is total divergence. It is also gauge invariant since it is equal to the gauge invariant version (3.25), in which $W[F, D\Phi]$ is by construction gauge invariant, while ϱ_G is gauge invariant by virtue of its definition, which is that of ϱ_0 with all partial derivatives replaced by covariant derivatives. Thus, ϱ is both gauge invariant and total divergence.

In the case of the $SO(2)$ gauged $O(4)$ sigma (Skyrme) model on $\mathbb{R}^3$ at hand, these two equivalent definitions of ϱ are

$$\varrho = \varrho_G + \frac{1}{2} \varepsilon_{ijk} F_{ij} \left(\phi^B \partial_k \phi^A \right) \tag{3.27}$$

$$= \varrho_0 + \varepsilon_{ijk} \partial_i \left(A_j \varepsilon^{AB} \phi^B \partial_k \phi^A \right) \tag{3.28}$$

in which ϕ^α, with $\alpha = 1, 2$ are the components of the $O(4)$ Skyrme scalar that are gauged with $SO(2)$ and ϕ^A, with $A = 3, 4$ are the components that are not gauged.

The quantity ϱ_0 (3.22) and (3.23) can alternatively be expressed as

$$\varrho_0 = \varepsilon_{ijk} \left[\left(\varepsilon^{\alpha\beta} \partial_i \phi^\alpha \partial_j \phi^\beta \right) \left(\varepsilon^{AB} \phi^B \partial_k \phi^A \right) + \left(\varepsilon^{AB} \partial_i \phi^A \partial_j \phi^B \right) \left(\varepsilon^{\alpha\beta} \phi^\beta \partial_k \phi^\alpha \right) \right] \tag{3.29}$$

and the density ϱ_G is defined by (3.29), with each partial derivative $\partial_i \phi^\alpha$ replaced by the covariant derivative $D_i \phi^\alpha$.

3.2. Definition of the (electric) Noether charge

Consider the Lagrangian density

$$\mathcal{L} = \frac{1}{4} |F_{\mu\nu}|^2 - \frac{1}{2} |D_\mu \phi^a|^2 + \frac{1}{8} |D_{[\mu} \phi^a D_{\nu]} \phi^b|^2 + V\left[\phi^4\right] + \kappa\, \Omega_{\mathrm{CW}}, \tag{3.30}$$

[9] Though for $n > 1$ the actual energy of the Skyrmion is known from the work of [15], to be somewhat less than that of the axially symmetric configuration (3.24).

$$\Omega_{\rm CW} = \varepsilon^{\tau\lambda\mu\nu} F_{\mu\nu} \left[-(g - \sin g)\, \partial_\tau \cos\Theta\, \partial_\lambda \Phi + A_\tau \left(\partial_\lambda g \cos\Theta + \sin g\, \partial_\lambda \cos\Theta \right) \right] \tag{3.31}$$

in which the anomaly term (3.31) is expressed explicitly in terms of the constraint compliant parametrisation (A.1) and (A.6) presented in appendix A. The quadratic and quartic Skyrme kinetic terms in this parametrisation are given by (A.7) and (A.8) respectively. Also,

$$\begin{aligned} \delta_{A_\tau}\mathcal{L} = & -\partial_\mu F^{\mu\tau} - 2\kappa\varepsilon^{\tau\lambda\mu\nu} \left[(1-\cos g)\, \partial_\lambda g\, \partial_\mu \cos\Theta\, (A_\nu - \partial_\nu \Phi) \right. \\ & \left. + F_{\mu\nu} \left(\partial_\lambda g \cos\Theta + \sin g\, \partial_\lambda \cos\Theta \right) \right] - \frac{1}{2}(1-\cos g) \sin^2\Theta\, (A^\tau - \partial^\tau \Phi) = 0 \end{aligned} \tag{3.32}$$

and the equation resulting from the variation w.r.t. Φ is

$$\delta_\Phi \mathcal{L} = -\kappa\, \varepsilon^{\tau\lambda\mu\nu}\, (1-\cos g)\, \partial_\tau g\, \partial_\lambda \cos\Theta\, F_{\mu\nu} - \frac{1}{2} \partial_\tau \left[(1-\cos g) \sin^2\Theta\, (A^\tau - \partial^\tau \Phi) \right] = 0\,. \tag{3.33}$$

(Note that in (3.32) and (3.33), the terms coming from the variations of the *quartic* kinetic term are omitted for typographic economy.)

We note that the divergence of the Maxwell equation (3.32) turns out to be precisely the Φ equation (3.33), so that the constraint imposed on the system by taking the divergence of the Maxwell equation is satisfied by the Φ equation (3.33).

The main difference between the gauged Skyrme models endowed with an Anomaly-type term as in (3.31) and those with CS terms is, that in the CS the term resulting from the variation w.r.t. the gauge field is *divergenceless* while, the corresponding term in (3.32) is **not** divergenceless.

Expressing the Maxwell equation (3.32) as

$$-\partial_\mu F^{\mu\tau} - \kappa\, \Omega^\tau = \frac{1}{2}(1-\cos g) \sin^2\Theta\, (A^\tau - \partial^\tau \Phi) \tag{3.34}$$

$$\Omega^\tau = 2\,\varepsilon^{\tau\lambda\mu\nu} \left[F_{\mu\nu} \left(\partial_\lambda g \cos\Theta + \sin g\, \partial_\lambda \cos\Theta \right) + (1-\cos g)\, \partial_\lambda g\, \partial_\mu \cos\Theta\, (A_\nu - \partial_\nu \Phi) \right] \tag{3.35}$$

it is clear that the current on the right hand side of (3.34) is not a conserved current since the divergence of Ω^θ does not vanish. Therefore the 0th component of that current cannot define the electric charge. Instead, the electric current must be defined using the Noether Theorem.

The symmetry of this system is the invariance under an Abelian gauge transformation

$$A_\mu \to \tilde{A}_\mu = A_\mu + \partial_\mu \Lambda \tag{3.36}$$

$$\Phi \to \tilde{\Phi} = \Phi + \Lambda \tag{3.37}$$

Λ being an arbitrary function.

One can now invoke the (first) Noether Theorem by considering a global transformation with $\Lambda = \text{const.}$. That gives a one parameter continuous symmetry of the action. Since

$$\frac{\delta \Phi}{\delta \Lambda} = 1\,,$$

the divergenceless Noether current is defined by

$$j^\mu_{(N)} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \tag{3.38}$$

which can be readily calculated for the Lagrangian (3.31) to give

$$j^{\tau}_{(N)} = -\frac{1}{2}\,(1-\cos g)\sin^2\Theta\,(A^{\tau}-\partial^{\tau}\Phi)+\kappa\,(g-\sin g)\,\varepsilon^{\tau\lambda\mu\nu}\,F_{\mu\nu}\,\partial_{\lambda}\cos\Theta\,. \tag{3.39}$$

By using the equation (3.32), one finds after some manipulations

$$j^{\tau}_{(N)} = \partial_{\mu}F^{\mu\tau}+2\kappa\,\varepsilon^{\tau\lambda\mu\nu}\,\partial_{\lambda}\left[(g-\sin g)\,\partial_{\mu}\cos\Theta\,(A_{\nu}-\partial_{\nu}\Phi)+F_{\mu\nu}\,g\,\cos\Theta\right] \tag{3.40}$$

which is manifestly a conserved current.

Equation (3.40) plays the role of the Maxwell equation, and its 0th component in the static limit, which plays the role of the Gauss Law equation

$$\partial_i F_{i0}-2\kappa\,\varepsilon_{ijk}\,\partial_k\left[(g-\sin g)\,\partial_i\cos\Theta\,(A_j-\partial_j\Phi)+F_{ij}\,g\,\cos\Theta\right]=j^{0}_{(N)} \tag{3.41}$$

in which $j^{0}_{(N)}$ plays the role of the zeroth component of the usual Maxwell current and the term with strength κ on the LHS plays the role of $\varepsilon_{ij\ldots mn}F_{ij}\ldots F_{mn}$ occurring in the usual case for CS in all odd dimensions.

