Universidade de Aveiro
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Tese apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática e Aplicações, realizada sob a orientação científica dos professores Ana Pilar Foulquié Moreno, Departamento de Matemática da Universidade de Aveiro, Hamza Chaggara, Universidade de Sousse e Amílcar Branquinho, Universidade de Coimbra.

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palavras-Chave

Problema de Riemann–Hilbert, Polinómios ortogonais de matrizes, Polinómios biortogonais de matrizes, Equação de Pearson, Equação de Painlevé.

resumo

O nosso objectivo nesta tese é o de expandir, até onde for possível, noções e resultados da teoria de polinómios escalares até à dos polinómios matriciais. O nosso estudo começa por investigar o caso de sucessões de polinómios ortogonais matriciais de tipo semiclássico seguindo tão próximo quanto possível o que se sabe no caso escalar, tendo-se conseguido alguns resultados neste sentido.

Além disso, ao impor certas restrições aos graus dos polinómios envolvidos na equação de Pearson associada à função peso, estabelecemos caracterizações para os chamados polinómios ortogonais semiclássicos matriciais bem como para as correspondentes funções de segunda espécie matriciais que lhes estão associadas. Demos particular atenção ao caso clássico conhecido na literatura como famílias de polinómios ortogonais matriciais tipo Hermite, Laguerre, Jacobi e Bessel.

Esta monografia teve como motivação um estudo anterior em [15], que se concentrou em funções peso e polinómios ortogonais matriciais tipo Hermite. Tendo por base esse trabalho, concentramos a investigação nos pesos matricias tipo Laguerre, Jacobi ou Bessel (*cf*. [11, 12, 13, 14]). Estes exemplos incluem (quando considerados juntamente com o caso Hermite), pelo menos no que nos é dado a conhecer, as famílias estudadas na literatura.

Estabelecemos problemas de Riemann–Hilbert para estas famílias de polinómios ortogonais matriciais. Esta abordagem manifestou-se como uma óptima técnica que permite obter propriedades diferenciais para estas famílias de polinómios ortogonais matriciais, bem como para as correspondentes funções de segunda espécie. Assim sendo estabelecemos fórmulas de estrutura de primeira ordem bem como relações diferencias de segunda ordem para esta famílias.

O segmento final desta tese está dedicado às aplicações destas famílias a operadores diferencias matriciais de segunda ordem bem como a equações matriciais de Painlevé discretas. As relações que encontramos colapsam, quando consideradas com o seu análogo escalar, nas equações conhecidas para as famílias de polinómios ortogonais de Hermite, Laguerre, Jacobi e Bessel. Conseguimos ainda obter relações análogas para as famílias de funções de segundo tipo associadas a estas famílias de polinómios ortogonais escalares. Além disso, encontramos equações Painlevé que governam os coeficientes matricias da relação de recorrência de certas famílias semiclássicas.

keywords Riemann–Hilbert problem, Matrix orthogonal polynomials, Matrix biorthogonal polynomials, Pearson equation, Painlevé equation.

abstract

In this thesis, our objective is to expand upon existing notions and results, transitioning from scalar to matrix concepts in a highly versatile framework. Our exploration begins by delving into the realm of semiclassical matrix orthogonal polynomials with respect to a regular matrix weight function that satisfies a Pearson equation. Through our research, we unveil various characterizations that shed light on these polynomials.

Furthermore, by imposing certain restrictions on the degrees of polynomials involved in the Pearson equation associated with the weight function, we are able to establish characterizations for the semiclassical matrix orthogonal polynomials and corresponding second kind functions. Here, special attention was given to the classical case known in the literature as families of matrix orthogonal polynomials Hermite, Laguerre, Jacobi and Bessel type.

Our research is motivated by a previous study in [15], which focused on matrix weight functions of Hermite type. Building upon this foundation, we are excited to expand our investigation by introducing matrix weight functions of Laguerre, Jacobi, and Bessel types (*cf*. [11, 12, 13, 14]). To the best of our knowledge, these definitions are the most comprehensive and, when combined with the Hermite type, encompass all examples documented in the literature.

We set up Riemann–Hilbert problems for these families of matrix orthogonal polynomials. This approach proved to be an excellent technique that allows us to obtain differential properties for these families of matrix orthogonal polynomials, as well as for the corresponding functions of the second kind. Therefore, we established first-order structural formulas as well as second-order differential relations for these families.

The final segment of this thesis is dedicated to the applications of these families to second-order matrix differential operators as well as discrete Painlevé matrix equations. The second order differential relations we found collapse, when considered with their scalar analogue, into the known differential equations for the Hermite, Laguerre, Jacobi and Bessel families of orthogonal polynomials. However, we were able to obtain analogous relations for the families of functions of the second type associated with these families of scalar orthogonal polynomials. Furthermore, we find the significant Painlevé equations that govern the matrix coefficients of the recurrence relation of certain semiclassical families.

Contents

CHAPTER 1

Definitions and preliminary results

1. Introduction

The theory of orthogonal polynomials has a rich history. Here we follow the seminal works of mathematicians like Szegő [87], Akhiezer [1], Chihara [28], Ismail [71] and Van Assche [90], in order to enter in this beautiful area of Mathematics. The subject of orthogonal polynomials covers a wide range of topics within mathematics and physical problems. These include the moment problem, numerical quadrature, rational and polynomial interpolation and approximation, electromagnetism, potential theory and many other fields as well as their applications in engineering.

As the theory progressed, researchers extended the scalar theory of orthogonal polynomials to the matrix one. Matrix extensions of real orthogonal polynomials were first discussed back in 1949 by Krein [74, 75] and thereafter were studied sporadically until the last decade of the XX century, being some relevant papers [3, 8, 59]. In that way a comprehensive theory of matrix-valued orthogonal polynomials has been developed.

Notably, significant results from the theory of scalar-valued orthogonal polynomials, such as Favard's Theorem and Markov's Theorem, have been extended to the matrix-valued case [43, 44, 45, 52, 53]. In 1984, Aptekarev and Nikishin, for a kind of discrete Sturm–Liouville operators, solved the corresponding scattering problem in [3], and found that the polynomials that satisfy a three term recurrence relation of the form

$$xP_k(x) = A_kP_{k+1}(x) + B_kP_k(x) + A^*_{k-1}P_{k-1}(x), \qquad k \in \mathbb{N},$$

are orthogonal with respect to a positive definite measure, *i.e.* they derived a matrix version of Favard's Theorem.

Between 1990 and 2010, it was discovered that in some cases, matrix orthogonal polynomials exhibit properties similar to classical orthogonal polynomials. Grünbaum provided the first explicit nontrivial example of matrix-valued orthogonal polynomials satisfying a second-order differential equation in [63], as a byproduct of [66, 67, 68]. Later, additional examples were found in a different manner in [46].

The theory of matrix valued orthogonal polynomials has been developed from different perspectives and found applications in several areas of mathematics and mathematical physics including spectral theory [61], scattering theory [59], tiling problems [41], integrable systems [2, 4, 5], stochastic processes [64, 70] and time-band limiting problem [54, 62]. Recently, we finished an incursion to the topic in [24].

The use of Riemann–Hilbert problems has proven to be beneficial in the analysis of orthogonal polynomials, special functions and various other applications. These techniques, also known as Riemann–Hilbert methods, have been applied extensively in the realm of orthogonal polynomials, special functions, and applications. There are numerous examples in the literature from mathematics and physics, including nonlinear waves and integrable systems theory, statistical mechanics, random matrix theory, integrable probability and quantum mechanics.

It was in 1992, when Fokas, Its and Kitaev, in the context of 2D quantum gravity, discovered that certain Riemann–Hilbert problem was solved in terms of orthogonal polynomials on the real line, [55]. Namely, it was found that the solution of a 2×2 Riemann–Hilbert problem can be expressed in terms of orthogonal polynomials on the real line and its Cauchy transforms. Later, Deift and Zhou combined these ideas with a nonlinear steepest descent analysis in a series of papers [36, 37, 39, 40] which was the seed for a large activity in the field. To mention just a few relevant results let us cite the study of strong asymptotic with applications in random matrix theory, [36, 38], the analysis of determinantal point processes [33, 34, 76, 77], orthogonal Laurent polynomials [81, 82] and Painlevé equations [35, 72].

One remarkable outcome of this development was the appearance of Painlevé equations. These nonlinear differential equations were first discovered by the French mathematician Paul Painlevé in the late 19th century. They arise naturally in the study of special functions, integrable systems, and orthogonal polynomials. The Painlevé equations have since become a subject of great interest and importance in both pure and applied mathematics.

Despite the extensive research on extending results from the scalar to matrix setting, there are still many unexplored aspects in the realm of matrix-valued orthogonal polynomials that require further investigation. Throughout this thesis, our aim is to contribute to the existing body of knowledge, pushing the boundaries of understanding in the field of matrix orthogonal polynomials and their associated weight functions.

Organization

This thesis is organized in four chapters. In the first one we present the matrix orthogonal polynomials theory we need for the sequel. Here we mainly use the general references [15, 20, 65] as well as [51], where the authors show that the matrix orthogonal polynomials generically satisfy structured relations. In fact, this was our departure point to the world of semiclassical matrix orthogonal polynomials that we study in Chapter 2.

To study the family of semiclassical matrix orthogonal polynomials we need a deep understanding of the concepts of matrix three term recurrence relation, and of their consequences, as the Christoffel–Darboux formulas. The Riemann–Hilbert problem that we state in the Chapter 1 (*cf.* for example [15, 20]) gives some insight for the results in this thesis, as can be seen in Chapters 3 and 4.

Nevertheless our approach in Chapter 2 was to see how far one can go using the orthogonality in context of semiclassical weights. The main results in this chapter are Theorems 2.2 and 2.3 that pave the way for the Magnus type interpretation of the matrix semiclassical orthogonal polynomials given in Theorem 2.7. Another interesting result that, so far as we know is new, is the characterization of the semiclassical matrix orthogonal polynomials in terms of structure formulas, given in Theorem 2.5. We also show that the zero curvature formula characterizes this families of matrix orthogonal polynomials in Theorem 2.8. These last two results give us the idea of studying the sequences of functions that are in the constant jump fundamental matrix. The result of this study is in Section 5 of Chapter 2, about Geronimus characterization for the classical matrix orthogonal polynomials. We end Chapter 2 with a characterization of semiclassical matrix orthogonal polynomials in terms of a Riccati differential equation.

The Chapter 3 is devoted to the analytic study of logarithm derivatives of the constant jump fundamental matrices for generic weights of types Hermite, Laguerre, Jacobi and Bessel. From this study we easily find structure formulas, as well as second order matrix differential relations for the matrix orthogonal polynomials and associated second kind functions. Our contributions are mainly to the study of the Laguerre, Jacobi and Bessel classes (*cf.* [11, 12, 14]). Even so, we present the Hermite case (*cf.* [15]) in order to make the work self contained.

In Chapter 4 we continue presenting the main results of works [11, 12, 13, 14] and determine explicitly the logarithm derivatives of the constant jump fundamental matrices for the weights defined in Chapter 3 and for some of their extensions. Within these matrices, and from the zero curvature formula, we get Painlevé type matrix equations for the three term recurrence relations coefficients. We also explain, how the differential matrix operators associated with the matrix orthogonal polynomials studied in Chapter 3 can be interpreted in the scalar setting in terms of a differential equation of Bochner type for the scalar orthogonal polynomials and associated functions of second kind.

Notation

We start with some notations that will be useful later. In what follows:

- By $\mathbb{N}$ we denote the set of positive integers including the zero, *i.e.* $\{0, 1, \ldots\}$ and by $\mathbb{Z}_+$ the set of positive integers, *i.e.* $\{1, 2, \ldots\}$.
- $\mathbb{C}^n$ will be the set of complex vectors with n components.
- $\mathbb{C}^{n\times m}$ the set of matrices with n rows, m columns and complex entries. Hence, $\mathbb{C}^{N\times N}$ is the set of square matrices having the same number N of rows as columns.
- Let us denote respectively $\mathbf{0}$ and I the zero and the identity matrices.

- For sake of simplicity we will denote $\mathbf{0}$ the zero linear functional that all outputs are zero.
- We shall denote by $\mathbb{C}^{N\times N}[z]$ the set of matrix polynomials of size N, *i.e.*
$$\mathbb{C}^{N\times N}[z] = \Big\{ \sum_{k=0}^{n} \alpha_k z^k, n \in \mathbb{N} \Big| \, \alpha_k \in \mathbb{C}^{N\times N}, k \in \mathbb{N} \Big\},$$
and by means of $\big(\mathbb{C}^{N\times N}[z]\big)'$ its algebraic dual space, that is the space of linear functionals defined on $\mathbb{C}^{N\times N}[z]$.
- $\mathbb{C}_n^{N\times N}[z]$ will be the subset of matrix polynomials of $\mathbb{C}^{N\times N}[z]$ with a degree not greater than n.
- The adjugate of a square matrix M denoted $\operatorname{adj}(M)$ is the transpose of its cofactor matrix.
- For a given matrix $M \in \mathbb{C}^{N\times N}$, we denote M^* its conjugate transpose.

2. Weights, moments, and orthogonality

We give a summary of basic results we use for the rest of this work. Let
$$W = \begin{bmatrix} W^{(1,1)} & \dots & W^{(1,N)} \\ \vdots & \ddots & \vdots \\ W^{(N,1)} & \dots & W^{(N,N)} \end{bmatrix} \in \mathbb{C}^{N\times N},$$
be a $N \times N$ weight matrix with support on a smooth oriented non self-intersecting curve γ, in the complex plane $\mathbb{C}$, *i.e.* $W^{(j,k)}$ is, for each $j, k \in \{1, \dots, N\}$, a complex weight with support on γ.

The weight matrix induces a *matrix inner product* in the set of matrix polynomials $\mathbb{C}^{N\times N}[z]$ given by
$$\langle P, Q \rangle_W := \int_\gamma P(z) W(z) Q^*(z) \, \frac{\mathrm{d}\, z}{2\pi\,\mathrm{i}},$$
such that $Q^*(z) := \sum_{k=0}^{p} q_k^* z^k$ for a given $Q(z) = \sum_{k=0}^{p} q_k z^k \in \mathbb{C}^{N\times N}[z]$.

This matrix inner product possesses the standard sesquilinear properties. The orthogonality with respect to W means the orthogonality with respect to the inner product.

Definition 1.1. *We define the moment of order k associated with a weight matrix, W, as*
$$w_k := \int_\gamma t^k W(t) \, \frac{\mathrm{d}\, t}{2\pi\,\mathrm{i}}, \qquad k \in \mathbb{N}.$$

Within a sequence of moments $\big(w_k\big)_{k\in\mathbb{N}}$, in $\mathbb{C}^{N\times N}$ we define a linear functional $u : \mathbb{C}^{N\times N}[z] \to \mathbb{C}^{N\times N}$, as $w_k = \big(t^k, u\big)$, $k \in \mathbb{N}$.

Definition 1.2. *Let W be a matrix weight function (respectively, a functional of moments, u) and w_n the moment of order n associated to W (respectively, to u). We say that W (respectively, u) is regular, quasi-definite, or nonsingular, if $\det\Delta_n \neq 0$, $n \in \mathbb{N}$, where Δ_n is the Hankel-block matrix*

$$\Delta_n = \begin{bmatrix} w_0 & \cdots & w_n \\ \vdots & \ddots & \vdots \\ w_n & \cdots & w_{2n} \end{bmatrix}, \qquad n \in \mathbb{N}.$$

Given a weight matrix W, we say that $\{P_n^{\mathsf{L}}\}_{n\in\mathbb{N}}$ respectively, $\{P_n^{\mathsf{R}}\}_{n\in\mathbb{N}}$, is a *sequence of matrix polynomials left orthogonal* (respectively, *sequences of matrix polynomials right orthogonal*), with respect to W, if $\deg P_n^{\mathsf{L}} = \deg P_n^{\mathsf{R}} = n$, $n \in \mathbb{N}$, with nonsingular leading coefficient, and

$$\frac{1}{2\pi\,\mathrm{i}}\int_\gamma P_n^{\mathsf{L}}(z)W(z)z^k\,\mathrm{d}\,z = \delta_{n,k}C_n^{-1}, \qquad \text{left orthogonality} \tag{1.1}$$

$$\frac{1}{2\pi\,\mathrm{i}}\int_\gamma z^kW(z)P_n^{\mathsf{R}}(z)\,\mathrm{d}\,z = \delta_{n,k}C_n^{-1}, \qquad \text{right orthogonality} \tag{1.2}$$

for $k = 0, 1, \dots, n$ and $n \in \mathbb{N}$, where C_n is, for each $n \in \mathbb{N}$, a nonsingular matrix.

We say that two sequences of matrix monic polynomials $\{P_n\}_{n\in\mathbb{N}}$ and $\{R_n\}_{n\in\mathbb{N}}$, are biorthogonal with respect to a regular matrix weight W, if there exists a nonsingular matrix $C_n \in \mathbb{C}^{N\times N}$ such that:

$$\int_\gamma P_n(t)W(t)R_m(t)\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} = \delta_{n,m}C_n^{-1}, \qquad n, m \in \mathbb{N}. \tag{1.3}$$

We have that $\{P_n^{\mathsf{L}}\}_{n\in\mathbb{N}}$ and $\{P_n^{\mathsf{R}}\}_{n\in\mathbb{N}}$, just defined, are biorthogonal with respect to W, *i.e.*

$$\int_\gamma P_n^{\mathsf{L}}(t)W(t)P_m^{\mathsf{R}}(t)\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} = \delta_{n,m}C_n^{-1}, \qquad n, m \in \mathbb{N}.$$

Theorem 1.1. *The weight matrix, W, is regular if and only if, there exists a sequence $\{P_n^{\mathsf{L}}\}_{n\in\mathbb{N}}$ of left orthogonal matrix polynomails and $\{P_n^{\mathsf{R}}\}_{n\in\mathbb{N}}$ right orthogonal matrix polynomials with respect to W. Moreover, the sequences $\{P_n^{\mathsf{L}}\}_{n\in\mathbb{N}}$ and $\{P_n^{\mathsf{R}}\}_{n\in\mathbb{N}}$ are unique up to nonsingular, left and right matrix factor, respectively.*

Proof. We can see that the sequence of monic polynomials $\{P_n^{\mathsf{L}}\}_{n\in\mathbb{N}}$ are defined by (1.1) with respect to a regular matrix weight W. In fact, taking into account a representation for P_n^{L} as

$$P_n^{\mathsf{L}}(z) = p_{\mathsf{L},n}^0 z^n + p_{\mathsf{L},n}^1 z^{n-1} + \cdots + p_{\mathsf{L},n}^{n-1} z + p_{\mathsf{L},n}^n,$$

such that for each $j = 0, 1, \dots, n-1$,

$$\int_\gamma P_n^{\mathsf{L}}(z)W(z)z^j\,\frac{\mathrm{d}\,z}{2\pi\,\mathrm{i}} = p_{\mathsf{L},n}^0 w_{n+j} + p_{\mathsf{L},n}^1 w_{n+j-1} + \cdots + p_{\mathsf{L},n}^{n-1} w_{j+1} + p_{\mathsf{L},n}^n w_j = \mathbf{0},$$

and with $j = n$

$$\int_\gamma P^{\mathsf{L}}_n(z)w(z)z^n \frac{\mathrm{d}\, z}{2\pi\,\mathrm{i}} = p^0_{\mathsf{L},n}w_{2n} + p^1_{\mathsf{L},n}w_{2n-1} + \cdots + p^{n-1}_{\mathsf{L},n}w_{n+1} + p^n_{\mathsf{L},n}w_n = C_n^{-1}.$$

Let us notice that

$$\Delta_n = \begin{bmatrix} w_0 & \cdots & w_n \\ \vdots & \ddots & \vdots \\ w_n & \cdots & w_{2n} \end{bmatrix} \qquad \text{is such that} \qquad \det \Delta_n \neq 0, \qquad n \in \mathbb{N}.$$

In matrix notation we have

$$\begin{bmatrix} p^n_{\mathsf{L},n} & p^{n-1}_{\mathsf{L},n} & \cdots & p^1_{\mathsf{L},n} & p^0_{\mathsf{L},n} \end{bmatrix} \Delta_n = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & C_n^{-1} \end{bmatrix}.$$

Since $\det \Delta_n \neq 0$, we know that the above linear system has a unique solution, *i.e.* there exist and are unique the matrices $p^n_{\mathsf{L},n}, p^{n-1}_{\mathsf{L},n}, \ldots, p^1_{\mathsf{L},n}, p^0_{\mathsf{L},n}$, and so the sequence $\{P^{\mathsf{L}}_n\}_{n\in\mathbb{N}}$ is uniquely defined up to a multiplicative nonsingular matrix defined by (1.1).

As a direct consequence of the nonsingularity of the last block of Δ_n^{-1}, *i.e.* the one in the position $(n+1),(n+1)$, of the matrix Δ_n^{-1}, as (see for instance [58])

$$\Delta_n^{-1} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

with

$$D = \left(w_{2n} - \begin{bmatrix} w_n & \cdots & w_{2n-1} \end{bmatrix} \Delta_{n-1}^{-1} \begin{bmatrix} w_n^\top & \cdots & w_{2n-1}^\top \end{bmatrix}^\top \right)^{-1},$$

and $\det D = \frac{\det \Delta_{n-1}}{\det \Delta_n}$ we get that $p^0_{\mathsf{L},n}$ is a nonsingular matrix. The same can be seen for $\{P^{\mathsf{R}}_n\}_{n\in\mathbb{N}}$. □

A sequence of matrix polynomials $\{P_n\}_{n\in\mathbb{N}}$, is said to be monic if the leading coefficient of P_n is equal to the identity matrix, I. We can normalize the corresponding matrix orthogonal polynomial by choosing the monic ones. In what follows we will assume that choice. So, a unique sequence of nonsingular matrices $\left(C_n^{-1}\right)_{n\in\mathbb{N}}$ with $C_n^{-1} = \left(z^n P^{\mathsf{L}}_n(z), u\right) = \displaystyle\int_\gamma z^n P^{\mathsf{L}}_n(z) W(z) \frac{\mathrm{d}\, z}{2\pi\,\mathrm{i}}$, is associated with a regular, linear functional u or weight matrix W.

Theorem 1.2 (Favard)**.** *Let $\{P^{\mathsf{L}}_n\}_{n\in\mathbb{N}}$ be a sequence of monic matrix polynomials. Then, the following are equivalent:*

(i) $\{P^{\mathsf{L}}_n\}_{n\in\mathbb{N}}$ *is left-orthogonal with respect to a linear functional u.*
(ii) *There are sequences of scalar matrices $(\beta^{\mathsf{L}}_n)_{n\in\mathbb{N}}$, and $(\gamma^{\mathsf{L}}_n)_{n\in\mathbb{N}}$ with γ^{L}_n nonsingular matrices for n $\in \mathbb{N}$, such that the sequence $\{P^{\mathsf{L}}_n\}_{n\in\mathbb{N}}$ satisfies*

$$zP^{\mathsf{L}}_n(z) = P^{\mathsf{L}}_{n+1}(z) + \beta^{\mathsf{L}}_n P^{\mathsf{L}}_n(z) + \gamma^{\mathsf{L}}_n P^{\mathsf{L}}_{n-1}(z), \qquad n \in \mathbb{N},$$

where $P^{\mathsf{L}}_{-1}(z) = \mathbf{0}$ and $P^{\mathsf{L}}_0(z) = \mathrm{I}$

Proof. We will prove the sufficient condition. Since $\left\{P_n^{\mathsf{L}}\right\}_{n\in\mathbb{N}}$ is a basis in the $\mathbb{C}^{N\times N}$-left-module of matrix polynomials then there exist constant matrices, A_k^n, such that:

$$zP_n^{\mathsf{L}}(z) = \sum_{k=0}^{n+1} A_k^n P_k^{\mathsf{L}}(z), \qquad n \in \mathbb{N}.$$

Multiplying the identity successively by z^j for $j \in \{0,\dots,n\}$ on the right and applying the functional u, we obtain:

$$\begin{aligned} A_j^n &= \mathbf{0}, && j = 0,\dots,n-2 \\ A_{n-1}^n &= \left(P_n^{\mathsf{L}}(z)z^n, u\right) = C_n^{-1}C_{n-1} && \text{nonsingular} \\ A_n^n &= \left(P_n^{\mathsf{L}}(z)z^{n+1}, u\right)C_n - A_{n-1}^n\left(P_{n-1}^{\mathsf{L}}(z)z^n, u\right)C_n, \end{aligned}$$

and $A_{n+1}^n = \mathrm{I}$ by comparison of the highest powers in the above identity. Taking $\beta_n = A_n^n$, $\gamma_n = A_{n-1}^n$ the result follows.

For the necessary condition, we define recursively the matrix moments associated with the linear functional u by the following conditions

$$w_0 = \left(P_0^{\mathsf{L}}(z), u\right) = C_0^{-1} \qquad \text{and} \qquad \left(P_n^{\mathsf{L}}(z), u\right) = \mathbf{0}, \qquad n \in \mathbb{Z}_+,$$

where C_0 is a nonsingular matrix. Now, the polynomial P_n^{L} can be written as

$$P_n^{\mathsf{L}}(z) = p_{\mathsf{L},n}^0 z^n + p_{\mathsf{L},n}^1 z^{n-1} + \dots + p_{\mathsf{L},n}^{n-1} z + p_{\mathsf{L},n}^n$$

where $p_{\mathsf{L},n}^0 = \mathrm{I}$. Then we have

$$w_n + p_{\mathsf{L},n}^1 w_{n-1} + p_{\mathsf{L},n}^2 w_{n-2} + \dots + p_{\mathsf{L},n}^n = \mathbf{0}$$

thus the moments are defined recursively by $w_n = \sum_{k=0}^{n-1} p_{\mathsf{L},n}^{n-k} w_k$. Let us show that

$$\begin{aligned} \left(P_n^{\mathsf{L}}(z)z^k, u\right) &= \mathbf{0}, && k = 0,\dots,n-1, \\ \left(P_n^{\mathsf{L}}(z)z^n, u\right) &= C_n^{-1}, && n \in \mathbb{N}. \end{aligned}$$

Using the recurrence relation, we get for all $n = 2,3,\dots$

$$\left(zP_n^{\mathsf{L}}(z), u\right) = \Big(\left(P_{n+1}^{\mathsf{L}}(z) + \beta_n^{\mathsf{L}}P_n^{\mathsf{L}}(z) + \gamma_n^{\mathsf{L}}P_{n-1}^{\mathsf{L}}(z)\right), u\Big) = \mathbf{0}.$$

Again by multiplying both sides of the recurrence relation by z we get

$$z^2P_n^{\mathsf{L}}(z) = zP_{n+1}^{\mathsf{L}}(z) + \beta_n^{\mathsf{L}}zP_n^{\mathsf{L}}(z) + \gamma_n^{\mathsf{L}}zP_{n-1}^{\mathsf{L}}(z)$$

and, as a consequence,

$$\left(z^2P_n^{\mathsf{L}}(z), u\right) = \mathbf{0}, \qquad n = 3,4,\dots$$

In an analogous way, we conclude that

$$\left(z^kP_n^{\mathsf{L}}(z), u\right) = \mathbf{0}, \qquad k = 0,\dots,n-1.$$

For $k = n$, we have:

$$\left(z^nP_n^{\mathsf{L}}(z), u\right) = \Big(\left(zP_n^{\mathsf{L}}(z)\right)z^{n-1}, u\Big)$$

$$= \Big(\big(P^{\mathsf{L}}_{n+1}(z) + \beta^{\mathsf{L}}_n P^{\mathsf{L}}_n(z) + \gamma^{\mathsf{L}}_n P^{\mathsf{L}}_{n-1}(z) \big) z^{n-1}, u \Big)$$
$$= \gamma_n \big(P^{\mathsf{L}}_{n-1}(z) z^{n-1}, u \big) = \gamma^{\mathsf{L}}_n \gamma^{\mathsf{L}}_{n-1} \cdots \gamma^{\mathsf{L}}_1 C_0^{-1},$$

which ends the proof. □

A similar result can be obtained for the right-orthogonality, *i.e.*

$$z P^{\mathsf{R}}_n(z) = P^{\mathsf{R}}_{n+1}(z) + P^{\mathsf{R}}_n(z)\beta^{\mathsf{R}}_n + P^{\mathsf{R}}_{n-1}(z)\gamma^{\mathsf{R}}_n, \qquad n \in \mathbb{N},$$

where $P^{\mathsf{R}}_{-1}(z) = \mathbf{0}$ and $P^{\mathsf{R}}_0(z) = \mathrm{I}$.

As the polynomials are chosen to be monic, we can write:

$$P^{\mathsf{L}}_n(z) = \mathrm{I}\, z^n + p^1_{\mathsf{L},n} z^{n-1} + p^2_{L,n} z^{n-2} + \cdots + p^n_{\mathsf{L},n}, \tag{1.4}$$
$$P^{\mathsf{R}}_n(z) = \mathrm{I}\, z^n + p^1_{\mathsf{R},n} z^{n-1} + p^2_{\mathsf{R},n} z^{n-2} + \cdots + p^n_{\mathsf{R},n}, \tag{1.5}$$

where $p^i_{\mathsf{L},n}, p^i_{\mathsf{R},n} \in \mathbb{C}^{N\times N}$, $i = 0, \ldots, n-1$ and $n \in \mathbb{N}$. Also, β^{L}_n and γ^{L}_n will denote the related recurrence relation

$$z P^{\mathsf{L}}_n(z) = P^{\mathsf{L}}_{n+1}(z) + \beta^{\mathsf{L}}_n P^{\mathsf{L}}_n(z) + \gamma^{\mathsf{L}}_n P^{\mathsf{L}}_{n-1}(z), \qquad n \in \mathbb{N}, \tag{1.6}$$

with $P^{\mathsf{L}}_{-1}(z) = \mathbf{0}$ and $P^{\mathsf{L}}_0(z) = \mathrm{I}$. The, β^{R}_n and γ^{R}_n will denote the related recurrence relation

$$z P^{\mathsf{R}}_n(z) = P^{\mathsf{R}}_{n+1}(z) + P^{\mathsf{R}}_n(z)\beta^{\mathsf{R}}_n + P^{\mathsf{R}}_{n-1}(z)\gamma^{\mathsf{R}}_n, \qquad n \in \mathbb{N}, \tag{1.7}$$

with $P^{\mathsf{R}}_{-1}(z) = \mathbf{0}$ and $P^{\mathsf{R}}_0(z) = \mathrm{I}$. Moreover, the coefficients in (1.6) and (1.7) are given in terms of the ones in (1.4) and (1.5) by

$$\beta^{\mathsf{L}}_n = p^1_{\mathsf{L},n} - p^1_{\mathsf{L},n+1}, \qquad \gamma^{\mathsf{L}}_n = C_n^{-1} C_{n-1}, \tag{1.8}$$
$$\beta^{\mathsf{R}}_n = C_n \beta^{\mathsf{L}}_n C_n^{-1}, \qquad \gamma^{\mathsf{R}}_n = C_n \gamma^{\mathsf{L}}_n C_n^{-1} = C_{n-1} C_n^{-1} \qquad n \in \mathbb{N}. \tag{1.9}$$

3. Second kind functions, Stieltjes function and associated polynomials

Given a regular weight matrix we define the sequence of second kind matrix functions by

$$Q^{\mathsf{L}}_n(z) \coloneqq \int_\gamma \frac{P^{\mathsf{L}}_n(t)}{t-z} W(t) \frac{\mathrm{d}\, t}{2\pi\,\mathrm{i}}, \qquad Q^{\mathsf{R}}_n(z) \coloneqq \int_\gamma W(t) \frac{P^{\mathsf{R}}_n(t)}{t-z} \frac{\mathrm{d}\, t}{2\pi\,\mathrm{i}}. \tag{1.10}$$

When, $n = 0$ we are in presence of a Stieltjes–Markov matrix function of the weight matrix W, *i.e.*

$$S_W(z) \coloneqq \int_\gamma \frac{W(t)}{t-z} \frac{\mathrm{d}\, t}{2\pi\,\mathrm{i}}. \tag{1.11}$$

Theorem 1.3. *Let a and b be the starting and end points of γ, respectively. Let C be a simple closed curve (circle for example) negatively oriented (clockwise), such that a and b are in the interior of C. Then the Stieltjes–Markov matrix function S_W is a complex measure of orthogonality for $\{P^{\mathsf{L}}_n\}_{n\in\mathbb{N}}$ and $\{P^{\mathsf{R}}_n\}_{n\in\mathbb{N}}$ over C.*

Proof. We have the following identities

$$\begin{aligned}\int_C P^{\mathsf{L}}_n(z) S_W(z) P^{\mathsf{R}}_m(z)\frac{\mathrm{d}\,z}{2\pi\,\mathrm{i}} &= \int_C P^{\mathsf{L}}_n(z)\left(\int_\gamma \frac{W(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}\right) P^{\mathsf{R}}_m(z)\frac{\mathrm{d}\,z}{2\pi\,\mathrm{i}}\\ &= \int_\gamma\left(\int_C \frac{P^{\mathsf{L}}_n(z)W(t)P^{\mathsf{R}}_m(z)}{t-z}\frac{\mathrm{d}\,z}{2\pi\,\mathrm{i}}\right)\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} \qquad \text{(Fubini's Theorem)}\\ &= \int_\gamma P^{\mathsf{L}}_n(t)W(t)P^{\mathsf{R}}_m(t)\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} \qquad \text{(Cauchy's integral formula)}\end{aligned}$$

and so we get the desired result. □

Sometimes in the literature some authors distinguish between Markov transforms and Stieltjes transform when we are dealing with a measure defined on a bounded or an unbounded interval, respectively, of the real line. Here we unify the notion as the scalar Markov convergence theorem (stated for the bounded case) is still valid for the unbounded case when the moment problem is determined.

Theorem 1.4. *The left and right sequences of second kind functions* $\{Q^{\mathsf{L}}_n\}_{n\in\mathbb{N}}$ *and* $\{Q^{\mathsf{R}}_n\}_{n\in\mathbb{N}}$ *verify*

$$zQ^{\mathsf{L}}_n = Q^{\mathsf{L}}_{n+1}(z) + \beta^{\mathsf{L}}_n Q^{\mathsf{L}}_n(z) + \gamma^{\mathsf{L}}_n Q^{\mathsf{L}}_{n-1}(z), \qquad n\in\mathbb{N}, \tag{1.12}$$

$$zQ^{\mathsf{R}}_n = Q^{\mathsf{R}}_{n+1}(z) + Q^{\mathsf{R}}_n(z)\beta^{\mathsf{R}}_n + Q^{\mathsf{R}}_{n-1}(z)\gamma^{\mathsf{R}}_n, \qquad n\in\mathbb{N}, \tag{1.13}$$

with initial conditions $Q^{\mathsf{L}}_{-1}(z) = Q^{\mathsf{R}}_{-1} = -C^{-1}_{-1}$ *and* $Q^{\mathsf{L}}_0(z) = Q^{\mathsf{R}}_0(z) = S_W(z)$.