The electric charge is then defined as

$$Q_e = \int j^{0}_{(N)}\,d^3x \stackrel{\text{def.}}{=} Q_{\text{Coulomb}}+Q_{\text{anomaly}} \tag{3.42}$$

in which Q_{Coulomb} is the Coulomb charge

$$Q_{\text{Coulomb}} = \int \partial_k F_{k0}\,d^3x = \int F_{k0}\,\mathrm{d}S^k\,, \tag{3.43}$$

which can be evaluated as a surface integral by Gauss' Theorem, and Q_{anomaly} is the contribution to the electric charge from the anomaly term of [6],

$$Q_{\text{anomaly}} = +2\kappa\int \varepsilon_{ijk}\partial_k\left[(g-\sin g)\,\partial_i\cos\Theta\,(A_j-\partial_j\Phi)+F_{ij}\,g\,\cos\Theta\right]d^3x\,. \tag{3.44}$$

4. Imposition of axial symmetry

Imposition of axial symmetry in the (x_1,x_2) plane on the Abelian connection $A_{\mu}=(A_{\bar\alpha},A_z,A_0)$ (with $\bar\alpha=1,2$) in the static limit is

$$A_{\bar\alpha} = \left(\frac{a(\rho,z)-m}{\rho}\right)(\varepsilon\hat{x})_i\,,\quad A_z = c(\rho,z)\,,\quad A_0 = b(\rho,z) \tag{4.45}$$

where $\rho=\sqrt{|x_{\bar\alpha}|^2}$, leading to the components of the curvature

$$F_{\bar\alpha\bar\beta} = -\frac{\partial_{\rho}a}{\rho}\,\varepsilon_{\bar\alpha\bar\beta}\,,\quad F_{\bar\alpha z} = \partial_{\rho}c\,\hat{x}_{\bar\alpha}-\frac{\partial_z a}{\rho}\,(\varepsilon\hat{x})_{\bar\alpha}\,,\quad F_{\bar\alpha 0} = \partial_{\rho}b\,\hat{x}_{\bar\alpha}\,,\quad F_{z0} = \partial_z b\,. \tag{4.46}$$

Imposition of symmetry on the scalar is expressed conveniently in terms of the constraint compliant parametrisation of $\phi^a = \phi^a[g(x_\mu), \Theta(x_\mu), \Phi(\Phi)]$ given in (A.1) and (A.6) used above, which can be combined in the form

$$\phi^a = \begin{pmatrix} \phi^\alpha \\ \phi^3 \\ \phi^4 \end{pmatrix} = \begin{pmatrix} \sin\frac{g(x_\mu)}{2}\sin\Theta(x_\mu)\, m^\alpha \\ \sin\frac{g(x_\mu)}{2}\cos\Theta(x_\mu) \\ \cos\frac{g(x_\mu)}{2} \end{pmatrix} , \quad \text{with} \quad m^\alpha[\Phi] = \begin{pmatrix} \cos\Phi(x_\mu) \\ \sin\Phi(x_\mu) \end{pmatrix} \tag{4.47}$$

in which notation imposition of axial symmetry on the Skyrme scalar is stated by

$$g(x_\mu) = g(\rho,z) \ , \quad \Theta(x_\mu) = \Theta(\rho,z) \ , \quad \Phi(x_\mu) = m\varphi \tag{4.48}$$

where φ is the azimuthal angle in the (x_1,x_2) plane and m is an integer.

It is useful to note that subject to this symmetry

$$(A_{\bar\alpha} - \partial_{\bar\alpha}\Phi) = \frac{a(\rho,z)}{\rho}(\varepsilon\hat{x})_{\bar\alpha} \ , \quad (A_z - \partial_z\Phi) = c(\rho,z) \ , \quad (A_0 - \partial_0\Phi) = b(\rho,z) \ . \tag{4.49}$$

Also, one can easily show that the magnetic potential $c(\rho,z)$ can be consistently set to zero, a choice we shall employ in what follows.

4.1. Gauge deformed baryon number

The question here is whether the baryon number can be altered due to the influence of the gauge field. For this, the relevant definition of the gauge deformed baryon number density is (3.28)

$$\varrho = \varrho_0 + \varrho_1$$

ϱ_0 being the baryon number density prior to gauging, and ϱ_1 given by

$$\varrho_1 = \varepsilon_{ijk}\,\partial_i\left(A_j\,\varepsilon^{AB}\phi^B\partial_k\phi^A\right) \tag{4.50}$$

which quantifies the departure of the baryon number due to gauging.

It is convenient to express these quantities in constraint compliant parametrisation given in appendix A in terms of the functions $g(x_\mu)$ and $\Theta(x_\mu)$, which after imposition of axial symmetry yield

$$\varrho_0 = \frac{1}{2\rho}m\,\partial_{[\rho}(g - \sin g)\,\partial_{z]}\cos\Theta \tag{4.51}$$

$$\varrho_1 = -\frac{1}{2\rho}\partial_{[\rho}\left[(a-m)\left(\partial_{z]}g\,\cos\Theta + \sin g\,\partial_{z]}\cos\Theta\right)\right] \tag{4.52}$$

in which the functions $g(\rho,z)$ and $\Theta(\rho,z)$ depend on (ρ,z) subject to (4.48), as does the function $a(\rho,z)$ due to imposition of symmetry (4.45).

The integrals of (4.51) and (4.52) yield the deformed baryon number $q = q_0 + q_1$, with

$$q_0 = 2\pi \int \varrho_0\, \rho\, \mathrm{d}\rho \mathrm{d}z = m\pi \int \partial_{[\rho}(g - \sin g)\, \partial_{z]} \cos\Theta) \mathrm{d}\rho\, \mathrm{d}z \tag{4.53}$$

$$q_1 = 2\pi \int \varrho_1\, \rho\, \mathrm{d}\rho \mathrm{d}z = -\pi \int \partial_{[\rho} \left[(a - m)\, \left(\partial_{z]} g \cos\Theta + \sin g\, \partial_{z]} \cos\Theta\right)\right] \mathrm{d}\rho\, \mathrm{d}z\,, \tag{4.54}$$

both being double integrals with 'curl' integrands, which are evaluated using Green's Theorem.

The integral (4.54) vanishes since $a(\rho = 0) = m$ all along the z-axis, while the integral (4.53) for q_0 is

$$q_0 = \pm m\pi \int (g - \sin g)\, \partial_z \cos\Theta)\big|_{\rho=0} \mathrm{d}z = \pm 4m\pi^2\,. \tag{4.55}$$

The conclusion is that the baryon number cannot be changed since all Skyrmion solutions are characterised by the asymptotic condition $a(\rho = 0) = m$.

4.2. Electric charge

The evaluation of the charge integral (3.43) for Q_{Coulomb} is standard, by applying Gauss Theorm, yielding

$$Q_{\mathrm{Coulomb}} = \int \partial_i F_{i0}\, d^3x = 4\pi\, \left(r^2 \partial_r b\right)\big|_{r=\infty} = 4\pi\, Q_0 \tag{4.56}$$

where Q_0 is the constant in the asymptotic behaviour $b_{r\to\infty} = \mathrm{const.} + Q_0/r^2 + \ldots$.

By contrast the Gauss Theorem cannot be applied to evaluate Q_{anomaly}, (3.44), since the function Θ in the integrand has a discontinuity in the (x_1, x_2) plane. We proceed to impose axial symmetry on the integrand I of (3.44), which yields the following two dimensional density

$$I = -\frac{2\kappa}{\rho} \partial_{[\rho} \left\{ cos\Theta \left[\left(a\partial_{z]} g - g\partial_{z]} a \right) - \partial_{z]} \left(a \sin g \right) \right] \right\}. \tag{4.57}$$

Note that after multiplying with ρ coming from the volume element, the total divergence in three dimensions reduces to a two dimensional **curl**.

After integration over the azimuthal angle φ, the two dimensional integral in (ρ, z) is

$$Q_{\mathrm{anomaly}} = -2\pi\kappa \int \partial_{[\rho} \left\{ \cos\Theta \left[\left(a\partial_{z]} g - g\partial_{z]} a \right) - \partial_{z]} \left(a \sin g \right) \right] \right\} \mathrm{d}\rho\, \mathrm{d}z \tag{4.58}$$

that can be integrated applying Green's Theorem, yielding

$$Q_{\mathrm{anomaly}} = -2\pi\kappa m \int_{z=-\infty}^{z=+\infty} \cos\Theta\, \partial_z \left(g - \sin g \right)\big|_{\rho=0} \mathrm{d}z \tag{4.59}$$

where the symmetry restriction (4.45) stating $a(\rho = 0, z) = m$ along the z-axis has been imposed.

Further, imposing the boundary conditions employed in the numerical construction of the solutions, (4.59) is evaluated to give

$$Q_{\mathrm{anomaly}} = 4\pi\,\kappa\,m \tag{4.60}$$

resulting in the following expression of the electric charge

$$Q_e = 4\pi\,(Q_0 + \kappa\,m) \tag{4.61}$$

4.3. Angular momentum

Imposition of axial symmetry (4.45), (4.46), (4.48) and (4.49) on $\mathcal{J}$, (2.18) results in

$$\begin{aligned}\mathcal{J} \simeq\ & (a_{,\rho}b_{,\rho} + a_{,z}b_{,z}) \\ & + ab\sin^2\frac{g}{2}\sin^2\Theta\left\{1 + \left[\frac{1}{4}\left(g_{,\rho}^2 + + g_{,z}^2\right) + \left(\Theta_{,\rho}^2 + \Theta_{,z}^2\right)\sin^2\frac{g}{2}\right]\right\}\end{aligned} \tag{4.62}$$

and the angular momentum is

$$J = \int \mathcal{J}\,d^3x = \int \mathcal{J}\,d^3x = 2\pi\int \mathcal{J}\,\rho\,\mathrm{d}\rho\mathrm{d}z\,. \tag{4.63}$$

The quantity (4.63) can be evaluated by extracting the partial derivatives on the function a,

$$\begin{aligned}J = 2\pi\int \mathrm{d}\rho\mathrm{d}z\,\Big(&\partial_\rho\left(\rho\,a\,b_{,\rho}\right) + \partial_z\left(\rho\,a\,b_{,z}\right) - a\left[\partial_\rho\left(\rho\,b_{,\rho}\right) + \partial_z\left(z\,b_{,z}\right)\right] \\ & + \rho ab\sin^2\frac{g}{2}\sin^2\Theta\left\{1 + \left[\frac{1}{4}\left(g_{,\rho}^2 + g_{,z}^2\right) + \left(\Theta_{,\rho}^2 + \Theta_{,z}^2\right)\sin^2\frac{g}{2}\right]\right\}\Big)\end{aligned} \tag{4.64}$$

and substituting the equations of motion of the function b in the second line in (4.64).

This analysis for the $SO(2)$ gauged Skyrme system in the absence of the anomaly term Ω_{CW} has been carried out previously in detail in [8], and since the latter does not enter the definition of the stress-energy tensor, the details are not repeated here.

Denoting the contribution of the anomaly free part of the system to the integral (4.64) by J_{Coulomb}, the result is, as per [8] ,

$$J_{\mathrm{Coulomb}} = 4\pi\,m\,Q_0 \tag{4.65}$$

in the notation of (4.56).

To calculate J_{anomaly}, the contribution of the anomaly term to (4.64), the b equation resulting from the variation of Ω_{CW} w.r.t. A_0 must be employed. This follows simply from setting the index $\tau = 0$ in $\delta_{A_\tau}\tilde{\Omega}_{\mathrm{CW}}$ equation (A.19).

After imposition of axial symmetry by (4.46), (4.48) and (4.49), one has

$$\delta_{A_0}\Omega_{\rm CW} = \frac{1}{\rho}\left[a\left(\partial_{[\rho}\left(g-\sin g\right)\partial_{z]}\cos\Theta\right) - 2\,\partial_{[\rho}a\left(\partial_{z}g\cos\Theta + \sin g\,\partial_{z]}\cos\Theta\right)\right]. \tag{4.66}$$

leading to

$$J_{\rm anomaly} = 2\pi\int {\rm d}\rho\,{\rm d}z\,\partial_{[\rho}\left[a^2\left(\partial_{z]}g\,\cos\Theta + \sin g\,\partial_{z]}\cos\Theta\right)\right], \tag{4.67}$$

which is evaluated using Green's Theorem to give

$$J_{\rm anomaly} = 2\pi\,m^2\int_{z=-\infty}^{z=+\infty}\left(\partial_z g\,\cos\Theta + \sin g\,\partial_z\cos\Theta\right)\Big|_{\rho=0}{\rm d}z \tag{4.68}$$

where we have used $a(\infty) = m$, and further using $\Theta = 0$ when $z<0$ and $\Theta = \pi$ at when $z>0$, finally resulting in

$$J_{\rm anomaly} = 4\pi\,m^2\,. \tag{4.69}$$

The total angular momentum $J = J_{\rm Coulomb} + \kappa J_{\rm anomaly}$ is finally

$$J = 4\pi\,m\,(Q_0+\kappa\,m) = m\,Q_e\,. \tag{4.70}$$

5. The solutions

5.1. The Ansatz, effective action and boundary solutions

The numerics are done in spherical coordinates (r,θ,φ), by employing an Ansatz in terms of five functions

$$\{\Phi_1(r,\theta),\ \Phi_2(r,\theta),\ \Phi_3(r,\theta)\}\quad\text{with}\quad \Phi_1^2+\Phi_2^2+\Phi_3^2=1,\ \ \text{and}\quad a(r,\theta),\ b(r,\theta), \tag{5.71}$$

the three real scalars Φ_a being related to the fields $\{\phi^I,\phi^4\}$ in the Skyrme field Ansatz (2.8), as follows

$$\phi^1 + i\,\phi^2 = \Phi_1(r,\theta)\,e^{i(m\varphi-\omega t)},\qquad \phi^3 = \Phi_2(r,\theta),\qquad \phi^4 = \Phi_3(r,\theta), \tag{5.72}$$

where m an integer. Also, to make contact with the previous work in the literature on the gauge decoupling limit, and to make more transparent the local $U(1)$ symmetry of the reduced action, we have introduced the harmonic frequency ω and the winding number m in the Skyrme field Ansatz, together with the gauge coupling constant e in the covariant derivative.

The gauge field Ansatz is

$$A_\mu{\rm d}x^\mu = a_\varphi\,(r,\theta)\,{\rm d}\varphi + b\,(r,\theta)\,{\rm d}t\,, \tag{5.73}$$

with a_φ and b the magnetic and electric potentials, respectively (with $a_\varphi = a-m$ in (4.45)).