Proof. Multiplying the relation (1.6) on the right by $\dfrac{W(t)}{t-z}$ and integrating we get,

$$\int_\gamma \frac{tP^{\mathsf{L}}_n(t)}{t-z}W(t)\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} = Q^{\mathsf{L}}_{n+1}(z) + \beta^{\mathsf{L}}_n Q^{\mathsf{L}}_n(z) + \gamma^{\mathsf{L}}_n Q^{\mathsf{L}}_{n-1}(z).$$

As $\dfrac{t}{t-z} = 1 + \dfrac{z}{t-z}$, from the orthogonality condition (1.1), the result follows. The proof of (1.13) is similar. □

From the orthogonality conditions (1.1) and (1.2) we have, for all $n\in\mathbb{N}$, the following asymptotic expansion when $z\to\infty$ for the sequence of functions of the second kind

$$Q^{\mathsf{L}}_n(z) = -C^{-1}_n\big(\mathrm{I}\,z^{-n-1} + q^1_{\mathsf{L},n}z^{-n-2}+\cdots\big), \tag{1.14}$$

$$Q^{\mathsf{R}}_n(z) = -\big(\mathrm{I}\,z^{-n-1} + q^1_{\mathsf{R},n}z^{-n-2}+\cdots\big)C^{-1}_n. \tag{1.15}$$

Assuming that the measures $W^{(j,k)}$, $j,k\in\{1,\dots,N\}$ are Hölder continuous, and using the Plemelj's formula, *cf.* [57], applied to (1.10), we get the following fundamental jump identities

$$\big(Q^{\mathsf{L}}_n(z)\big)_+ - \big(Q^{\mathsf{L}}_n(z)\big)_- = P^{\mathsf{L}}_n(z)W(z), \tag{1.16}$$

$$\big(Q^{\mathsf{R}}_n(z)\big)_+ - \big(Q^{\mathsf{R}}_n(z)\big)_- = W(z)P^{\mathsf{R}}_n(z), \tag{1.17}$$

$z \in \gamma$, where, $\big(f(z)\big)_{\pm} = \lim_{\epsilon \to 0^{\pm}} f(z + i\epsilon)$. Here $\pm$ sign indicates the positive/negative region according to the orientation of the curve γ.

Definition 1.3. *We define the sequences of left associated polynomials* $\big\{P_n^{\mathsf{L},(1)}\big\}_{n\in\mathbb{N}}$ *with respect to* $\big\{P_n^{\mathsf{L}}\big\}_{n\in\mathbb{N}}$ *and* W *by*

$$P_{n-1}^{\mathsf{L},(1)}(z) = \int_{\gamma} \frac{P_n^{\mathsf{L}}(t) - P_n^{\mathsf{L}}(z)}{t - z} W(t) \frac{\mathrm{d}\, t}{2\pi\,\mathrm{i}}, \qquad n \in \mathbb{Z}_+,$$

Similarly, for the right situation we have the right associated polynomials $\big\{P_n^{\mathsf{R},(1)}\big\}_{n\in\mathbb{N}}$ *with respect to* $\big\{P_n^{\mathsf{R}}\big\}_{n\in\mathbb{N}}$ *and* W *by*

$$P_{n-1}^{\mathsf{R},(1)}(z) = \int_{\gamma} W(t) \frac{P_n^{\mathsf{R}}(t) - P_n^{\mathsf{R}}(z)}{t - z} \frac{\mathrm{d}\, t}{2\pi\,\mathrm{i}}, \qquad n \in \mathbb{Z}_+.$$

Theorem 1.5. *The associated polynomials verify*

$$zP_{n-1}^{\mathsf{L},(1)}(z) = P_n^{\mathsf{L},(1)}(z) + \beta_n^{\mathsf{L}} P_{n-1}^{\mathsf{L},(1)}(z) + \gamma_n^{\mathsf{L}} P_{n-2}^{\mathsf{L},(1)}(z), \qquad n \in \mathbb{N}, \tag{1.18}$$

with $P_{-2}^{\mathsf{L},(1)}(z) = -C_{-1}^{-1}$ *and* $P_{-1}^{\mathsf{L},(1)}(z) = \mathbf{0}$.

$$zP_{n-1}^{\mathsf{R},(1)}(z) = P_n^{\mathsf{R},(1)}(z) + P_{n-1}^{\mathsf{R},(1)}(z)\beta_n^{\mathsf{R}} + P_{n-2}^{\mathsf{R},(1)}(z)\gamma_n^{\mathsf{R}}, \qquad n \in \mathbb{N}, \tag{1.19}$$

with $P_{-2}^{\mathsf{R},(1)}(z) = -C_{-1}^{-1}$ *and* $P_{-1}^{\mathsf{R},(1)}(z) = \mathbf{0}$.

Proof. We will prove (1.18):

$$\begin{aligned} zP_{n-1}^{\mathsf{L},(1)}(z) &= z\int_{\gamma} \frac{P_n^{\mathsf{L}}(t) - P_n^{\mathsf{L}}(z)}{t - z} W(t) \frac{\mathrm{d}\, t}{2\pi\,\mathrm{i}} \\ &= \int_{\gamma} \frac{(z - t + t)P_n^{\mathsf{L}}(t) - zP_n^{\mathsf{L}}(z)}{t - z} W(t) \frac{\mathrm{d}\, t}{2\pi\,\mathrm{i}} \\ &= -\int_{\gamma} P_n^{\mathsf{L}}(t) W(t) \frac{\mathrm{d}\, t}{2\pi\,\mathrm{i}} + \int_{\gamma} \frac{tP_n^{\mathsf{L}}(t) - zP_n^{\mathsf{L}}(z)}{t - z} W(t) \frac{\mathrm{d}\, t}{2\pi\,\mathrm{i}} \end{aligned}$$

using now (1.6) and the orthogonality condition (1.1), we get

$$zP_{n-1}^{\mathsf{L},(1)}(z) = P_n^{\mathsf{L},(1)}(z) + \beta_n^{\mathsf{L}} P_{n-1}^{\mathsf{L},(1)}(z) + \gamma_n^{\mathsf{L}} P_{n-2}^{\mathsf{L},(1)}(z), \qquad n \in \mathbb{N}.$$

The proof of (1.19) follows by similar arguments. □

Theorem 1.6. *The Hermite–Padé formula for the left-orthogonal polynomials is given by,*

$$P_n^{\mathsf{L}}(z) S_W(z) + P_{n-1}^{\mathsf{L},(1)}(z) = Q_n^{\mathsf{L}}(z), \qquad n \in \mathbb{N}, \tag{1.20}$$

and the Hermite–Padé formula for the right-orthogonal polynomials is given by,

$$S_W(z) P_n^{\mathsf{R}}(z) + P_{n-1}^{\mathsf{R},(1)}(z) = Q_n^{\mathsf{R}}(z), \qquad n \in \mathbb{N}. \tag{1.21}$$

Proof. Using (1.10) we successively get

$$Q^{\mathsf{L}}_n(z)=\int_\gamma \frac{P^{\mathsf{L}}_n(t)}{t-z}W(t)\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}=\int_\gamma \frac{P^{\mathsf{L}}_n(t)-P^{\mathsf{L}}_n(z)+P^{\mathsf{L}}_n(z)}{t-z}W(t)\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}$$
$$=P^{\mathsf{L},(1)}_{n-1}(z)+P^{\mathsf{L}}_n(z)\int_\gamma\frac{W(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}=P^{\mathsf{L},(1)}_{n-1}(z)+P^{\mathsf{L}}_n(z)S_W(z).$$

In the same way we prove the Hermite–Padé formula for the right-orthogonal polynomials. □

We consider two possible reductions from biorthogonality to orthogonality:

(1) When the weight matrix, W, with support on γ is symmetric, *i.e.* $(W(z))^\top=W(z)$, $z\in\gamma$, then

$$P^{\mathsf{R}}_n(z)=\big(P^{\mathsf{L}}_n(z)\big)^\top, \qquad Q^{\mathsf{R}}_n(z)=\big(Q^{\mathsf{L}}_n(z)\big)^\top, \qquad z\in\mathbb{C}.$$

Moreover, $\langle P^{\mathsf{L}}_n,P^{\mathsf{L}}_n\rangle_W=\int_\gamma P^{\mathsf{L}}_n(x)W(x)(P^{\mathsf{L}}_n(x))^\top\frac{\mathrm{d}\,x}{2\pi\,\mathrm{i}}$.

(2) When the weight matrix W is Hermitian positive definite with support on $\gamma\subset\mathbb{R}$, *i.e.* $(W(x))^*=W(x)$, $x\in\mathbb{R}$, then

$$P^{\mathsf{R}}_n(z)=\big(P^{\mathsf{L}}_n(\bar z)\big)^*, \qquad Q^{\mathsf{R}}_n(z)=\big(Q^{\mathsf{L}}_n(\bar z)\big)^*, \qquad z\in\mathbb{C}.$$

In this case we have $\langle P^{\mathsf{L}}_n,P^{\mathsf{L}}_n\rangle_W=\int_{\mathbb{R}} P^{\mathsf{L}}_n(x)W(x)(P^{\mathsf{L}}_n(x))^*\frac{\mathrm{d}\,x}{2\pi\,\mathrm{i}}$.

4. Fundamental and transfer matrices

We can summarize the left three term recurrence relation (1.6) and (1.12) as follows

$$\begin{bmatrix}P^{\mathsf{L}}_{n+1}(z) & Q^{\mathsf{L}}_{n+1}(z)\\ -C_nP^{\mathsf{L}}_n(z) & -C_nQ^{\mathsf{L}}_n(z)\end{bmatrix}=\begin{bmatrix}z\,\mathrm{I}-\beta^{\mathsf{L}}_n & C_n^{-1}\\ -C_n & \mathbf{0}\end{bmatrix}\begin{bmatrix}P^{\mathsf{L}}_n(z) & Q^{\mathsf{L}}_n(z)\\ -C_{n-1}P^{\mathsf{L}}_{n-1}(z) & -C_{n-1}Q^{\mathsf{L}}_{n-1}(z)\end{bmatrix};$$

and by (1.18)

$$\begin{bmatrix}P^{\mathsf{L},(1)}_n(z)\\ -C_nP^{\mathsf{L},(1)}_{n-1}(z)\end{bmatrix}=\begin{bmatrix}z\,\mathrm{I}-\beta^{\mathsf{L}}_n & C_n^{-1}\\ -C_n & \mathbf{0}\end{bmatrix}\begin{bmatrix}P^{\mathsf{L},(1)}_{n-1}(z)\\ -C_{n-1}P^{\mathsf{L},(1)}_{n-2}(z)\end{bmatrix}.$$

In terms of the left fundamental matrix $Y^{\mathsf{L}}_n(z)$ and the left transfer matrix $T^{\mathsf{L}}_n(z)$,

$$Y^{\mathsf{L}}_n(z):=\begin{bmatrix}P^{\mathsf{L}}_n(z) & Q^{\mathsf{L}}_n(z)\\ -C_{n-1}P^{\mathsf{L}}_{n-1}(z) & -C_{n-1}Q^{\mathsf{L}}_{n-1}(z)\end{bmatrix}, \qquad T^{\mathsf{L}}_n(z):=\begin{bmatrix}z\,\mathrm{I}-\beta^{\mathsf{L}}_n & C_n^{-1}\\ -C_n & \mathbf{0}\end{bmatrix}, \tag{1.22}$$

we rewrite the above identities as follows

$$Y^{\mathsf{L}}_{n+1}(z)=T^{\mathsf{L}}_n(z)Y^{\mathsf{L}}_n(z), \qquad n\in\mathbb{N}.$$

From this relation and taking into account that $\det T_n^{\mathsf{L}} = 1$, one can see that $\det Y_n^{\mathsf{L}}(z) = \det Y_0^{\mathsf{L}}(z) = 1$ on $\mathbb{C} \setminus \gamma$ for $n \in \mathbb{N}$. For the right orthogonality, we similarly obtain from (1.7) and (1.13) that

$$\begin{bmatrix} P_{n+1}^{\mathsf{R}}(z) & -P_n^{\mathsf{R}}(z)C_n \\ Q_{n+1}^{\mathsf{R}}(z) & -Q_n^{\mathsf{R}}(z)C_n \end{bmatrix} = \begin{bmatrix} P_n^{\mathsf{R}}(z) & -P_{n-1}^{\mathsf{R}}(z)C_{n-1} \\ Q_n^{\mathsf{R}}(z) & -Q_{n-1}^{\mathsf{R}}(z)C_{n-1} \end{bmatrix} \begin{bmatrix} z\,\mathrm{I} - \beta_n^{\mathsf{R}} & -C_n \\ C_n^{-1} & \mathbf{0} \end{bmatrix}$$

and also by (1.19)

$$\begin{bmatrix} P_n^{\mathsf{R},(1)}(z) & -P_{n-1}^{\mathsf{R},(1)}(z)C_n \end{bmatrix} = \begin{bmatrix} P_{n-1}^{\mathsf{R},(1)}(z) & -P_{n-2}^{\mathsf{R},(1)}(z)C_n \end{bmatrix} \begin{bmatrix} z\,\mathrm{I} - \beta_n^{\mathsf{R}} & -C_n \\ C_n^{-1} & \mathbf{0} \end{bmatrix}$$

as we have the Hermite–Padé like formula for the right orthogonal polynomials,

$$Q_0^{\mathsf{R}}(z)\, P_n^{\mathsf{R}}(z) + P_{n-1}^{\mathsf{R},(1)}(z) = Q_n^{\mathsf{R}}(z).$$

Taking the right versions of fundamental matrix $Y_n^{\mathsf{R}}(z)$ and transfer matrix $T_n^{\mathsf{R}}(z)$,

$$Y_n^{\mathsf{R}}(z) := \begin{bmatrix} P_n^{\mathsf{R}}(z) & -P_{n-1}^{\mathsf{R}}(z)C_{n-1} \\ Q_n^{\mathsf{R}}(z) & -Q_{n-1}^{\mathsf{R}}(z)C_{n-1} \end{bmatrix}, \qquad T_n^{\mathsf{R}}(z) := \begin{bmatrix} z\,\mathrm{I} - \beta_n^{\mathsf{R}} & -C_n \\ C_n^{-1} & \mathbf{0} \end{bmatrix}, \tag{1.23}$$

we see that $\det Y_n^{\mathsf{R}}(z) = \det\, Y_0^{\mathsf{R}}(z) = 1$, because $\det T_n^{\mathsf{R}} = 1$ on $\mathbb{C} \setminus \gamma$ for $n \in \mathbb{N}$. Note that,

$$T_n^{\mathsf{R}}(z) = \begin{bmatrix} C_n & \mathbf{0} \\ \mathbf{0} & -C_n^{-1} \end{bmatrix} T_n^{\mathsf{L}}(z) \begin{bmatrix} C_n & \mathbf{0} \\ \mathbf{0} & -C_n^{-1} \end{bmatrix}^{-1}, \qquad n \in \mathbb{N}.$$

Lemma 1.1. *For any fixed integers s, r, m, M, the following holds, for all $n \in \mathbb{N}$:*

$$Y_{n-m}^{\mathsf{L}} = \prod_{i=0}^{r-m-1} (T_{n-m-i}^{\mathsf{L}}) Y_{n-r}^{\mathsf{L}}, \qquad m < r,$$

$$Y_{n-r-M}^{\mathsf{L}} = \prod_{i=0}^{M-1} (T_{n-r-i}^{\mathsf{L}})^{-1} Y_{n-r}^{\mathsf{L}}, \qquad M \geq 1.$$

For all n in $\mathbb{N}$ and k an integer such that $1 \leq k \leq n$ we have

$$Y_{n+k}^{\mathsf{L}} = \prod_{i=1}^{k} T_{n+i}^{\mathsf{L}} Y_n^{\mathsf{L}} \qquad \textit{and} \qquad Y_{n-k}^{\mathsf{L}} = \prod_{i=0}^{k-1} (T_{n+1-k+i}^{\mathsf{L}})^{-1} Y_n^{\mathsf{L}} = \left(\prod_{i=0}^{k-1} T_{n-i}^{\mathsf{L}} \right)^{-1} Y_n^{\mathsf{L}}.$$

Theorem 1.7 (Christoffel–Darboux formulas)**.** *For all $n \in \mathbb{N}$, we have that:*

$$(z-t) \sum_{k=0}^{n} P_k^{\mathsf{R}}(t) C_k P_k^{\mathsf{L}}(z) = P_n^{\mathsf{R}}(t) C_n P_{n+1}^{\mathsf{L}}(z) - P_{n+1}^{\mathsf{R}}(t) C_n P_n^{\mathsf{L}}(z), \tag{1.24}$$

$$(z-t) \sum_{k=0}^{n} Q_k^{\mathsf{R}}(t) C_k Q_k^{\mathsf{L}}(z) = Q_n^{\mathsf{R}}(t) C_n Q_{n+1}^{\mathsf{L}}(z) - Q_{n+1}^{\mathsf{R}}(t) C_n Q_n^{\mathsf{L}}(z), \tag{1.25}$$

$$(z-t) \sum_{k=0}^{n} Q_k^{\mathsf{R}}(t) C_k P_k^{\mathsf{L}}(z) = Q_n^{\mathsf{R}}(t) C_n P_{n+1}^{\mathsf{L}}(z) - Q_{n+1}^{\mathsf{R}}(t) C_n P_n^{\mathsf{L}}(z) + \mathrm{I}, \tag{1.26}$$

$$(z-t) \sum_{k=0}^{n} P_k^{\mathsf{R}}(t) C_k Q_k^{\mathsf{L}}(z) = P_n^{\mathsf{R}}(t) C_n Q_{n+1}^{\mathsf{L}}(z) - P_{n+1}^{\mathsf{R}}(t) C_n Q_n^{\mathsf{L}}(z) - \mathrm{I}, \tag{1.27}$$

$$(z-t)\sum_{k=0}^{n} P_k^{\mathsf{R},(1)}(t)C_kP_k^{\mathsf{L},(1)}(z) = P_{n-2}^{\mathsf{R},(1)}(t)C_{n-1}P_{n-1}^{\mathsf{R},(1)}(z) - P_{n-1}^{\mathsf{R},(1)}(t)C_{n-1}P_{n-2}^{\mathsf{L},(1)}(z), \tag{1.28}$$

$$(z-t)\sum_{k=0}^{n} P_{k-1}^{\mathsf{R},(1)}(t)C_kP_k^{\mathsf{L}}(z) = P_{n-1}^{\mathsf{R},(1)}(t)C_nP_{n+1}^{\mathsf{L}}(z) - P_{n}^{\mathsf{R},(1)}(t)C_nP_n^{\mathsf{L}}(z) + \mathrm{I}, \tag{1.29}$$

$$(z-t)\sum_{k=0}^{n} P_k^{\mathsf{R}}(t)C_kP_{k-1}^{\mathsf{L},(1)}(z) = P_n^{\mathsf{R}}(t)C_nP_n^{\mathsf{L},(1)}(z) - P_{n+1}^{\mathsf{R}}(t)C_nP_{n-1}^{\mathsf{L},(1)}(z) - \mathrm{I}\,. \tag{1.30}$$

Proof. To prove (1.24), we multiply (1.6) on the left by $P_n^{\mathsf{R}}(t)C_n$ and (1.7) on the right by $C_nP_n^{\mathsf{L}}(z)$ and taking into account (1.9), then summing up, we arrive after applying telescoping rule to the result. Proceeding in the same way with $\{Q_n^{\mathsf{L}}\}_{n\in\mathbb{N}}$ and $\{Q_n^{\mathsf{R}}\}_{n\in\mathbb{N}}$, $\{P_n^{\mathsf{L},(1)}\}_{n\in\mathbb{N}}$ and $\{P_n^{\mathsf{R},(1)}\}_{n\in\mathbb{N}}$ in place of $\{P_n^{\mathsf{L}}\}_{n\in\mathbb{N}}$ and $\{P_n^{\mathsf{R}}\}_{n\in\mathbb{N}}$ and using (1.12), (1.13), (1.18) and (1.19), the results (1.25) and (1.28) follow.

For the proof of (1.26), (1.27), (1.29) and (1.30) applying the same procedure mixing the P's, the Q's and first kind associated polynomials. □

If we take $t=z$ in the previous theorem, we get

$$P_n^{\mathsf{R}}(z)C_nP_{n+1}^{\mathsf{L}}(z) - P_{n+1}^{\mathsf{R}}(z)C_nP_n^{\mathsf{L}}(z) = \mathbf{0}, \qquad n\in\mathbb{N}, \tag{1.31}$$

$$Q_n^{\mathsf{R}}(z)C_nQ_{n+1}^{\mathsf{L}}(z) - Q_{n+1}^{\mathsf{R}}(z)C_nQ_n^{\mathsf{L}}(z) = \mathbf{0}, \qquad n\in\mathbb{N}, \tag{1.32}$$

$$Q_{n+1}^{\mathsf{R}}(z)C_nP_n^{\mathsf{L}}(z) - Q_n^{\mathsf{R}}(z)C_nP_{n+1}^{\mathsf{L}}(z) = \mathrm{I}, \qquad n\in\mathbb{N}, \tag{1.33}$$

$$P_n^{\mathsf{R}}(z)C_nQ_{n+1}^{\mathsf{L}}(z) - P_{n+1}^{\mathsf{R}}(z)C_nQ_n^{\mathsf{L}}(z) = \mathrm{I}, \qquad n\in\mathbb{N}, \tag{1.34}$$

$$P_{n-2}^{\mathsf{R},(1)}(z)C_{n-1}P_{n-1}^{\mathsf{R},(1)}(z) - P_{n-1}^{\mathsf{R},(1)}(z)C_{n-1}P_{n-2}^{\mathsf{L},(1)}(z) = \mathbf{0}, \qquad n\in\mathbb{N}, \tag{1.35}$$

$$P_n^{\mathsf{R},(1)}(t)C_nP_n^{\mathsf{L}}(z) - P_{n-1}^{\mathsf{R},(1)}(t)C_nP_{n+1}^{\mathsf{L}}(z) = \mathrm{I}, \qquad n\in\mathbb{N}, \tag{1.36}$$

$$P_n^{\mathsf{R}}(t)C_nP_n^{\mathsf{L},(1)}(z) - P_{n+1}^{\mathsf{R}}(t)C_nP_{n-1}^{\mathsf{L},(1)}(z) = \mathrm{I}, \qquad n\in\mathbb{N}. \tag{1.37}$$

Corollary 1.1. *The previous equations could be summarized as*

$$\big(Y_n^{\mathsf{L}}(z)\big)^{-1} = \begin{bmatrix}\mathbf{0} & \mathrm{I}\\ -\mathrm{I} & \mathbf{0}\end{bmatrix} Y_n^{\mathsf{R}}(z)\begin{bmatrix}\mathbf{0} & -\mathrm{I}\\ \mathrm{I} & \mathbf{0}\end{bmatrix}, \qquad n\in\mathbb{N}. \tag{1.38}$$

Proof. From (1.31), (1.32), (1.33), and (1.34) we get

$$\begin{bmatrix}-Q_{n-1}^{\mathsf{R}}(z)C_{n-1} & -Q_n^{\mathsf{R}}(z)\\ P_{n-1}^{\mathsf{R}}(z)C_{n-1} & P_n^{\mathsf{R}}(z)\end{bmatrix} Y_n^{\mathsf{L}}(z) = \mathrm{I}, \qquad n\in\mathbb{N},$$

straightforward calculation shows that

$$\begin{bmatrix}-Q_{n-1}^{\mathsf{R}}(z)C_{n-1} & -Q_n^{\mathsf{R}}(z)\\ P_{n-1}^{\mathsf{R}}(z)C_{n-1} & P_n^{\mathsf{R}}(z)\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \mathrm{I}\\ -\mathrm{I} & \mathbf{0}\end{bmatrix} Y_n^{\mathsf{R}}(z)\begin{bmatrix}\mathbf{0} & -\mathrm{I}\\ \mathrm{I} & \mathbf{0}\end{bmatrix}, \qquad n\in\mathbb{N},$$

Hence, we get (1.38), for all $n\in\mathbb{N}$. □

Corollary 1.2. *For all $n\in\mathbb{N}$, we have*

$$Q_n^{\mathsf{L}}(z)P_{n-1}^{\mathsf{R}} - P_n^{\mathsf{L}}Q_{n-1}^{\mathsf{R}}(z) = C_{n-1}^{-1} \tag{1.39}$$

$$P^{\mathsf{L}}_{n-1}(z)Q^{\mathsf{R}}_n - Q^{\mathsf{L}}_{n-1}P^{\mathsf{R}}_n(z) = C^{-1}_{n-1} \tag{1.40}$$

$$Q^{\mathsf{L}}_n(z)P^{\mathsf{R}}_n - P^{\mathsf{L}}_nQ^{\mathsf{R}}_n(z) = \mathbf{0}. \tag{1.41}$$

Proof. As we have already proved that the matrix

$$\begin{bmatrix} -Q^{\mathsf{R}}_{n-1}(z)C_{n-1} & -Q^{\mathsf{R}}_n(z) \\ P^{\mathsf{R}}_{n-1}(z)C_{n-1} & P^{\mathsf{R}}_n(z) \end{bmatrix}$$

is the inverse of $Y^{\mathsf{L}}_n(z)$, so

$$Y^{\mathsf{L}}_n(z)\begin{bmatrix} -Q^{\mathsf{R}}_{n-1}(z)C_{n-1} & -Q^{\mathsf{R}}_n(z) \\ P^{\mathsf{R}}_{n-1}(z)C_{n-1} & P^{\mathsf{R}}_n(z) \end{bmatrix} = \mathrm{I}$$

by identifying the components of this product, we get the result. □

Denoting $\widehat{Y}^{\mathsf{L}}_{n+1}(z) := \begin{bmatrix} P^{\mathsf{L}}_{n+1}(z) & Q^{\mathsf{L}}_{n+1}(z)W^{-1}(z) \\ -C_nP^{\mathsf{L}}_n(z) & -C_nQ^{\mathsf{L}}_n(z)W^{-1}(z) \end{bmatrix}$, then $\widehat{Y}^{\mathsf{L}}_{n+1}$ is invertible and

$$\left(\widehat{Y}^{\mathsf{L}}_{n+1}\right)^{-1} = \begin{bmatrix} -Q^{\mathsf{R}}_n(z)C_n & -Q^{\mathsf{R}}_{n+1}(z) \\ W(z)P^{\mathsf{R}}_n(z)C_n & W(z)P^{\mathsf{R}}_{n+1}(z) \end{bmatrix}.$$

Moreover, if we denote

$$\widetilde{Y}^{\mathsf{L}}_n(z) := \begin{bmatrix} P^{\mathsf{L}}_n(z) & P^{\mathsf{L},(1)}_{n-1}(z) \\ -C_{n-1}P^{\mathsf{L}}_{n-1}(z) & -C_{n-1}P^{\mathsf{L},(1)}_{n-2}(z) \end{bmatrix}, \quad \widetilde{Y}^{\mathsf{R}}_n(z) := \begin{bmatrix} P^{\mathsf{R}}_n(z) & -P^{\mathsf{R}}_{n-1}(z)C_{n-1} \\ P^{\mathsf{R},(1)}_{n-1}(z) & -P^{\mathsf{R},(1)}_{n-2}(z)C_{n-1} \end{bmatrix},$$

then, from (1.35), (1.36) and (1.37) it follows that $\widetilde{Y}^{\mathsf{R}}_n$ is invertible and,

$$\widetilde{Y}^{\mathsf{R}}_n(z)\begin{bmatrix} -C_{n-1}P^{\mathsf{L},(1)}_{n-2}(z) & C_{n-1}P^{\mathsf{L}}_{n-1}(z) \\ -P^{\mathsf{L},(1)}_{n-1}(z) & P^{\mathsf{L}}_n(z) \end{bmatrix} = \mathrm{I},$$

From this matrix equation we get

$$\begin{bmatrix} -C_{n-1}P^{\mathsf{L},(1)}_{n-2}(z) & C_{n-1}P^{\mathsf{L}}_{n-1}(z) \\ -P^{\mathsf{L},(1)}_{n-1}(z) & P^{\mathsf{L}}_n(z) \end{bmatrix}\widetilde{Y}^{\mathsf{R}}_n(z) = \mathrm{I},$$

or equivalently, the following Christoffel–Darboux formulas holds

$$P^{\mathsf{L},(1)}_{n-1}(z)P^{\mathsf{R}}_n(z) - P^{\mathsf{L}}_n(z)P^{\mathsf{R},(1)}_{n-1}(z) = \mathbf{0} \tag{1.42}$$

$$P^{\mathsf{L},(1)}_{n-1}(z)P^{\mathsf{R}}_{n-1}(z) - P^{\mathsf{L}}_n(z)P^{\mathsf{R},(1)}_{n-2}(z) = C^{-1}_{n-1}, \tag{1.43}$$

$$P^{\mathsf{L}}_{n-1}(z)P^{\mathsf{R},(1)}_{n-1}(z) - P^{\mathsf{L},(1)}_{n-2}(z)P^{\mathsf{R}}_n(z) = C^{-1}_{n-1}, \tag{1.44}$$

Furthermore, we gather the Hermite–Padé formulas (1.20), (1.21) in such way

$$\widetilde{Y}^{\mathsf{L}}_n(z)\begin{bmatrix} \mathrm{I} & S_W(z) \\ \mathbf{0} & \mathrm{I} \end{bmatrix} = Y^{\mathsf{L}}_n(z), \qquad \begin{bmatrix} \mathrm{I} & \mathbf{0} \\ S_W(z) & \mathrm{I} \end{bmatrix}\widetilde{Y}^{\mathsf{R}}_n(z) = Y^{\mathsf{R}}_n(z), \qquad n \in \mathbb{N}.$$

Theorem 1.8 (Riemann–Hilbert problem)**.** *Given a regular weight matrix W with support on γ, then the matrix function Y^{L}_n and Y^{R}_n, defined respectively by* (1.22) *and* (1.23) *satisfies, for each $n \in \mathbb{N}$, the following properties:*

(i) *Y^{L}_n and Y^{R}_n is holomorphic in $\mathbb{C} \setminus \gamma$.*

(ii) *Satisfies the jump condition*

$$\big(Y_n^{\mathsf{L}}(z)\big)_+ = \big(Y_n^{\mathsf{L}}(z)\big)_- \begin{bmatrix} \mathrm{I} & W(z) \\ \mathbf{0} & \mathrm{I} \end{bmatrix}, \qquad \big(Y_n^{\mathsf{R}}(z)\big)_+ = \begin{bmatrix} \mathrm{I} & \mathbf{0} \\ W(z) & \mathrm{I} \end{bmatrix} \big(Y_n^{\mathsf{R}}(z)\big)_-, \qquad z \in \gamma.$$

(iii) *Has the following asymptotic behavior, as* $z \to \infty$

$$Y_n^{\mathsf{L}}(z) = (\mathrm{I} + \mathrm{O}(1/z)) \begin{bmatrix} z^n\,\mathrm{I} & \mathbf{0} \\ \mathbf{0} & z^{-n}\,\mathrm{I} \end{bmatrix}, \qquad Y_n^{\mathsf{R}}(z) = \begin{bmatrix} \mathrm{I}\, z^n & \mathbf{0} \\ \mathbf{0} & \mathrm{I}\, z^{-n} \end{bmatrix} \Big(\mathrm{I} + \mathrm{O}(1/z)\Big).$$

Proof. Conditions ii) and iii) are direct consequences of the representation of the second kind functions (1.14), (1.15) and the inverse formulas (1.16), (1.17), respectively. □

We define the family of normalized left fundamental matrices $\big\{S_n^{\mathsf{L}}(z)\big\}_{n\in\mathbb{N}}$ associated with $\big\{Y_n^{\mathsf{L}}(z)\big\}_{n\in\mathbb{N}}$ by means of

$$S_n^{\mathsf{L}}(z) := Y_n^{\mathsf{L}}(z) \begin{bmatrix} \mathrm{I}\, z^{-n} & \mathbf{0} \\ \mathbf{0} & \mathrm{I}\, z^n \end{bmatrix}, \qquad n \in \mathbb{N}.$$

Taking into account the representation of $\big\{P_n^{\mathsf{L}}(z)\big\}_{n\in\mathbb{N}}$ and $\big\{Q_n^{\mathsf{L}}(z)\big\}_{n\in\mathbb{N}}$ in (1.6) and (1.12), we arrive to the asymptotic representation for the normalized fundamental matrices

$$S_n^{\mathsf{L}}(z) = \mathrm{I} + \begin{bmatrix} p^1_{\mathsf{L},n} & -C_n^{-1} \\ -C_{n-1} & q^1_{\mathsf{L},n-1} \end{bmatrix} \frac{1}{z} + \begin{bmatrix} p^2_{\mathsf{L},n} & -C_n^{-1} q^1_{\mathsf{L},n} \\ -C_{n-1} p^1_{\mathsf{L},n-1} & q^2_{\mathsf{L},n-1} \end{bmatrix} \frac{1}{z^2} + O(z^{-3}),$$

for $z \to \infty$, where

$$\begin{aligned} p^1_{\mathsf{L},n} - p^1_{\mathsf{L},n+1} &= \beta_n^{\mathsf{L}}, \\ p^2_{\mathsf{L},n} - p^2_{\mathsf{L},n+1} &= \beta_n^{\mathsf{L}} p^1_{\mathsf{L},n} + C_n^{-1} C_{n-1}, \\ p^3_{\mathsf{L},n} - p^3_{\mathsf{L},n+1} &= \beta_n^{\mathsf{L}} p^2_{\mathsf{L},n} + C_n^{-1} C_{n-1} p^1_{\mathsf{L},n-1}, \end{aligned}$$

and

$$\begin{aligned} q^1_{\mathsf{L},n} - q^1_{\mathsf{L},n-1} &= \beta_n^{\mathsf{R}}, \\ q^2_{\mathsf{L},n} - q^2_{\mathsf{L},n-1} &= \beta_n^{\mathsf{R}} q^1_{\mathsf{L},n} + C_n C_{n+1}^{-1}. \end{aligned}$$

Observe that we will also have the following asymptotics for $z \to \infty$,

$$\begin{aligned} \big(S_n^{\mathsf{L}}(z)\big)^{-1} = \mathrm{I} - &\begin{bmatrix} p^1_{\mathsf{L},n} & -C_n^{-1} \\ -C_{n-1} & q^1_{\mathsf{L},n-1} \end{bmatrix} \frac{1}{z} \\ &+ \left(\begin{bmatrix} p^1_{\mathsf{L},n} & -C_n^{-1} \\ -C_{n-1} & q^1_{\mathsf{L},n-1} \end{bmatrix}^2 - \begin{bmatrix} p^2_{\mathsf{L},n} & -C_n^{-1} q^1_{\mathsf{L},n} \\ -C_{n-1} p^1_{\mathsf{L},n-1} & q^2_{\mathsf{L},n-1} \end{bmatrix} \right) \frac{1}{z^2} + O(z^{-3}). \end{aligned}$$

For the right version we have normalized right fundamental matrices $\big\{S_n^{\mathsf{R}}(z)\big\}_{n\in\mathbb{N}}$ associated with $\big\{Y_n^{\mathsf{R}}(z)\big\}_{n\in\mathbb{N}}$

$$S_n^{\mathsf{R}}(z) = \begin{bmatrix} \mathrm{I}\ z^{-n} & \mathbf{0} \\ \mathbf{0} & \mathrm{I}\ z^n \end{bmatrix} Y_m^{\mathsf{R}}(z),$$

with asymptotic behavior at infinity given by

$$S_n^{\mathsf{R}}(z) = \mathrm{I} + \begin{bmatrix} p^1_{\mathsf{R},n} & -C_{n-1} \\ -C_n^{-1} & q^1_{\mathsf{R},n-1} \end{bmatrix} \frac{1}{z} + \begin{bmatrix} p^2_{\mathsf{R},n} & -p^1_{\mathsf{R},n-1} C_{n-1} \\ -q^1_{\mathsf{R},n} C_n^{-1} & q^2_{\mathsf{R},n-1} \end{bmatrix} \frac{1}{z^2} + O(z^{-3}),$$

for $z \to \infty$, and the asymptotics for the inverse matrix is

$$\begin{aligned}\big(S_n^{\mathsf{R}}(z)\big)^{-1} = \mathrm{I} - \begin{bmatrix} p^1_{\mathsf{R},n} & -C_{n-1} \\ -C_n^{-1} & q^1_{\mathsf{R},n-1} \end{bmatrix} \frac{1}{z} \\ + \left(\begin{bmatrix} p^1_{\mathsf{R},n} & -C_{n-1} \\ -C_n^{-1} & q^1_{\mathsf{R},n-1} \end{bmatrix}^2 - \begin{bmatrix} p^2_{\mathsf{R},n} & -p^1_{\mathsf{R},n-1} C_{n-1} \\ -q^1_{\mathsf{R},n} C_n^{-1} & q^2_{\mathsf{R},n-1} \end{bmatrix} \right) \frac{1}{z^2} + O(z^{-3}).\end{aligned}$$

Here

$$\begin{aligned} p^1_{\mathsf{R},n} - p^1_{\mathsf{R},n+1} &= \beta^{\mathsf{R}}_n, \\ p^2_{\mathsf{R},n} - p^2_{\mathsf{R},n+1} &= p^1_{\mathsf{R},n} \beta^{\mathsf{R}}_n + C_{n-1} C_n^{-1}, \\ p^3_{\mathsf{R},n} - p^3_{\mathsf{R},n+1} &= p^2_{\mathsf{R},n} \beta^{\mathsf{R}}_n + p^1_{\mathsf{L},n-1} C_{n-1} C_n^{-1}, \end{aligned}$$

and

$$\begin{aligned} q^1_{\mathsf{R},n} - q^1_{\mathsf{R},n-1} &= \beta^{\mathsf{L}}_n, \\ q^2_{\mathsf{R},n} - q^2_{\mathsf{R},n-1} &= q^1_{\mathsf{R},n} \beta^{\mathsf{L}}_n + C_{n+1}^{-1} C_n. \end{aligned}$$

5. Dual sequences and Riccati equation

We begin this section with the definition of dual sequence.