This axially symmetric Ansatz results in the following two-dimensional effective action

$$S_{\it eff} = \int_0^\infty {\rm d}r\int_0^\pi {\rm d}\theta L_{\it eff}\,, \tag{5.74}$$

with

$$L_{\it eff} = r^2\sin\theta\left(\frac{1}{4}\mathcal{F}^2 + \lambda_1\frac{1}{2}L_2^{(S)} + \lambda_2\frac{1}{4}L_4^{(S)} + \mu^2\left(1-\Phi_3\right)\right) + \kappa L_{CW}, \tag{5.75}$$

where

$$\mathcal{F}^2 = \left(a_{\varphi,r}^2 + \frac{a_{\varphi,\theta}^2}{r^2}\right) - \left(b_{,r}^2 + \frac{b_{,\theta}^2}{r^2}\right), \tag{5.76}$$

$$L_2^{(S)} = \Phi_{1,r}^2 + \Phi_{2,r}^2 + \Phi_{3,r}^2 + \frac{1}{r^2}\left(\Phi_{1,\theta}^2 + \Phi_{2,\theta}^2 + \Phi_{3,\theta}^2\right) + \phi_1^2\left(\frac{(m+ea_\varphi)^2}{r^2\sin^2\theta} - (\omega - eb)^2\right), \tag{5.77}$$

$$L_4^{(S)} = \frac{4}{r^2}\Bigg[\left(\Phi_{3,\theta}\Phi_{2,r} - \Phi_{2,\theta}\Phi_{3,r}\right)^2 + \left(\Phi_{2,\theta}\Phi_{1,r} - \Phi_{1,\theta}\Phi_{2,r}\right)^2 + \left(\Phi_{3,\theta}\Phi_{1,r} - \Phi_{1,\theta}\Phi_{3,r}\right)^2 \tag{5.78}$$
$$+ r^2\phi_1^2\left(\frac{(m+ea_\varphi)^2}{r^2\sin^2\theta} - (\omega - eb)^2\right)\left(\Phi_{1,r}^2 + \Phi_{2,r}^2 + \Phi_{3,r}^2 + \frac{1}{r^2}\left(\Phi_{1,\theta}^2 + \Phi_{2,\theta}^2 + \Phi_{3,\theta}^2\right)\right)\Bigg],$$

while the CW term reads

$$L_{CW} = 2\Phi_1\left(\omega\left(a_\varphi + \frac{m}{e}\right) + m\left(b - \frac{\omega}{e}\right)\right)$$
$$\times\left(\Phi_1\left(\Phi_{2,\theta}\Phi_{3,r} - \Phi_{3,\theta}\Phi_{2,r}\right) + \Phi_2\left(\Phi_{3,\theta}\Phi_{1,r} - \Phi_{1,\theta}\Phi_{3,r}\right) + \Phi_3\left(\Phi_{1,\theta}\Phi_{2,r} - \Phi_{2,\theta}\Phi_{1,r}\right)\right)$$
$$\times e\Bigg[\Phi_2\left(\left(b - \frac{\omega}{e}\right)\left(a_{\varphi,\theta}\Phi_{3,r} - a_{\varphi,r}\Phi_{3,\theta}\right) + \left(a_\varphi + \frac{m}{e}\right)b_{,r}\Phi_{3,\theta} - b_{,\theta}\Phi_{3,r}\right), \tag{5.79}$$
$$+ \Phi_3\left(\left(b - \frac{\omega}{e}\right)\left(a_{\varphi,r}\Phi_{2,\theta} - a_{\varphi,\theta}\Phi_{2,r}\right) + \left(a_\varphi + \frac{m}{e}\right)\left(b_{,\theta}\Phi_{2,r} - b_{,r}\Phi_{2,\theta}\right)\right)\Bigg]$$
$$+ \left(\Phi_3\Phi_{2,\theta} - \Phi_2\Phi_{3,\theta}\right)\left(\omega a_{\varphi,r} + mb_{,r}\right) + \left(\Phi_2\Phi_{3,r} - \Phi_3\Phi_{2,r}\right)\left(\omega a_{\varphi,\theta} + mb_{,\theta}\right).$$

The constants λ_1, λ_2, λ_M, μ and κ in the above relations are input parameters (with μ corresponding to the mass term for the Skyrme field). The main reason for introducing these constants (in particular λ_2, μ and κ) is to keep track of several limits of interest. The precise values used in the numerics will be specified in section 5.2 (and they will be chosen to match equations (2.1) and (2.2)).

Also, let us remark on the existence of a residual gauge symmetry, with the gauge field potentials always appearing in the combination

$$(\omega - eb) \quad \text{and} \quad (m + ea_\varphi)\ . \tag{5.80}$$

This freedom is fixed by imposing $\omega = 0$, and setting the magnetic gauge potential to zero at infinity.

The solutions are found numerically, by solving a boundary value problem. The boundary conditions imposed are

$$\Phi_1\big|_{r=0} = 0,\ \Phi_2\big|_{r=0} = 0,\ \Phi_3\big|_{r=0} = -1,\ a_\varphi\big|_{r=0} = 0,\ \partial_r b\big|_{r=0} = 0, \tag{5.81}$$

$$\Phi_1\big|_{r=\infty} = 0,\ \Phi_2\big|_{r=\infty} = 0,\ \Phi_3\big|_{r=\infty} = 1,\ a_\varphi\big|_{r=\infty} = 0,\ b\big|_{r=\infty} = V, \tag{5.82}$$

with V a constant corresponding to the electrostatic potential at infinity (which should not be confused with the potential term in (2.1)), and

$$\Phi_1\big|_{\theta=0,\pi} = 0,\ \partial_\theta\Phi_2\big|_{\theta=0,\pi} = 0,\ \partial_\theta\Phi_3\big|_{\theta=0,\pi} = 0,\ a_\varphi\big|_{\theta=0,\pi} = 0,\ \partial_\theta b\big|_{\theta=0,\pi} = 0. \tag{5.83}$$

The solutions also possess a reflexion symmetry *w.r.t.* the equatorial plane, where we impose

$$\partial_\theta \Phi_1\big|_{\theta=\pi/2}=0,\ \Phi_2\big|_{\theta=\pi/2}=0,\ \partial_\theta\Phi_3\big|_{\theta=\pi/2}=0,\ \partial_\theta a_\varphi\big|_{\theta=\pi/2}=0,\ \partial_\theta b\big|_{\theta=\pi/2}=0. \quad (5.84)$$

In this approach the sigma-model constraint is not imposed directly, but rather emerges within the numerical scheme, where we use the Lagrange multiplier method [16]. In this approach, the effective Lagrangian of the model is supplemented with the constraint

$$L_{eff}\to L_{eff}^{(0)}+\sqrt{-g}\left(\sum_a \Phi^a\Phi^a-1\right)\mathcal{C}, \quad (5.85)$$

with $L_{eff}^{(0)}$ the initial effective lagrangian (5.75) and $\mathcal{C}$ is *a new function*. Then the equations for Φ_a get an extra-contribution,

$$\mathbb{E}q_a=\mathbb{E}q_a^{(0)}+2\mathcal{C}\Phi_a\sqrt{-g}=0,\ \text{ with }\ \mathbb{E}q_a^{(0)}=\partial_r\left(\frac{\partial L_{eff}}{\partial(\partial_r\Phi_a)}\right)+\partial_\theta\left(\frac{L_{eff}}{\partial(\partial_\theta\Phi_a)}\right)-\frac{\partial L_{eff}}{\partial\Phi_a}, \quad (5.86)$$

while the variation *w.r.t.* $\mathcal{C}$ imposes the sigma-model contraint as an extra equation. Then the equation (5.86) are multiplied each one with Φ_a and one takes their sum. After using also the equation for $\mathcal{C}$, one finds

$$\mathcal{C}=-\frac{1}{2\sqrt{-g}}\Phi_a\left(\mathbb{E}q_a^{(0)}\right). \quad (5.87)$$

Thus in the numerics, the equations for the Skyrme fields Φ_a are still (5.86), with $\mathcal{C}$ given by (5.87).

All numerical calculations in the axially symmetric case have been performed by using a professional package, which uses a Newton-Raphson finite difference method with an arbitrary grid and arbitrary consistency order [17]. In the numerics, a compactified radial coordinate x was introduced, with

$$x=\frac{r}{1+r}, \quad (5.88)$$

the equations being discretized on a $(x,\ \theta)$ grid with around 150×50 points. Then the resulting system is solved iteratively until convergence is achieved. The typical numerical error for the solutions reported in this work is estimated to be of the order of 10^{-3} (also, the order of the difference formulae was six).