Definition 1.4. *The sequences of matrix functions* $\{\alpha_n^{\mathsf{L}}\}_{n\in\mathbb{N}}$, $\{\alpha_n^{\mathsf{R}}\}_{n\in\mathbb{N}}$ *defined on* γ *are said to be dual if*

$$\int_\gamma \alpha_n^{\mathsf{L}}(t) \alpha_m^{\mathsf{R}}(t) \frac{\mathrm{d}\, t}{2\pi\,\mathrm{i}} = C_n^{-1} \delta_{n,m}, \qquad n, m \in \mathbb{N},$$

where C_n *is nonsingular matrix.*

Given a regular weight matrix W, factorized in terms of two weight matrices W^{L} and W^{R}, in such way $W = W^{\mathsf{L}} W^{\mathsf{R}}$, we define the dual sequences

$$\alpha_n^{\mathsf{L}}(t) = P_n^{\mathsf{L}}(t) W^{\mathsf{L}}(t) \qquad \text{and} \qquad \alpha_n^{\mathsf{R}}(t) = W^{\mathsf{R}}(t) P_n^{\mathsf{R}}(t), \qquad n \in \mathbb{N}, \qquad t \in \gamma.$$

In fact,

$$\int_\gamma \alpha_n^{\mathsf{L}}(t) \alpha_m^{\mathsf{R}}(t) \frac{\mathrm{d}\, t}{2\pi\,\mathrm{i}} = \int_\gamma P_n^{\mathsf{L}}(t) W(t) P_m^{\mathsf{R}}(t) \frac{\mathrm{d}\, t}{2\pi\,\mathrm{i}} = C_n^{-1} \delta_{n,m}, \qquad n, m \in \mathbb{N}.$$

We underline that for a given weight matrix W we will have many possible factorization $W(z) = W^{\mathsf{L}}(z) W^{\mathsf{R}}(z)$. Indeed, if we define an equivalence relation

$$(W^{\mathsf{L}}, W^{\mathsf{R}}) \sim (\widetilde{W}^{\mathsf{L}}, \widetilde{W}^{\mathsf{R}}) \qquad \text{if and only if,} \qquad W^{\mathsf{L}} W^{\mathsf{R}} = \widetilde{W}^{\mathsf{L}} \widetilde{W}^{\mathsf{R}},$$

then each weight matrix W can be though as a class of equivalence, and can be described by the orbit

$$\big\{(W^{\mathsf{L}}\Sigma, \Sigma^{-1}W^{\mathsf{R}}),\ \Sigma(z)\ \text{ is a nonsingular matrix of entire functions}\big\}.$$

It is straightforward consequence of (1.6) and (1.7), and taking into account (1.8) and (1.9), that the dual sequences just defined satisfy the same three term recurrence relations as $\{P_n^{\mathsf{L}}\}_{n\in\mathbb{N}}$ and $\{P_n^{\mathsf{R}}\}_{n\in\mathbb{N}}$, respectively, *i.e.*

$$\begin{aligned} z\alpha_n^{\mathsf{L}}(z) &= \alpha_{n+1}^{\mathsf{L}}(z) + \beta_n^{\mathsf{L}}\alpha_n^{\mathsf{L}}(z) + \gamma_n^{\mathsf{L}}\alpha_{n-1}^{\mathsf{L}}(z) \\ z\alpha_n^{\mathsf{R}}(z) &= \alpha_{n+1}^{\mathsf{R}}(z) + \alpha_n^{\mathsf{R}}(z)\beta_n^{\mathsf{R}} + \alpha_{n-1}^{\mathsf{R}}(z)\gamma_n^{\mathsf{R}}, \end{aligned}$$

with initial conditions, $\alpha_{-1}^{\mathsf{L}} = \mathbf{0}$, $\alpha_0^{\mathsf{L}} = W^{\mathsf{L}}$, and $\alpha_{-1}^{\mathsf{R}} = \mathbf{0}$, $\alpha_0^{\mathsf{R}} = W^{\mathsf{R}}$.

Theorem 1.9. *Let $\{\alpha_n^{\mathsf{L}}\}_{n\in\mathbb{N}}$, $\{\alpha_n^{\mathsf{R}}\}_{n\in\mathbb{N}}$ be dual sequences defined on γ. Then, the sequence $\{\alpha_n^{\mathsf{L}}\}_{n\in\mathbb{N}}$ satisfies a three term recurrence relation of type* (1.6) *if and only if, the sequence $\{\alpha_n^{\mathsf{R}}\}_{n\in\mathbb{N}}$ satisfies a three term recurrence relation of type* (1.7)*.*

Proof. Let's consider, for example, that $\{\alpha_n^{\mathsf{L}}\}_{n\in\mathbb{N}}$ are such that

$$z\alpha_n^{\mathsf{L}}(z) = \alpha_{n+1}^{\mathsf{L}}(z) + \beta_n^{\mathsf{L}}\alpha_n^{\mathsf{L}}(z) + \gamma_n^{\mathsf{L}}\alpha_{n-1}^{\mathsf{L}}(z), \qquad n\in\mathbb{N}.$$

Then, let's write

$$z\alpha_n^{\mathsf{R}} = \sum_{k=0}^{\infty}\alpha_k^{\mathsf{R}}\,A_k^n, \tag{1.45}$$

where by duality,

$$\begin{aligned} C_m^{-1}A_m^n &= \int_\gamma z\alpha_m^{\mathsf{L}}(z)\,\alpha_n^{\mathsf{R}}(z)\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} = \int_\gamma \big(\alpha_{m+1}^{\mathsf{L}}(z) + \beta_m^{\mathsf{L}}\alpha_m^{\mathsf{L}}(z) + \gamma_m^{\mathsf{L}}\alpha_{m-1}^{\mathsf{L}}(z)\big)\,\alpha_n^{\mathsf{R}}(z)\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} \\ &= \begin{cases} C_n^{-1}, & m+1=n \\ \beta_n^{\mathsf{L}}C_n^{-1}, & m=n \\ \gamma_{n+1}^{\mathsf{L}}C_n^{-1}, & m-1=n \\ \mathbf{0}, & \text{if not} \end{cases} \end{aligned}$$

Hence, the equation (1.45) can be written as,

$$z\alpha_n^{\mathsf{R}}(z) = \alpha_{n+1}^{\mathsf{R}}(z) + \alpha_n^{\mathsf{R}}(z)\beta_n^{\mathsf{R}} + \alpha_{n-1}^{\mathsf{R}}(z)\gamma_n^{\mathsf{R}}, \qquad n\in\mathbb{N},$$

where, $\beta_n^{\mathsf{R}} = C_n\beta_n^{\mathsf{L}}C_n^{-1}$, $\gamma_n^{\mathsf{R}} = C_{n-1}C_n^{-1}$ (we observe that $C_{n+1}\gamma_{n+1}^{\mathsf{L}}C_n^{-1} = \mathrm{I}$).

On doing the same procedure departing from $\{\alpha_n^{\mathsf{R}}\}_{n\in\mathbb{N}}$ defining by (1.7) we get a three term recurrence relation of type (1.6) for $\{\alpha_n^{\mathsf{L}}\}_{n\in\mathbb{N}}$. □

Theorem 1.10. *Let a and b be the starting and end points of γ. Let C be a simple closed curve (circle for example) negatively oriented (clockwise), such that a and b are in the interior of C. Then, the sequence $\{P_n^{\mathsf{L}}W^{\mathsf{L}}\}_{n\in\mathbb{N}}$ (respectively,*

$\{Q^{\mathsf{L}}_n(W^{\mathsf{R}})^{-1}\}_{n\in\mathbb{N}}$*), is left dual of* $\{(W^{\mathsf{L}})^{-1}Q^{\mathsf{R}}_n\}_{n\in\mathbb{N}}$ *(respectively,* $\{W^{\mathsf{R}}P^{\mathsf{R}}_n\}_{n\in\mathbb{N}}$*), over* C*, i.e for all* $n,m\in\mathbb{N}$*, we have*

$$\int_C Q^{\mathsf{L}}_n(z)P^{\mathsf{R}}_m(z)\frac{\mathrm{d}\,z}{2\pi\,\mathrm{i}}=C_n^{-1}\delta_{n,m},\qquad \int_C P^{\mathsf{L}}_n(z)Q^{\mathsf{R}}_m(z)\frac{\mathrm{d}\,z}{2\pi\,\mathrm{i}}=C_n^{-1}\delta_{n,m}.$$

Proof. The result follows by analogous arguments as the one used in the proof of Theorem 1.3. In fact,

$$\begin{aligned}\int_C Q^{\mathsf{L}}_n(z)P^{\mathsf{R}}_m(z)\frac{\mathrm{d}\,z}{2\pi\,\mathrm{i}}&=\int_C\left(\int_\gamma\frac{P^{\mathsf{L}}_n(t)W(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}\right)P^{\mathsf{R}}_m(z)\frac{\mathrm{d}\,z}{2\pi\,\mathrm{i}}\\&=\int_\gamma P^{\mathsf{L}}_n(t)W(t)\left(\int_C\frac{P^{\mathsf{R}}_m(z)}{t-z}\frac{\mathrm{d}\,z}{2\pi\,\mathrm{i}}\right)\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}\qquad\text{(Fubini's Theorem)}\\&=\int_\gamma P^{\mathsf{L}}_n(t)W(t)P^{\mathsf{R}}_m(t)\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}\qquad\text{(Cauchy's integral formula)}\\&=C_n^{-1}\delta_{n,m},\qquad n,m\in\mathbb{N}.\end{aligned}$$

Similarly we obtain the result between $\{P^{\mathsf{L}}_nW^{\mathsf{L}}\}_{n\in\mathbb{N}}$ and $\{(W^{\mathsf{L}})^{-1}Q^{\mathsf{R}}_n\}_{n\in\mathbb{N}}$. □

In this section we are inspired in the work of Durán and Ismail [51].

Lemma 1.2. *For any polynomials* P *and any matrix function* F *with* $\mathcal{C}^1$ *entries, we have*

$$\int_\gamma F(t)W'(t)P(t)\,\mathrm{d}\,t=F(t)W(t)P(t)\Big]_{\partial\gamma}-\int_\gamma F'(t)W(t)P(t)\,\mathrm{d}\,t-\int_\gamma F(t)W(t)P'(t)\,\mathrm{d}\,t.$$

Proof. We first expand P as $\int_\gamma F(t)W'(t)P(t)\,\mathrm{d}\,t$ as a linear combination of t with matrix coefficients and joint the power of t to the function F to the left-hand side of W'. It is now enough to apply an integration by parts and then to recover the polynomial P and its derivative P' by moving the powers of t to the right-hand side of W. □

Theorem 1.11. *Let* $\{P^{\mathsf{L}}_n\}_{n\in\mathbb{N}}$*,* $\{P^{\mathsf{R}}_n\}_{n\in\mathbb{N}}$ *be sequences of biorthogonal polynomials with respect to* W*. Then* $P^{\mathsf{L}}_n(z)$ *satisfies the following differential recurrence relations (lowering and raising operators, respectively).*

$$\begin{aligned}(P^{\mathsf{L}}_n)'(z)&=\mathfrak{A}_n(z)C_{n-1}P^{\mathsf{L}}_{n-1}(z)-\mathfrak{B}_n(z)C_{n-1}P^{\mathsf{L}}_n(z)\\(P^{\mathsf{L}}_n)'(z)&=\big(\mathfrak{A}_n(z)C_n(zI-\beta_n)-\mathfrak{B}_n(z)C_{n-1}\big)P^{\mathsf{L}}_n-\mathfrak{A}_n(z)C_nP^{\mathsf{L}}_{n+1}\end{aligned}\tag{1.46}$$

where

$$\begin{aligned}\mathfrak{A}_n(z)&=\frac{P^{\mathsf{L}}_n(t)W(t)P^{\mathsf{R}}_n(t)}{t-z}\Big]_{\partial\gamma}-\int_\gamma\frac{P^{\mathsf{L}}_n(t)W'(t)P^{\mathsf{R}}_n(t)}{t-z}\,\mathrm{d}\,t,\\\mathfrak{B}_n(z)&=\frac{P^{\mathsf{L}}_n(t)W(t)P^{\mathsf{R}}_{n-1}(t)}{t-z}\Big]_{\partial\gamma}-\int_\gamma\frac{P^{\mathsf{L}}_n(t)W'(t)P^{\mathsf{R}}_{n-1}(t)}{t-z}\,\mathrm{d}\,t.\end{aligned}$$

Proof. We first prove (1.46). We start by stating the yielding relation

$$(P_n^{\mathsf{L}})'(z) = \sum_{k=0}^{n-1} A_k^n P_k^{\mathsf{L}}(z)$$

where

$$A_k^n = \int_\gamma (P_n^{\mathsf{L}})'(t) W(t) P_k^{\mathsf{R}}(t) \, \frac{\mathrm{d}\, t}{2\pi\, \mathrm{i}} \, C_k, \qquad\qquad k = 0, \dots, n-1$$

In fact, we can write

$$\big(P_n^{\mathsf{L}}\big)'(z) = \sum_{k=0}^{n-1} A_k^n P_k^{\mathsf{L}}(z) \qquad \text{and find} \qquad A_k^n.$$

Multiplying the previous equality on the right by $W P_m^{\mathsf{R}}$, $m = 0, \dots, n-1$, and integrate:

$$\begin{aligned}
\int_\gamma (P_n^{\mathsf{L}})'(t) W(t) P_m^{\mathsf{R}}(t) \, \frac{\mathrm{d}\, t}{2\pi\, \mathrm{i}} &= \int_\gamma \sum_{k=0}^{n-1} A_k^n P_k^{\mathsf{L}}(t) W(t) P_m^{\mathsf{R}}(t) \, \frac{\mathrm{d}\, t}{2\pi\, \mathrm{i}} \\
&= \sum_{k=0}^{n-1} A_k^n \int_\gamma P_k^{\mathsf{L}}(t) W(t) P_m^{\mathsf{R}}(t) \, \frac{\mathrm{d}\, t}{2\pi\, \mathrm{i}} \\
&= A_n^m C_m^{-1}. \qquad\qquad \text{(biorthogonality condition (1.3))}
\end{aligned}$$

Let $k = 0, \dots, n-1$

$$\begin{aligned}
\int_\gamma (P_n^{\mathsf{L}})'(t) W(t) P_k^{\mathsf{R}}(t) \; \mathrm{d}\, t &= P_n^{\mathsf{L}}(t) W(t) P_k^{\mathsf{R}}(t) \Big]_{\partial\gamma} - \int_\gamma P_n^{\mathsf{L}}(t) W'(t) P_k^{\mathsf{R}}(t) \; \mathrm{d}\, t \\
&\quad - \int_\gamma P_n^{\mathsf{L}}(t) W(t) (P_k^{\mathsf{R}})'(t) \; \mathrm{d}\, t \quad \text{(integration by parts)} \\
&= P_n^{\mathsf{L}}(t) W(t) P_k^{\mathsf{R}}(t) \Big]_{\partial\gamma} - \int_\gamma P_n^{\mathsf{L}}(t) W'(t) P_k^{\mathsf{R}}(t) \; \mathrm{d}\, t
\end{aligned}$$

So

$$\begin{aligned}
(P_n^{\mathsf{L}})'(z) = \sum_{k=0}^{n-1} A_k^n P_k^{\mathsf{L}}(z) &= \sum_{k=0}^{n-1} \left(P_n^{\mathsf{L}}(t) W(t) P_k^{\mathsf{R}}(t) \Big]_{\partial\gamma} - \int_\gamma P_n^{\mathsf{L}}(t) W'(t) P_k^{\mathsf{R}}(t) \; \mathrm{d}\, t \right) P_k^{\mathsf{L}}(z) \\
&= \sum_{k=0}^{n-1} P_n^{\mathsf{L}}(t) W(t) P_k^{\mathsf{R}}(t) \Big]_{\partial\gamma} - \int_\gamma P_n^{\mathsf{L}}(t) W'(t) \sum_{k=0}^{n-1} P_k^{\mathsf{R}}(t) C_k P_k^{\mathsf{L}}(z) \; \mathrm{d}\, t.
\end{aligned}$$

Using the fact that

$$(t - z) \sum_{k=0}^{n-1} P_k^{\mathsf{R}}(t) C_k P_k^{\mathsf{L}}(z) = P_n^{\mathsf{R}}(t) C_{n-1} P_{n-1}^{\mathsf{L}}(z) - P_{n-1}^{\mathsf{R}}(t) C_{n-1} P_n^{\mathsf{L}}(z)$$

we obtain the result.

To establish the raising differential relation, eliminate $C_{n-1} P_{n-1}^{\mathsf{L}}$ between (1.46) and (1.6). This completes the proof. □

Corollary 1.3. *Let* $\{P_n^{\mathsf{L}}\}_{n\in\mathbb{N}}$ *be a sequence of left orthogonal polynomials with respect to* W*, and define* $f_n = P_{n+1}^{\mathsf{L}}(P_n^{\mathsf{L}})^{-1}$*,* $n \in \mathbb{N}$*. Then,* $\{f_n\}_{n\in\mathbb{N}}$ *verifies a Riccati type matrix equation such that:*

$$f_n'(z) = \mathfrak{A}_{\mathfrak{n}+\mathfrak{1}}(z)C_n + f_n(z)\big(\mathfrak{B}_{\mathfrak{n}}(z)C_{n-1} - \mathfrak{A}_{\mathfrak{n}}(z)C_n(zI - \beta_n)\big) \\ - \mathfrak{B}_{\mathfrak{n}+\mathfrak{1}}(z)C_nf_n(z) + f_n(z)\mathfrak{A}_{\mathfrak{n}}(z)C_nf_n(z)$$

Proof. After replacing n by $n+1$ in (1.46) and multiplying on the right by $(P_n^{\mathsf{L}})^{-1}$, we get

$$(P_{n+1}^{\mathsf{L}})'(z)(P_n^{\mathsf{L}}(z))^{-1} = \mathfrak{A}_{\mathfrak{n}+\mathfrak{1}}(z)C_n - \mathfrak{B}_{\mathfrak{n}+\mathfrak{1}}(z)C_nf_n(z)\ , \tag{1.47}$$

then, multiplying (1.46) on the left by $-f_n$ and on the right by $(P_n^{\mathsf{L}})^{-1}$,

$$-P_{n+1}^{\mathsf{L}}(z)(P_n^{\mathsf{L}}(z))^{-1}(P_n^{\mathsf{L}})'(z)(P_n^{\mathsf{L}}(z))^{-1} \\ = f_n(z)\mathfrak{B}_{\mathfrak{n}}(z)C_{n-1} - f_n(z)\mathfrak{A}_{\mathfrak{n}}(z)C_{n-1}P_{n-1}^{\mathsf{L}}(z)(P_n^{\mathsf{L}}(z))^{-1} \tag{1.48}$$

So that

$$P_{n+1}^{\mathsf{L}}(z)\Big((P_n^{\mathsf{L}})^{-1}\Big)'(z) = f_n(z)\mathfrak{B}_{\mathfrak{n}}(z)C_{n-1} - f_n(z)\mathfrak{A}_{\mathfrak{n}}(z)C_{n-1}f_{n-1}^{-1},$$

by summing up (1.47) and (1.48), we get

$$f_n'(z) = \mathfrak{A}_{\mathfrak{n}+\mathfrak{1}}(z)C_n - \mathfrak{B}_{\mathfrak{n}+\mathfrak{1}}(z)C_nf_n(z) - f_n(z)\mathfrak{A}_{\mathfrak{n}}C_{n-1}f_{n-1}^{-1}(z) + f_n(z)\mathfrak{B}_{\mathfrak{n}}C_{n-1} \tag{1.49}$$

Using now (1.6) we obtain: $f_{n-1}^{-1}(z) = (\gamma_n^{\mathsf{L}})^{-1}\big(zI - \beta_n^{\mathsf{L}} - f_n(z)\big)$. By replacing $f_{n-1}^{-1}(z)$ in (1.49), and using $C_{n-1}\gamma_n^{-1} = C_n$ the result follows. □

CHAPTER 2

Semiclassical monic orthogonal polynomials

1. Introduction

One approach to study various families of matrix orthogonal polynomials is to extend the analysis of their differential properties. A natural progression in this line of research involves extending the theory of semiclassical scalar orthogonal polynomials to the matrix setting. In [18] semiclassical matrix orthogonal polynomials were defined using a Pearson type equation that relates to matrix functionals, *i.e.* for a given linear functional u, there exists a nonzero scalar polynomial ϕ and a matrix polynomial ψ, such that

$$\mathrm{D}(\phi u) = \psi u,$$

where D denotes the distributional derivative operator on the space $\left(\mathbb{C}^{N\times N}[z]\right)'$

$$\mathrm{D} : \left(\mathbb{C}^{N\times N}[z]\right)' \to \left(\mathbb{C}^{N\times N}[z]\right)' \qquad \text{with} \qquad (P, \mathrm{D}\, u) := -\left(P', u\right).$$

Similar to the scalar case, several characterizations were obtained, including a structural relation and a differential recurrence relations for the matrix orthogonal polynomials.

In [17], classical matrix orthogonal polynomials are defined as a specific instance of the semiclassical ones by imposing restrictions on the polynomial degrees in the Pearson type equation described above. However, unlike the scalar case, it was observed that the differential recurrence relations satisfied by semiclassical matrix orthogonal polynomials does not simply reduce to a differential equation in the classical scenario.

In this chapter, we delve into a broader context that highlights the non-commutativity aspect in the Pearson equation fulfilled by the regular matrix weight function W. More precisely, a noncommutative matrix Pearson type equation (called equivalently Sylvester type differential equation) is satisfied, *i.e.*

$$\phi(z)W'(z) = \psi_1(z)W(z) + W(z)\psi_2(z),$$

where ϕ is a nonzero scalar polynomial and ψ_1 and ψ_2 are two matrix polynomials. The current context expands upon the one explored in [17, 18]. In fact, the weight function W defines a linear functional u in the space $\mathbb{C}^{N\times N}[z]$ in the following way

$$(P, u) = \int_\gamma P(z)W(z)\,\frac{\mathrm{d}\,z}{2\pi\,\mathrm{i}}.$$

If some boundary conditions are taken at the endpoints of γ, a and b,

$$\lim_{z\to a}\phi(z)W(z)=\mathbf{0} \qquad \text{and} \qquad \lim_{z\to b}\phi(z)W(z)=\mathbf{0}, \tag{2.1}$$

then the linear functional u defined above satisfies the following equation

$$\phi(z)\,\mathrm{D}(u)=\psi_1(z)u+u\psi_2(z).$$

In fact, for all P in $\mathbb{C}^{N\times N}[z]$

$$\begin{aligned}\big(P,\phi\,\mathrm{D}(u)\big)&=-\big(\,(P\phi)'\,,u\big)=-\int_\gamma\big(P(z)\phi(z)\big)'W(z)\,\frac{\mathrm{d}\,z}{2\pi\,\mathrm{i}}\\&=\int_\gamma P(z)\phi(z)W'(z)\,\frac{\mathrm{d}\,z}{2\pi\,\mathrm{i}}\quad\text{(Integration by parts)}\\&=\int_\gamma P(z)\big(\psi_1(z)W(z)+W(z)\psi_2(z)\big)\,\frac{\mathrm{d}\,z}{2\pi\,\mathrm{i}}\\&=\big(P,\psi_1u\big)+\big(P,u\psi_2\big).\end{aligned}$$

Within this general framework, we establish various characterizations for the semiclassical matrix orthogonal polynomials. These characterizations encompass structural relations for the orthogonal polynomials, second kind functions, associated polynomials, and the Stieltjes–Markov matrix function. Moreover, these findings provide a characterization in terms of a Sylvester type equation for the fundamental matrix, and prove another characterization that involves the constant jump fundamental matrix when the weight matrix function can be factorized using two weights that satisfy left and right Pearson equations. Additionally, inspired by the Corollary of Theorem 1.11 we discover a new type of Riccati equation that characterizes these families of matrix orthogonal polynomials.

2. Pearson type equation

A particularly interesting family of matrix functionals is given by the ones which satisfy a matrix differential equation of Pearson type.

Definition 2.1. *We say that a regular matrix weight function W is semi-classical, if there exists a scalar polynomial ϕ and two matrix polynomials ψ_1 and ψ_2, with $\deg\phi\geq 0$ and $\deg\psi_1$, $\deg\psi_2\geq 1$ such that W satisfies the matrix Pearson type equation*

$$\phi(z)W'(z)=\psi_1(z)W(z)+W(z)\psi_2(z), \tag{2.2}$$

together with the boundary conditions (2.1)*.*

In this way, the corresponding sequence of left or right orthogonal matrix polynomials are called semiclassical. Moreover, if W satisfies (2.2) with

$$0\leq\deg\phi\leq 2 \qquad \text{and} \qquad \deg\psi_1=\deg\psi_2=1,$$

we say that W is a classical weight, and the corresponding sequence of left or right orthogonal matrix polynomials are called classical.

So far we discussed the properties of biorthogonal families of matrix polynomials and the fundamental matrices for a given weight matrix W. In what follows, to derive different characterization theorems for these families of matrix polynomials we will assume that the weight matrix defined in Definition 2.1 factors out as

$$W(z) = W^{\mathsf{L}}(z)W^{\mathsf{R}}(z), \qquad z \in \gamma.$$

We suppose that W is a semiclassical weight matrix, admitting the factorization $W = W^{\mathsf{L}}W^{\mathsf{R}}$, such that

$$\phi(z)\left(W^{\mathsf{L}}\right)'(z) = \psi_1(z)W^{\mathsf{L}}(z), \qquad \phi(z)\left(W^{\mathsf{R}}\right)'(z) = W^{\mathsf{R}}(z)\psi_2(z). \tag{2.3}$$

Theorem 2.1 (Pearson type differential equation)**.** *In the setting just described, any solution of the Pearson equation for the weight, W,* (2.2)*, is of the form $W(z) = W^{\mathsf{L}}(z)W^{\mathsf{R}}(z)$ where the matrix factors W^{L} and W^{R} are solutions of* (2.3)*.*

Proof. Given solutions W^{L} and W^{R} of (2.3), it follows intermediately, just using the Leibniz law for derivatives, that $W = W^{\mathsf{L}}W^{\mathsf{R}}$ fulfills (2.2). Moreover, given a solution W of (2.2) we pick a solution W^{L} of the first equation in (2.3), then it is easy to see that $(W^{\mathsf{L}})^{-1}W$ satisfies the second equation in (2.3). □

Now, we prove a characterization using Stieltjes–Markov matrix function.

Theorem 2.2. *Let ϕ be a scalar polynomials and W be a regular matrix weight function together with the boundary conditions* (2.1)*. The following are equivalent:*

(i) *W is semiclassical (*cf. *Definition 2.1).*
(ii) *There exists a matrix polynomial η such that, its Stieltjes–Markov matrix function, S_W, cf.* (1.11)*, satisfies*

$$\phi(z)S_W'(z) = \psi_1(z)S_W(z) + S_W(z)\psi_2(z) + \eta(z) \tag{2.4}$$

Proof. Assume that W satisfies

$$\phi(z)W'(z) = \psi_1(z)W(z) + W(z)\psi_2(z).$$

Then write

$$\begin{aligned}\phi(z)S_W'(z) &= \phi(z)\int_\gamma \frac{W(t)}{(t-z)^2}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}\\ &= \int_\gamma \frac{\phi(z)-\phi(t)}{(t-z)^2}W(t)\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} + \int_\gamma \frac{\phi(t)}{(t-z)^2}W(t)\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}\end{aligned}$$

Hence

$$\begin{aligned}\phi(z)S_W'(z) &= \int_\gamma \frac{\phi(z)-\phi(t)}{(t-z)^2}W(t)\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} - \frac{\phi(t)W(t)}{2\pi\,\mathrm{i}(t-z)}\Big]_{\partial\gamma} + \int_\gamma \frac{(\phi(t)W(t))'}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}\\ &= \int_\gamma \frac{\phi(z)-\phi(t)}{(t-z)^2}W(t)\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} + \int_\gamma \frac{\phi'(t)W(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} + \int_\gamma \frac{\phi(t)W'(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}\\ &= \int_\gamma \frac{\phi(z)-\phi(t)+\phi'(t)(t-z)}{(t-z)^2}W(t)\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} + \int_\gamma \frac{\phi(t)W'(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}.\end{aligned}$$

Now, using (2.2), we get

$$
\begin{aligned}
\phi(z)S_W'(z) &= \int_\gamma \frac{\phi(z)-\phi(t)+\phi'(t)(t-z)}{(t-z)^2} W(t)\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} \\
&\qquad + \int_\gamma \frac{\psi_1(t)W(t)}{t-z}\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} + \int_\gamma \frac{W(t)\psi_2(t)}{t-z}\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} \\
&= \int_\gamma \frac{\phi(z)-\phi(t)+\phi'(t)(t-z)}{(t-z)^2} W(t)\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} \\
&\qquad + \frac{\psi_1(z)}{2\pi\,\mathrm{i}}\int_\gamma \frac{W(t)}{t-z}\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} + \int_\gamma \frac{(\psi_1(t)-\psi_1(z))W(t)}{t-z}\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} \\
&\qquad + \int_\gamma W(t)\frac{\psi_2(t)-\psi_2(z)}{t-z}\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} + \int_\gamma \frac{W(t)}{t-z}\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}\psi_2(z).
\end{aligned}
$$

So, we arrive to

$$
\phi(z)S_W'(z) = \psi_1(z)S_W(z) + S_W(z)\psi_2(z) + \eta(z),
$$

with

$$
\begin{aligned}
\eta(z) &= \int_\gamma \frac{\phi(z)-\phi(t)+\phi'(t)(t-z)}{(t-z)^2} W(t)\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} \\
&\qquad + \int_\gamma \frac{\psi_1(t)-\psi_1(z)}{t-z}W(t)\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} + \int_\gamma W(t)\frac{\psi_2(t)-\psi_2(z)}{t-z}\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}.
\end{aligned} \tag{2.5}
$$

We notice that, η is a matrix polynomial. In fact, we only have to use the Taylor expansion of ϕ centered at t, *i.e.*

$$
\phi(z) = \phi(t) + \phi'(t)\,(z-t) + \frac{\phi''(t)}{2!}\,(z-t)^2 + \cdots + \frac{\phi^{(\deg\phi)}(t)}{(\deg\phi)!}\,(z-t)^{\deg\phi};
$$

and that $(t-z)$ divide $\big(P(t)-P(z)\big)$, $P\in\mathbb{C}^{N\times N}[z]$, to get the desired result.

Let us prove the reciprocal. From the calculation done in the first part of the proof we know

$$
\phi(z)S_W'(z) = \widetilde{\mathrm{pol}}_1(z) + \int_\gamma \frac{\phi(t)W'(t)}{t-z}\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}},
$$

and

$$
\psi_1(z)S_W(z) + S_W(z)\psi_2(z) = \int_\gamma \frac{\psi_1(t)W(t)}{t-z}\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} + \int_\gamma \frac{W(t)\psi_2(t)}{t-z}\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} + \widetilde{\mathrm{pol}}_2(z).
$$

where $\widetilde{\mathrm{pol}}_1$ and $\widetilde{\mathrm{pol}}_2$ are matrix polynomials. Hence (2.4) is equivalent to

$$
\int_\gamma \frac{\phi(t)W'(t)-\psi_1(t)W(t)-W(t)\psi_2(t)}{t-z}\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} = \widetilde{\mathrm{pol}}(z)
$$

where $\widetilde{\mathrm{pol}}$ is a matrix polynomial. Taking into account the behavior at infinity, and using Liouville's Theorem, we can assert that

$$\int_\gamma \frac{\phi(t)W'(t)-\psi_1(t)W(t)-W(t)\psi_2(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}=\mathbf{0}$$

Now, denoting $v(t):=\phi(t)W'(t)-\psi_1(t)W(t)-W(t)\psi_2(t)$ we get

$$\int_\gamma \frac{v(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}=\sum_{k=0}^{+\infty}\frac{v_k}{z^{k+1}},\qquad \left|\frac{t}{z}\right|<1,\qquad \text{with}\qquad v_k=-\int_\gamma v(t)t^k\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}.$$

Hence, $\phi(t)W'(t)-\psi_1(t)W(t)-W(t)\psi_2(t)=\mathbf{0}$. □

Corollary 2.1. *A regular matrix weight W together with the boundary conditions* (2.1) *is classical if and only if,*

$$\phi(z)S_W'(z)=\psi_1(z)S_W(z)+S_W(z)\psi_2(z)+\psi_1'(z)C_0^{-1}+C_0^{-1}\psi_2'(z)-\frac{1}{2}\phi''(z)C_0^{-1}.$$

Proof. It is enough to write

$$\psi_1(t)=\psi_1(z)+\psi_1'(z)(t-z),\qquad \psi_2(t)=\psi_2(z)+\psi_2'(z)(t-z),$$
$$\phi(t)=\phi(z)+\phi'(z)(t-z)+\frac{\phi''(z)}{2}(t-z)^2,$$

and substitute these expressions in (2.5), to get the result. □

3. Structure relation and differential recurrence relations

In this section we need the following technical Lemma.