The total mass-energy and angular momentum of the configurations is defined as the volume integral over all space of the ${T_0}^0$ and $T_\varphi 0$ components of the stress-energy tensor, with $M=\int d^3x{T_0}^0$ and $J=\int d^3x{T_\varphi}^0$. In addition, the configurations possess also a nonzero magnetic dipole moment μ_{mag} [7], which is read from the asymptotics of the magnetic potential, $a_\varphi\to\mu_{mag}\sin^2\theta/r+\ldots$.

5.2. Numerical results

Since the model is invariant under the change of sign of $A_0\equiv b$ and κ, we shall restrict our study to the case $\kappa\geqslant 0$. Also, all numerics is done for unit winding number, $m=1$, *i.e.* a unit topological charge. Moreover, the value of the gauge coupling constant is set to one.

Concerning the choice of other parameters in the reduced Lagrangian (5.75), the numerics is done for the choice

$$\mu=\lambda_1=2\lambda_2=\lambda_M=e=1, \quad (5.89)$$

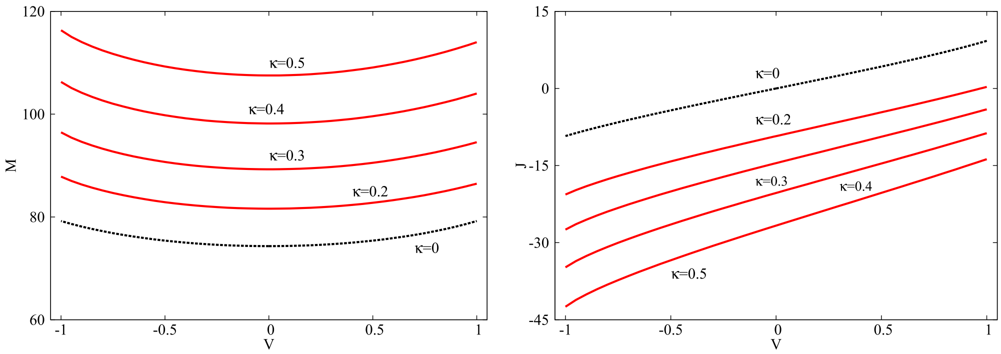

Figure 1. Mass (left panel) and angular momentum (right panel) for gauged Skyrmions are shown in terms of electrostatic potential at infinity V for several values of the constant κ multiplying the Callan–Witten term.

unless specified otherwise. Also, we report results for several values of κ ranging from zero to 1/2 (note that solutions are likely to exist for arbitrary κ; however, the numerical errors increase for larger κ).

Some of the properties of the solutions are similar to those found in the literature for $\kappa = 0$. The solutions exist for a finite range of the electrostatic potential at infinity,

$$e^2V^2 \leqslant \mu^2, \tag{5.90}$$

as found from a study of the far field behaviour of the scalar functions. We also mention that the limiting configuration with $eV = \pm\mu$ are still localized, with finite global charges. The solutions carry nonzero angular momentum densities, while their intrinsic shape (as found *e.g.* from the study of surfaces of constant energy density) corresponds to deformed spheres.

Other features are new, being found for $\kappa \neq 0$ only. First, as seen in figure 1 the symmetry of solutions under the change of sign $V \to -V$ (or $Q_e \to -Q_e$) is lost in the presence of a CW term. Moreover, configurations with $J = 0$ and still possessing a non-zero angular momentum density exist as well.

Also, for a fixed value of the electrostatic potential at infinity, the mass, angular momentum (and thus also the electric charge) increase with κ. At the same time, for a given model with some $\kappa > 0$, we notice a non-monotonic behaviour of the mass of the solutions as a function of electric charge, with a minimal value of M for some critical Q_e, see figure 2.

For $\kappa = 0$, the solutions with a vanishing electric potential at infinity are static, while the electric field vanishes, $b \equiv 0$. This no longer is the case for a nonzero κ, where we have found spinning, electrically charged configurations with $V = 0$, see figure 3 (left panel). Therefore, in this case the presence of a pion-mass term in the model is not necessary, and configurations with finite charges exist for $\mu = 0$ as well.

Another specific feature consists in the existence of solutions with finite mass and angular momentum in the absence of a quartic Skyrme term in the Lagrangian (5.75), $\lambda_2 = 0$. That is, such configurations are supported by the contribution of the CW-term, which provides the supplementary contribution to fulfill the Derrick-type scaling requirement. Otherwise, the main properties of these solutions are similar to those found for configurations in the full model, see figure 3 (right panel). Note that, as expected, for a given V, the mass and angular momentum of the $\lambda_2 = 0$ solutions is smaller than the corresponding solutions with a quartic term.

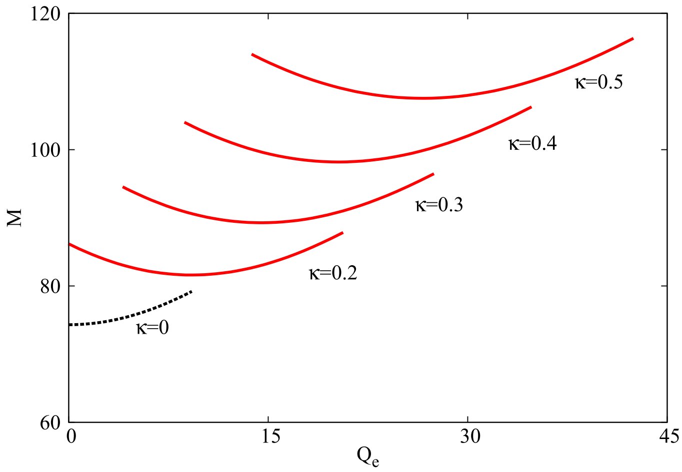

Figure 2. The mass of gauged Skyrmions is shown as a function of electric charge for several values of the input parameter κ.

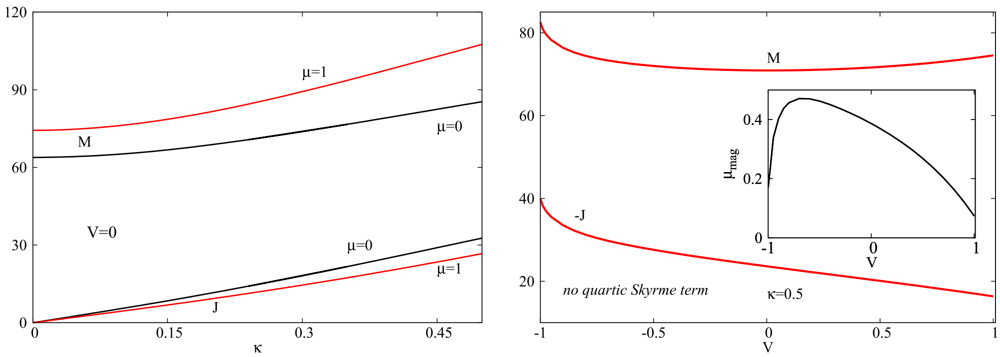

Figure 3. Left panel: the mass and the angular momentum are shown as functions of the constant κ for the special solutions with vanishing electric potential at infinity, and for two values of the pion mass μ. Right panel: mass, angular momentum and magnetic dipole moment are shown in terms of electrostatic potential at infinity V for gauged solutions in a model without a quartic Skyrme term.

6. Conclusions

The main purpose of this work was to present a preliminary investigation of $U(1)$-gauged Skyrmions in $3+1$ dimensions, in a model featuring the Callan–Witten (CW) anomaly term. To our knowledge such results have not appeared in the literature[10] to date. The main thrust of the work is to reveal the qualitative similarity of the dynamical effects of the CW density on the gauged Skyrmion, to that of the CS density observed in Abelian gauged Skyrmions (in odd dimensions).

[10] The CW term has been employed previously in the study [18] of charged black hole solutions with pion hair. However, the results there are found for a non-topological Ansatz for the Skyrme field.