Lemma 2.1. *Let W be a regular matrix weight with boundary conditions* (2.1)*, and $\left\{P_n^{\mathsf{L}}\right\}_{n\in\mathbb{N}}$, $\left\{Q_n^{\mathsf{L}}\right\}_{n\in\mathbb{N}}$, be the sequences of left monic orthogonal polynomials and functions of second kind, respectively. Then, we have for all $n\geq\max\left\{\deg\phi-1,\deg\psi_2\right\}$,*

$$\int_\gamma P_n^{\mathsf{L}}(t)\phi(t)\frac{W(t)}{(t-z)^2}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}=\phi(z)\big(Q_n^{\mathsf{L}}\big)'(z)+\phi'(z)Q_n^{\mathsf{L}}(z),\tag{2.6}$$

$$\int_\gamma P_n^{\mathsf{L}}(t)\phi'(t)\frac{W(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}=\phi'(z)Q_n^{\mathsf{L}}(z),\tag{2.7}$$

$$\int_\gamma P_n^{\mathsf{L}}(t)\frac{W(t)}{t-z}\psi_2(t)\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}=Q_n^{\mathsf{L}}(z)\psi_2(z).\tag{2.8}$$

Proof. We start with the proof of (2.6),

$$\int_\gamma P_n^{\mathsf{L}}(t)\phi(t)\frac{W(t)}{(t-z)^2}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}$$
$$=\int_\gamma\frac{P_n^{\mathsf{L}}(t)\big(\phi(t)-\phi(z)\big)W(t)}{(t-z)^2}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}+\phi(z)\int_\gamma\frac{P_n^{\mathsf{L}}(t)W(t)}{(t-z)^2}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}$$

$$= \int_\gamma \frac{P_n^{\mathsf{L}}(t)\big(\phi(t)-\phi(z)\big)W(t)}{(t-z)^2}\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} + \phi(z)\big(Q_n^{\mathsf{L}}\big)'(z).$$

Using the Taylor expansion of ϕ centered at z, *i.e.*

$$\phi(t) = \phi(z) + \phi'(z)\,(t-z) + \frac{\phi''(z)}{2!}\,(t-z)^2 + \cdots + \frac{\phi^{(\deg\phi)}(z)}{(\deg\phi)!}\,(t-z)^{\deg\phi};$$

or, equivalentely,

$$\frac{\phi(t)-\phi(z)}{(t-z)^2} = \phi'(z)\frac{1}{t-z} + \frac{\phi''(z)}{2!} + \cdots + \frac{\phi^{(\deg\phi)}(z)}{(\deg\phi)!}\,(t-z)^{\deg\phi-2},$$

then

$$\int_\gamma \frac{P_n^{\mathsf{L}}(t)\big(\phi(t)-\phi(z)\big)W(t)}{(t-z)^2}\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} = \phi'(z)Q_n^{\mathsf{L}}(z) + \sum_{k=2}^{\deg\phi}\frac{\phi^{(k)}(z)}{k!}\int_\gamma P_n^{\mathsf{L}}(t)W(t)(t-z)^{k-2}\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}$$

Now, applying the orthogonality (1.1), of $\{P_n^{\mathsf{L}}\}_{n\in\mathbb{N}}$ with respect to W we have,

$$\int_\gamma P_n^{\mathsf{L}}(t)W(t)(t-z)^{k-2}\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} = \mathbf{0}, \qquad n\geq \deg\phi-1.$$

And so (2.6) is true. Now we will prove (2.7),

$$\begin{aligned}&\int_\gamma P_n^{\mathsf{L}}(t)\phi'(t)\frac{W(t)}{t-z}\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}\\ &\qquad= \int_\gamma P_n^{\mathsf{L}}(t)\frac{\phi'(t)-\phi'(z)}{t-z}W(t)\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} + \phi'(z)\int_\gamma P_n^{\mathsf{L}}(t)\frac{W(t)}{t-z}\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}\\ &\qquad= \phi'(z)Q_n^{\mathsf{L}}(z), \qquad n\geq\deg\phi-1.\end{aligned}$$

We finish with the proof of (2.8),

$$\begin{aligned}&\int_\gamma P_n^{\mathsf{L}}(t)\frac{W(t)}{t-z}\psi_2(t)\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}\\ &\qquad= \int_\gamma P_n^{\mathsf{L}}(t)W(t)\frac{\psi_2(t)-\psi_2(z)}{t-z}\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} + Q_n^{\mathsf{L}}(z)\psi_2(z)\\ &\qquad= Q_n^{\mathsf{L}}(z)\psi_2(z), \qquad n\geq\deg\psi_2\end{aligned}$$

As we wanted to prove. □

Theorem 2.3. *Let W be a semiclassical matrix weight function, and $\{P_n^{\mathsf{L}}\}_{n\in\mathbb{N}}$, $\{Q_n^{\mathsf{L}}\}_{n\in\mathbb{N}}$, and $\{P_{n-1}^{\mathsf{L},(1)}\}_{n\in\mathbb{N}}$ the corresponding sequences of left matrix orthogonal polynomial, second kind function, and of associated polynomials, respectively. Then, for all $n\geq\max\{\deg\phi-1,\deg\psi_2\}$, we have*

$$\phi(z)(P_n^{\mathsf{L}})'(z) + P_n^{\mathsf{L}}(z)\psi_1(z) = \sum_{k=n-r}^{n+s} A_k^n P_k^{\mathsf{L}}, \tag{2.9}$$

$$\phi(z)(Q_n^{\mathsf{L}})'(z) - Q_n^{\mathsf{L}}(z)\psi_2(z) = \sum_{k=n-r}^{n+s} A_k^n Q_k^{\mathsf{L}}, \tag{2.10}$$

$$\phi(z)(P^{\mathsf{L},(1)}_{n-1})'(z) - P^{\mathsf{L},(1)}_{n-1}\psi_2(z) + P^{\mathsf{L}}_n\eta = \sum_{k=n-r}^{n+s} A^n_k P^{\mathsf{L},(1)}_{k-1}, \tag{2.11}$$

where $s = \max\big\{\deg\phi - 1, \deg\psi_1\big\}$ *and* $r = \max\big\{\deg\phi - 1, \deg\psi_2\big\}$.

Proof. We suppose W is semiclassical and we start with the proof of (2.9). As $\big\{P^{\mathsf{L}}_n\big\}_{n\in\mathbb{N}}$ is a basis in the linear space of matrix polynomials, there exist $A^n_k \in \mathbb{C}^{N\times N}$ such that

$$\phi(z)(P^{\mathsf{L}}_n)'(z) + P^{\mathsf{L}}_n(z)\psi_1(z) = \sum_{k=0}^{n+s} A^n_k P^{\mathsf{L}}_k.$$

Multiplying on the right by Wz^j for $j = 0, \dots, n+s$ and integrating, we get

$$\begin{multline}\int_\gamma \phi(z)(P^{\mathsf{L}}_n)'(z)W(z)z^j\,\frac{\mathrm{d}\,z}{2\pi\,\mathrm{i}} + \int_\gamma P^{\mathsf{L}}_n(z)\psi_1(z)W(z)z^j\,\frac{\mathrm{d}\,z}{2\pi\,\mathrm{i}} \\ = \sum_{k=0}^{n+s} A^n_k \int_\gamma P^{\mathsf{L}}_k(z)W(z)z^j\,\frac{\mathrm{d}\,z}{2\pi\,\mathrm{i}}.\end{multline} \tag{2.12}$$

Begin with

$$\begin{aligned}
\int_\gamma \phi(z)(P^{\mathsf{L}}_n)'(z)W(z)z^j\,\frac{\mathrm{d}\,z}{2\pi\,\mathrm{i}} &= \frac{P^{\mathsf{L}}_n(z)\phi(z)W(z)z^j}{2\pi\,\mathrm{i}}\Bigg]_{\partial\gamma} - \int_\gamma P^{\mathsf{L}}_n(z)\big(\phi(z)z^jW(z)\big)'\,\frac{\mathrm{d}\,z}{2\pi\,\mathrm{i}} \\
\text{(boundary conditions (2.1))} \quad &= -\int_\gamma P^{\mathsf{L}}_n(z)\big(\phi(z)z^j\big)'W(z)\,\frac{\mathrm{d}\,z}{2\pi\,\mathrm{i}} - \int_\gamma P^{\mathsf{L}}_n\phi(z)z^jW'(z)\,\frac{\mathrm{d}\,z}{2\pi\,\mathrm{i}} \\
\text{(orthogonality (1.1))} \quad &= -\int_\gamma P^{\mathsf{L}}_n(z)\phi(z)z^jW'(z)\,\frac{\mathrm{d}\,z}{2\pi\,\mathrm{i}}, \qquad n \geq j + \deg\phi \\
\text{(Pearson equation (2.2))} \quad &= -\int_\gamma P^{\mathsf{L}}_n(z)\psi_1(z)W(z)z^j\,\frac{\mathrm{d}\,z}{2\pi\,\mathrm{i}} - \int_\gamma P^{\mathsf{L}}_nW\psi_2(z)z^j\,\frac{\mathrm{d}\,z}{2\pi\,\mathrm{i}} \\
\text{(orthogonality (1.1))} \quad &= -\int_\gamma P^{\mathsf{L}}_n(z)\psi_1(z)W(z)z^j\,\frac{\mathrm{d}\,z}{2\pi\,\mathrm{i}}, \qquad n \geq j + 1 + \deg\psi_2.
\end{aligned}$$

Together, the previous calculation with (2.12) leads to

$$\mathbf{0} = \sum_{k=0}^{n+s} A^n_k \int_\gamma P^{\mathsf{L}}_k(z)W(z)z^j\,\frac{\mathrm{d}\,z}{2\pi\,\mathrm{i}} = A^n_jC^{-1}_j, \qquad j \leq \min\big\{n - \deg\phi, n - 1 - \deg\psi_2\big\}.$$

Then

$$\phi(z)(P^{\mathsf{L}}_n)'(z) + P^{\mathsf{L}}_n(z)\psi_1(z) = \sum_{k=\min\{n-\deg\phi, n-1-\deg\psi_2\}+1}^{n+s} A^n_k P^{\mathsf{L}}_k(z).$$

Since $\min\big\{n-\deg\phi, n-1-\deg\psi_2\big\}+1 = n-\max\big\{\deg\phi-1, \deg\psi_2\big\}$ we get the result. To derive (2.10), we multiply (2.9) on the right by $\dfrac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}\dfrac{W(t)}{t-z}$ and integrate

$$\int_\gamma \phi(t)(P^{\mathsf{L}}_n)'(t)\frac{W(t)}{t-z}\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} + \int_\gamma P^{\mathsf{L}}_n(t)\psi_1(t)\frac{W(t)}{t-z}\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} = \sum_{k=n-r}^{n+s} A^n_k Q^{\mathsf{L}}_k(z). \tag{2.13}$$

Let us analyze the left-hand side of this equality,

$$\begin{aligned}
&\int_\gamma \phi(t)(P^{\mathsf{L}}_n)'(t)\frac{W(t)}{t-z}\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}\\
&\quad= \frac{1}{2\pi\,\mathrm{i}}\phi(t)P^{\mathsf{L}}_n(t)\frac{W(t)}{t-z}\bigg]_{\partial\gamma} - \int_\gamma P^{\mathsf{L}}_n(t)\left(\phi(t)\frac{W(t)}{t-z}\right)'\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}\\
&\quad= -\int_\gamma P^{\mathsf{L}}_n(t)\phi'(t)\frac{W(t)}{t-z}\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} - \int_\gamma P^{\mathsf{L}}_n(t)\phi(t)\frac{W'(t)}{t-z}\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} + \int_\gamma P^{\mathsf{L}}_n(t)\phi(t)\frac{W(t)}{(t-z)^2}\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}
\end{aligned}$$

Now, using the previous Lemma 2.1, in particular identity (2.6), we get

$$\begin{aligned}
\int_\gamma \phi(t)(P^{\mathsf{L}}_n)'(t)\frac{W(t)}{t-z}\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} &= -\phi'(z)Q^{\mathsf{L}}_n(z) - \int_\gamma P^{\mathsf{L}}_n(t)\psi_1(t)\frac{W(t)}{t-z}\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}\\
&\quad - \int_\gamma P^{\mathsf{L}}_n(t)\frac{W(t)}{t-z}\psi_2(t)\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} + \big(\phi(z)\big(Q^{\mathsf{L}}_n\big)'(z) + \phi'(z)Q^{\mathsf{L}}_n(z)\big)\\
&= -\int_\gamma P^{\mathsf{L}}_n(t)\psi_1(t)\frac{W(t)}{t-z}\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} - Q^{\mathsf{L}}_n(z)\psi_2(z) + \phi(z){Q^{\mathsf{L}}_n}'(z),
\end{aligned}$$

then

$$\int_\gamma \phi(t)(P^{\mathsf{L}}_n)'(t)\frac{W(t)}{t-z}\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} + \int_\gamma P^{\mathsf{L}}_n(t)\psi_1(t)\frac{W(t)}{t-z}\,\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} = \phi(z)\big(Q^{\mathsf{L}}_n\big)'(z) - Q^{\mathsf{L}}_n(z)\psi_2(z). \tag{2.14}$$

Comparing the equations (2.13) and (2.14), we get (2.10) as we wanted to prove. To get (2.11) we substitute Q^{L}_n in (2.10) by the left Hermite–Padé formula (1.20),

$$\begin{aligned}
&\sum_{k=n-r}^{n+s} A^n_k\left(P^{\mathsf{L}}_k(z)S_W(z) + P^{\mathsf{L},(1)}_{k-1}(z)\right)\\
&\qquad= \phi(z)\left(P^{\mathsf{L}}_n(z)S_W(z) + P^{\mathsf{L},(1)}_{n-1}(z)\right)' - \left(P^{\mathsf{L}}_n(z)S_W(z) + P^{\mathsf{L},(1)}_{n-1}(z)\right)\psi_2(z).
\end{aligned}$$

Now replace $\phi(P^{\mathsf{L}}_n)'$ and $\phi S'_W$ from (2.9) and (2.4), respectively, to get (2.11). ☐

In the next theorem we go to a block matrix scenario.

Theorem 2.4. *Let W be a semiclassical matrix weight function, and $\big\{Y^{\mathsf{L}}_n\big\}_{n\in\mathbb{N}}$, $\big\{Y^{\mathsf{R}}_n\big\}_{n\in\mathbb{N}}$ be the corresponding sequences of left and right fundamental matrices. Then, there exists L_n, $\mathsf{R}_n \in \mathbb{C}^{2N\times 2N}[z]$ (whose degree does not depend on n), such* *that the left and right fundamental matrices* Y_n^{L}, Y_n^{R} *satisfies the following Sylvester matrix differential equations, for all* $n \geq \max\big\{\deg\phi - 1, \deg\psi_2\big\}$*,*

$$\phi\big(Y_n^{\mathsf{L}}\big)'(z) + Y_n^{\mathsf{L}}(z)\begin{bmatrix}\psi_1(z) & \mathbf{0}\\ \mathbf{0} & -\psi_2(z)\end{bmatrix} = \mathsf{L}_n(z)Y_n^{\mathsf{L}}(z). \tag{2.15}$$

$$\phi\big(Y_n^{\mathsf{R}}\big)'(z) + \begin{bmatrix}\psi_2(z) & \mathbf{0}\\ \mathbf{0} & -\psi_1(z)\end{bmatrix}Y_n^{\mathsf{R}}(z) = Y_n^{\mathsf{R}}(z)\mathsf{R}_n(z), \tag{2.16}$$

where

$$\mathsf{R}_n(z) = \begin{bmatrix}\mathbf{0} & \mathrm{I}\\ -\mathrm{I} & \mathbf{0}\end{bmatrix}\mathsf{L}_n(z)\begin{bmatrix}\mathbf{0} & \mathrm{I}\\ -\mathrm{I} & \mathbf{0}\end{bmatrix}. \tag{2.17}$$

Proof. We begin the proof by establishing the equivalence between (2.15) and (2.16). In fact, taking derivative in (1.38) and multiplying by ϕ, we arrive to,

$$-\big(Y_n^{\mathsf{L}}\big)^{-1}(z)\phi(z)\big(Y_n^{\mathsf{L}}\big)'(z)\big(Y_n^{\mathsf{L}}\big)^{-1}(z) = \begin{bmatrix}\mathbf{0} & \mathrm{I}\\ -\mathrm{I} & \mathbf{0}\end{bmatrix}\phi(z)\big(Y_n^{\mathsf{R}}\big)'(z)\begin{bmatrix}\mathbf{0} & -\mathrm{I}\\ \mathrm{I} & \mathbf{0}\end{bmatrix},$$

or, equivalently,

$$\phi(z)\big(Y_n^{\mathsf{R}}\big)'(z) = \begin{bmatrix}\mathbf{0} & \mathrm{I}\\ -\mathrm{I} & \mathbf{0}\end{bmatrix}\big(Y_n^{\mathsf{L}}\big)^{-1}(z)\phi(z)\big(Y_n^{\mathsf{L}}\big)'(z)\big(Y_n^{\mathsf{L}}\big)^{-1}(z)\begin{bmatrix}\mathbf{0} & \mathrm{I}\\ -\mathrm{I} & \mathbf{0}\end{bmatrix}.$$

Now, replacing $\phi(z)\big(Y_n^{\mathsf{L}}\big)'(z)$ or $\phi(z)\big(Y_n^{\mathsf{R}}\big)'(z)$ by (2.15), respectively by (2.16), we get

$$\phi\big(Y_n^{\mathsf{R}}\big)'(z) + \begin{bmatrix}\psi_2(z) & \mathbf{0}\\ \mathbf{0} & -\psi_1(z)\end{bmatrix}Y_n^{\mathsf{R}}(z) = Y_n^{\mathsf{R}}(z)\begin{bmatrix}\mathbf{0} & \mathrm{I}\\ -\mathrm{I} & \mathbf{0}\end{bmatrix}\mathsf{L}_n(z)\begin{bmatrix}\mathbf{0} & \mathrm{I}\\ -\mathrm{I} & \mathbf{0}\end{bmatrix},$$
$$\phi\big(Y_n^{\mathsf{L}}\big)'(z) + Y_n^{\mathsf{L}}(z)\begin{bmatrix}\psi_1(z) & \mathbf{0}\\ \mathbf{0} & -\psi_2(z)\end{bmatrix} = \begin{bmatrix}\mathbf{0} & -\mathrm{I}\\ \mathrm{I} & \mathbf{0}\end{bmatrix}\mathsf{R}_n(z)\begin{bmatrix}\mathbf{0} & -\mathrm{I}\\ \mathrm{I} & \mathbf{0}\end{bmatrix}Y_n^{\mathsf{L}}(z),$$

which prove also (2.17), *i.e.* the conexion between L_n and R_n.

Now, supposing that W is semiclassical, from Theorem 2.3, we know that

$$\phi\big(Y_n^{\mathsf{L}}\big)'(z) + Y_n^{\mathsf{L}}(z)\begin{bmatrix}\psi_1(z) & \mathbf{0}\\ \mathbf{0} & -\psi_2(z)\end{bmatrix} = \begin{bmatrix}\displaystyle\sum_{k=n-r}^{n+s} A_k^n P_k^{\mathsf{L}}(z) & \displaystyle\sum_{k=n-r}^{n+s} A_k^n Q_k^{\mathsf{L}}(z)\\ \displaystyle -C_{n-1}\sum_{k=n-1-r}^{n-1+s} A_k^{n-1} P_k^{\mathsf{L}}(z) & \displaystyle -C_{n-1}\sum_{k=n-1-r}^{n-1+s} A_k^{n-1} Q_k^{\mathsf{L}}(z)\end{bmatrix}.$$

Using the three term recurrence relation (1.6), it can be proven the existence of $\mathsf{L}_{1,1}^n$, $\mathsf{L}_{1,2}^n \in \mathbb{C}^{N\times N}[z]$ such that

$$\sum_{k=n-r}^{n+s} A_k^n P_k^{\mathsf{L}}(z) = \mathsf{L}_{1,1}^n(z)P_n^{\mathsf{L}}(z) - \mathsf{L}_{1,2}^n(z)C_{n-1}^{-1}P_{n-1}^{\mathsf{L}}(z).$$

The same happens to be for the linear combinations of $\left\{Q_n^{\mathsf{L}}\right\}_{n\in\mathbb{N}}$ or $\left\{P_n^{\mathsf{L},(1)}\right\}_{n\in\mathbb{N}}$. Hence we get (2.15), and by the discussion made at the beginning (2.16). Alternatively, we can apply Lemma 1.1 to explicitly determine the matrix polynomials $\mathsf{L}^n_{1,1}$, $\mathsf{L}^n_{1,2}$, $\mathsf{L}^n_{2,1}$ and $\mathsf{L}^n_{2,2}$. □

Denoting $\mathsf{L}_n(z)=\begin{bmatrix}\mathsf{L}_n^{11}(z) & \mathsf{L}_n^{12}(z)\\ \mathsf{L}_n^{21}(z) & \mathsf{L}_n^{22}(z)\end{bmatrix}$ then from (2.17) follows

$$\mathsf{R}_n(z)=\begin{bmatrix}-\mathsf{L}_n^{22}(z) & \mathsf{L}_n^{21}(z)\\ \mathsf{L}_n^{12}(z) & -\mathsf{L}_n^{11}(z)\end{bmatrix}. \tag{2.18}$$

In the next result we present a characterization for the semiclassical matrix weights.

Theorem 2.5. *Let W be a regular matrix weight together with the boundary conditions* (2.1)*. Then, W is semiclassical if and only if, there exist matrix polynomials L_n^{11}, $\mathsf{L}_n^{12},\mathsf{L}_n^{21},\mathsf{L}_n^{22},\mathsf{R}_n^{11},\mathsf{R}_n^{12},\mathsf{R}_n^{21},\mathsf{R}_n^{22}$ such that for all $n\geq\max\{\deg\phi-1,\deg\psi_2\}$,*

$$\phi\left(P_n^{\mathsf{L}}W^{\mathsf{L}}\right)'=\mathsf{L}_n^{11}\left(P_n^{\mathsf{L}}W^{\mathsf{L}}\right)-\mathsf{L}_n^{12}C_{n-1}\left(P_{n-1}^{\mathsf{L}}W^{\mathsf{L}}\right), \tag{2.19}$$

$$\phi\left(Q_n^{\mathsf{L}}(W^{\mathsf{R}})^{-1}\right)'=\mathsf{L}_n^{11}\left(Q_n^{\mathsf{L}}(W^{\mathsf{R}})^{-1}\right)-\mathsf{L}_n^{12}C_{n-1}\left(Q_{n-1}^{\mathsf{L}}(W^{\mathsf{R}})^{-1}\right), \tag{2.20}$$

$$\phi\left(W^{\mathsf{R}}P_n^{\mathsf{R}}\right)'=-\left(W^{\mathsf{R}}P_n^{\mathsf{R}}\right)\mathsf{L}_n^{22}-\left(W^{\mathsf{R}}P_{n-1}^{\mathsf{R}}\right)C_{n-1}\mathsf{L}_n^{12}, \tag{2.21}$$

$$\phi\left((W^{\mathsf{L}})^{-1}Q_n^{\mathsf{R}}\right)'=-\left((W^{\mathsf{L}})^{-1}Q_n^{\mathsf{R}}\right)\mathsf{L}_n^{22}-\left((W^{\mathsf{L}})^{-1}Q_{n-1}^{\mathsf{R}}\right)C_{n-1}\mathsf{L}_n^{12}, \tag{2.22}$$

$$-\phi C_{n-1}\left(P_{n-1}^{\mathsf{L}}W^{\mathsf{L}}\right)'=\mathsf{L}_n^{21}\left(P_n^{\mathsf{L}}W^{\mathsf{L}}\right)-\mathsf{L}_n^{22}C_{n-1}\left(P_{n-1}^{\mathsf{L}}W^{\mathsf{L}}\right), \tag{2.23}$$

$$-\phi C_{n-1}\left(Q_{n-1}^{\mathsf{L}}(W^{\mathsf{R}})^{-1}\right)'=\mathsf{L}_n^{21}\left(Q_n^{\mathsf{L}}(W^{\mathsf{R}})^{-1}\right)-\mathsf{L}_n^{22}C_{n-1}\left(Q_{n-1}^{\mathsf{L}}(W^{\mathsf{R}})^{-1}\right), \tag{2.24}$$

$$-\phi\left(W^{\mathsf{R}}P_{n-1}^{\mathsf{R}}\right)'C_{n-1}=\left(W^{\mathsf{R}}P_n^{\mathsf{R}}\right)\mathsf{L}_n^{21}+\left(W^{\mathsf{R}}P_{n-1}^{\mathsf{R}}\right)C_{n-1}\mathsf{L}_n^{11}, \tag{2.25}$$

$$-\phi\left((W^{\mathsf{L}})^{-1}Q_{n-1}^{\mathsf{R}}\right)'C_{n-1}=\left((W^{\mathsf{L}})^{-1}Q_n^{\mathsf{R}}\right)\mathsf{L}_n^{21}+\left((W^{\mathsf{L}})^{-1}Q_{n-1}^{\mathsf{R}}\right)C_{n-1}\mathsf{L}_n^{11}. \tag{2.26}$$

Proof. It is a straightforward fact from (2.15) and (2.16), and using (2.18), that

$$\phi(z)\left(P_n^{\mathsf{L}}\right)'(z)+P_n^{\mathsf{L}}(z)\psi_1(z)=\mathsf{L}_n^{11}(z)P_n^{\mathsf{L}}(z)-\mathsf{L}_n^{12}(z)C_{n-1}P_{n-1}^{\mathsf{L}}(z), \tag{2.27}$$

$$\phi(z)\left(Q_n^{\mathsf{L}}\right)'(z)-Q_n^{\mathsf{L}}(z)\psi_2(z)=\mathsf{L}_n^{11}(z)Q_n^{\mathsf{L}}(z)-\mathsf{L}_n^{12}(z)C_{n-1}Q_{n-1}^{\mathsf{L}}(z), \tag{2.28}$$

$$\phi(z)\left(P_n^{\mathsf{R}}\right)'(z)+\psi_2(z)P_n^{\mathsf{R}}(z)=-P_n^{\mathsf{R}}(z)\mathsf{L}_n^{22}(z)-P_{n-1}^{\mathsf{R}}(z)C_{n-1}\mathsf{L}_n^{12}(z),$$

$$\phi(z)\left(Q_n^{\mathsf{R}}\right)'(z)-\psi_1(z)Q_n^{\mathsf{R}}(z)=-Q_n^{\mathsf{R}}(z)\mathsf{L}_n^{22}(z)-Q_{n-1}^{\mathsf{R}}(z)C_{n-1}\mathsf{L}_n^{12}(z),$$

$$-\phi(z)C_{n-1}\left(P_{n-1}^{\mathsf{L}}\right)'(z)-C_{n-1}P_{n-1}^{\mathsf{L}}(z)\psi_1(z)=\mathsf{L}_n^{21}(z)P_n^{\mathsf{L}}(z)-\mathsf{L}_n^{22}(z)C_{n-1}P_{n-1}^{\mathsf{L}}(z), \tag{2.29}$$

$$-\phi(z)C_{n-1}\left(Q_{n-1}^{\mathsf{L}}\right)'(z)-C_{n-1}Q_{n-1}^{\mathsf{L}}(z)\psi_1(z)=\mathsf{L}_n^{21}(z)Q_n^{\mathsf{L}}(z)-\mathsf{L}_n^{22}(z)C_{n-1}Q_{n-1}^{\mathsf{L}}(z), \tag{2.30}$$

$$-\phi(z)\left(P_{n-1}^{\mathsf{R}}\right)'(z)C_{n-1}-\psi_2(z)P_{n-1}^{\mathsf{R}}(z)C_{n-1}=P_n^{\mathsf{R}}(z)\mathsf{L}_n^{21}(z)+P_{n-1}^{\mathsf{R}}(z)C_{n-1}\mathsf{L}_n^{11}(z),$$

$$-\phi(z)\left(Q_{n-1}^{\mathsf{R}}\right)'(z)C_{n-1}-\psi_1(z)Q_{n-1}^{\mathsf{R}}(z)C_{n-1}=Q_n^{\mathsf{R}}(z)\mathsf{L}_n^{21}(z)+Q_{n-1}^{\mathsf{R}}(z)C_{n-1}\mathsf{L}_n^{11}(z).$$

At this moment we only have to:

- multiply the first equation on the write by W^{L}, and applying (2.3) to get (2.19);
- multiply the second equation on the write by $(W^{\mathsf{R}})^{-1}$, and applying (2.3) to get (2.20);

- multiply the third equation on the left by W^{R}, and applying (2.3) to get (2.21);
- multiply the fourth equation on the left by $(W^{\mathsf{L}})^{-1}$, and applying (2.3) to get (2.22);

Doing the same with the 5-th till 8-th equations we get (2.23)–(2.26).

We will see now, how to derive equations (2.3), for W^{L} and W^{R}, from (2.19)–(2.26). Substitute n by $n-1$ in (1.33) and multiply on the left by $\big(W^{\mathsf{L}}\big)^{-1}$, to get

$$\big(W^{\mathsf{L}}\big)^{-1}=\big(W^{\mathsf{L}}\big)^{-1}Q^{\mathsf{R}}_n(z)C_{n-1}P^{\mathsf{L}}_{n-1}(z)-\big(W^{\mathsf{L}}\big)^{-1}Q^{\mathsf{R}}_{n-1}(z)C_{n-1}P^{\mathsf{L}}_n(z).$$

Taking derivative in this equation, i.e.

$$\begin{aligned}\big((W^{\mathsf{L}})^{-1}\big)'=\big((W^{\mathsf{L}})^{-1}Q^{\mathsf{R}}_n\big)'C_{n-1}P^{\mathsf{L}}_{n-1}+(W^{\mathsf{L}})^{-1}Q^{\mathsf{R}}_nC_{n-1}(P^{\mathsf{L}}_{n-1})'\\ -\big((W^{\mathsf{L}})^{-1}Q^{\mathsf{R}}_{n-1}\big)'C_{n-1}P^{\mathsf{L}}_n-(W^{\mathsf{L}})^{-1}Q^{\mathsf{R}}_{n-1}C_{n-1}(P^{\mathsf{L}}_n)'.\end{aligned}\tag{2.31}$$

multiplying by ϕ, applying equations (2.22), (2.26), together with (2.27) and (2.29), in (2.31), we obtain

$$\begin{aligned}\phi\big((W^{\mathsf{L}})^{-1}\big)'=(W^{\mathsf{L}})^{-1}\big(Q^{\mathsf{R}}_n(R^{12}_n-L^{21}_n)P^{\mathsf{L}}_n-Q^{\mathsf{R}}_{n-1}C_{n-1}(R^{22}_n+L^{11}_n)P^{\mathsf{L}}_n\\ +Q^{\mathsf{R}}_{n-1}C_{n-1}(L^{12}_n-R^{21}_n)C_{n-1}P^{\mathsf{L}}_{n-1}+Q^{\mathsf{R}}_n(R^{11}_n+L^{22}_n)C_{n-1}P^{\mathsf{L}}_{n-1}\\ +(Q^{\mathsf{R}}_{n-1}C_{n-1}P^{\mathsf{L}}_{n-1}-Q^{\mathsf{R}}_nC_{n-1}P^{\mathsf{L}}_{n-1})\psi_1\big).\end{aligned}$$

Using the connection between L_n and R_n in (2.18) and (1.33), in this equation to get

$$\phi(z)\big((W^{\mathsf{L}}(z))^{-1}\big)'=-(W^{\mathsf{L}}(z))^{-1}\psi_1(z)\qquad\textit{i.e.}\qquad\phi(z)\big(W^{\mathsf{L}}(z)\big)'=\psi_1(z)W^{\mathsf{L}}(z).$$

In a very similar way we get $\phi\left(W^{\mathsf{R}}\right)'=W^{\mathsf{R}}\psi_2$. In fact, departing from equation (1.34) written in $n-1$, multiplying on the right by W^{R}, taking derivatives, and multiplying by ϕ we get

$$\begin{aligned}\phi(W^{\mathsf{R}})'=\phi\left(W^{\mathsf{R}}P^{\mathsf{R}}_{n-1}\right)'C_{n-1}Q^{\mathsf{L}}_n+W^{\mathsf{R}}P^{\mathsf{R}}_{n-1}C_{n-1}\phi(Q^{\mathsf{L}}_n)'\\ -\phi\left(W^{\mathsf{R}}P^{\mathsf{R}}_n\right)'C_{n-1}Q^{\mathsf{L}}_{n-1}-W^{\mathsf{R}}P^{\mathsf{R}}_n\phi C_{n-1}\left(Q^{\mathsf{L}}_{n-1}\right)'.\end{aligned}\tag{2.32}$$

Combining (2.21) and (2.25) with (2.28) and (2.30), and taking into account (1.34), we get from (2.32) that

$$\phi(z)\left(W^{\mathsf{R}}\right)'(z)=W^{\mathsf{R}}(z)\psi_2(z),\qquad\textit{i.e.}\qquad\text{we arrive to (2.3).}$$

From Theorem 2.1 we get that W is semiclassical. □

What we have just proven in Theorem 2.5 is that (2.15) or, equivalently, (2.16) characterizes the sequences of semiclassical matrix orthogonal polynomials. From the last theorem we see that the following matrices are instrumental in the study of the semiclassical matrix orthogonal polynomial theory. Associated with a regular weight matrix, W, factorized as $W=W^{\mathsf{L}}W^{\mathsf{R}}$, we define the constant jump fundamental matrices, in terms of its fundamental matrices, Y^{L}_n, Y^{R}_n, by means of,

$$Z^{\mathsf{L}}_n(z):=Y^{\mathsf{L}}_n(z)\begin{bmatrix}W^{\mathsf{L}}(z)&\mathbf{0}\\\mathbf{0}&(W^{\mathsf{R}}(z))^{-1}\end{bmatrix},\tag{2.33}$$

$$Z_n^{\mathsf{R}}(z) := \begin{bmatrix} W^{\mathsf{R}}(z) & \mathbf{0} \\ \mathbf{0} & (W^{\mathsf{L}}(z))^{-1} \end{bmatrix} Y_n^{\mathsf{R}}(z), \qquad n \in \mathbb{N}. \tag{2.34}$$

Taking inverse on (2.33) and applying (1.38) we see that Z_n^{R} given in (2.34) admits the representation

$$Z_n^{\mathsf{R}}(z) = \begin{bmatrix} \mathbf{0} & -\mathrm{I} \\ \mathrm{I} & \mathbf{0} \end{bmatrix} (Z_n^{\mathsf{L}}(z))^{-1} \begin{bmatrix} \mathbf{0} & \mathrm{I} \\ -\mathrm{I} & \mathbf{0} \end{bmatrix}, \qquad n \in \mathbb{N}. \tag{2.35}$$

Moreover, it's easy to check that they satisfy the same recurrence relations as Y_n^{L} and Y_n^{R}, *i.e.*

$$T_n^{\mathsf{L}}(z) Z_n^{\mathsf{L}}(z) = Z_{n+1}^{\mathsf{L}}(z), \qquad Z_n^{\mathsf{R}}(z) T_n^{\mathsf{R}}(z) = Z_{n+1}^{\mathsf{R}}(z), \qquad n \in \mathbb{N}. \tag{2.36}$$

Now, we state a matrix reinterpretation of Theorem 2.5.

Theorem 2.6. *Let W be a regular matrix weight function together with the boundary conditions* (2.1)*, admitting the factorization $W = W^{\mathsf{L}} W^{\mathsf{R}}$ and the constant jump fundamental matrices, Z_n^{L}, Z_n^{R}. The following are equivalent:*

(i) W *is semiclassical,* i.e. (2.2) *takes place;*
(ii) *there exists a polynomial matrix (whose degree does not depend on n), L_n, such that the corresponding left constant jump fundamental matrix Z_n^{L} satisfies, the following Sylvester matrix differential equations,*

$$\phi(z) \big(Z_n^{\mathsf{L}}\big)'(z) = \mathsf{L}_n(z) Z_n^{\mathsf{L}}(z), \qquad n \geq \max\{\deg \phi - 1, \deg \psi_2\}; \tag{2.37}$$

(iii) *there exists a polynomial matrix (whose degree does not depend on n), R_n, such that the corresponding right constant jump fundamental matrix Z_n^{R} satisfies, the following Sylvester matrix differential equations,*

$$\phi(z) \big(Z_n^{\mathsf{R}}\big)'(z) = Z_n^{\mathsf{R}}(z) \mathsf{R}_n(z), \qquad n \geq \max\{\deg \phi - 1, \deg \psi_2\}, \tag{2.38}$$

where the matrices L_n and R_n are connected by (2.17).

Proof. It is easy to see from Theorem 2.5, that the equations (2.19), (2.20), (2.23) and (2.24) colapse into (2.37). In the same way, equations (2.21), (2.22), (2.25) and (2.26) colapse into (2.38). Moreover, (2.37) and (2.38) are equivalent to the ones in Theorem 2.4, *i.e.* (2.15) and (2.16), respectively. Hence, we get the result. □

4. Characterization using Magnus procedure

In an inspiring work, *cf.* [78], Alphonse Magnus explains a procedure to derive structure relations for the sequence of scalar monic orthogonal polynomials associated with a semiclassical weight function. This procedure, uses the Hermite–Padé formula connecting the monic orthogonal polynomials, functions of second kind and associated monic polynomials, instead of using the left and right orthogonality. Using Theorem 2.2 we know that the Stieltjes–Markov matrix function associated with a semiclassical weight (*cf.* Definition 2.1) satisfies (2.4).

Theorem 2.7. *Let W be a regular matrix weight function together with the boundary conditions* (2.1)*, S_W is its Stieltjes–Markov matrix function, and $\{Y_n^{\mathsf{L}}\}_{n\in\mathbb{N}}$, $\{Y_n^{\mathsf{R}}\}_{n\in\mathbb{N}}$ the corresponding left and right fundamental matrices, respectively. Then, the following are equivalent:*

(i) *There exist a scalar polynomial ϕ and two matrix polynomials ψ_1, ψ_2 such that, the Stieltjes–Markov matrix function S_W satisfies* (2.4).
(ii) *There exists a matrix polynomial L_n (whose degree does not depend on n) such that the Sylvester matrix differential equation* (2.15) *holds for all $n \in \mathbb{N}$.*
(iii) *There exists a matrix polynomial R_n (whose degree does not depend on n) such that the Sylvester matrix differential equation* (2.16) *holds for all $n \in \mathbb{N}$.*

Moreover, the matrices L_n and R_n are connected by (2.17)*, for all $n \in \mathbb{N}$.*

Proof. From Theorem 2.4 we know that (ii) and (iii) are equivalent propositions. Moreover, we can see that (ii) implies (i). In fact, taking $n = 0$ in the (1.2) entry of (2.15), *i.e.*

$$\phi(z)(Q_0^{\mathsf{L}})'(z) - Q_0^{\mathsf{L}}(z)\psi_2(z) = \mathsf{L}_0^{11} Q_0^{\mathsf{L}} - \mathsf{L}_0^{12} C_{-1} Q_{-1}^{\mathsf{L}}.$$

Now, taking into account that $Q_0^{\mathsf{L}}(z) = S_W(z)$ and $Q_{-1}^{\mathsf{L}}(z) = C_{-1}^{-1}$ we get

$$\phi(z) S_W'(z) - S_W(z)\psi_2(z) = \psi_1(z) S_W + \eta(z),$$

where $\psi_1(z) = \mathsf{L}_0^{11}$ and $\eta = -\mathsf{L}_0^{12}$.

From here we see that, to finish the proof, we only have to show that (i) implies (ii). Multiply the equation (2.4) from the left by P_n^{L} and on the right by P_n^{R}, *i.e.*

$$\phi P_n^{\mathsf{L}} S_W' P_n^{\mathsf{R}} = P_n^{\mathsf{L}} \psi_1 S_W P_n^{\mathsf{R}} + P_n^{\mathsf{L}} S_W \psi_2 P_n^{\mathsf{R}} + P_n^{\mathsf{L}} \eta P_n^{\mathsf{R}}. \tag{2.39}$$

We write the left hand-side of (2.39) as

$$\phi(z) P_n^{\mathsf{L}} S_W'(z) P_n^{\mathsf{R}}(z) = \phi(z)\big((P_n^{\mathsf{L}}(z) S_W(z))' - (P_n^{\mathsf{L}})'(z) S_W(z)\big) P_n^{\mathsf{R}}(z),$$

then using the Hermite–Padé formulas (1.20) and (1.21), we transforms (2.39) into,

$$\big(\phi(Q_n^{\mathsf{L}})' - Q_n^{\mathsf{L}}\psi_2\big) P_n^{\mathsf{R}} - \big(P_n^{\mathsf{L}}\psi_1 + \phi(P_n^{\mathsf{L}})'\big) Q_n^{\mathsf{R}} = \Theta_n^{\mathsf{L}} \tag{2.40}$$

where

$$\Theta_n^{\mathsf{L}} \coloneqq \big(\phi(P_{n-1}^{\mathsf{L},(1)})' - P_{n-1}^{\mathsf{L},(1)}\psi_2 + P_n^{\mathsf{L}}\eta\big) P_n^{\mathsf{R}} - \big(\phi(P_n^{\mathsf{L}})' + P_n^{\mathsf{L}}\psi_1\big) P_{n-1}^{\mathsf{R},(1)}, \tag{2.41}$$

is a polynomial whose degree does not depend on n, as the left hand side of (2.40) shows. In fact, by using the assymptotic expansions near infinity (1.14) and (1.15) of Q_n^{L} and Q_n^{R}, respectively, the expansion of P_n^{L} and P_n^{R} in powers of z and considering the fact that $S_W = Q_0^{\mathsf{L}}$ it follows that,

$$\frac{(2n+1)\phi(z) C_n^{-1}}{z^2} + \frac{\psi_1(z) C_n^{-1}}{z} + \frac{C_n^{-1}\psi_2(z)}{z} + \text{ higher power of } \frac{1}{z}; \qquad |z| \to +\infty.$$

Then, the highest degree in the expression of Θ_n^{L} comes from

$$\frac{(2n+1)\phi(z) C_n^{-1}}{z^2} + \frac{\psi_1(z) C_n^{-1}}{z} + \frac{C_n^{-1}\psi_2(z)}{z}.$$

If we denote $k = \max\{\deg\phi - 2, \deg\psi_1 - 1, \deg\psi_2 - 1\}$ then $\deg\Theta_n^{\mathsf{L}} \leq k$. Now, using (1.40), and writing $\Theta_n^{\mathsf{L}} = \Theta_n^{\mathsf{L}} C_{n-1}\big(P_{n-1}^{\mathsf{L}} Q_n^{\mathsf{R}} - Q_{n-1}^{\mathsf{L}} P_n^{\mathsf{R}}\big)$ in (2.40), it follows that,

$$\big(\phi(Q_n^{\mathsf{L}})' - Q_n^{\mathsf{L}}\psi_2 + \Theta_n^{\mathsf{L}} C_{n-1} Q_{n-1}^{\mathsf{L}}\big) P_n^{\mathsf{R}} = \big(\phi(P_n^{\mathsf{L}})' + P_n^{\mathsf{L}}\psi_1 + \Theta_n^{\mathsf{L}} C_{n-1} P_{n-1}^{\mathsf{L}}\big) Q_n^{\mathsf{R}},$$

and taking into account (1.41) we arrive to

$$\begin{aligned}\big(\phi(Q_n^{\mathsf{L}})' - Q_n^{\mathsf{L}}\psi_2 + \Theta_n^{\mathsf{L}} C_{n-1} Q_{n-1}^{\mathsf{L}}\big)\big(Q_n^{\mathsf{L}}\big)^{-1} \\ = \big(\phi(P_n^{\mathsf{L}})' + P_n^{\mathsf{L}}\psi_1 + \Theta_n^{\mathsf{L}} C_{n-1} P_{n-1}^{\mathsf{L}}\big)\big(P_n^{\mathsf{L}}\big)^{-1} =: \Omega_n^{\mathsf{L}}.\end{aligned} \tag{2.42}$$

In the same way, using (1.44) and writing $\Theta_n^{\mathsf{L}} = \Theta_n^{\mathsf{L}} C_{n-1}\big(P_{n-1}^{\mathsf{L}} P_{n-1}^{\mathsf{R},(1)} - P_{n-2}^{\mathsf{L},(1)} P_n^{\mathsf{R}}\big)$ in (2.41) to get,

$$\begin{aligned}\Big(\phi\big(P_{n-1}^{\mathsf{L},(1)}\big)' - P_{n-1}^{\mathsf{L},(1)}\psi_2 + P_n^{\mathsf{L}}\eta + \Theta_n^{\mathsf{L}} C_{n-1} P_{n-2}^{\mathsf{L},(1)}\Big) P_n^{\mathsf{R}} \\ = \Big(\phi\big(P_n^{\mathsf{L}}\big)' + P_n^{\mathsf{L}}\psi_1 + \Theta_n^{\mathsf{L}} C_{n-1} P_{n-1}^{\mathsf{L}}\Big) P_{n-1}^{\mathsf{R},(1)},\end{aligned}$$

and by (1.42), we arrive to

$$\begin{aligned}\Big(\phi\big(P_{n-1}^{\mathsf{L},(1)}\big)' - P_{n-1}^{\mathsf{L},(1)}\psi_2 + P_n^{\mathsf{L}}\eta + \Theta_n^{\mathsf{L}} C_{n-1} P_{n-2}^{\mathsf{L},(1)}\Big)\big(P_{n-1}^{\mathsf{L},(1)}\big)^{-1} \\ = \Big(\phi\big(P_n^{\mathsf{L}}\big)' + P_n^{\mathsf{L}}\psi_1 + \Theta_n^{\mathsf{L}} C_{n-1} P_{n-1}^{\mathsf{L}}\Big)\big(P_n^{\mathsf{L}}\big)^{-1} = \Omega_n^{\mathsf{L}}.\end{aligned} \tag{2.43}$$

With Ω_n^{L} defined by (2.42). Hence, equations (2.42) and (2.43) reads as

$$\phi\big(P_n^{\mathsf{L}}\big)' + P_n^{\mathsf{L}}\psi_1 + \Theta_n^{\mathsf{L}} C_{n-1} P_{n-1}^{\mathsf{L}} = \Omega_n^{\mathsf{L}} P_n^{\mathsf{L}}, \tag{2.44}$$

$$\phi(Q_n^{\mathsf{L}})' - Q_n^{\mathsf{L}}\psi_2 + \Theta_n^{\mathsf{L}} C_{n-1} Q_{n-1}^{\mathsf{L}} = \Omega_n^{\mathsf{L}} Q_n^{\mathsf{L}}, \tag{2.45}$$

$$\phi\big(P_{n-1}^{\mathsf{L},(1)}\big)' - P_{n-1}^{\mathsf{L},(1)}\psi_2 + P_n^{\mathsf{L}}\eta + \Theta_n^{\mathsf{L}} C_{n-1} P_{n-2}^{\mathsf{L},(1)} = \Omega_n^{\mathsf{L}} P_{n-1}^{\mathsf{L},(1)}. \tag{2.46}$$

To see that Ω_n^{L} is a matrix polynomial whose degree does not depend on n, we multiply equation (2.44) and (2.46) on the right by $-P_{n-2}^{\mathsf{R},(1)}$ and P_{n-1}^{R}, respectively, adding the resulting expressions and applying (1.43) to get

$$\begin{aligned}\Omega_n^{\mathsf{L}} C_{n-1}^{-1} = \big(\phi\big(P_{n-1}^{\mathsf{L},(1)}\big)' - P_{n-1}^{\mathsf{L},(1)}\psi_2 + P_n^{\mathsf{L}}\eta + \Theta_n^{\mathsf{L}} C_{n-1} P_{n-2}^{\mathsf{L},(1)}\big) P_{n-1}^{\mathsf{R}} \\ - \big(\phi\big(P_n^{\mathsf{L}}\big)' + P_n^{\mathsf{L}}\psi_1 + \Theta_n^{\mathsf{L}} C_{n-1} P_{n-1}^{\mathsf{L}}\big) P_{n-2}^{\mathsf{R},(1)},\end{aligned}$$

as well as, multiplying (2.44) by $-Q_{n-1}^{\mathsf{R}}$ and (2.45) by P_{n-1}^{R}, respectively, adding the resulting expressions and applying (1.39)

$$\Omega_n^{\mathsf{L}} C_{n-1}^{-1} = \big(\phi(Q_n^{\mathsf{L}})' - Q_n^{\mathsf{L}}\psi_2 + \Theta_n^{\mathsf{L}} C_{n-1} Q_{n-1}^{\mathsf{L}}\big) P_{n-1}^{\mathsf{R}} - \big(\phi\big(P_n^{\mathsf{L}}\big)' + P_n^{\mathsf{L}}\psi_1 + \Theta_n^{\mathsf{L}} C_{n-1} P_{n-1}^{\mathsf{L}}\big) Q_{n-1}^{\mathsf{R}}.$$

From the two last equations we conclude that the degree of Ω_n^{L} are bounded by $t = \max\{\deg\phi - 1, \deg\Theta_n^{\mathsf{L}} - 1, \deg\psi_1, \deg\psi_2 - 2\}$. To end the proof we only have to recall Theorem 2.4. □

From Theorem 2.7 we see that the Theorems 2.4 and 2.6 the equations are valid for all $n \in \mathbb{N}$. This is a great achievement that will open new avenues for these systems of semiclassical matrix orthogonal polynomials, as will be seen in Chapter 4. Now, we will see a characterization of semiclassical matrix orthogonal polynomials using zero curvature formula.

Theorem 2.8. *Let W be a regular matrix weight function together with the boundary conditions* (2.1)*, admitting the factorization $W = W^{\mathsf{L}}W^{\mathsf{R}}$ and the constant jump fundamental matrices, Z_n^{L}, Z_n^{R}. The following are equivalent:*

(i) W *is semiclassical* i.e. (2.3) *takes place.*
(ii) *The left zero curvature formula*

$$\phi(z)\big(T_n^{\mathsf{L}}\big)'(z) = \mathsf{L}_{n+1}(z)T_n^{\mathsf{L}}(z) - T_n^{\mathsf{L}}(z)\mathsf{L}_n(z), \qquad n \in \mathbb{N}, \tag{2.47}$$

holds, with initial condition $\mathsf{L}_0(z) = \begin{bmatrix} \psi_1(z) & \eta(z) \\ \mathbf{0} & -\psi_2(z) \end{bmatrix}$.

(iii) *The right zero curvature formula*

$$\phi(z)\big(T_n^{\mathsf{R}}\big)'(z) = T_n^{\mathsf{R}}(z)\,\mathsf{R}_{n+1}(z) - \mathsf{R}_n(z)T_n^{\mathsf{R}}(z), \qquad n \in \mathbb{N}, \tag{2.48}$$

holds, with initial condition $\mathsf{R}_0(z) = \begin{bmatrix} \psi_2(z) & \mathbf{0} \\ \eta(z) & -\psi_1(z) \end{bmatrix}$.

Proof. We know, by Theorem 2.6 that W is semiclassical if and only if, we have (2.37) and (2.38) for $n \geq \max\{\deg\phi - 1, \deg\psi_2\}$. Moreover, from Theorem 2.7 we know that (2.37) and (2.38) holds true for $n \in \mathbb{N}$. Hence, if we prove that (2.37) is equivalent to (2.47), and (2.38) is equivalent to (2.48), for $n \in \mathbb{N}$ we finish the proof.

We begin with the direct proof. Taking (2.37) for $n+1$, and applying the recurrence relation (2.36), we get

$$\phi(z)\big(T_n^{\mathsf{L}}(z)Z_n^{\mathsf{L}}(z)\big)' = \mathsf{L}_{n+1}(z)T_n^{\mathsf{L}}(z)Z_n^{\mathsf{L}}(z),$$

hence

$$\phi(z)\big(T_n^{\mathsf{L}}\big)'(z)Z_n^{\mathsf{L}}(z) + T_n^{\mathsf{L}}(z)\phi(z)\big(Z_n^{\mathsf{L}}\big)'(z) = \mathsf{L}_{n+1}(z)T_n^{\mathsf{L}}(z)Z_n^{\mathsf{L}}(z).$$

Now, apply (2.37) to find

$$\big\{\phi(z)\big(T_n^{\mathsf{L}}\big)'(z) + T_n^{\mathsf{L}}(z)\mathsf{L}_n(z) - \mathsf{L}_{n+1}(z)T_n^{\mathsf{L}}(z)\big\}Z_n^{\mathsf{L}}(z) = \mathbf{0},$$

and because $\det Z_n^{\mathsf{L}}(z) = \frac{\det W^{\mathsf{L}}(z)}{\det W^{\mathsf{R}}(z)} \neq 0$, we arrive to (2.47). Using the same ideas we derive (2.48). In fact, taking (2.38) for $n+1$, and applying the second recurrence relation in (2.36), we get

$$\phi(z)\big(Z_n^{\mathsf{R}}(z)T_n^{\mathsf{R}}(z)\big)' = Z_n^{\mathsf{R}}(z)T_n^{\mathsf{R}}(z)\mathsf{R}_{n+1}(z),$$

hence

$$\phi(z)\big(Z_n^{\mathsf{R}}\big)'(z)T_n^{\mathsf{R}}(z) + Z_n^{\mathsf{R}}(z)\phi(z)\big(T_n^{\mathsf{R}}\big)'(z) = Z_n^{\mathsf{R}}(z)T_n^{\mathsf{R}}(z)\mathsf{R}_{n+1}(z).$$

Now, apply (2.38) to find

$$Z_n^{\mathsf{R}}(z)\Big\{\phi(z)\big(T_n^{\mathsf{L}}\big)'(z)+\mathsf{R}_n(z)T_n^{\mathsf{R}}(z)-T_n^{\mathsf{R}}(z)\mathsf{R}_{n+1}(z)\Big\}=\mathbf{0},$$

and because $\det Z_n^{\mathsf{L}}(z)=\frac{\det W^{\mathsf{R}}(z)}{\det W^{\mathsf{L}}(z)}\neq 0$, we arrive to (2.48).

For the initial condition of L_n and R_n, we take $n=0$ in (2.37) and in (2.38), respectively.

We analyze the reciprocal. Multiplying (2.47) from the right by Z_n^{L}, applying the recurrence relation (2.36), and then adding $\phi(z)T_n^{\mathsf{L}}(z)\left(Z_n^{\mathsf{L}}\right)'(z)$ on both sides of the resulting expression, we obtain

$$\phi(z)\big(Z_{n+1}^{\mathsf{L}}\big)'(z)-\mathsf{L}_{n+1}(z)Z_{n+1}^{\mathsf{L}}(z)=T_n^{\mathsf{L}}(z)\big\{\phi(z)(Z_n^{\mathsf{L}})'(z)-\mathsf{L}_n(z)Z_n^{\mathsf{L}}(z)\big\}.$$

Thus

$$\phi(z)(Z_{n+1}^{\mathsf{L}})'(z)-\mathsf{L}_{n+1}(z)Z_{n+1}^{\mathsf{L}}(z)=T_n^{\mathsf{L}}(z)\cdots T_0^{\mathsf{L}}(z)\big(\phi(z)(Z_0^{\mathsf{L}})'(z)-\mathsf{L}_0(z)Z_1^{\mathsf{L}}(z)\big).$$

Taking into account that $\left(Z_0^{\mathsf{L}}\right)'(z)=\mathsf{L}_0(z)Z_0^{\mathsf{L}}(z)$, we arrive to (2.37).

In the same manner, multiplying (2.48) from the left by Z_n^{R}, applying the recurrence relation (2.36), and then adding $\phi(z)\big(Z_n^{\mathsf{R}}\big)'(z)T_n^{\mathsf{R}}(z)$ on both sides of the resulting expression, we obtain

$$\phi(z)(Z_{n+1}^{\mathsf{R}})'(z)-Z_{n+1}^{\mathsf{R}}(z)\mathsf{R}_{n+1}(z)=\big\{\phi(z)(Z_n^{\mathsf{R}})'(z)-Z_n^{\mathsf{R}}(z)\mathsf{R}_n(z)\big\}T_n^{\mathsf{R}}(z).$$

Thus

$$\phi(z)(Z_{n+1}^{\mathsf{R}})'(z)-Z_{n+1}^{\mathsf{R}}(z)\mathsf{R}_{n+1}(z)=\big(\phi(z)(Z_0^{\mathsf{R}})'(z)-Z_0^{\mathsf{R}}(z)\mathsf{R}_0(z)\big)T_0^{\mathsf{R}}(z)\cdots T_n^{\mathsf{R}}(z).$$

Taking into account that $\left(Z_0^{\mathsf{R}}\right)'(z)=Z_0^{\mathsf{R}}(z)\mathsf{R}_0(z)$, we arrive to (2.38). □

Corollary 2.2. *Let W be a semiclassical matrix weight function, admitting the factorization $W=W^{\mathsf{L}}W^{\mathsf{R}}$ and the constant jump fundamental matrices, Z_n^{L}, Z_n^{R}. Then, we have, for all $n\in\mathbb{N}$, the second order zero curvature formulas*

$$\phi(z)\Big\{\big(T_n^{\mathsf{L}}\big)'(z)\mathsf{L}_n(z)+\mathsf{L}_{n+1}(z)\big(T_n^{\mathsf{L}}\big)'(z)\Big\}=\mathsf{L}_{n+1}^2(z)T_n^{\mathsf{L}}(z)-T_n^{\mathsf{L}}(z)\mathsf{L}_n^2(z),\tag{2.49}$$

$$\phi(z)\Big\{\big(T_n^{\mathsf{R}}\big)'(z)\mathsf{R}_{n+1}(z)+\mathsf{R}_n(z)\big(T_n^{\mathsf{R}}\big)'(z)\Big\}=T_n^{\mathsf{R}}(z)\mathsf{R}_{n+1}^2(z)-\mathsf{R}_n^2(z)T_n^{\mathsf{R}}(z).\tag{2.50}$$

Proof. By Theorem 2.8 we know that (2.47) and (2.48) takes place. Hence multiplying (2.47) on the right by L_n (respectively, (2.48) on the left by R_n), we get

$$\phi(z)\big(T_n^{\mathsf{L}}\big)'(z)\mathsf{L}_n(z)=\mathsf{L}_{n+1}(z)T_n^{\mathsf{L}}(z)\mathsf{L}_n(z)-T_n^{\mathsf{L}}(z)\mathsf{L}_n^2(z),$$
$$\phi(z)\mathsf{R}_n(z)\big(T_n^{\mathsf{R}}\big)'(z)=\mathsf{R}_n(z)T_n^{\mathsf{R}}(z)\mathsf{R}_{n+1}(z)-\mathsf{R}_n^2(z)T_n^{\mathsf{R}}(z),$$

and applying again (2.47) we get (2.49) (respectively, (2.48) to get (2.50)). □

5. Geronimus characterization for classical matrix orthogonal polynomials

It was shown in [19], analogously to the scalar situation (*cf.* [80]), the orthogonality of the derivatives is equivalent to a Pearson type equation $\phi W' = \psi_1 W$ for the corresponding weight matrix, as well as, an expression of P_n in terms of a linear combination of the polynomials derivatives P'_n, P'_{n-1} and P'_{n+1} takes place.

Below we describe the features in a more general context. More precisely, if the Pearson equation (2.3) is satisfied and so non-commutativity is strongly involved. In this setting, we prove that the orthogonality of sequence of derivatives $\big\{\big(P^{\mathsf{L}}_{n+1}\big)'\big\}_{n\in\mathbb{N}}$ with respect to ϕW is replaced by the orthogonality of $\big\{\,\big(P^{\mathsf{L}}_{n+1}W^{\mathsf{L}}\big)'\,\big\}_{n\in\mathbb{N}}$ with respect to ϕW^{R}.

We start by stating the following theorem.

Theorem 2.9. *Let C be a simple closed curve (circle for example) negatively oriented (clockwise where a and b the starting and end points of γ, respectively), such that a and b are in the interior of C. Let W be a semiclassical matrix weight that admits the factorization $W = W^{\mathsf{L}}W^{\mathsf{R}}$ such that* (2.3) *takes place, and $\{Z^{\mathsf{L}}_n\}_{n\in\mathbb{N}}$, $\{Z^{\mathsf{R}}_n\}_{n\in\mathbb{N}}$, are its sequence of constant jump fundamental matrices. Then,*

$$\int_\gamma \big(P^{\mathsf{L}}_{n+1}W^{\mathsf{L}}\big)'\phi W^{\mathsf{R}}P^{\mathsf{R}}_k \frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} = \mathbf{0}, \qquad k = 0,1,\dots,n-p, \tag{2.51}$$

$$\int_\gamma \phi P^{\mathsf{L}}_k W^{\mathsf{L}}\big(W^{\mathsf{R}}P^{\mathsf{R}}_{n+1}\big)' \frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} = \mathbf{0}, \qquad k = 0,1,\dots,n-q, \tag{2.52}$$

$$\int_C \big(Q^{\mathsf{L}}_{n+1}(W^{\mathsf{R}})^{-1}\big)'\phi W^{\mathsf{R}}P^{\mathsf{R}}_k \frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} = \mathbf{0}, \qquad k = 0,1,\dots,n-p, \tag{2.53}$$

$$\int_C \phi P^{\mathsf{L}}_k W^{\mathsf{L}}\big((W^{\mathsf{L}})^{-1}Q^{\mathsf{R}}_{n+1}\big)' \frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} = \mathbf{0}, \qquad k = 0,1,\dots,n-q, \tag{2.54}$$

where $p = \max\{\deg\phi - 1, \deg\psi_2\}$ and $q = \max\{\deg\phi - 1, \deg\psi_1\}$.

Proof. We will prove (2.51) and (2.53), then the proof of (2.52) and (2.54) are very similar. Using boundary condition, orthogonality (1.1) and Pearson equations (2.3)

$$\int_\gamma \big(P^{\mathsf{L}}_{n+1}W^{\mathsf{L}}\big)'\phi W^{\mathsf{R}}t^k \frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} = -\int_\gamma P^{\mathsf{L}}_{n+1}W\big(\psi_2 t^k + (\phi t^k)'\big) \frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} = \mathbf{0},$$

for $k = 0,1,\dots,n-p$ and $n\in\mathbb{N}$. The quasi-orthogonality follows for $\big\{(P^{\mathsf{L}}_{n+1}W^{\mathsf{L}})'\big\}_{n\in\mathbb{N}}$.

$$\begin{aligned}\int_C \big(Q^{\mathsf{L}}_{n+1}(W^{\mathsf{R}})^{-1}\big)'\phi W^{\mathsf{R}}t^k \frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} &= -\int_C Q^{\mathsf{L}}_{n+1}(W^{\mathsf{R}})^{-1}\big(\phi W^{\mathsf{R}}t^k\big)' \frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}\\ &= -\int_C Q^{\mathsf{L}}_{n+1}\big(\psi_2 t^k + (\phi t^k)'\big) \frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}\\ &= -\int_C \Big(\int_\gamma \frac{P^{\mathsf{L}}_{n+1}(\epsilon)}{\epsilon - t}W(\epsilon)\,\frac{\mathrm{d}\,\epsilon}{2\pi\,\mathrm{i}}\Big)\big(\psi_2 t^k + (\phi t^k)'\big) \frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}\end{aligned}$$

We use now Fubini's Theorem then Cauchy's integral formula,

$$\begin{aligned}
&\int_C \big(Q^{\mathsf{L}}_{n+1}(W^{\mathsf{R}})^{-1}\big)'\phi W^{\mathsf{R}} t^k \,\frac{\mathrm{d}\, t}{2\pi\,\mathrm{i}}\\
&\qquad = -\int_\gamma \Big(\int_C \frac{P^{\mathsf{L}}_{n+1}(\epsilon)}{\epsilon - t} W(\epsilon)\big(\psi_2 t^k + (\phi t^k)'\big)\,\frac{\mathrm{d}\, t}{2\pi\,\mathrm{i}}\Big)\frac{\mathrm{d}\,\epsilon}{2\pi\,\mathrm{i}}\\
&\qquad = -\int_\gamma P^{\mathsf{L}}_{n+1}(\epsilon) W(\epsilon)\Big(\int_C \frac{\psi_2 t^k + (\phi t^k)'}{\epsilon - t}\,\frac{\mathrm{d}\, t}{2\pi\,\mathrm{i}}\Big)\frac{\mathrm{d}\,\epsilon}{2\pi\,\mathrm{i}}\\
&\qquad = -\int_\gamma P^{\mathsf{L}}_{n+1}(\epsilon) W(\epsilon)\big(\psi_2(\epsilon)\epsilon^k + (\phi(\epsilon)\epsilon^k)'\big)\frac{\mathrm{d}\,\epsilon}{2\pi\,\mathrm{i}}\\
&\qquad = \mathbf{0}, \qquad k = 0,\dots,n-p, \quad n\in\mathbb{N},
\end{aligned}$$

as we wanted to prove. □

Corollary 2.3. *Let C be a simple closed curve (circle for example) negatively oriented (clockwise where a and b are the starting and end points of γ, respectively), such that a and b are in the interior of C. Let W be a classical matrix weight that admits the factorization $W = W^{\mathsf{L}} W^{\mathsf{R}}$ such that* (2.3) *takes place, and $\{Z^{\mathsf{L}}_n\}_{n\in\mathbb{N}}$, $\{Z^{\mathsf{R}}_n\}_{n\in\mathbb{N}}$, are its sequence of constant jump fundamental matrices. Then, the following are dual sequences:*

(i) $\big\{\phi P^{\mathsf{L}}_n W^{\mathsf{L}}\big\}_{n\in\mathbb{N}}$ *and* $\big\{\big(W^{\mathsf{R}} P^{\mathsf{R}}_{n+1}\big)'\big\}_{n\in\mathbb{N}}$, *defined on γ,*
(ii) $\big\{\phi P^{\mathsf{L}}_n W^{\mathsf{L}}\big\}_{n\in\mathbb{N}}$ *and* $\big\{\big((W^{\mathsf{L}})^{-1} Q^{\mathsf{R}}_{n+1}\big)'\big\}_{n\in\mathbb{N}}$, *defined on C,*
(iii) $\big\{\big(P^{\mathsf{L}}_{n+1} W^{\mathsf{L}}\big)'\big\}_{n\in\mathbb{N}}$ *and* $\big\{\phi W^{\mathsf{R}} P^{\mathsf{R}}_n\big\}_{n\in\mathbb{N}}$, *defined on γ,*
(iv) $\big\{\big(Q^{\mathsf{L}}_{n+1}(W^{\mathsf{R}})^{-1}\big)'\big\}_{n\in\mathbb{N}}$ *and* $\big\{\phi W^{\mathsf{R}} P^{\mathsf{R}}_n\big\}_{n\in\mathbb{N}}$, *defined on C,*

are dual sequences.

Proof. Under the assymptions above, we get $p = q = 1$. Equations (2.52) and (2.54) leads to the left dual while equations (2.51) and (2.53) gives the right dual. □

This result together with Theorem 1.9 we get that the dual sequences $\big\{\big(W^{\mathsf{R}} P^{\mathsf{R}}_{n+1}\big)'\big\}_{n\in\mathbb{N}}$, $\big\{\big((W^{\mathsf{L}})^{-1} Q^{\mathsf{R}}_{n+1}\big)'\big\}_{n\in\mathbb{N}}$, $\big\{\big(P^{\mathsf{L}}_{n+1} W^{\mathsf{L}}\big)'\big\}_{n\in\mathbb{N}}$, and $\big\{\big(Q^{\mathsf{L}}_{n+1}(W^{\mathsf{R}})^{-1}\big)'\big\}_{n\in\mathbb{N}}$, also satisfies three term recurrence relations.

Theorem 2.10. *Let C be a simple closed curve (circle for example) negatively oriented (clockwise where a and b are the starting and end points of γ, respectively), such that a and b are in the interior of C. Let W be a regular matrix weight, together with the boundary conditions* (2.1)*, that admits the factorization $W = W^{\mathsf{L}} W^{\mathsf{R}}$, and $\{Z^{\mathsf{L}}_n\}_{n\in\mathbb{N}}$, $\{Z^{\mathsf{R}}_n\}_{n\in\mathbb{N}}$, are its sequence of constant jump fundamental matrices. Then, the following are equivalent:*

(i) *The weight matrix W is classical.*
(ii) *The following linear relations holds*

$$P^{\mathsf{L}}_n W^{\mathsf{L}} = \sum_{k=n-1}^{n+1} A^n_k \big(P^{\mathsf{L}}_k W^{\mathsf{L}}\big)', \qquad Q^{\mathsf{L}}_n (W^{\mathsf{R}})^{-1} = \sum_{k=n-1}^{n+1} A^n_k \big(Q^{\mathsf{L}}_k (W^{\mathsf{R}})^{-1}\big)'. \tag{2.55}$$

Proof. (i)$\Longrightarrow$(ii). As we have said before this theorem, we know from Corollary 2.3 that the sequence $\big\{\big(P^{\mathsf{L}}_{n+1}W^{\mathsf{L}}\big)'\big\}_{n\in\mathbb{N}}$ satisfies a three term recurrence relation of type

$$z\left(P^{\mathsf{L}}_{n+1}W^{\mathsf{L}}\right)' = \widetilde{\epsilon}^{\mathsf{L}}_{n}\left(P^{\mathsf{L}}_{n+1}W^{\mathsf{L}}\right)' + \widetilde{\beta}^{\mathsf{L}}_{n}\left(P^{\mathsf{L}}_{n+1}W^{\mathsf{L}}\right)' + \widetilde{\gamma}^{\mathsf{L}}_{n}\left(P^{\mathsf{L}}_{n}W^{\mathsf{L}}\right)', \tag{2.56}$$

for some sequences of matrix $(\widetilde{\epsilon}^{\mathsf{L}}_{n})$, $(\widetilde{\beta}^{\mathsf{L}}_{n})$, $(\widetilde{\gamma}^{\mathsf{L}}_{n})$, with $\widetilde{\epsilon}^{\mathsf{L}}_{n}$, $\widetilde{\gamma}^{\mathsf{L}}_{n}$, $n\in\mathbb{N}$ invertible matrices. Moreover, Theorem 1.9 asserts that the sequence $\{P^{\mathsf{L}}_{n}\}_{n\in\mathbb{N}}$ is defined by (1.6) (by its left orthogonality). Multiplying, (1.6) on the right by W^{L} and taking derivatives

$$\left(P^{\mathsf{L}}_{n+1}W^{\mathsf{L}}\right)' = P^{\mathsf{L}}_{n}W^{\mathsf{L}} + (z-\beta^{\mathsf{L}}_{n})\left(P^{\mathsf{L}}_{n}W^{\mathsf{L}}\right)' - \gamma^{\mathsf{L}}_{n}\left(P^{\mathsf{L}}_{n-1}W^{\mathsf{L}}\right)'.$$

From this relation, we see that $\big\{\big(P^{\mathsf{L}}_{n+1}W^{\mathsf{L}}\big)'\big\}_{n\in\mathbb{N}}$ satisfies (2.56) if and only if, there exists A^{n}_{k}, $k\in\{n-1,n,n+1\}$ such that,

$$P^{\mathsf{L}}_{n}W^{\mathsf{L}} = \sum_{k=n-1}^{n+1} A^{n}_{k}\left(P^{\mathsf{L}}_{k}W^{\mathsf{L}}\right)', \tag{2.57}$$

where $\left(A^{n}_{n-1}-\gamma^{\mathsf{L}}_{n}\right)$ are for each $n\in\mathbb{N}$, invertible matrices.