It turns out that the presence of the CW term in the Abelian gauged Skyrme system in $3+1$ dimensions has most of the dynamical effects that the CS term has on the Abelian gauged Skyrme systems in $2+1$ and $4+1$ dimensions, observed previously in [10, 11] and [12, 13] respectively. Specifically, these effects are the unusual dependence of the mass/energy M on the electric charge Q_e and angular momentum J, contrasting with the usual monotonically increasing behaviour of M on (Q_e, J) such that M displays a minimum for a critical value of (Q_e, J). In other words, the slope of M *vs*. (Q_e, J) can be positive and negative. This is the salient message here.

The exception is the effect of CW term (in $3+1$ dimensions) on the 'baryon number" q as opposed to the effect of the CS term (in $2+1$ and $4+1$ dimensions) on q, which differs for different solutions characterised by a_∞. In the case of the CW term, q does not change. This may be a feature of Abelian gauging here, and it is conceivable that the situation may be different for non-Abelian gauged models, *e.g*. in [19, 20].

Another effect of the CW term in this model is that $SO(2)$ gauged Skyrmion persists even when the quartic kinetic term of the Skyrme scalar is suppressed, which contrasts with the usual case when the CW term is absent.

Finally it should be remarked that the CW density differs qualitatively from the CS density in that its definition includes the Skyrme scalar in addition to the gauge field while the CS density is defined exclusively in terms of the gauge field. This results in the definition of electric charge (and angular momentum) as a Noether charge. In this respect, the CW density is akin to the Skyrme–Chern–Simons (SCS) densities discussed most recently in [21]. In this context, it may be worth remarking that a SCS density can be defined also in odd spatial dimensions, and in particular in $2+1$ dimensions we have verified that the salient effects of that SCS term are the same as those resulting from the CS term. This work is in active development will appear in the near future.

All numerical results reported in this work are for unit winding number, $m=1$. However, similar (axially symmetric) solutions should exist for arbitrary $m>1$. Moreover, it would be interesting to extend these results for non-axisymmetric configurations which are known to exist in the ungauged case.

Data availability statement

No new data were created or analysed in this study.

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Appendix A. The $3+1$ constraint compliant parametrisation

The constraint compliant parametrisation employed here is

$$U=\cos\frac{g}{2}\mathbb{1}+i\,m^I\sigma_I\sin\frac{g}{2}\,,\quad I=1,2,3\,;\quad\text{and}\quad |m^I|^2=1\,,\tag{A.1}$$

in which according to the gauging prescription (2.11) and (2.12), the function g is gauge-inert and only the m^I gauge-rotate.

In the parametrisation (A.1),

$$U^{-1}\partial_\mu U=X_\mu^K\sigma_K\tag{A.2}$$

$$U\partial_\mu U^{-1}=Y_\mu^K\sigma_K\,,\quad\text{or}\quad \partial_\mu UU^{-1}=-Y_\mu^K\sigma_K\tag{A.3}$$

where

$$X_\mu^K=\ \frac{i}{2}\left[\partial_\mu g\,m^K+\sin g\,\partial_\mu m^K+(1-\cos g)\,\varepsilon^{IJK}m_I\partial_\mu m_J\right]\sigma_K\tag{A.4}$$

$$Y_\mu^K=\ -\frac{i}{2}\left[\partial_\mu g\,m^K+\sin g\,\partial_\mu m^K-(1-\cos g)\,\varepsilon^{IJK}m_I\partial_\mu m_J\right]\sigma_K\,,\tag{A.5}$$

and we further employ the constraint compliant polar parametrisation of $m^I=(m^\alpha,m^3)$

$$m^I=\begin{pmatrix}\sin\Theta\,(x_\mu)\cos\Phi\,(x_\mu)\\ \sin\Theta\,(x_\mu)\sin\Phi\,(x_\mu)\\ \cos\Theta\,(x_\mu)\end{pmatrix}\tag{A.6}$$

in which the function $\Theta(x_\mu)$ is gauge inert and the function $\Phi(x_\mu)$ translates with the $SO(2)$ gauge transformation.

The quadratic kinetic Skyrme term in this parametrisation takes the form

$$|D_\mu\phi^a|^2=\frac{1}{4}|\partial_\mu g|^2+\frac{1}{2}(1-\cos g)\left[\partial_\mu\Theta|^2+\sin^2\Theta|\,(A_\mu-\partial_\mu\Phi)\,|^2\right]\tag{A.7}$$

and the quartic kinetic Skyrme term in this parametrisation takes the form

$$|D_{[\mu}\phi^aD_{\nu]}\phi^b|^2=\frac{1}{4}(1-\cos g)\,|\partial_{[\mu}g\,\partial_{\nu]}\Theta|^2+\frac{1}{2}(1-\cos g)\sin^2\Theta\left[\frac{1}{2}|\partial_{[\mu}g\,\left(A_{\nu]}-\partial_{\nu]}\Phi\right)|^2\right.$$
$$\left.+(1-\cos g)\,\partial_{[\mu}\Theta\,|\left(A_{\nu]}-\partial_{\nu]}\Phi\right)|^2\right]\tag{A.8}$$

which are both gauge invariant as they stand, since the Maxwell connection A_μ appears only in the form

$$A_\mu-\partial_\mu\Phi\,.$$

Next we calculate the CW density, to which end we start by evaluating the traces

$$\varepsilon^{\tau\lambda\mu\nu}\mathrm{Tr}\,Q\left(U^{-1}\partial_\lambda U\right)\left(U^{-1}\partial_\mu U\right)\left(U^{-1}\partial_\nu U\right)\ =\ \frac{1}{4!}\varepsilon^{\tau\lambda\mu\nu}\varepsilon_{IJK}X_\lambda^KX_\mu^IX_\nu^J\tag{A.9}$$

$$\varepsilon^{\tau\lambda\mu\nu}\mathrm{Tr}\,Q\left(U\partial_\lambda U^{-1}\right)\left(U\partial_\mu U^{-1}\right)\left(U\partial_\nu U^{-1}\right)\ =\ -\frac{1}{4!}\varepsilon^{\tau\lambda\mu\nu}\varepsilon_{IJK}Y_\lambda^KY_\mu^IY_\nu^J\,.\tag{A.10}$$

Substituting (A.9) and (A.10) in (2.5) yields the expression for $W^{\tau}_{(1)}$

$$W^{\tau}_{(1)} = \frac{1}{2}\,\partial_\lambda\,(g-\sin g)\;\varepsilon^{\tau\lambda\mu\nu}\,\varepsilon_{IJK}\,m^I\partial_\mu m^J\partial_\nu m^K\,. \tag{A.11}$$

Substituting (A.4) and (A.5) in (2.6), taking account of (2.20), it follows that

$$W^{\tau\mu\nu}_{(2)} = -\frac{1}{2}\,\varepsilon^{\tau\lambda\mu\nu}\left(m^3\,\partial_\lambda g + \sin g\;\partial_\lambda m^3\right) \tag{A.12}$$

In the parametrisation (A.6), (A.11) and (A.12) are expressed as

$$W^{\tau}_{(1)} = 2\,\varepsilon^{\tau\lambda\mu\nu}\,\partial_\lambda\,(g-\sin g)\;\partial_\mu\cos\Theta\,\partial_\nu\Phi \tag{A.13}$$

$$W^{\tau\mu\nu}_{(2)} = \varepsilon^{\tau\lambda\mu\nu}\,(\partial_\lambda g\,\cos\Theta + \sin g\,\partial_\lambda\cos\Theta) \tag{A.14}$$

leading to the action densities

$$\Omega^{(1)}_{\rm CW} = 2\,\varepsilon^{\tau\lambda\mu\nu}A_\tau\,\partial_\lambda\,(g-\sin g)\;\partial_\mu\cos\Theta\,\partial_\nu\Phi \tag{A.15}$$

$$= -\,\varepsilon^{\tau\lambda\mu\nu}F_{\mu\nu}\,(g-\sin g)\,\partial_\tau\cos\Theta\,\partial_\lambda\Phi \tag{A.16}$$

$$\Omega^{(2)}_{\rm CW} = \varepsilon^{\tau\lambda\mu\nu}A_\tau\,F_{\mu\nu}\,(\partial_\lambda g\,\cos\Theta + \sin g\,\partial_\lambda\cos\Theta) \tag{A.17}$$

where (A.16) is equivalent to (A.15) up to a total divergence term., *i.e.*,

$$\Omega_{\rm CW} = \varepsilon^{\tau\lambda\mu\nu}F_{\mu\nu}\,[-(g-\sin g)\,\partial_\tau\cos\Theta\,\partial_\lambda\Phi \\ + A_\tau\,(\partial_\lambda g\,\cos\Theta + \sin g\,\partial_\lambda\cos\Theta)]\;. \tag{A.18}$$