We will prove that the second expression in (2.55) takes place. In fact, multiplying (2.57) from the right by $\dfrac{\phi(t)W^{\mathsf{R}}(t)}{2\pi\,\mathrm{i}(t-z)}$ then integrating,

$$\begin{aligned}\int_{\gamma}\frac{\phi(t)P^{\mathsf{L}}_{n}(t)W(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} &= \sum_{k=n-1}^{n+1} A^{n}_{k}\int_{\gamma}\left(P^{\mathsf{L}}_{k}(t)W^{\mathsf{L}}(t)\right)'\frac{\phi(t)W^{\mathsf{R}}(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}\\ &= -\sum_{k=n-1}^{n+1} A^{n}_{k}\int_{\gamma}P^{\mathsf{L}}_{k}(t)W^{\mathsf{L}}(t)\Big(\frac{\phi(t)W^{\mathsf{R}}(t)}{t-z}\Big)'\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}\\ &= -\sum_{k=n-1}^{n+1} A^{n}_{k}\Big(\int_{\gamma}\frac{P^{\mathsf{L}}_{k}W}{t-z}(\phi'+\psi_2)\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} - \int_{\gamma}\frac{\phi P^{\mathsf{L}}_{k}W}{(t-z)^2}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}\Big).\end{aligned} \tag{2.58}$$

We will simplify separately the integrals,

$$\begin{aligned}\int_{\gamma}\frac{P^{\mathsf{L}}_{k}W}{t-z}\big(\phi'+\psi_2\big)\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} &= \int_{\gamma}\frac{P^{\mathsf{L}}_{k}W}{t-z}\big\{\left(\phi'+\psi_2\right)(t) - (\phi'+\psi_2)(z)\big\}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} + Q^{\mathsf{L}}_{k}(\phi'+\psi_2)(z)\\ &= Q^{\mathsf{L}}_{k}\big(\phi'+\psi_2\big)(z), \qquad k\geq 1.\end{aligned}$$

In a similar way, we prove that

$$\int_{\gamma}\frac{P^{\mathsf{L}}_{k}(t)\phi(t)W(t)}{(t-z)^2}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} = \phi'(z)Q^{\mathsf{L}}_{k}(z) + \phi(z)\big(Q^{\mathsf{L}}_{k}\big)'(z), \qquad k\geq 1.$$

Then, follows

$$\int_{\gamma}\frac{\phi(t)P^{\mathsf{L}}_{n}(t)W(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} = \phi(z)Q^{\mathsf{L}}_{n}(z), \qquad n\geq 2, \tag{2.59}$$

now, replacing (2.59) in (2.58) we get

$$\phi(z)Q^{\mathsf{L}}_{n}(z) = -\sum_{k=n-1}^{n+1} A^{n}_{k}\left(Q^{\mathsf{L}}_{k}(z)\psi_2 - \phi Q^{\mathsf{L}}_{k}(z)\right)$$

$$= \sum_{k=n-1}^{n+1} A_k^n \left(\phi Q_k^{\mathsf{L}}(z) - Q_k^{\mathsf{L}}(z)\left(\phi (W^{\mathsf{R}})^{-1}(W^{\mathsf{R}})'\right)\right)$$

Simplify by ϕ then multiply this relation from the right by $(W^{\mathsf{R}})^{-1}$, it follows that

$$Q_n^{\mathsf{L}}(W^{\mathsf{R}})^{-1} = \sum_{k=n-1}^{n+1} A_k^n \left(Q_k^{\mathsf{L}}(W^{\mathsf{R}})^{-1}\right)'.$$

Now, we will show that (ii)$\Longrightarrow$(i). Equations (2.55) could be written as,

$$\begin{aligned} Z_{n+1}^{\mathsf{L}} &= \begin{bmatrix} A_{n+2}^{n+1} & -A_{n+1}^{n+1}C_{n+1}^{-1} \\ \mathbf{0} & C_nA_{n+1}^nC_{n+1}^{-1}\end{bmatrix}\left(Z_{n+2}^{\mathsf{L}}\right)' + \begin{bmatrix} A_n^{n+1} & \mathbf{0} \\ -C_nA_n^n & C_nA_{n-1}^nC_{n-1}^{-1}\end{bmatrix}\left(Z_n^{\mathsf{L}}\right)' \\ &= \begin{bmatrix} A_{n+2}^{n+1} & -A_{n+1}^{n+1}C_{n+1}^{-1} \\ \mathbf{0} & C_nA_{n+1}^nC_{n+1}^{-1}\end{bmatrix}\left(T_{n+1}^{\mathsf{L}}Z_{n+1}^{\mathsf{L}}\right)' + \begin{bmatrix} A_n^{n+1} & \mathbf{0} \\ -C_nA_n^n & C_nA_{n-1}^nC_{n-1}^{-1}\end{bmatrix}\left(\left(T_n^{\mathsf{L}}\right)^{-1}Z_{n+1}^{\mathsf{L}}\right)'\end{aligned}$$

After simplification we get,

$$\begin{aligned} &\left\{\mathrm{I} - \begin{bmatrix} A_{n+2}^{n+1} & -A_{n+1}^{n+1}C_{n+1}^{-1} \\ \mathbf{0} & C_nA_{n+1}^nC_{n+1}^{-1}\end{bmatrix}\left(T_{n+1}^{\mathsf{L}}\right)' - \begin{bmatrix} A_n^{n+1} & \mathbf{0} \\ -C_nA_n^n & C_nA_{n-1}^nC_{n-1}^{-1}\end{bmatrix}\left((T_n^{\mathsf{L}})^{-1}\right)'\right\}Z_{n+1}^{\mathsf{L}} \\ &\quad = \left\{\begin{bmatrix} A_{n+2}^{n+1} & -A_{n+1}^{n+1}C_{n+1}^{-1} \\ \mathbf{0} & C_nA_{n+1}^nC_{n+1}^{-1}\end{bmatrix}T_{n+1}^{\mathsf{L}} + \begin{bmatrix} A_n^{n+1} & \mathbf{0} \\ -C_nA_n^n & C_nA_{n-1}^nC_{n-1}^{-1}\end{bmatrix}(T_n^{\mathsf{L}})^{-1}\right\}\left(Z_{n+1}^{\mathsf{L}}\right)'\end{aligned} \tag{2.60}$$

Taking,

$$\begin{aligned} H_n^{\mathsf{L}} &:= \mathrm{I} - \begin{bmatrix} A_{n+2}^{n+1} & -A_{n+1}^{n+1}C_{n+1}^{-1} \\ \mathbf{0} & C_nA_{n+1}^nC_{n+1}^{-1}\end{bmatrix}\left(T_{n+1}^{\mathsf{L}}\right)' - \begin{bmatrix} A_n^{n+1} & \mathbf{0} \\ -C_nA_n^n & C_nA_{n-1}^nC_{n-1}^{-1}\end{bmatrix}(T_n^{\mathsf{L}})^{-1} \\ G_n^{\mathsf{L}} &:= \begin{bmatrix} A_{n+2}^{n+1} & -A_{n+1}^{n+1}C_{n+1}^{-1} \\ \mathbf{0} & C_nA_{n+1}^nC_{n+1}^{-1}\end{bmatrix}T_{n+1}^{\mathsf{L}} + \begin{bmatrix} A_n^{n+1} & \mathbf{0} \\ -C_nA_n^n & C_nA_{n-1}^nC_{n-1}^{-1}\end{bmatrix}(T_n^{\mathsf{L}})^{-1} \\ \phi_n &\coloneqq \det(G_n^{\mathsf{L}}), \qquad \mathcal{L}_{n,1} \coloneqq \operatorname{adj}(G_n^{\mathsf{L}})H_n^{\mathsf{L}},\end{aligned}$$

where $\operatorname{adj}(G_n^{\mathsf{L}})$ is the adjugate of the matrix G_n^{L}, we get that (2.60) becomes,

$$\phi_n\left(Z_{n+1}^{\mathsf{L}}\right)' = \mathcal{L}_{n,1}Z_{n+1}^{\mathsf{L}} \tag{2.61}$$

We will prove now that ϕ_n does not depend on n. In fact, by one hand

$$\phi_{n+1}\left(Z_{n+2}^{\mathsf{L}}\right)' = \phi_{n+1}\left(T_{n+1}^{\mathsf{L}}Z_{n+1}^{\mathsf{L}}\right)' = \phi_{n+1}\left\{T_{n+1}^{\mathsf{L}}\left(Z_{n+1}^{\mathsf{L}}\right)' + \left(T_{n+1}^{\mathsf{L}}\right)'Z_{n+1}^{\mathsf{L}}\right\}$$

and using (2.61) we get that

$$\phi_{n+1}\left(Z_{n+1}^{\mathsf{L}}\right)' = \mathcal{L}_{n,2}Z_{n+1}^{\mathsf{L}} \quad \text{with} \quad \mathcal{L}_{n,2} = \left(T_{n+1}^{\mathsf{L}}\right)^{-1}\left\{\mathcal{L}_{n+1,1}T_{n+1}^{\mathsf{L}} - \phi_{n+1}\left(T_{n+1}^{\mathsf{L}}\right)'\right\}. \tag{2.62}$$

Since the first order differential equation for Z_{n+1}^{L} is unique, up to a multiplicative factor, the equations (2.61) and (2.62) implies the existence of a scalar polynomial v_n such that $\phi_{n+1} = v_n\phi_n$. Thus, $\phi_n = v_n \times \cdots \times v_1 \times \phi_1$. Since the degree of ϕ_n is bounded by a number independent of n, then the degree of the v_n's are zero and we get

$$\phi_1\left(Z_{n+1}^{\mathsf{L}}\right)' = \mathcal{L}_{n+1}Z_{n+1}^{\mathsf{L}}$$

where, $\mathcal{L}_{n+1} = (v_n \times \cdots \times v_1)^{-1}\mathcal{L}_{n,1}$. □

6. Riccati type equation

Now we state a characterization for the semiclassical matrix orthogonal polynomials in terms of a Riccati differential equation for the ratio of matrix orthogonal polynomials and second kind matrix functions. In some sense we reinterpret the Corollary of the Theorem 1.11 for the case that the weights belongs to the semiclassical class.

Theorem 2.11. *Let W be a regular matrix weight, together with the boundary conditions* (2.1)*, that admits the factorization $W = W^{\mathsf{L}}W^{\mathsf{R}}$, and $\{Z_n^{\mathsf{L}}\}_{n\in\mathbb{N}}$ its sequence of left constant jump fundamental matrix. The following are equivalent:*

i) W *is semiclassical,* i.e. (2.2) *takes place.*
ii) *there exists a polynomial matrix (whose degree does not depend on n), L_n, such that the corresponding left constant jump fundamental matrix Z_n^{L} satisfies, for each $n \in \mathbb{N}$, the Sylvester matrix differential equations,* (2.37)*.*
iii) *The sequence of functions $f_n^{\mathsf{L}} = P_n^{\mathsf{L}}\big(P_{n-1}^{\mathsf{L}}\big)^{-1}$, satisfies, for each $n \in \mathbb{Z}_+$, the Riccati type equation*

$$\phi\left(f_n^{\mathsf{L}}\right)' = \mathsf{L}_n^{1,1}f_n^{\mathsf{L}} - f_n^{\mathsf{L}}C_{n-1}^{-1}\mathsf{L}_n^{2,2}C_{n-1} + f_n^{\mathsf{L}}C_{n-1}^{-1}\mathsf{L}_n^{2,1}f_n^{\mathsf{L}} - \mathsf{L}_n^{1,2}C_{n-1}. \tag{2.63}$$

iv) *The sequence of functions $g_n^{\mathsf{L}} = Q_n^{\mathsf{L}}\big(Q_{n-1}^{\mathsf{L}}\big)^{-1}$ satisfies, for each $n \in \mathbb{Z}_+$, the Riccati type equation*

$$\phi\left(g_n^{\mathsf{L}}\right)' = \mathsf{L}_n^{1,1}g_n^{\mathsf{L}} - g_n^{\mathsf{L}}C_{n-1}^{-1}\mathsf{L}_n^{2,2}C_{n-1} + g_n^{\mathsf{L}}C_{n-1}^{-1}\mathsf{L}_n^{2,1}g_n^{\mathsf{L}} - \mathsf{L}_n^{1,2}C_{n-1}. \tag{2.64}$$

Proof. We know, from Theorem 2.8 that i) is equivalent to ii). Lets start by ii) $\Longrightarrow$ iii). We recall that (2.37) is the matrix reinterpretation of (2.19), (2.23), (2.20), (2.24). Multiply (2.19) from the right by $\big(P_{n-1}^{\mathsf{L}}W^{\mathsf{L}}\big)^{-1}$ and (2.23) from the right by $\big(P_{n-1}^{\mathsf{L}}W^{\mathsf{L}}\big)^{-1}$ and from the left by $-f_n^{\mathsf{L}}C_{n-1}^{-1}$ the sum, we obtain (2.63). Now, multiply (2.20) from the right by $\big(Q_{n-1}^{\mathsf{L}}(W^{\mathsf{R}})^{-1}\big)^{-1}$ and (2.24) from the right by $\big(Q_{n-1}^{\mathsf{L}}(W^{\mathsf{R}})^{-1}\big)^{-1}$ and from the left by $-g_n^{\mathsf{L}}C_{n-1}^{-1}$ then sum, we obtain (2.64), and so i) $\Longrightarrow$ iii).

To prove iii) $\Longrightarrow$ ii). Suppose that the equations (2.63) and (2.64) holds. Using the the three terms recurrence relation (1.6) we get,

$$z\,\mathrm{I} - \beta_{n-1}^{\mathsf{L}} = f_n^{\mathsf{L}} + \gamma_{n-1}^{\mathsf{L}}\big(f_{n-1}^{\mathsf{L}}\big)^{-1} \tag{2.65}$$

Taking derivative in (2.65) leads to

$$\mathrm{I} = \left(f_n^{\mathsf{L}}\right)' - \gamma_{n-1}^{\mathsf{L}}\left(f_{n-1}^{\mathsf{L}}\right)^{-1}\left(f_{n-1}^{\mathsf{L}}\right)'\left(f_{n-1}^{\mathsf{L}}\right)^{-1}.$$

Hence,

$$\begin{aligned}
\phi\big(f_{n+1}^{\mathsf{L}}\big)' &= \phi\,\mathrm{I} + \gamma_n^{\mathsf{L}}\big(f_n^{\mathsf{L}}\big)^{-1}\big(\phi(f_n^{\mathsf{L}})'\big)\big(f_n^{\mathsf{L}}\big)^{-1} \\
&= \phi\,\mathrm{I} + \gamma_n^{\mathsf{L}}\left(f_n^{\mathsf{L}}\right)^{-1}\left(\mathsf{L}_n^{1,1}f_n^{\mathsf{L}} - f_n^{\mathsf{L}}C_{n-1}^{-1}\mathsf{L}_n^{2,2}C_{n-1} + f_n^{\mathsf{L}}C_{n-1}^{-1}\mathsf{L}_n^{2,1}f_n^{\mathsf{L}} - \mathsf{L}_n^{1,2}C_{n-1}\right)\left(f_n^{\mathsf{L}}\right)^{-1} \\
&= \phi\,\mathrm{I} + \gamma_n^{\mathsf{L}}\left(f_n^{\mathsf{L}}\right)^{-1}\mathsf{L}_n^{1,1} - \gamma_n^{\mathsf{L}}C_{n-1}^{-1}\mathsf{L}_n^{2,2}C_{n-1}\left(f_n^{\mathsf{L}}\right)^{-1} + \gamma_n^{\mathsf{L}}C_{n-1}^{-1}\mathsf{L}_n^{2,1} \\
&\quad - \gamma_n^{\mathsf{L}}\left(f_n^{\mathsf{L}}\right)^{-1}\mathsf{L}_n^{1,2}C_{n-1}\left(f_n^{\mathsf{L}}\right)^{-1}.
\end{aligned}$$

Replace $\gamma_n^{\mathsf{L}}\left(f_n^{\mathsf{L}}\right)^{-1}$ from (2.65),

$$\begin{aligned}\phi\left(f_{n+1}^{\mathsf{L}}\right)' = \phi\,\mathrm{I} + C_n^{-1}\mathsf{L}_n^{2,1} - C_n^{-1}\mathsf{L}_n^{2,2}C_n\left(z\,\mathrm{I}-\beta_n^{\mathsf{L}}\right) + \left(z\,\mathrm{I}-\beta_n^{\mathsf{L}}\right)\mathsf{L}_n^{1,1}\\ - \left(z\,\mathrm{I}-\beta_n^{\mathsf{L}}\right)\mathsf{L}_n^{1,2}C_n\left(z\,\mathrm{I}-\beta_n^{\mathsf{L}}\right) - f_{n+1}^{\mathsf{L}}\left\{\mathsf{L}_n^{1,1} - \mathsf{L}_n^{1,2}C_n\left(z\,\mathrm{I}-\beta_n^{\mathsf{L}}\right)\right\}\\ - f_{n+1}^{\mathsf{L}}\mathsf{L}_n^{1,2}C_n f_{n+1}^{\mathsf{L}} + \left\{C_n^{-1}\mathsf{L}_n^{2,2}C_n + \left(z\,\mathrm{I}-\beta_n^{\mathsf{L}}\right)\mathsf{L}_n^{1,2}C_n\right\}f_{n+1}^{\mathsf{L}}\end{aligned}$$

So f_{n+1}^{L} verifies a Riccati equation. Using (2.63) we obtain by compatibility the system,

$$\begin{aligned}\phi\,\mathrm{I} - \mathsf{L}_{n+1}^{1,2}C_n &= C_n^{-1}\mathsf{L}_n^{2,1} - C_n^{-1}\mathsf{L}_n^{2,2}C_n\left(z\,\mathrm{I}-\beta_n^{\mathsf{L}}\right) + \left(z\,\mathrm{I}-\beta_n^{\mathsf{L}}\right)\mathsf{L}_n^{1,1} - \left(z\,\mathrm{I}-\beta_n^{\mathsf{L}}\right)\mathsf{L}_n^{1,2}C_n\left(z\,\mathrm{I}-\beta_n^{\mathsf{L}}\right)\\ -\mathsf{L}_{n+1}^{2,2}C_n &= -\mathsf{L}_n^{1,1} + \mathsf{L}_n^{1,2}C_n\left(z\,\mathrm{I}-\beta_n^{\mathsf{L}}\right)\\ -\mathsf{L}_n^{1,2}C_n &= C_n^{-1}\mathsf{L}_{n+1}^{2,1}\\ \mathsf{L}_{n+1}^{1,1} &= C_n^{-1}\mathsf{L}_n^{2,2}C_n + \left(z\,\mathrm{I}-\beta_n^{\mathsf{L}}\right)\mathsf{L}_n^{1,2}C_n.\end{aligned}$$

This system, after simplification, collapses to the curvature formula (2.47). Now, from Theorem 2.8 we get that W is semiclassical. In the same way we can see that iii) implies i). □

Similar considerations lead to Riccati equations for the right case.

Theorem 2.12. *Let W be a regular matrix weight, together with the boundary conditions* (2.1)*, that admits the factorization $W = W^{\mathsf{L}}W^{\mathsf{R}}$, and $\{Z_n^{\mathsf{R}}\}_{n\in\mathbb{N}}$ its sequence of right constant jump fundamental matrices. The following are equivalent:*

i) *W is semiclassical,* i.e. (2.2) *takes place.*
ii) *there exists a polynomial matrix (whose degree does not depend on n), R_n, such that the corresponding left constant jump fundamental matrix Z_n^{R} satisfies, for each $n\in\mathbb{N}$, the Sylvester matrix differential equations,* (2.38)*.*
iii) *The functions $f_n^{\mathsf{R}} = \left(P_{n-1}^{\mathsf{R}}\right)^{-1}P_n^{\mathsf{R}}$ satisfies, for each $n\in\mathbb{Z}_+$, the Riccati type equation*

$$\phi\left(f_n^{\mathsf{R}}\right)' = f_n^{\mathsf{R}}\mathsf{R}_{1,1}^n - C_{n-1}\mathsf{R}_{2,2}^nC_{n-1}^{-1}f_n^{\mathsf{R}} + f_n^{\mathsf{R}}\mathsf{R}_{1,2}^nC_{n-1}^{-1}f_n^{\mathsf{R}} - C_{n-1}\mathsf{R}_{2,1}^n.$$

iv) *The functions $g_n^{\mathsf{R}} = \left(Q_{n-1}^{\mathsf{R}}\right)^{-1}\left(Q_n^{\mathsf{R}}\right)$ satisfies, for each $n\in\mathbb{Z}_+$, the Riccati type equation*

$$\phi\left(g_n^{\mathsf{R}}\right)' = g_n^{\mathsf{R}}\mathsf{R}_{1,1}^n - C_{n-1}\mathsf{R}_{2,2}^nC_{n-1}^{-1}g_n^{\mathsf{R}} + g_n^{\mathsf{R}}\mathsf{R}_{1,2}^nC_{n-1}^{-1}g_n^{\mathsf{R}} - C_{n-1}\mathsf{R}_{2,1}^n.$$

CHAPTER 3

Riemann–Hilbert analysis

1. Introduction

In Chapter 2 we have studied different characterizations of semiclassical matrix orthogonal polynomials with regard to a matrix weight function satisfying a matrix Pearson type equation (2.2), where ψ_1 and ψ_2 are matrix polynomials.

Depending on the degree and the roots of ϕ, we get that every classical matrix weight function belongs, up to afine transformations of z, to one of the following canonical types:

(1) Hermite: $\phi(z) = 1$.
(2) Laguerre: $\phi(z) = z$.
(3) Jacobi: $\phi(z) = z(1-z)$.
(4) Bessel: $\phi(z) = z^2$.

Let us mention that for matrix versions of Laguerre, Hermite and Jacobi polynomials, the scalar-type Rodrigues' formula [47, 48] and a second order differential equation [10, 42, 46] has been discussed. It also has been proven [50] that operators of the form $D = \partial^2 F_2(t) + \partial^1 F_1(t) + \partial^0 F_0$ have as eigenfunctions different infinite families of matrix orthogonal polynomials. A new family of matrix orthogonal polynomials satisfying second order differential equations, whose three term recurrence relation coefficients do not behave asymptotically as the identity matrix, was found in [10]; see also [19].

In this chapter, we will look at matrix orthogonal polynomials with regard to a weight matrix, which will satisfy a more generic Pearson type equation, *i.e.* ϕ is one of the four cases mentioned above, ψ_1 and ψ_2 are replaced by entire functions h^{L} and h^{R}, respectively. To be more specific, we suppose W to be a $N \times N$ weight matrix with support on a smooth oriented non self-intersecting unbounded curve γ in the complex plane $\mathbb{C}$, *i.e.* $W^{(j,k)}$ is, for each $j,k \in \{1,\dots,N\}$, a complex weight with support on γ. In addition to boundary conditions (2.1), we assume that the right and left logarithmic derivative

$$h^{\mathsf{L}}(z) \coloneqq \phi(z)\big(W^{\mathsf{L}}(z)\big)'\big(W^{\mathsf{L}}(z)\big)^{-1}, \qquad h^{\mathsf{R}}(z) \coloneqq \phi(z)\big(W^{\mathsf{R}}(z)\big)^{-1}\big(W^{\mathsf{R}}(z)\big)' \tag{3.1}$$

exist and are entire functions. In such a setting, the weight matrix factors out as $W(z) = W^{\mathsf{L}}W^{\mathsf{R}}$. As well, (3.1) holds, and so by Theorem 2.1 we have

$$\phi(z)W'(z) = h^{\mathsf{L}}(z)W(z) + W(z)h^{\mathsf{R}}(z). \tag{3.2}$$

We obtain Sylvester systems of differential equations for the orthogonal polynomials and its second kind functions, directly from a Riemann–Hilbert problem, with jumps supported on appropriate curves in the complex plane. The differential properties for the weight function are fundamental. In this case we consider a Pearson type differential equation for the weight matrix. In order to get first and second order matrix differential operators we present, in addition to the left and right constant jump fundamental matrix defined in (2.33) and (2.34), the important structure matrices.

For each factorization $W = W^{\mathsf{L}} W^{\mathsf{R}}$, we recall the constant jump matrices defined in (2.33) and (2.34). In the next theorem it is explained the name we gave to these matrix functions.

Theorem 3.1. *For each factorization $W = W^{\mathsf{L}} W^{\mathsf{R}}$, the constant jump fundamental matrices Z_n^{L} and Z_n^{R} are, for each $n \in \mathbb{N}$, characterized by the following properties:*

(i) *They are holomorphic on $\mathbb{C} \setminus \gamma$.*
(ii) *We have the following asymptotic behaviors for $z \to \infty$,*

$$Z_n^{\mathsf{L}}(z) = \big(\mathrm{I} + \mathrm{O}(z^{-1})\big) \begin{bmatrix} z^n W^{\mathsf{L}}(z) & \mathbf{0} \\ \mathbf{0} & \mathrm{I}\, z^{-n} (W^{\mathsf{R}}(z))^{-1} \end{bmatrix},$$
$$Z_n^{\mathsf{R}}(z) = \begin{bmatrix} z^n W^{\mathsf{R}}(z) & \mathbf{0} \\ \mathbf{0} & (W^{\mathsf{L}}(z))^{-1} z^{-n} \end{bmatrix} \big(\mathrm{I} + \mathrm{O}(z^{-1})\big).$$

(iii) *They present the following* constant jump condition *on γ*

$$\big(Z_n^{\mathsf{L}}(z)\big)_+ = \big(Z_n^{\mathsf{L}}(z)\big)_- \begin{bmatrix} \mathrm{I} & \mathrm{I} \\ \mathbf{0} & \mathrm{I} \end{bmatrix}, \qquad \big(Z_n^{\mathsf{R}}(z)\big)_+ = \begin{bmatrix} \mathrm{I} & \mathbf{0} \\ \mathrm{I} & \mathrm{I} \end{bmatrix} \big(Z_n^{\mathsf{R}}(z)\big)_-,$$

for all $z \in \gamma$ in the support on the weight matrix.

Proof. We only give the proofs for the left case because their right ones follows from (2.35).

(i) As the W^{L} and W^{R} are matrices of entire functions the holomorphity properties of Z_n^{L} is inherit from that of the fundamental matrices Y_n^{L}.
(ii) It follows from the asymptotic of the fundamental matrices.
(iii) From the definition of Z_n^{L} we have

$$\big(Z_n^{\mathsf{L}}(z)\big)_+ = \big(Y_n^{\mathsf{L}}(z)\big)_+ \begin{bmatrix} W^{\mathsf{L}}(z) & \mathbf{0} \\ \mathbf{0} & (W^{\mathsf{R}}(z))^{-1} \end{bmatrix},$$

and taking into account Theorem 1.8 we arrive to

$$\big(Z_n^{\mathsf{L}}(z)\big)_+ = \big(Y_n^{\mathsf{L}}(z)\big)_- \begin{bmatrix} \mathrm{I} & W^{\mathsf{L}}(z) W^{\mathsf{R}}(z) \\ \mathbf{0} & \mathrm{I} \end{bmatrix} \begin{bmatrix} W^{\mathsf{L}}(z) & \mathbf{0} \\ \mathbf{0} & (W^{\mathsf{R}}(z))^{-1} \end{bmatrix};$$

now, as

$$\begin{bmatrix} \mathrm{I} & W^{\mathsf{L}}(z) W^{\mathsf{R}}(z) \\ \mathbf{0} & \mathrm{I} \end{bmatrix} \begin{bmatrix} W^{\mathsf{L}}(z) & \mathbf{0} \\ \mathbf{0} & (W^{\mathsf{R}}(z))^{-1} \end{bmatrix} = \begin{bmatrix} W^{\mathsf{L}}(z) & \mathbf{0} \\ \mathbf{0} & (W^{\mathsf{R}}(z))^{-1} \end{bmatrix} \begin{bmatrix} \mathrm{I} & \mathrm{I} \\ \mathbf{0} & \mathrm{I} \end{bmatrix},$$

we get the desired *constant* jump condition for Z_n^{L}. □

Now, we introduce what we call structure matrices given in terms of the left and right logarithmic derivatives, respectively by,

$$M^{\mathsf{L}}_n(z) := \big(Z^{\mathsf{L}}_n(z)\big)'\big(Z^{\mathsf{L}}_n(z)\big)^{-1}, \qquad M^{\mathsf{R}}_n(z) := \big(Z^{\mathsf{R}}_n(z)\big)^{-1}\big(Z^{\mathsf{R}}_n(z)\big)', \qquad n\in\mathbb{N}. \tag{3.3}$$

We will study, for each Riemann–Hilbert problem, the analytic properties of these functions. As can be seen from their definition in (3.3), these functions are pretty much connected to the left and right matrices, L_n and R_n, studied in Chapter 2 (*cf.* for example Theorem 2.6).

As the constant jump fundamental matrices, Z^{L}_n and Z^{R}_n, are connected by (2.35) it can be shown that,

$$M^{\mathsf{R}}_n(z) = \begin{bmatrix} \mathbf{0} & \mathrm{I} \\ -\mathrm{I} & \mathbf{0}\end{bmatrix} M^{\mathsf{L}}_n(z) \begin{bmatrix} \mathbf{0} & \mathrm{I} \\ -\mathrm{I} & \mathbf{0}\end{bmatrix}, \qquad n\in\mathbb{N}. \tag{3.4}$$

Theorem 3.2. *The following formulas hold:*

i) *The zero curvature formulas*

$$\begin{bmatrix} \mathrm{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix} = M^{\mathsf{L}}_{n+1}(z)T^{\mathsf{L}}_n(z) - T^{\mathsf{L}}_n(z)M^{\mathsf{L}}_n(z), \qquad n\in\mathbb{N}, \tag{3.5}$$

$$\begin{bmatrix} \mathrm{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix} = T^{\mathsf{R}}_n(z)\, M^{\mathsf{R}}_{n+1}(z) - M^{\mathsf{R}}_n(z)T^{\mathsf{R}}_n(z), \qquad n\in\mathbb{N}. \tag{3.6}$$

ii) *The* second order *zero curvature formulas holds for all* $n\in\mathbb{N}$*,*

$$\begin{bmatrix} \mathrm{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix} M^{\mathsf{L}}_n(z) + M^{\mathsf{L}}_{n+1}(z)\begin{bmatrix} \mathrm{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix} = \big(M^{\mathsf{L}}_{n+1}(z)\big)^2 T^{\mathsf{L}}_n(z) - T^{\mathsf{L}}_n(z)\big(M^{\mathsf{L}}_n(z)\big)^2, \tag{3.7}$$

$$\begin{bmatrix} \mathrm{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix} M^{\mathsf{R}}_{n+1}(z) + M^{\mathsf{R}}_n(z)\begin{bmatrix} \mathrm{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix} = T^{\mathsf{R}}_n(z)\big(M^{\mathsf{R}}_{n+1}(z)\big)^2 - \big(M^{\mathsf{R}}_n(z)\big)^2 T^{\mathsf{R}}_n(z).$$

Proof. It follows from the definition of Z^{L}_n that

$$T^{\mathsf{L}}_n(z) = Y^{\mathsf{L}}_{n+1}(z)\big(Y^{\mathsf{L}}_n(z)\big)^{-1} = Z^{\mathsf{L}}_{n+1}(z)\big(Z^{\mathsf{L}}_n(z)\big)^{-1}.$$

Taking derivatives on $T_n(z)$ we get

$$\big(T^{\mathsf{L}}_n(z)\big)' = \big(Z^{\mathsf{L}}_{n+1}(z)\big)'\big(Z^{\mathsf{L}}_n(z)\big)^{-1} - Z^{\mathsf{L}}_{n+1}(z)\big(Z^{\mathsf{L}}_n(z)\big)^{-1}\big(Z^{\mathsf{L}}_n(z)\big)'\big(Z^{\mathsf{L}}_n(z)\big)^{-1}, \qquad n\in\mathbb{N},$$

and so, taking into account that

$$\big(Z^{\mathsf{L}}_{n+1}(z)\big)'\big(Z^{\mathsf{L}}_n(z)\big)^{-1} = \big(Z^{\mathsf{L}}_{n+1}(z)\big)'\big(Z^{\mathsf{L}}_{n+1}(z)\big)^{-1}Z^{\mathsf{L}}_{n+1}(z)\big(Z^{\mathsf{L}}_n(z)\big)^{-1} = M^{\mathsf{L}}_{n+1}(z)T^{\mathsf{L}}_n(z),$$

we get (3.5). Using the same ideas we derive (3.6).

Now, multiplying (3.5) on the left by M^{L}_{n+1} we get

$$M^{\mathsf{L}}_{n+1}(z)\begin{bmatrix} \mathrm{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix} = \big(M^{\mathsf{L}}_{n+1}(z)\big)^2 T^{\mathsf{L}}_n(z) - \big(M^{\mathsf{L}}_{n+1}(z)T^{\mathsf{L}}_n(z)\big)M^{\mathsf{L}}_n(z),$$

and again by (3.5) applied to the term $M^{\mathsf{L}}_{n+1}T^{\mathsf{L}}_n$ we get (3.7). □

2. Hermite type weights

In [15], the Hermite case is thoroughly examined, providing us with inspiration and motivation to delve into the study of Laguerre, Jacobi, and Bessel weights. As a starting point for our own discoveries, this section aims to present a comprehensive overview of the main definitions and results from that reference.

In this section, we suppose that the support γ of W has no finite end points. Special attention is paid to non-Abelian Hermite biorthogonal polynomials on the real line, understood as those whose weight matrix is a solution of a Pearson type equation with given first order matrix polynomial coefficients, *i.e.* W satisfies (3.1) with $\phi = 1$, or equivalently

$$\left(W^{\mathsf{L}}\right)'(z) = h^{\mathsf{L}} W^{\mathsf{L}}(z), \qquad\qquad \left(W^{\mathsf{R}}\right)'(z) = W^{\mathsf{R}}(z) h^{\mathsf{R}}(z),$$

then the matrix Pearson type equation (3.2) transforms into

$$W'(z) = h^{\mathsf{L}}(z) W(z) + W(z) h^{\mathsf{R}}(z),$$

with boundary conditions (2.1), *i.e.*

$$\lim_{z\to+\infty} W(z) = \mathbf{0}, \qquad \text{and} \qquad \lim_{z\to-\infty} W(z) = \mathbf{0}.$$

Now, we state a theorem on Riemann–Hilbert problem for the Hermite type weights. The results in this Section 2 for the Hermite case are taken from [15].

2.1. Riemann–Hilbert problem for the Hermite type weights.

Theorem 3.3. *The matrix function*

$$Y_n^{\mathsf{L}}(z) := \begin{bmatrix} P_n^{\mathsf{L}}(z) & Q_n^{\mathsf{L}}(z) \\ -C_{n-1} P_{n-1}^{\mathsf{L}}(z) & -C_{n-1} Q_{n-1}^{\mathsf{L}}(z) \end{bmatrix}$$

$$\textit{respectively,} \qquad Y_n^{\mathsf{R}}(z) := \begin{bmatrix} P_n^{\mathsf{R}}(z) & -P_{n-1}^{\mathsf{R}}(z) C_{n-1} \\ Q_n^{\mathsf{R}}(z) & -Q_{n-1}^{\mathsf{R}}(z) C_{n-1} \end{bmatrix},$$

is, for each $n \in \mathbb{N}$, the unique solution of the Riemann–Hilbert problem; which consists in the determination of a $2N \times 2N$ complex matrix function such that:

(RH1): $Y_n^{\mathsf{L}}(z)$ *(respectively, $Y_n^{\mathsf{R}}(z)$) is holomorphic in $\mathbb{C} \setminus \gamma$;*

(RH2): *has the following asymptotic behavior when $z \to \infty$,*

$$Y_n^{\mathsf{L}}(z) = \left(\mathrm{I} + \mathrm{O}(z^{-1})\right) \begin{bmatrix} \mathrm{I}\, z^n & \mathbf{0} \\ \mathbf{0} & \mathrm{I}\, z^{-n} \end{bmatrix};$$

$$\textit{respectively,} \qquad Y_n^{\mathsf{R}}(z) = \begin{bmatrix} \mathrm{I}\, z^n & \mathbf{0} \\ \mathbf{0} & \mathrm{I}\, z^{-n} \end{bmatrix} \left(\mathrm{I} + \mathrm{O}(z^{-1})\right);$$

(RH3): *satisfies the jump condition for all $z \in \gamma$,*

$$\left(Y_n^{\mathsf{L}}(z)\right)_+ = \left(Y_n^{\mathsf{L}}(z)\right)_- \begin{bmatrix} \mathrm{I} & W(z) \\ \mathbf{0} & \mathrm{I} \end{bmatrix},$$

$$\textit{respectively,}\qquad \big(Y_n^{\mathsf{R}}(z)\big)_+ = \begin{bmatrix} \mathrm{I} & \mathbf{0} \\ W(z) & \mathrm{I} \end{bmatrix} \big(Y_n^{\mathsf{R}}(z)\big)_-.$$

2.2. Sylvester differential equations for the fundamental matrices. Now we will study the analytic character for the structure matrices.