A.1. Gauge invariance of the equations of the CW term

Unlike the quadratic and quartic Skyrme kinetic terms (A.7) and (A.8) which are gauge invariant as they stand, the CW density (A.18) is not explicitly gauge invariant, so the gauge invariance of the Euler–Lagrange equations arising from it must be checked. This is done explicitly by subjecting (A.18) to variations w.r.t. A_τ g, Θ and Φ. The results are

$$\delta_{A_\tau}\Omega_{\rm CW} = -\,2\varepsilon^{\tau\lambda\mu\nu}\,[(1-\cos g)\,\partial_\lambda g\,\partial_\mu\cos\Theta\,(A_\nu - \partial_\nu\Phi) \\ + F_{\mu\nu}\,(\partial_\lambda g\,\cos\Theta + \sin g\,\partial_\lambda\cos\Theta)] \tag{A.19}$$

$$\delta_g\Omega_{\rm CW} = -\,\varepsilon^{\tau\lambda\mu\nu}F_{\mu\nu}\left[(1-\cos g)\,\partial_\tau\cos\Theta\,(A_\lambda - \partial_\lambda\Phi) + \frac{1}{2}\,\cos\Theta\,F_{\tau\lambda}\right] \tag{A.20}$$

$$\delta_\Theta\Omega_{\rm CW} = \varepsilon^{\tau\lambda\mu\nu}F_{\mu\nu}\left[(1-\cos g)\,\partial_\tau g\,(A_\lambda - \partial_\lambda\Phi) - \frac{1}{2}\,\sin g\,F_{\tau\lambda}\right] \tag{A.21}$$

$$\delta_\Phi\Omega_{\rm CW} = -\,\varepsilon^{\tau\lambda\mu\nu}\,(1-\cos g)\,\partial_\tau g\;\partial_\lambda\cos\Theta\;F_{\mu\nu}\,, \tag{A.22}$$

which are gauge invariant since the connection A_μ appears only in the combination

$$A_\mu - \partial_\mu\Phi\,,$$

with all other terms there being by definition gauge invariant.

Appendix B. The $2+2$ constraint compliant parametrisation

The analysis carried out in the above subsection is carried out here in the formulation of [14], where instead of parametrising the Skyrme scalar by the $SU(2)$ group element U, the real valued four component field $\phi^a = (\phi^\alpha, \phi^A)$ of the $O(4)$ sigma model, satisfying the constraint

$$|\phi^a|^2 = |\phi^\alpha|^2 + |\phi^A|^2 = 1\,, \quad \alpha = 1,2\,,\ A = 3,4\,,$$

is employed, and the definition of the covariant derivative of ϕ^a given by (2.11) and (2.12), is precisely equivalent with the covariant derivative (2.7) of U.

This is much the more natural parametrisation of the $SO(2)$ gauged $O(4)$ sigma model scalar ϕ^a, but we have eschewed this choice and opted for the parametrisation of appendix A in the main text because of technicalities associated with the numerical construction.

The relation of the $SU(2)$ valued field U to the scalar ϕ^a is natural and is given by

$$U = \phi^a\,\tilde{\Sigma}_a\,, \quad \text{and} \quad U^\dagger = U^{-1} = \phi^a\,\Sigma_a \tag{B.1}$$

where

$$\Sigma_\alpha = i\sigma_\alpha \ \ , \Sigma_3 = i\sigma_3\,,\ \Sigma_4 = \mathbb{1} \quad \text{and} \quad \tilde{\Sigma}_\alpha = -i\sigma_\alpha \ \ , \tilde{\Sigma}_3 = -i\sigma_3\,,\ \tilde{\Sigma}_4 = \mathbb{1} \tag{B.2}$$

in terms of the Pauli spin matrices $\sigma_\alpha = (\sigma_1, \sigma_2)$ and σ_3.

In this notation, one can conveniently express the $su(2)$ algebra valued quantities

$$U^{-1}\partial_\mu U = -2\,\phi^a\,\Sigma^{(+)}_{ab}\,\phi^b \quad \text{and} \quad \partial_\mu U U^{-1} = 2\,\phi^a\,\Sigma^{(-)}_{ab}\,\phi^b \tag{B.3}$$

where $\Sigma^{(\pm)}_{ab}$ are the positive and negative chiral representations of the $SO(4)$ algebra, $\gamma_{ab} = -\frac{1}{4}[\gamma_a, \gamma_b]$,

$$\Sigma^{(\pm)}_{ab} = \frac{1}{2}\,(\mathbb{1} \pm \gamma_5)\,\gamma_{ab}$$

such that

$$\Sigma^{(+)}_{ab} = -\frac{1}{4}\,\Sigma_{[a}\tilde{\Sigma}_{b]} \quad \text{and} \quad \Sigma^{(-)}_{ab} = -\frac{1}{4}\,\tilde{\Sigma}_{[a}\Sigma_{b]}\,. \tag{B.4}$$

It is useful to note that $\Sigma^{(\pm)}_{ab}$ are self-dual and anti–self-dual respectively,

$$\Sigma^{(+)}_{ab} = \frac{1}{2}\,\varepsilon_{abcd}\,\Sigma^{(+)}_{cd} \quad \text{and} \quad \Sigma^{(-)}_{ab} = -\frac{1}{2}\,\varepsilon_{abcd}\,\Sigma^{(-)}_{cd}\,. \tag{B.5}$$

The natural parametrisation of ϕ^α and ϕ^A is

$$\phi^\alpha = \sin\frac{f}{2}\,n^\alpha\,, \qquad n^\alpha = \begin{pmatrix}\cos\psi\\ \sin\psi\end{pmatrix} \tag{B.6}$$

$$\phi^A = \cos\frac{f}{2}\,n^A\,, \qquad n^A = \begin{pmatrix}\cos\chi\\ \sin\chi\end{pmatrix}\,, \tag{B.7}$$

in which according to the gauging prescription (2.11) and (2.12), the functions f and $n^A[\chi]$ are gauge-inert and only the unit doublet $n^\alpha[\psi]$ gauge-rotates.

In this parametrisation the quadratic kinetic Skyrme term is calculated immediately to be

$$|D_\mu\phi^a|^2 = \frac{1}{4}|\partial_\mu f|^2 + \cos^2\frac{f}{2}|\partial_\mu\chi|^2 + \sin^2\frac{f}{2}|\,(A_\mu-\partial_\mu\psi)\,|^2 \tag{B.8}$$

and the quartic kinetic Skyrme term

$$\begin{aligned}|D_{[\mu}\phi^a D_{\nu]}\phi^b|^2 = \cos^2\frac{f}{2}&\left\{|\partial_\mu f|^2\,|\partial_\mu\chi|^2-(\partial_\mu f\partial_\mu\chi)^2\right\}\\ &+\sin^2\frac{f}{2}\left\{|\partial_\mu f|^2\,|\,(A_\nu-\partial_\nu\psi)\,|^2-[\partial_\mu f(A_\mu-\partial_\mu\psi)]^2\right\}\\ &+\sin^2\frac{f}{2}\left\{|\partial_\mu\chi|^2\,|(A_\nu-\partial_\nu\psi)|^2-[\partial_\mu\chi(A_\mu-\partial_\mu\psi)]^2)\right\}\end{aligned} \tag{B.9}$$

which are both gauge invariant.