Theorem 3.4. *The structure matrices M_n^{L} and M_n^{R}, defined in* (3.3) *are, for each $n \in \mathbb{N}$, matrices of entire functions in the complex plane.*

Proof. We only give the proofs for the left case: $(M_n^{\mathsf{L}})_+ = \big((Z_n^{\mathsf{L}})'\big)_+\big((Z_n^{\mathsf{L}})^{-1}\big)_+$, and applying the *constant* jump condition we get

$$(M_n^{\mathsf{L}}(z))_+ = \big((Z_n^{\mathsf{L}})'\big)_- \begin{bmatrix} \mathrm{I} & \mathrm{I} \\ \mathbf{0} & \mathrm{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathrm{I} & \mathrm{I} \\ \mathbf{0} & \mathrm{I} \end{bmatrix} \big((Z_n^{\mathsf{L}})^{-1}\big)_- = (M_n^{\mathsf{L}}(z))_-,$$

and so the result follows. □

The differential structure determined by the Pearson equation for the weight matrix induces a corresponding Sylvester differential equations for the fundamental matrices as follows.

Theorem 3.5 (Sylvester differential linear equations)**.** *In the conditions of Proposition 2.1, the left fundamental matrix Y_n^{L} and the right fundamental matrix Y_n^{R} satisfy, for each $n \in \mathbb{N}$, the following Sylvester matrix differential equations,*

$$\big(Y_n^{\mathsf{L}}(z)\big)' = M_n^{\mathsf{L}}(z)Y_n^{\mathsf{L}}(z) - Y_n^{\mathsf{L}}(z)\begin{bmatrix} h^{\mathsf{L}}(z) & \mathbf{0} \\ \mathbf{0} & -h^{\mathsf{R}}(z) \end{bmatrix}, \tag{3.8}$$

$$\big(Y_n^{\mathsf{R}}(z)\big)' = Y_n^{\mathsf{R}}(z)M_n^{\mathsf{R}}(z) - \begin{bmatrix} h^{\mathsf{R}}(z) & \mathbf{0} \\ \mathbf{0} & -h^{\mathsf{L}}(z) \end{bmatrix} Y_n^{\mathsf{R}}(z), \tag{3.9}$$

respectively.

Proof. As $M_n^{\mathsf{L}}(z) = \big(Z_n^{\mathsf{L}}(z)\big)'\big(Z_n^{\mathsf{L}}(z)\big)^{-1}$ is the right derivative of the constant jump structure matrix from (2.33) we get (3.8). Now, (3.9) is proven analogously. □

2.3. Second order differential operators. We firstly derive, as a consequence of the Sylvester differential linear systems, second order differential equations fulfilled by the fundamental matrices. We reinterpret these results in terms of matrix biorthogonal polynomials as well as in terms of the corresponding second kind functions.

Following the standard use in Soliton Theory, given a matrix of holomorphic functions A we define its Miura transform by

$$\mathcal{M}(A) = A'(z) + (A(z))^2. \tag{3.10}$$

Observe that when A is a logarithmic derivative, $A = w'w^{-1}$ or $A = w^{-1}w'$, we have $\mathcal{M}(A) = w''w^{-1}$ or $\mathcal{M}(A) = w^{-1}w''$, respectively.

Theorem 3.6 (Second order linear differential equations)**.** *In the conditions of Theorem 2.1, the sequence of fundamental matrices,* $\{Y_n^{\mathsf{L}}\}_{n\in\mathbb{Z}_+}$ *and* $\{Y_n^{\mathsf{R}}\}_{n\in\mathbb{Z}_+}$*, satisfy*

$$\big(Y_n^{\mathsf{L}}(z)\big)''+2\big(Y_n^{\mathsf{L}}(z)\big)'\begin{bmatrix}h^{\mathsf{L}}(z) & \mathbf{0}\\ \mathbf{0} & -h^{\mathsf{R}}(z)\end{bmatrix}+Y_n^{\mathsf{L}}(z)\begin{bmatrix}\mathcal{M}\big(h^{\mathsf{L}}(z)\big) & \mathbf{0}\\ \mathbf{0} & \mathcal{M}\big(-h^{\mathsf{R}}(z)\big)\end{bmatrix}=\mathcal{M}\big(M_n^{\mathsf{L}}(z)\big)Y_n^{\mathsf{L}}(z), \tag{3.11}$$

$$\big(Y_n^{\mathsf{R}}(z)\big)''+2\begin{bmatrix}h^{\mathsf{R}}(z) & \mathbf{0}\\ \mathbf{0} & -h^{\mathsf{L}}(z)\end{bmatrix}\big(Y_n^{\mathsf{R}}(z)\big)'+\begin{bmatrix}\mathcal{M}\big(h^{\mathsf{R}}(z)\big) & \mathbf{0}\\ \mathbf{0} & \mathcal{M}\big(-h^{\mathsf{L}}(z)\big)\end{bmatrix}Y_n^{\mathsf{L}}(z)=Y_n^{\mathsf{R}}(z)\mathcal{M}\big(M_n^{\mathsf{R}}(z)\big), \tag{3.12}$$

where $\mathcal{M}$ *is defined by* (3.10)*.*

Proof. We prove (3.11). First, let us take a derivative of (3.8) to get

$$\big(Y_n^{\mathsf{L}}(z)\big)''+\big(Y_n^{\mathsf{L}}(z)\big)'\begin{bmatrix}h^{\mathsf{L}}(z) & \mathbf{0}\\ \mathbf{0} & -h^{\mathsf{R}}(z)\end{bmatrix}+Y_n^{\mathsf{L}}(z)\begin{bmatrix}\big(h^{\mathsf{L}}(z)\big)' & \mathbf{0}\\ \mathbf{0} & -\big(h^{\mathsf{R}}(z)\big)'\end{bmatrix}=\big(M_n^{\mathsf{L}}(z)\big)'Y_n^{\mathsf{L}}(z)+M_n^{\mathsf{L}}(z)\big(Y_n^{\mathsf{L}}(z)\big)'$$

but again by (3.8)

$$M_n^{\mathsf{L}}(z)\big(Y_n^{\mathsf{L}}(z)\big)'=\big(M_n^{\mathsf{L}}(z)\big)^2Y_n^{\mathsf{L}}(z)-M_n^{\mathsf{L}}(z)Y_n^{\mathsf{L}}(z)\begin{bmatrix}h^{\mathsf{L}}(z) & \mathbf{0}\\ \mathbf{0} & -h^{\mathsf{R}}(z)\end{bmatrix}$$

and if we substitute

$$M_n^{\mathsf{L}}(z)Y_n^{\mathsf{L}}(z)=\big(Y_n^{\mathsf{L}}(z)\big)'+Y_n^{\mathsf{L}}(z)\begin{bmatrix}h^{\mathsf{L}}(z) & \mathbf{0}\\ \mathbf{0} & -h^{\mathsf{R}}(z)\end{bmatrix}$$

we finally get

$$M_n^{\mathsf{L}}(z)\big(Y_n^{\mathsf{L}}(z)\big)'=\big(M_n^{\mathsf{L}}(z)\big)^2Y_n^{\mathsf{L}}(z)-\big(Y_n^{\mathsf{L}}(z)\big)'\begin{bmatrix}h^{\mathsf{L}}(z) & \mathbf{0}\\ \mathbf{0} & -h^{\mathsf{R}}(z)\end{bmatrix}-Y_n^{\mathsf{L}}(z)\begin{bmatrix}h^{\mathsf{L}}(z) & \mathbf{0}\\ \mathbf{0} & -h^{\mathsf{R}}(z)\end{bmatrix}^2,$$

and the result follows. □

Definition 3.1. *For the next corollary we need to introduce the following* $\mathbb{C}^{2N\times 2N}$ *valued functions in terms of the difference of two Miura maps*

$$\mathsf{H}_n^{\mathsf{L}}(z)=\begin{bmatrix}\mathsf{H}_{1,1,n}^{\mathsf{L}}(z) & \mathsf{H}_{1,2,n}^{\mathsf{L}}(z)\\ \mathsf{H}_{2,1,n}^{\mathsf{L}}(z) & \mathsf{H}_{2,2,n}^{\mathsf{L}}(z)\end{bmatrix}=\mathcal{M}(M_n^{\mathsf{L}}(z))-\mathcal{M}\left(\begin{bmatrix}h^{\mathsf{L}}(z) & \mathbf{0}\\ \mathbf{0} & -h^{\mathsf{R}}(z)\end{bmatrix}\right),$$

$$\mathsf{H}_n^{\mathsf{R}}(z)=\begin{bmatrix}\mathsf{H}_{1,1,n}^{\mathsf{R}}(z) & \mathsf{H}_{1,2,n}^{\mathsf{R}}(z)\\ \mathsf{H}_{2,1,n}^{\mathsf{R}}(z) & \mathsf{H}_{2,2,n}^{\mathsf{R}}(z)\end{bmatrix}=\mathcal{M}(M_n^{\mathsf{R}}(z))-\mathcal{M}\left(\begin{bmatrix}h^{\mathsf{R}}(z) & \mathbf{0}\\ \mathbf{0} & -h^{\mathsf{L}}(z)\end{bmatrix}\right).$$

Corollary 3.1. *The second order matrix differential equations* (3.11) *and* (3.12) *split in the following differential relations*

$$\begin{aligned}\big(P_n^{\mathsf{L}}\big)''(z)+2\big(P_n^{\mathsf{L}}\big)'(z)h^{\mathsf{L}}(z)+P_n^{\mathsf{L}}(z)\mathcal{M}(h^{\mathsf{L}}(z))&\\ =\big(\mathcal{M}(h^{\mathsf{L}}(z))+\mathsf{H}_{1,1,n}^{\mathsf{L}}(z)\big)P_n^{\mathsf{L}}(z)-\mathsf{H}_{1,2,n}^{\mathsf{L}}(z)C_{n-1}P_{n-1}^{\mathsf{L}}(z)&,\\ \big(Q_n^{\mathsf{L}}\big)''(z)-2\big(Q_n^{\mathsf{L}}\big)'(z)h^{\mathsf{R}}(z)+Q_n^{\mathsf{L}}(z)\mathcal{M}(-h^{\mathsf{R}}(z))&\\ =\big(\mathcal{M}(h^{\mathsf{L}}(z))+\mathsf{H}_{1,1,n}^{\mathsf{L}}(z)\big)Q_n^{\mathsf{L}}(z)-\mathsf{H}_{1,2,n}^{\mathsf{L}}(z)C_{n-1}Q_{n-1}^{\mathsf{L}}(z)&,\end{aligned}$$

$$
\begin{aligned}
&\big(P^{\mathsf R}_n\big)''(z)+2h^{\mathsf R}(z)\big(P^{\mathsf R}_n(z)\big)'(z)+\mathcal M(h^{\mathsf R}(z))P^{\mathsf R}_n(z)\\
&\qquad\qquad\qquad=P^{\mathsf R}_n(z)\big(\mathcal M(h^{\mathsf R}(z))+\mathsf H^{\mathsf R}_{1,1,n}(z)\big)-P^{\mathsf R}_{n-1}(z)C_{n-1}\mathsf H^{\mathsf R}_{2,1,n}(z),\\
&\big(Q^{\mathsf R}_n\big)''(z)-2h^{\mathsf L}(z)\big(Q^{\mathsf R}_n\big)'(z)+\mathcal M(-h^{\mathsf L}(z))Q^{\mathsf R}_n(z)\\
&\qquad\qquad\qquad=Q^{\mathsf R}_n(z)\big(\mathcal M(h^{\mathsf R}(z))+\mathsf H^{\mathsf R}_{1,1,n}(z)\big)-Q^{\mathsf R}_{n-1}(z)C_{n-1}\mathsf H^{\mathsf R}_{2,1,n}(z).
\end{aligned}
$$

Proof. Is a direct consequence of Theorem 3.6. □

3. Laguerre type weights

In this section the Riemann–Hilbert problem, with jump supported on an appropriate curve on the complex plane with a finite end point at the origin, is used for the study of the corresponding matrix biorthogonal polynomials associated with Laguerre type matrices of weights, which are constructed in terms of matrix Pearson equation (3.1) with $\phi(z)=z$, *i.e.*

$$z\left(W^{\mathsf L}\right)'(z)=h^{\mathsf L}(z)W^{\mathsf L}(z), \qquad z\left(W^{\mathsf R}\right)'(z)=W^{\mathsf R}(z)h^{\mathsf R}(z), \tag{3.13}$$

then the matrix Pearson type equation (3.2) transforms to

$$zW'(z)=h^{\mathsf L}(z)(z)W(z)+W(z)h^{\mathsf R}(z) \tag{3.14}$$

Definition 3.2 (Laguerre type weights)**.** *We say that a regular weight matrix,*

$$W=\begin{bmatrix}W^{(1,1)} & \dots & W^{(1,N)}\\ \vdots & \ddots & \vdots\\ W^{(N,1)} & \dots & W^{(N,N)}\end{bmatrix}\in\mathbb C^{N\times N} \qquad \textit{is of Laguerre type if}$$

1) *The support of W is a non self-intersecting smooth curve on the complex plane with beginning point at 0 and ending point at ∞, and such that it intersects the circles $|z|=R$, $R\in(0,+\infty)$, once and only once (*i.e.*, it can be taken as a determination curve for $\arg(z)$).*
2) *The entries $W^{(j,k)}$ of the matrix weight W can be written as*

$$W^{(j,k)}(z)=\sum_{m\in I_{j,k}}A_m(z)z^{\alpha_m}\log^{p_m}(z), \qquad z\in\gamma, \tag{3.15}$$

where $I_{j,k}$ denotes a finite set of indexes, $\operatorname{Re}(\alpha_m)>-1$, $p_m\in\mathbb N$ and A_m is Hölder continuous and bounded. Here the determination of logarithm and the powers are taken along γ. We will request, in the development of the theory, that the functions A_m have an holomorphic extension to the whole complex plane.

In this work, for the sake of simplicity, $\gamma=(0,+\infty)$ and the finite end point of the curve γ is taken at the origin, $c=0$, with no loss of generality, as a similar arguments apply for $c\neq 0$. In [46] different examples of Laguerre weights for the matrix orthogonal polynomials on the real line are studied.

3.1. The Riemann–Hilbert problem for the Laguerre type weights. We begin this subsection stating a general theorem on Riemann–Hilbert problem for the Laguerre general weights. A preliminary version of this can be found in [16].

Theorem 3.7. *Given a regular Laguerre type weight matrix W with support on γ we have: The matrix function*

$$Y_n^{\mathsf{L}}(z) := \begin{bmatrix} P_n^{\mathsf{L}}(z) & Q_n^{\mathsf{L}}(z) \\ -C_{n-1}P_{n-1}^{\mathsf{L}}(z) & -C_{n-1}Q_{n-1}^{\mathsf{L}}(z) \end{bmatrix},$$

$$\textit{respectively,} \qquad Y_n^{\mathsf{R}}(z) := \begin{bmatrix} P_n^{\mathsf{R}}(z) & -P_{n-1}^{\mathsf{R}}(z)C_{n-1} \\ Q_n^{\mathsf{R}}(z) & -Q_{n-1}^{\mathsf{R}}(z)C_{n-1} \end{bmatrix}$$

is, for each $n \in \mathbb{N}$, the unique solution of the Riemann–Hilbert problem, which consists in the determination of a $2N \times 2N$ complex matrix function such that:

(RH1) $Y_n^{\mathsf{L}}(z)$ *(respectively Y_n^{R}) is holomorphic in $\mathbb{C} \setminus \gamma$.*

(RH2) *Has the following asymptotic behavior when $z \to \infty$,*

$$Y_n^{\mathsf{L}}(z) \sim \Big(\mathrm{I} + \sum_{j=1}^{\infty}(z^{-j})Y_n^{j,\mathsf{L}}\Big) \begin{bmatrix} \mathrm{I}\, z^n & \mathbf{0} \\ \mathbf{0} & \mathrm{I}\, z^{-n} \end{bmatrix},$$

$$\textit{respectively,} \qquad Y_n^{\mathsf{R}}(z) \sim \begin{bmatrix} \mathrm{I}\, z^n & \mathbf{0} \\ \mathbf{0} & \mathrm{I}\, z^{-n} \end{bmatrix} \Big(\mathrm{I} + \sum_{j=1}^{\infty}(z^{-j})Y_n^{j,\mathsf{R}}\Big).$$

(RH3) *Satisfies the jump condition on $z \in \gamma \setminus \{0\}$,*

$$\big(Y_n^{\mathsf{L}}(z)\big)_+ = \big(Y_n^{\mathsf{L}}(z)\big)_- \begin{bmatrix} \mathrm{I} & W(z) \\ \mathbf{0} & \mathrm{I} \end{bmatrix}, \quad \textit{respectively,} \quad \big(Y_n^{\mathsf{R}}(z)\big)_+ = \begin{bmatrix} \mathrm{I} & \mathbf{0} \\ W(z) & \mathrm{I} \end{bmatrix} \big(Y_n^{\mathsf{R}}(z)\big)_-.$$

(RH4) $Y_n^{\mathsf{L}}(z) = \begin{bmatrix} \mathrm{O}(1) & s_1^{\mathsf{L}}(z) \\ \mathrm{O}(1) & s_2^{\mathsf{L}}(z) \end{bmatrix}$*, respectively,* $Y_n^{\mathsf{R}}(z) = \begin{bmatrix} \mathrm{O}(1) & \mathrm{O}(1) \\ s_1^{\mathsf{R}}(z) & s_2^{\mathsf{R}}(z) \end{bmatrix}$*, as $z \to 0$, $\lim_{z\to 0} z s_j^{\mathsf{L}}(z) = \mathbf{0}$, $\lim_{z\to 0} z s_j^{\mathsf{L}}(z) = \mathbf{0}$, $\lim_{z\to 0} z s_j^{\mathsf{R}}(z) = \mathbf{0}$, $j = 1, 2$, and the O conditions are understood entrywise.*

Proof. Using Theorem 1.8 it follows that the matrices Y_n^{L} and Y_n^{R} satisfy $(\mathrm{RH1})$–$(\mathrm{RH3})$. The entries $W^{j,k}$ of the matrix weight W are given in (3.15). It holds (*cf.* [57]) that in a neighborhood of $z = 0$ the Cauchy transform

$$\phi_m(z) = \frac{1}{2\pi\,\mathrm{i}} \int_{\gamma} \frac{p(\zeta)A_m(\zeta)\zeta^{\alpha_m}\log^{p_m}(\zeta)}{\zeta - z}\,\mathrm{d}\,\zeta,$$

where $p(\zeta)$ denotes any polynomial in ζ, that satisfies $\lim_{z\to 0} z\phi_m(z) = 0$. Then, $(\mathrm{RH4})$ is fulfilled by the matrices Y_n^{L} and Y_n^{R}, respectively. To prove the unicity of both Riemann–Hilbert problems let us consider the matrix function

$$G(z) = Y_n^{\mathsf{L}}(z) \begin{bmatrix} \mathbf{0} & \mathrm{I} \\ -\mathrm{I} & \mathbf{0} \end{bmatrix} Y_n^{\mathsf{R}}(z) \begin{bmatrix} \mathbf{0} & -\mathrm{I} \\ \mathrm{I} & \mathbf{0} \end{bmatrix}.$$

It can easily be proved that $G(z)$ has no jump or discontinuity on the curve γ and that its behavior at the end point 0 is given by

$$G(z) \sim \begin{bmatrix} \mathrm{O}(1)s_1^{\mathsf{L}}(z) + \mathrm{O}(1)s_2^{\mathsf{R}}(z) & \mathrm{O}(1)s_1^{\mathsf{L}}(z) + \mathrm{O}(1)s_1^{\mathsf{R}}(z) \\ \mathrm{O}(1)s_2^{\mathsf{L}}(z) + \mathrm{O}(1)s_2^{\mathsf{R}}(z) & \mathrm{O}(1)s_2^{\mathsf{L}}(z) + \mathrm{O}(1)s_1^{\mathsf{R}}(z) \end{bmatrix}, \qquad z \to 0,$$

so, it holds that, $\lim_{z\to 0} z\,G(z) = 0$ and we conclude that the end point 0 is a removable singularity of G. Now, from the behavior for $z \to \infty$,

$$G(z) \sim \begin{bmatrix} \mathrm{I}\,z^n & \mathbf{0} \\ \mathbf{0} & \mathrm{I}\,z^{-n} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathrm{I} \\ -\mathrm{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathrm{I}\,z^n & \mathbf{0} \\ \mathbf{0} & \mathrm{I}\,z^{-n} \end{bmatrix} \begin{bmatrix} \mathbf{0} & -\mathrm{I} \\ \mathrm{I} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathrm{I} & \mathbf{0} \\ \mathbf{0} & \mathrm{I} \end{bmatrix},$$

hence the Liouville theorem implies that $G(z) = \mathrm{I}$. To prove the unicity of the solution of $(\mathrm{RH1})$–$(\mathrm{RH3})$ let $\widetilde{Y}_n^{\mathsf{L}}$ be any solution of the left Riemann–Hilbert problem. Then

$$\widetilde{Y}_n^{\mathsf{L}}(z) = \left(\begin{bmatrix} \mathbf{0} & \mathrm{I} \\ -\mathrm{I} & \mathbf{0} \end{bmatrix} Y_n^{\mathsf{R}}(z) \begin{bmatrix} \mathbf{0} & -\mathrm{I} \\ \mathrm{I} & \mathbf{0} \end{bmatrix} \right)^{-1}.$$

Hence any solution of this left Riemann–Hilbert problem is equal to the inverse of a fixed matrix, and the uniqueness follows. We obtain the uniqueness of the solution of the right Riemann–Hilbert in a similar way. □

We can give the following result from the literature [91].

Theorem 3.8 (Solution at a regular singular point)**.** *Let the matrix function $h^{\mathsf{L}}(z)$ be entire. Then, for the solutions of the Pearson equation* (3.13) *we have:*

(1) *If $A^{\mathsf{L}} := h^{\mathsf{L}}(0)$ has no eigenvalues that differs from each other by positive integers then, the solution of the left matrix differential equation in* (3.13) *can be written as $W^{\mathsf{L}}(z) = H^{\mathsf{L}}(z)z^{A^{\mathsf{L}}}W_0^{\mathsf{L}}$, where H^{L} is an entire and nonsingular matrix function such that $H^{\mathsf{L}}(0) = \mathrm{I}$, and W_0^{L} is a constant nonsingular matrix.*
(2) *If the matrix function A^{L} has eigenvalues that differs from each other by positive integers, then the solution of the left matrix differential equation in* (3.13) *can be written as*

$$W^{\mathsf{L}}(z) = H^{\mathsf{L}}(z)z^{\widetilde{A}^{\mathsf{L}}}W_0^{\mathsf{L}}, \qquad \text{where, in this case,} \qquad H^{\mathsf{L}}(z) = \widetilde{S}^{\mathsf{L}}(z)\Pi^{\mathsf{L}}(z),$$

and $\widetilde{S}^{\mathsf{L}}$ is a finite product of factors of the form $T_iS_i^{\mathsf{L}}$, with T_i a nonsingular matrix and S_i^{L} is a shearing matrix, i.e. *a matrix given by blocks as*

$$S_i^{\mathsf{L}}(z) = \begin{bmatrix} \mathrm{I}_{n_i} & \mathbf{0} \\ \mathbf{0} & z\,\mathrm{I}_{m_i} \end{bmatrix},$$

for some positive integers n_i, m_i, and Π^{L} is an entire and nonsingular matrix function such that $\Pi^{\mathsf{L}}(0) = I$, $\widetilde{A}^{\mathsf{L}}$ is a constant matrix built from the matrix A^{L}, where the eigenvalues of this matrix are decreased in such a way that the eigenvalues of the resulting matrix do not differ by a positive integer and W_0^{L} is a constant nonsingular matrix.

We can get analogous results for the right matrix differential equation in (3.13) and we will denote the solution as

$$W^{\mathsf{R}}(z) = W_0^{\mathsf{R}} z^{A^{\mathsf{R}}} H^{\mathsf{R}}(z).$$

Notice that given a matrix A, and the oriented curve γ, the matrix of functions $z^A = \mathrm{e}^{A\log(z)}$ is a matrix of holomorphic functions in $\mathbb{C}\setminus\gamma$, and

$$\big(z^A\big)_- = \big(z^A\big)_+ \mathrm{e}^{2\pi\,\mathrm{i}\,A} = \mathrm{e}^{2\pi\,\mathrm{i}\,A}\big(z^A\big)_+, \qquad z\in\gamma.$$

We also adopt the convention that $\big(W^{\mathsf{L}}(z)W^{\mathsf{R}}(z)\big)_+ = W(z)$, *i.e.* the weight matrix is obtained from the limit behavior of the right side of the curve γ of the matrix function $W^{\mathsf{L}}W^{\mathsf{R}}$.

It is necessary, in other to consider the Riemann–Hilbert problem related to the weight matrix W satisfying (3.14), to study the behavior of W around the origin. For that aim, let us consider J, the Jordan matrix similar to the matrix A, *i.e.* there exists a nonsingular matrix P such that $A = PJP^{-1}$. It holds $z^A = Pz^JP^{-1}$ so if

$$J = (\lambda_1\,\mathrm{I}_{m_1} + N_1)\oplus(\lambda_2\,\mathrm{I}_{m_2} + N_2)\oplus\cdots\oplus(\lambda_s\,\mathrm{I}_{m_s} + N_s)$$

where m_k is the order of the nilpotent matrix N_k, we have that

$$z^J = z^{\lambda_1\,\mathrm{I}_{m_1}+N_1}\oplus z^{\lambda_2\,\mathrm{I}_{m_2}+N_2}\oplus\cdots\oplus z^{\lambda_s\,\mathrm{I}_{m_s}+N_s}$$

where $z^{\lambda_k\,\mathrm{I}_{m_k}+N_k} = z^{\lambda_k\,\mathrm{I}_{m_k}}z^{N_k}$. It is straightforward that $z^{\lambda_k\,\mathrm{I}_{m_k}} = z^{\lambda_k}\,\mathrm{I}_{m_k}$ and

$$z^{N_k} = \mathrm{e}^{N_k\log(z)} = \mathrm{I}_{m_k} + \log(z)N_k + \frac{\log^2(z)}{2!}N_k^2 + \cdots + \frac{\log^{m_k-1}(z)}{(m_k-1)!}N_k^{m_k-1},$$

where we have used the nilpotency of $N_k^j = \mathbf{0}$ for $j\geq m_k$. So we can conclude that the entries of z^A are linear combinations of z^{λ_j} with polynomials coefficients in the variable $\log(z)$.

3.2. Constant jump fundamental matrices and structure matrices.

The following theorem explicit the *constant jump condition* for the constant jump fundamental matrices Z_n^{L} and Z_n^{R} defined in (2.33) and (2.34), respectively.

Theorem 3.9. *The constant jump fundamental matrices Z_n^{L} and Z_n^{R} satisfies the following* constant jump condition *on γ*

$$\big(Z_n^{\mathsf{L}}(z)\big)_+ = \big(Z_n^{\mathsf{L}}(z)\big)_-\begin{bmatrix}(W_0^{\mathsf{L}})^{-1}\,\mathrm{e}^{-2\pi\,\mathrm{i}\,A^{\mathsf{L}}}\,W_0^{\mathsf{L}} & (W_0^{\mathsf{L}})^{-1}\,\mathrm{e}^{-2\pi\,\mathrm{i}\,A^{\mathsf{L}}}\,W_0^{\mathsf{L}}\\ \mathbf{0} & W_0^{\mathsf{R}}\,\mathrm{e}^{2\pi\,\mathrm{i}\,A^{\mathsf{R}}}(W_0^{\mathsf{R}})^{-1}\end{bmatrix},$$

$$\big(Z_n^{\mathsf{R}}(z)\big)_+ = \begin{bmatrix}W_0^{\mathsf{R}}\,\mathrm{e}^{-2\pi\,\mathrm{i}\,A^{\mathsf{R}}}(W_0^{\mathsf{R}})^{-1} & \mathbf{0}\\ W_0^{\mathsf{R}}\,\mathrm{e}^{-2\pi\,\mathrm{i}\,A^{\mathsf{R}}}(W_0^{\mathsf{R}})^{-1} & {W_0^{\mathsf{L}}}^{-1}\,\mathrm{e}^{2\pi\,\mathrm{i}\,A^{\mathsf{L}}}\,W_0^{\mathsf{L}}\end{bmatrix}\big(Z_n^{\mathsf{R}}(z)\big)_-,$$

for all $z\in\gamma$.

Proof. From the definition of Z_n^{L} we have

$$\big(Z_n^{\mathsf{L}}(z)\big)_+ = \big(Y_n^{\mathsf{L}}(z)\big)_+\begin{bmatrix}\big(W^{\mathsf{L}}(z)\big)_+ & \mathbf{0}\\ \mathbf{0} & \big(W^{\mathsf{R}}(z)\big)_+^{-1}\end{bmatrix},$$

and taking into account Theorem 1.8 we successively get

$$\begin{aligned}\big(Z_n^{\mathsf{L}}(z)\big)_+ &= \big(Y_n^{\mathsf{L}}(z)\big)_- \begin{bmatrix}\mathrm{I} & \big(W^{\mathsf{L}}(z)W^{\mathsf{R}}(z)\big)_+\\ \mathbf{0} & \mathrm{I}\end{bmatrix}\begin{bmatrix}\big(W^{\mathsf{L}}(z)\big)_+ & \mathbf{0}\\ \mathbf{0} & \big(W^{\mathsf{R}}(z)\big)_+^{-1}\end{bmatrix}\\
&= \big(Y_n^{\mathsf{L}}(z)\big)_- \begin{bmatrix}\big(W^{\mathsf{L}}(z)\big)_- & \mathbf{0}\\ \mathbf{0} & \big(W^{\mathsf{R}}(z)\big)_-^{-1}\end{bmatrix}\begin{bmatrix}\big(W^{\mathsf{L}}(z)\big)_-^{-1} & \mathbf{0}\\ \mathbf{0} & \big(W^{\mathsf{R}}(z)\big)_-\end{bmatrix}\begin{bmatrix}\big(W^{\mathsf{L}}(z)\big)_+ & \big(W^{\mathsf{L}}(z)\big)_+\\ \mathbf{0} & \big(W^{\mathsf{R}}(z)\big)_+^{-1}\end{bmatrix}\\
&= \big(Z_n^{\mathsf{L}}(z)\big)_- \begin{bmatrix}\big(W^{\mathsf{L}}(z)\big)_-^{-1}\big(W^{\mathsf{L}}(z)\big)_+ & \big(W^{\mathsf{L}}(z)\big)_-^{-1}\big(W^{\mathsf{L}}(z)\big)_+\\ \mathbf{0} & \big(W^{\mathsf{R}}(z)\big)_-\big(W^{\mathsf{R}}(z)\big)_+^{-1}\end{bmatrix}\\
&= \big(Z_n^{\mathsf{L}}(z)\big)_- \begin{bmatrix}(W_0^{\mathsf{L}})^{-1}\,\mathrm{e}^{-2\pi\,\mathrm{i}\,A^{\mathsf{L}}}\,W_0^{\mathsf{L}} & (W_0^{\mathsf{L}})^{-1}\,\mathrm{e}^{-2\pi\,\mathrm{i}\,A^{\mathsf{L}}}\,W_0^{\mathsf{L}}\\ \mathbf{0} & W_0^{\mathsf{R}}\,\mathrm{e}^{2\pi\,\mathrm{i}\,A^{\mathsf{L}}}(W_0^{\mathsf{R}})^{-1}\end{bmatrix}.\end{aligned}$$

Hence, we get the desired *constant* jump condition for Z_n^{L}. To complete the proof we only have to use that (2.35) holds. □

Now, we discuss the holomorphic properties of the structure matrices already introduced in (3.3).

Theorem 3.10. *The structure matrices M_n^{L} and M_n^{R} are, for each $n \in \mathbb{N}$, meromorphic on $\mathbb{C}$, with singularity located at $z = 0$, which happens to be a removable singularity or a simple pole.*

Proof. Let us prove the statement for M_n^{L}, for M_n^{R} one should proceed similarly. From (3.3) it follows that M_n^{L} is holomorphic in $\mathbb{C} \setminus \gamma$. Due to the fact that Z_n^{L} has a constant jump on the curve γ, the matrix function $\big(Z_n^{\mathsf{L}}\big)'$ has the same constant jump on the curve γ, so the matrix M_n^{L} has no jump on the curve γ, and it follows that at the origin M_n^{L} has an isolated singularity. From (3.3) and (2.33) it holds

$$\begin{aligned}M_n^{\mathsf{L}}(z) &= \big(Z_n^{\mathsf{L}}\big)'(z)\big(Z_n^{\mathsf{L}}(z)\big)^{-1}\\ &= \big(Y_n^{\mathsf{L}}\big)'(z)\big(Y_n^{\mathsf{L}}(z)\big)^{-1} + \frac{1}{z}Y_n^{\mathsf{L}}(z)\begin{bmatrix}h^{\mathsf{L}}(z) & \mathbf{0}\\ \mathbf{0} & -h^{\mathsf{R}}(z)\end{bmatrix}\big(Y_n^{\mathsf{L}}(z)\big)^{-1},\end{aligned} \tag{3.16}$$

where

$$Y_n^{\mathsf{L}}(z) = \begin{bmatrix}P_n^{\mathsf{L}}(z) & Q_n^{\mathsf{L}}(z)\\ -C_{n-1}P_{n-1}^{\mathsf{L}}(z) & -C_{n-1}Q_{n-1}^{\mathsf{L}}(z)\end{bmatrix}.$$

Each entry of the matrix Q_n^{L} is the Cauchy transform of certain function f, where $f(z) = \sum_{i\in I}\phi_i(z)z^{\alpha_i}\log^{p_i}(z)$, ϕ_i is an entire function, $\operatorname{Re}(\alpha_i) > -1$, $p_i \in \mathbb{N}\cup\{0\}$, and I is a finite set of indices. It is clear that $\lim_{z\to 0} zf(z) = 0$. Now, see [57, §8.3-8.6] and [83], its Cauchy transform $g(z) = \int_\gamma \frac{f(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}$ also satisfies the same property

$\lim_{z\to 0} zg(z) = 0$. We can also see that $\lim_{z\to 0} z^2 g'(z) = 0$. Indeed,

$$\begin{aligned} zg'(z) &= \int_\gamma \frac{zf(t)}{(t-z)^2}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} = \int_\gamma \frac{(z-t)f(t)}{(t-z)^2}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} + \int_\gamma \frac{tf(t)}{(t-z)^2}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}, \\ &= -\int_\gamma \frac{f(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} - \frac{tf(t)}{2\pi\,\mathrm{i}(t-z)}\Big]_{\partial\gamma} + \int_\gamma \frac{(tf(t))'}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} \\ &= -\frac{tf(t)}{2\pi\,\mathrm{i}(t-z)}\Big]_{\partial\gamma} + \int_\gamma \frac{tf'(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}. \end{aligned}$$

From the boundary conditions, the first term is zero and we get $zg'(z) = \int_\gamma \frac{tf'(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}$. and from the definition of f we get that $tf'(t)$ is a function in the class of f, that we denote by v and, consequently, $\lim_{z\to 0} z^2 g'(z) = 0$. From these considerations it follows,

$$\big(Y_n^{\mathsf{L}}\big)'(z) = \begin{bmatrix} \mathrm{O}(1) & r_1^{\mathsf{L}}(z) \\ \mathrm{O}(1) & r_2^{\mathsf{L}}(z) \end{bmatrix}, \qquad \big(Y_n^{\mathsf{L}}(z)\big)^{-1} = \begin{bmatrix} r_3^{\mathsf{L}}(z) & r_4^{\mathsf{L}}(z) \\ \mathrm{O}(1) & \mathrm{O}(1) \end{bmatrix}, \qquad z \to 0,$$

where $\lim_{z\to 0} z^2 r_i^{\mathsf{L}}(z) = \mathbf{0}$, for $i = 1, 2$, and $\lim_{z\to 0} z r_i^{\mathsf{R}}(z) = \mathbf{0}$, for $i = 3, 4$, so it holds that

$$\lim_{z\to 0} z^2 \big(Y_n^{\mathsf{L}}\big)'(z)\big(Y_n^{\mathsf{L}}\big)^{-1} = \lim_{z\to 0} z^2 \begin{bmatrix} \mathrm{O}(1)r_1^{\mathsf{L}}(z) + \mathrm{O}(1)r_3^{\mathsf{L}}(z) & \mathrm{O}(1)r_1^{\mathsf{L}}(z) + \mathrm{O}(1)r_4^{\mathsf{L}}(z) \\ \mathrm{O}(1)r_2^{\mathsf{L}}(z) + \mathrm{O}(1)r_3^{\mathsf{L}}(z) & \mathrm{O}(1)r_2^{\mathsf{L}}(z) + \mathrm{O}(1)r_4^{\mathsf{L}}(z) \end{bmatrix} = \mathbf{0}.$$

Similar considerations leads us to the result that

$$\lim_{z\to 0} z Y_n^{\mathsf{L}}(z) \begin{bmatrix} h^{\mathsf{L}}(z) & \mathbf{0} \\ \mathbf{0} & -h^{\mathsf{R}}(z) \end{bmatrix} \big(Y_n^{\mathsf{L}}(z)\big)^{-1} = \mathbf{0},$$

so we obtain that $\lim_{z\to 0} z^2 M_n^{\mathsf{L}}(z) = \mathbf{0}$, and hence the matrix function M_n^{L} has at most a simple pole at the point $z = 0$. □

3.3. Differential relations from the Riemann–Hilbert problem. We are interested in the differential equations fulfilled by the biorthogonal matrix polynomials determined by Laguerre type matrices of weights. Here we use the Riemann–Hilbert problem approach in order to derive these differential relations. We use the notation for the structure matrices

$$\widetilde{M}_n^{\mathsf{L}}(z) = zM_n^{\mathsf{L}}(z), \qquad \widetilde{M}_n^{\mathsf{R}}(z) = zM_n^{\mathsf{R}}(z),$$

with $\widetilde{M}_n^{\mathsf{L}}$ and $\widetilde{M}_n^{\mathsf{R}}$ matrices of entire functions.