Next, we proceed to calculate the CW density in this parametrisation. Substituting (B.6) and (B.7) into (B.3), and exploiting the (anti)–self-duality relations (B.5), one finds (after some lengthy calculation)

$$U^{-1}\,\partial_\mu U = i\,(X_\mu\,\sigma_3+Y_\mu\,\sigma_1+Z_\mu\,\sigma_2)\,, \tag{B.10}$$

$$\partial_\mu U\,U^{-1} = -\,i\,(\bar{X}_\mu\,\sigma_3+\bar{Y}_\mu\,\sigma_1+\bar{Z}_\mu\,\sigma_2)\,, \tag{B.11}$$

in which

$$\begin{aligned}X_\mu &= \frac{1}{2}\,[\partial_\mu\,(\psi+\chi)-\cos f\partial_\mu\,(\psi-\chi)]\,,\\ Y_\mu &= -\,\frac{1}{2}\,[\partial_\mu f\sin(\psi+\chi)+\sin f\cos(\psi+\chi)\,\partial_\mu\,(\psi-\chi)]\,,\\ Z_\mu &= \frac{1}{2}\,[\partial_\mu f\cos(\psi+\chi)-\sin f\sin(\psi+\chi)\,\partial_\mu\,(\psi-\chi)]\,,\end{aligned} \tag{B.12}$$

and

$$\begin{aligned}\bar{X}_\mu &= \frac{1}{2}\,[\partial_\mu\,(\psi-\chi)-\cos f\partial_\mu\,(\psi+\chi)]\,,\\ \bar{Y}_\mu &= -\,\frac{1}{2}\,[\partial_\mu f\sin(\psi-\chi)+\sin f\cos(\psi-\chi)\,\partial_\mu\,(\psi+\chi)]\,,\\ \bar{Z}_\mu &= \frac{1}{2}\,[\partial_\mu f\cos(\psi-\chi)-\sin f\sin(\psi-\chi)\,\partial_\mu\,(\psi+\chi)]\,.\end{aligned} \tag{B.13}$$

The result is,

$$\varepsilon^{\tau\lambda\mu\nu}\,\mathrm{Tr}Q[\left(U^{-1}\partial_\mu U\right)\left(U^{-1}\partial_\nu U\right)\left(U^{-1}\partial_\lambda U\right] = \frac{1}{2}\,\varepsilon^{\tau\lambda\mu\nu}\partial_\lambda f\sin f\partial_\mu\,\psi\,\partial\chi,$$

$$\varepsilon^{\tau\lambda\mu\nu}\,\mathrm{Tr}Q\left[\left(U\partial_\mu U^{-1}\right)\left(U\partial_\nu U^{-1}\right)\left(U\partial_\lambda U^{-1}\right)\right]) = \frac{1}{2}\,\varepsilon^{\tau\lambda\mu\nu}\partial_\lambda f\sin f\partial_\mu\,\psi\,\partial\chi,$$

which when substituted in (2.5) yield the density $\Omega^{(1)}_{\mathrm{CW}} = A_\tau\,W^\tau_{(1)}$ appearing in (2.3) and (2.4)

$$\Omega^{(1)}_{\mathrm{CW}} = \varepsilon^{\tau\lambda\mu\nu}A_\tau\,\partial_\lambda f\sin f\partial_\mu\,\psi\,\partial_\nu\chi\,. \tag{B.14}$$

Next, we dispose of the term $\Omega^{(2)}_{\mathrm{CW}} = W^{\tau\mu\nu}_{(2)}\,A_\tau F_{\mu\nu}$ in (2.3) and (2.4). For this, one calculates $W^{\tau\mu\nu}_{(2)}$ in (2.6). It can be checked that the second term in (2.6) yields exactly the same expression as the first term, which, can be readily calculated, using (B.10) and (B.11), to give

$$\Omega^{(2)}_{\mathrm{CW}} = \frac{i^2}{2}\,\varepsilon^{\tau\lambda\mu\nu}A_\tau F_{\mu\nu}\,\left(1+\cos f\right)\,\partial_\lambda\chi\,, \tag{B.15}$$

which allows the concise expression of $\Omega_{\rm CW} = \Omega^{(1)}_{\rm CW} + \Omega^{(2)}_{\rm CW}$

$$\begin{aligned}\Omega_{\rm CW} &= \varepsilon^{\tau\lambda\mu\nu}\left[A_\tau\,\partial_\lambda f \sin f \partial_\mu\,\psi\,\partial_\nu\chi - \frac{1}{2}A_\tau F_{\mu\nu}\,(1+\cos f)\,\partial_\lambda\chi\right] \\ &= -\varepsilon^{\tau\lambda\mu\nu}A_\tau\,\partial_\lambda\chi\left[-(\partial_\mu\cos f)\,\partial_\nu\psi + \frac{1}{2}F_{\mu\nu}\,(1+\cos f)\right].\end{aligned} \tag{B.16}$$

To test the gauge transformation properties of the resulting Euler–Lagrange equations of (B.16), one needs the transformation properties of the functions A_μ, ψ, χ and f. It can be seen from the definition of the covariant derivatives (2.11) and (2.12) and the parametrisations (B.6) and (B.7), that under the $SO(2)$ gauge transformation they transform the following way

$$A_\tau \to A_\tau + \partial_\tau\Lambda \tag{B.17}$$

$$\psi \to \psi + \Lambda \tag{B.18}$$

$$\chi \to \chi \tag{B.19}$$

$$f \to f, \tag{B.20}$$

where the functions χ and f are inert under the Abelian gauge transformation.

The resulting equations of motion w.r.t. variations of A_τ, f, ψ and χ, in that order are

$$\varepsilon^{\tau\lambda\mu\nu}\partial_\lambda\chi\left[\frac{1}{2}F_{\mu\nu}\,(1+\cos f) + (\partial_\mu\cos f)\,(A_\nu - \partial_\nu\psi)\right] = 0, \tag{B.21}$$

$$\frac{1}{2}\varepsilon^{\tau\lambda\mu\nu}F_{\mu\nu}\,\partial_\lambda\chi\,(A_\tau - \partial_\tau\psi) = 0, \tag{B.22}$$

$$-\frac{1}{2}\varepsilon^{\tau\lambda\mu\nu}F_{\mu\nu}\,\partial_\lambda\chi\,(\partial_\tau\cos f) = 0, \tag{B.23}$$

$$\varepsilon^{\tau\lambda\mu\nu}\left[\frac{1}{4}(1+\cos f)\,F_{\mu\nu}F_{\tau\lambda} - \frac{1}{2}F_{\mu\nu}\,(\partial_\lambda\cos f)\,(A_\tau - \partial_\tau\psi)\right] = 0, \tag{B.24}$$

each of which is gauge invariant according to (B.17)–(B.20).

The analysis carried out in section 3 employing the constraint compliant parametrisation of appendix A can be carried out readily using the parametrisation given here in appendix B. In particular the Maxwell equation analogous with (3.32) there and the corresponding Φ equation with (3.33), follow from (B.21) and (B.23) respectively for the variations of A_μ and ψ. The role played there by Φ in the definition of the Noether (electric) charge is played by the function ψ.

As for imposition of axial symmetry, here in analogy with (4.48) we have

$$f(x_\mu) = f(\rho,z)\ , \quad \chi(x_\mu) = \chi(\rho,z)\ , \quad \psi(x_\mu) = n\,\varphi \tag{B.25}$$

where φ is the azimuthal angle in the (x_1,x_2) plane and n is an integer.

The rest of the analysis in section 4 follows systematically with qualitatively similar results. The numerical work in section 4 however is carried out using exclusively the $3+1$ parametrisation of appendix A. This is because the numerical construction even of the (ungauged) 1-Skyrmion in the $2+2$ parametrisation here is far from straightforward.

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