Theorem 3.11 (First order differential equation for the fundamental matrices Y_n^{L} and Y_n^{R})**.** *It holds that*

$$z\big(Y_n^{\mathsf{L}}\big)'(z) + Y_n^{\mathsf{L}}(z) \begin{bmatrix} h^{\mathsf{L}}(z) & \mathbf{0} \\ \mathbf{0} & -h^{\mathsf{R}}(z) \end{bmatrix} = \widetilde{M}_n^{\mathsf{L}}(z) Y_n^{\mathsf{L}}(z), \tag{3.17}$$

$$z\big(Y_n^{\mathsf{R}}\big)'(z)+\begin{bmatrix}h^{\mathsf{R}}(z) & \mathbf{0}\\ \mathbf{0} & -h^{\mathsf{L}}(z)\end{bmatrix}Y_n^{\mathsf{R}}(z)=Y_n^{\mathsf{R}}(z)\widetilde{M}_n^{\mathsf{R}}(z). \tag{3.18}$$

Proof. Equations (3.17) and (3.18) follows immediately from the definition of the matrices M_n^{L} and M_n^{R} in (3.3). □

We introduce the $\mathcal{N}$ transform, $\mathcal{N}(F(z))=F'(z)+\dfrac{F^2(z)}{z}$.

Theorem 3.12 (Second order differential equation for the fundamental matrices)**.** *It holds*

$$z\big(Y_n^{\mathsf{L}}\big)''+\big(Y_n^{\mathsf{L}}\big)'\begin{bmatrix}2h^{\mathsf{L}}+\mathrm{I} & \mathbf{0}\\ \mathbf{0} & -2h^{\mathsf{R}}+\mathrm{I}\end{bmatrix}+Y_n^{\mathsf{L}}(z)\begin{bmatrix}\mathcal{N}(h^{\mathsf{L}}) & \mathbf{0}\\ \mathbf{0} & \mathcal{N}(-h^{\mathsf{R}})\end{bmatrix}=\mathcal{N}(\widetilde{M}_n^{\mathsf{L}})Y_n^{\mathsf{L}}, \tag{3.19}$$

$$z\big(Y_n^{\mathsf{R}}\big)''+\begin{bmatrix}2h^{\mathsf{R}}+\mathrm{I} & \mathbf{0}\\ \mathbf{0} & -2h^{\mathsf{L}}+\mathrm{I}\end{bmatrix}\big(Y_n^{\mathsf{R}}\big)'+\begin{bmatrix}\mathcal{N}(h^{\mathsf{R}}) & \mathbf{0}\\ \mathbf{0} & \mathcal{N}(-h^{\mathsf{L}})\end{bmatrix}{Y_n}^{\mathsf{R}}(z)=Y_n^{\mathsf{R}}\mathcal{N}(\widetilde{M}_n^{\mathsf{R}}). \tag{3.20}$$

Proof. Differentiating in (3.3) we get successively

$$\big(Z_n^{\mathsf{L}}\big)''\big(Z_n^{\mathsf{L}}\big)^{-1}=\frac{\big(\widetilde{M}_n^{\mathsf{L}}\big)'}{z}-\frac{\widetilde{M}_n^{\mathsf{L}}}{z^2}+\frac{(\widetilde{M}_n^{\mathsf{L}})^2}{z^2},$$

$$z\big(Z_n^{\mathsf{L}}\big)''\big(Z_n^{\mathsf{L}}\big)^{-1}+\big(Z_n^{\mathsf{L}}\big)'\big(Z_n^{\mathsf{L}}\big)^{-1}=\big(\widetilde{M}_n^{\mathsf{L}}\big)'+\frac{(\widetilde{M}_n^{\mathsf{L}})^2}{z}.$$

Now, using (2.33) and (3.13), we get the stated result (3.19). The equation (3.20) follows in a similar way from definition of M_n^{R} in (3.3). □

We introduce the following $\mathbb{C}^{2N\times 2N}$ valued functions

$$\mathsf{H}_n^{\mathsf{L}}=\begin{bmatrix}\mathsf{H}_{1,1,n}^{\mathsf{L}} & \mathsf{H}_{1,2,n}^{\mathsf{L}}\\ \mathsf{H}_{2,1,n}^{\mathsf{L}} & \mathsf{H}_{2,2,n}^{\mathsf{L}}\end{bmatrix}:=\mathcal{N}(\widetilde{M}_n^{\mathsf{L}}),\qquad \mathsf{H}_n^{\mathsf{R}}=\begin{bmatrix}\mathsf{H}_{1,1,n}^{\mathsf{R}} & \mathsf{H}_{1,2,n}^{\mathsf{R}}\\ \mathsf{H}_{2,1,n}^{\mathsf{R}} & \mathsf{H}_{2,2,n}^{\mathsf{R}}\end{bmatrix}:=\mathcal{N}(\widetilde{M}_n^{\mathsf{R}}).$$

It holds that the second order matrix differential equations (3.11) and (3.12) split in the following differential relations

$$\begin{aligned}
z\big(P_n^{\mathsf{L}}\big)''+\big(P_n^{\mathsf{L}}\big)'\big(2h^{\mathsf{L}}+\mathrm{I}\big)+P_n^{\mathsf{L}}\mathcal{N}(h^{\mathsf{L}})&=\mathsf{H}_{1,1,n}^{\mathsf{L}}P_n^{\mathsf{L}}-\mathsf{H}_{1,2,n}^{\mathsf{L}}C_{n-1}P_{n-1}^{\mathsf{L}},\\
z\big(Q_n^{\mathsf{L}}\big)''+\big(Q_n^{\mathsf{L}}\big)'\big(-2h^{\mathsf{R}}+\mathrm{I}\big)+Q_n^{\mathsf{L}}\mathcal{N}(-h^{\mathsf{R}})&=\mathsf{H}_{1,1,n}^{\mathsf{L}}Q_n^{\mathsf{L}}-\mathsf{H}_{1,2,n}^{\mathsf{L}}C_{n-1}Q_{n-1}^{\mathsf{L}},\\
z\big(P_n^{\mathsf{R}}\big)''+\big(2h^{\mathsf{R}}+\mathrm{I}\big)\big(P_n^{\mathsf{R}}\big)'+\mathcal{N}(h^{\mathsf{R}})P_n^{\mathsf{R}}&=P_n^{\mathsf{R}}\mathsf{H}_{1,1,n}^{\mathsf{R}}-P_{n-1}^{\mathsf{R}}C_{n-1}\mathsf{H}_{2,1,n}^{\mathsf{R}},\\
z\big(Q_n^{\mathsf{R}}\big)''+\big(-2h^{\mathsf{L}}+\mathrm{I}\big)\big(Q_n^{\mathsf{R}}\big)'+\mathcal{N}(-h^{\mathsf{L}})Q_n^{\mathsf{R}}&=Q_n^{\mathsf{R}}\mathsf{H}_{1,1,n}^{\mathsf{R}}-Q_{n-1}^{\mathsf{R}}C_{n-1}\mathsf{H}_{2,1,n}^{\mathsf{R}}.
\end{aligned}$$

4. Jacobi type weights

In this part we deal with *regular weight matrix* W where its support, is a non self-intersecting smooth curve, γ, on the complex plane with two end points at $a,b\in\mathbb{C}$, and such that it intersects the circles $|z|=R$, $R\in\mathbb{R}^+$, once and only once (*i.e.*, it can be taken as a determination curve for $\arg(z)$).

In the current paragraph, we focus on Jacobi type examples.

Definition 3.3. *We say that a* $N\times N$ weight matrix W *with support* γ *is of* Jacobi type *if the entries* $W^{(j,k)}$ *of the matrix weight* W *can be written as*

$$W^{(j,k)}(z)=\sum_{m\in I_{j,k}}\varphi_m(z)(a+z)^{\alpha_m}(b-z)^{\beta_m}\log^{p_m}(a+z)\log^{q_m}(b-z),\qquad z\in\gamma,\tag{3.21}$$

where $I_{j,k}$ *denotes a finite set of indexes,* $\operatorname{Re}(\alpha_m)$, $\operatorname{Re}(\beta_m)>-1$, p_m, $q_m\in\mathbb{N}$, $a\neq b$ *real numbers and* φ_m *is Hölder continuous, bounded and non-vanishing on* γ.

We assume that the determination of logarithm and the powers are taken along γ. We will request, in the development of the theory, that the functions φ_m have a holomorphic extension to the whole complex plane.

This definition includes the non scalar examples of Jacobi type weights given in the literature [2, 23, 25, 26, 68, 86], and as far as we know it was not been studied elsewhere in all its generality.

In this work, for the sake of simplicity, the finite end points of the curve γ is taken at the origin, $a=0$ and $b=1$ with no loss of generality, as a similar arguments apply for $a\neq 0$ or $b\neq 1$. In [46] different examples of Jacobi matrix weights for the matrix orthogonal polynomials on $(0,1)$ are studied.

4.1. Riemann–Hilbert problem for the Jacobi type weights. Now, we state a theorem on Riemann–Hilbert problem for the Jacobi type weights.

Theorem 3.13. *Given a regular Jacobi type weight matrix,* W, *with support on* γ *we have the matrix function* Y_n^{L} *and* Y_n^{R}, *defined by* (1.22) *and* (1.23) *is, for each* $n\in\mathbb{N}$, *the unique solution of the following Riemann–Hilbert problem, which consists, respectively, in the determination of a* $2N\times 2N$ *complex matrix function such that:*

(RH1): Y_n^{L} *and* Y_n^{R} *is holomorphic in* $\mathbb{C}\setminus\gamma$.

(RH2): *Satisfies the jump condition*

$$\big(Y_n^{\mathsf{L}}(z)\big)_+=\big(Y_n^{\mathsf{L}}(z)\big)_-\begin{bmatrix}\mathrm{I} & W(z)\\ \mathbf{0} & \mathrm{I}\end{bmatrix},\qquad \big(Y_n^{\mathsf{R}}(z)\big)_+=\begin{bmatrix}\mathrm{I} & \mathbf{0}\\ W(z) & \mathrm{I}\end{bmatrix}\big(Y_n^{\mathsf{R}}(z)\big)_-,\qquad z\in\gamma.$$

(RH3): *Has the following asymptotic behavior, as* $|z|\to\infty$

$$Y_n^{\mathsf{L}}(z)=(\mathrm{I}+\mathrm{O}(1/z))\begin{bmatrix}z^n\,\mathrm{I} & \mathbf{0}\\ \mathbf{0} & z^{-n}\,\mathrm{I}\end{bmatrix},\qquad Y_n^{\mathsf{R}}(z)=\begin{bmatrix}\mathrm{I}\,z^n & \mathbf{0}\\ \mathbf{0} & \mathrm{I}\,z^{-n}\end{bmatrix}\Big(\mathrm{I}+\mathrm{O}(1/z)\Big).$$

(RH4): $Y_n^{\mathsf{L}}(z)=\begin{bmatrix}\mathrm{O}(1) & s_1^{\mathsf{L}}(z)\\ \mathrm{O}(1) & s_2^{\mathsf{L}}(z)\end{bmatrix}$, $Y_n^{\mathsf{R}}(z)=\begin{bmatrix}\mathrm{O}(1) & \mathrm{O}(1)\\ s_1^{\mathsf{R}}(z) & s_2^{\mathsf{R}}(z)\end{bmatrix}$, *as* $z\to 0$, *with* $\lim_{z\to 0}zs_j^{\mathsf{L}}(z)=\mathbf{0}$ *and* $\lim_{z\to 0}zs_j^{\mathsf{R}}(z)=\mathbf{0}$, $j=1,2$.

(RH5): $Y_n^{\mathsf{L}}(z)=\begin{bmatrix}\mathrm{O}(1) & r_1^{\mathsf{L}}(z)\\ \mathrm{O}(1) & r_2^{\mathsf{L}}(z)\end{bmatrix}$, $Y_n^{\mathsf{L}}(z)=\begin{bmatrix}\mathrm{O}(1) & \mathrm{O}(1)\\ r_1^{\mathsf{R}}(z) & r_2^{\mathsf{R}}(z)\end{bmatrix}$, *as* $z\to 1$, *with* $\lim_{z\to 1}(1-z)r_j^{\mathsf{L}}(z)=\mathbf{0}$ *and* $\lim_{z\to 1}(1-z)r_j^{\mathsf{R}}(z)=\mathbf{0}$, $j=1,2$. *The* s_i^{L}, s_i^{R} *(respectively,* r_i^{L} *and* r_i^{R}*) could be replaced by* $\mathrm{o}(1/z)$, *as* $z\to 0$ *(respectively,* $\mathrm{o}(1/(1-z))$, *as* $z\to 1$*). The* O *and* o *conditions are understood entrywise.*

Proof. Very similar to the proof of Theorem 3.7. ☐

4.2. Pearson equation and constant jump fundamental matrices. Here we consider a weight matrix W satisfying a matrix Pearson type equation

$$z(1-z)W'(z) = h^{\mathsf{L}}(z)W(z) + W(z)h^{\mathsf{R}}(z), \tag{3.22}$$

with entire matrix functions h^{L}, h^{R}. If we take a matrix function W^{L} such that

$$z(1-z)(W^{\mathsf{L}})'(z) = h^{\mathsf{L}}(z)W^{\mathsf{L}}(z), \tag{3.23}$$

then there exists a matrix function W^{R} such that $W(z) = W^{\mathsf{L}}(z)W^{\mathsf{R}}(z)$ with

$$z(1-z)(W^{\mathsf{R}})'(z) = W^{\mathsf{R}}(z)h^{\mathsf{R}}(z). \tag{3.24}$$

The reciprocal is also true as shown in Theorem 2.1.

The solution of (3.23) and (3.24) will have possibly branch points at 0 and 1, *cf.* [91]. This means that there exists constant matrices, $\mathsf{C}_j^{\mathsf{L}}$, $\mathsf{C}_j^{\mathsf{R}}$, with $j=0,1$, such that

$$(W^{\mathsf{L}}(z))_- = (W^{\mathsf{L}}(z))_+\mathsf{C}_0^{\mathsf{L}}, \qquad (W^{\mathsf{R}}(z))_- = \mathsf{C}_0^{\mathsf{R}}(W^{\mathsf{R}}(z))_+, \qquad \text{in } (0,1), \tag{3.25}$$

$$(W^{\mathsf{L}}(z))_- = (W^{\mathsf{L}}(z))_+\mathsf{C}_1^{\mathsf{L}}, \qquad (W^{\mathsf{R}}(z))_- = \mathsf{C}_1^{\mathsf{R}}(W^{\mathsf{R}}(z))_+, \qquad \text{in } (1,+\infty). \tag{3.26}$$

The constant jump fundamental matrices Z_n^{L} and Z_n^{R} satisfy, for each $n \in \mathbb{N}$, the following properties:

- Are holomorphic on $\mathbb{C} \setminus [0,+\infty)$.
- Present the following *constant jump condition* on $(0,1)$

$$\big(Z_n^{\mathsf{L}}(z)\big)_+ = \big(Z_n^{\mathsf{L}}(z)\big)_- \begin{bmatrix} \mathsf{C}_0^{\mathsf{L}} & \mathsf{C}_0^{\mathsf{L}} \\ \mathbf{0} & \mathrm{I} \end{bmatrix}, \qquad \big(Z_n^{\mathsf{R}}(z)\big)_+ = \begin{bmatrix} \mathrm{I} & \mathbf{0} \\ \mathsf{C}_0^{\mathsf{R}} & \mathsf{C}_0^{\mathsf{R}} \end{bmatrix} \big(Z_n^{\mathsf{R}}(z)\big)_-.$$

- Present the following *constant jump condition* on $(1,+\infty)$

$$\big(Z_n^{\mathsf{L}}(z)\big)_+ = \big(Z_n^{\mathsf{L}}(z)\big)_- \begin{bmatrix} \mathsf{C}_1^{\mathsf{L}} & \mathbf{0} \\ \mathbf{0} & \mathsf{C}_1^{\mathsf{R}} \end{bmatrix}, \qquad \big(Z_n^{\mathsf{R}}(z)\big)_+ = \begin{bmatrix} \mathsf{C}_1^{\mathsf{R}} & \mathbf{0} \\ \mathbf{0} & \mathsf{C}_1^{\mathsf{L}} \end{bmatrix} \big(Z_n^{\mathsf{R}}(z)\big)_-.$$

Now, we will explicit the constant jump matrix in the special case when we have the following decompositions for the weight matrix, $W(z) = W^{\mathsf{L}}(z)W^{\mathsf{R}}(z)$, with:

$$z\left(W^{\mathsf{L}}\right)'(z) = \widetilde{h}^{\mathsf{L}}(z)W^{\mathsf{L}}(z), \qquad (1-z)\left(W^{\mathsf{R}}\right)'(z) = W^{\mathsf{R}}(z)\widetilde{h}^{\mathsf{R}}(z), \tag{3.27}$$

where h^{L} and h^{R} are entire functions. Therefore, the matrix $W(z) = W^{\mathsf{L}}(z)W^{\mathsf{R}}(z)$ is such that,

$$z(1-z)W'(z) = h^{\mathsf{L}}(z)W(z) + W(z)h^{\mathsf{R}}(z),$$

where $h^{\mathsf{L}}(z) = (1-z)\widetilde{h}^{\mathsf{L}}(z)$ and $h^{\mathsf{R}}(z) = z\widetilde{h}^{\mathsf{R}}(z)$.

General solutions W^{L} and W^{R} of (3.27) are given explicitly (*cf.* [91]) by

$$W^{\mathsf{L}}(z) = H^{\mathsf{L}}(z)z^{\alpha}W_0^{\mathsf{L}}, \qquad W^{\mathsf{R}}(z) = W_0^{\mathsf{R}}(1-z)^{\beta}H^{\mathsf{R}}(z), \tag{3.28}$$

where H^{L} and H^{R} are entire and nonsingular matrix functions, and α, β are constant matrices, as well as W_0^{L} and W_0^{R} are constant nonsingular matrices.

It is easy to see that W, within this decomposition, is a Jacobi type weight matrix defined by (3.21). From (3.28), the constant jump fundamental matrices Z_n^{L} and Z_n^{R} have the following *constant jump condition* on $(0,1)$

$$\big(Z_n^{\mathsf{L}}(z)\big)_+ = \big(Z_n^{\mathsf{L}}(z)\big)_- \begin{bmatrix} (W_0^{\mathsf{L}})^{-1}\,\mathrm{e}^{-2\pi\,\mathrm{i}\,\alpha}\,W_0^{\mathsf{L}} & (W_0^{\mathsf{L}})^{-1}\,\mathrm{e}^{-2\pi\,\mathrm{i}\,\alpha}\,W_0^{\mathsf{L}} \\ \mathbf{0} & \mathrm{I} \end{bmatrix},$$

$$\big(Z_n^{\mathsf{R}}(z)\big)_+ = \begin{bmatrix} \mathrm{I} & \mathbf{0} \\ \mathrm{I} & (W_0^{\mathsf{L}})^{-1}\,\mathrm{e}^{2\pi\,\mathrm{i}\,\alpha}\,W_0^{\mathsf{L}} \end{bmatrix} \big(Z_n^{\mathsf{R}}(z)\big)_-,$$

as well as, the *constant jump condition* on $(1,+\infty)$

$$\big(Z_n^{\mathsf{L}}(z)\big)_+ = \big(Z_n^{\mathsf{L}}(z)\big)_- \begin{bmatrix} (W_0^{\mathsf{L}})^{-1}\,\mathrm{e}^{-2\pi\,\mathrm{i}\,\alpha}\,W_0^{\mathsf{L}} & \mathbf{0} \\ \mathbf{0} & W_0^{\mathsf{R}}\,\mathrm{e}^{2\pi\,\mathrm{i}\,\beta}(W_0^{\mathsf{R}})^{-1} \end{bmatrix},$$

$$\big(Z_n^{\mathsf{R}}(z)\big)_+ = \begin{bmatrix} W_0^{\mathsf{R}}\,\mathrm{e}^{-2\pi\,\mathrm{i}\,\beta}(W_0^{\mathsf{R}})^{-1} & \mathbf{0} \\ \mathbf{0} & \big(W_0^{\mathsf{L}}\big)^{-1}\,\mathrm{e}^{2\pi\,\mathrm{i}\,\alpha}\,W_0^{\mathsf{L}} \end{bmatrix} \big(Z_n^{\mathsf{R}}(z)\big)_-.$$

In fact, from the definition of Z_n^{L} we have

$$\big(Z_n^{\mathsf{L}}(z)\big)_+ = \big(Y_n^{\mathsf{L}}(z)\big)_+ \begin{bmatrix} (W^{\mathsf{L}}(z))_+ & \mathbf{0} \\ \mathbf{0} & (W^{\mathsf{R}}(z))_+^{-1} \end{bmatrix},$$

and taking into account Theorem 3.13 we successively get

$$\begin{aligned}
\big(Z_n^{\mathsf{L}}(z)\big)_+ &= \big(Y_n^{\mathsf{L}}(z)\big)_- \begin{bmatrix} \mathrm{I} & (W^{\mathsf{L}}(z)W^{\mathsf{R}}(z))_+ \\ \mathbf{0} & \mathrm{I} \end{bmatrix} \begin{bmatrix} (W^{\mathsf{L}}(z))_+ & \mathbf{0} \\ \mathbf{0} & (W^{\mathsf{R}}(z))_+^{-1} \end{bmatrix} \\
&= \big(Y_n^{\mathsf{L}}(z)\big)_- \begin{bmatrix} (W^{\mathsf{L}}(z))_- & \mathbf{0} \\ \mathbf{0} & (W^{\mathsf{R}}(z))_-^{-1} \end{bmatrix} \begin{bmatrix} (W^{\mathsf{L}}(z))_-^{-1} & \mathbf{0} \\ \mathbf{0} & (W^{\mathsf{R}}(z))_- \end{bmatrix} \begin{bmatrix} (W^{\mathsf{L}}(z))_+ & (W^{\mathsf{L}}(z))_+ \\ \mathbf{0} & (W^{\mathsf{R}}(z))_+^{-1} \end{bmatrix} \\
&= \big(Z_n^{\mathsf{L}}(z)\big)_- \begin{bmatrix} (W^{\mathsf{L}}(z))_-^{-1}(W^{\mathsf{L}}(z))_+ & (W^{\mathsf{L}}(z))_-^{-1}(W^{\mathsf{L}}(z))_+ \\ \mathbf{0} & W^{\mathsf{R}}(z)_-(W^{\mathsf{R}}(z))_+^{-1} \end{bmatrix}.
\end{aligned}$$

Similarly over $(1,+\infty)$ we have

$$\begin{aligned}
\big(Z_n^{\mathsf{L}}(z)\big)_+ &= \big(Y_n^{\mathsf{L}}(z)\big)_- \begin{bmatrix} (W^{\mathsf{L}}(z))_+ & \mathbf{0} \\ \mathbf{0} & (W^{\mathsf{R}}(z))_+^{-1} \end{bmatrix} \\
&= \big(Y_n^{\mathsf{L}}(z)\big)_- \begin{bmatrix} (W^{\mathsf{L}}(z))_- & \mathbf{0} \\ \mathbf{0} & (W^{\mathsf{R}}(z))_-^{-1} \end{bmatrix} \begin{bmatrix} (W^{\mathsf{L}}(z))_-^{-1} & \mathbf{0} \\ \mathbf{0} & (W^{\mathsf{R}}(z))_- \end{bmatrix} \begin{bmatrix} (W^{\mathsf{L}}(z))_+ & \mathbf{0} \\ \mathbf{0} & (W^{\mathsf{R}}(z))_+^{-1} \end{bmatrix} \\
&= \big(Z_n^{\mathsf{L}}(z)\big)_- \begin{bmatrix} (W^{\mathsf{L}}(z))_-^{-1}(W^{\mathsf{L}}(z))_+ & \mathbf{0} \\ \mathbf{0} & W^{\mathsf{R}}(z)_-(W^{\mathsf{R}}(z))_+^{-1} \end{bmatrix}.
\end{aligned}$$

To complete the proof we only have to see that

$$\big(W^{\mathsf{L}}\big)_- = H^{\mathsf{L}}\,\mathrm{e}^{2\pi\,\mathrm{i}\,\alpha}\,z^{\alpha}W_0^{\mathsf{L}}, \qquad \big(W^{\mathsf{R}}\big)_- = W_0^{\mathsf{R}}\,\mathrm{e}^{2\pi\,\mathrm{i}\,\alpha}(1-z)^{\beta}H^{\mathsf{R}},$$

and then use (2.34).

Now, we discuss the holomorphic properties of the structure matrices (3.3).

Theorem 3.14. *Let W be a regular Jacobi matrix weight that satisfies a Pearson type equation* (3.22) *that admits a factorization $W(z) = W^{\mathsf{L}}(z)W^{\mathsf{R}}(z)$, where W^{L}*

and W^{R} satisfies (3.23) *and* (3.24)*. Then, the structure matrices M_n^{L} and M_n^{R} are, for each $n \in \mathbb{N}$, meromorphic on $\mathbb{C}$, with singularities located at $z=0$ and $z=1$, which happens to be a removable singularity or a simple pole.*

Proof. Let us prove the statement for M_n^{L}. The matrix function M_n^{L} is holomorphic in $\mathbb{C}\setminus[0,+\infty)$ by definition, *cf.* (3.3). Due to the fact that Z_n^{L} has a constant jump on $(0,1)\cup(1,+\infty)$, *cf.* (3.25) and (3.26), the matrix function $\big(Z_n^{\mathsf{L}}\big)'$ has the same constant jump on $(0,1)\cup(1,+\infty)$, so that the matrix M_n^{L} has no jump on $(0,1)\cup(1,+\infty)$, and it follows that at $z=0$ and $z=1$, M_n^{L} has an isolated singularity.

From (2.33) and (3.3) it holds

$$
\begin{aligned}
M_n^{\mathsf{L}}(z) &= \big(Z_n^{\mathsf{L}}\big)'(z)\big(Z_n^{\mathsf{L}}(z)\big)^{-1} \\
&= \big(Y_n^{\mathsf{L}}\big)'(z)\big(Y_n^{\mathsf{L}}(z)\big)^{-1} + \frac{1}{z(z-1)} Y_n^{\mathsf{L}}(z)\begin{bmatrix} h^{\mathsf{L}}(z) & \mathbf{0} \\ \mathbf{0} & -h^{\mathsf{R}}(z)\end{bmatrix}\big(Y_n^{\mathsf{L}}(z)\big)^{-1},
\end{aligned}
\tag{3.29}
$$

where Y_n^{L} is given in (1.22). Each entry of the matrix Q_n^{L} is the Cauchy transform of certain function, f, of type

$$
f(z) = \sum_{j\in I}\varphi_j(z) z^{\alpha_j}(1-z)^{\beta_j}\log^{p_j}(z)\log^{q_j}(1-z),
$$

where φ_j is, for each $j\in I$, an entire function with $\operatorname{Re}(\alpha_j)$, $\operatorname{Re}(\beta_j) > -1$, p_j, $q_j\in\mathbb{N}$, and I is a finite set of indices. It's clear that

$$
\lim_{z\to 0} z f(z) = 0 \qquad \text{and} \qquad \lim_{z\to 1}(1-z)f(z) = 0.
$$

By [57, §8.3-8.6] and [83], we deduce that the Cauchy transform of f have the same properties:

$$
\lim_{z\to 0} z\int_0^1 \frac{f(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} = 0 \qquad \text{and} \qquad \lim_{z\to 1}(1-z)\int_0^1 \frac{f(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} = 0.
\tag{3.30}
$$

Now, we will prove that

$$
\lim_{z\to 0} z^2\left(\int_0^1 \frac{f(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}\right)' = 0 \qquad \text{and} \qquad \lim_{z\to 1}(1-z)^2\left(\int_0^1 \frac{f(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}\right)' = 0.
\tag{3.31}
$$

In fact,

$$
\begin{aligned}
z(1-z)\left(\int_0^1 \frac{f(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}\right)' &= \int_0^1 \frac{z(1-z)f(t)}{(t-z)^2}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} \\
&= \int_0^1 \frac{(t-z)(t+z-1)f(t)}{(t-z)^2}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} + \int_0^1 \frac{t(1-t)f(t)}{(t-z)^2}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}, \\
&= \int_0^1 \frac{t+z-1}{t-z}f(t)\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} - \frac{t(1-t)f(t)}{2\pi\,\mathrm{i}(t-z)}\bigg]_0^1 + \int_0^1 \frac{\big(t(1-t)f(t)\big)'}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}.
\end{aligned}
$$

From the boundary conditions, the first term is zero and we get

$$
z(1-z)\left(\int_0^1 \frac{f(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}\right)' = -\int_0^1 f(t)\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} + \int_0^1 \frac{t(1-t)f'(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}.
$$

We return back to (3.31), and see that this is equivalent to prove that

$$\lim_{z\to 0} z^2(1-z)\left(\int_0^1 \frac{f(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}\right)' = 0.$$

This follows from the fact that the Stieltjes–Markov matrix function of $z(1-z)f'(z)$, *i.e.*

$$\int_0^1 \frac{t(1-t)f'(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}},$$

is of the same type of f. Then,

$$\lim_{z\to 0} z\left(\int_0^1 \frac{t(1-t)f'(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}\right) = 0, \qquad \lim_{z\to 1}(1-z)\left(\int_0^1 \frac{t(1-t)f'(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}\right) = 0,$$

and (3.31) follows.

Now, as each entry of the matrix Q_n^{L} is a Cauchy transform of certain function f described previously, by using (3.30) and (3.31) we have that,

$$\left(Y_n^{\mathsf{L}}\right)'(z) = \begin{bmatrix} \mathrm{O}(1) & \mathrm{o}(\frac{1}{z^2}) \\ \mathrm{O}(1) & \mathrm{o}(\frac{1}{z^2}) \end{bmatrix}, \qquad \left(Y_n^{\mathsf{L}}(z)\right)^{-1} = \begin{bmatrix} \mathrm{o}(\frac{1}{z}) & \mathrm{o}(\frac{1}{z}) \\ \mathrm{O}(1) & \mathrm{O}(1) \end{bmatrix}, \qquad z\to 0,$$

$$\left(Y_n^{\mathsf{L}}\right)'(z) = \begin{bmatrix} \mathrm{O}(1) & \mathrm{o}(\frac{1}{(1-z)^2}) \\ \mathrm{O}(1) & \mathrm{o}(\frac{1}{(1-z)^2}) \end{bmatrix}, \qquad \left(Y_n^{\mathsf{L}}(z)\right)^{-1} = \begin{bmatrix} \mathrm{o}(\frac{1}{1-z}) & \mathrm{o}(\frac{1}{1-z}) \\ \mathrm{O}(1) & \mathrm{O}(1) \end{bmatrix}, \qquad z\to 1.$$

This implies that

$$\lim_{z\to 0} z^2\left(Y_n^{\mathsf{L}}\right)'(z)\left(Y_n^{\mathsf{L}}(z)\right)^{-1} = \lim_{z\to 0} z^2 \begin{bmatrix} \mathrm{o}(\frac{1}{z})+\mathrm{o}(\frac{1}{z^2}) & \mathrm{o}(\frac{1}{z})+\mathrm{o}(\frac{1}{z^2}) \\ \mathrm{o}(\frac{1}{z^2})+\mathrm{o}(\frac{1}{z}) & \mathrm{o}(\frac{1}{z^2})+\mathrm{o}(\frac{1}{z}) \end{bmatrix} = \lim_{z\to 0} z^2 \begin{bmatrix} \mathrm{o}(\frac{1}{z^2}) & \mathrm{o}(\frac{1}{z^2}) \\ \mathrm{o}(\frac{1}{z^2}) & \mathrm{o}(\frac{1}{z^2}) \end{bmatrix} = \mathbf{0},$$

and $\lim_{z\to 1}(1-z)^2\left(Y_n^{\mathsf{L}}\right)'(z)\left(Y_n^{\mathsf{L}}(z)\right)^{-1} = \lim_{z\to 1}(1-z)^2 \begin{bmatrix} \mathrm{o}(\frac{1}{(1-z)^2}) & \mathrm{o}(\frac{1}{(1-z)^2}) \\ \mathrm{o}(\frac{1}{(1-z)^2}) & \mathrm{o}(\frac{1}{(1-z)^2}) \end{bmatrix} = \mathbf{0}.$

Straightforward calculation and similar considerations lead us to

$$\lim_{z\to 0} zY_n^{\mathsf{L}}(z)\begin{bmatrix} h^{\mathsf{L}}(z) & \mathbf{0} \\ \mathbf{0} & -h^{\mathsf{R}}(z) \end{bmatrix}\left(Y_n^{\mathsf{L}}(z)\right)^{-1} = \mathbf{0},$$

$$\lim_{z\to 1}(1-z)Y_n^{\mathsf{L}}(z)\begin{bmatrix} h^{\mathsf{L}}(z) & \mathbf{0} \\ \mathbf{0} & -h^{\mathsf{R}}(z) \end{bmatrix}\left(Y_n^{\mathsf{L}}(z)\right)^{-1} = \mathbf{0}.$$

Finally we arrive to

$$\lim_{z\to 0} z^2 M_n^{\mathsf{L}}(z) = \mathbf{0} \qquad \text{and} \qquad \lim_{z\to 1}(1-z)^2 M_n^{\mathsf{L}}(z) = \mathbf{0}.$$

By analogous arguments we get the results for M_n^{R}. □

4.3. Differential relations from the Riemann–Hilbert problem. Our objective is to derive differential equations satisfied by the biorthogonal matrix polynomials associated to regular Jacobi type matrices of weights. Here we use the Riemann–Hilbert problem approach in order to derive these differential relations.

Let us define a new matrix functions,

$$\widetilde{M}_n^{\mathsf{L}}(z) = z(1-z)M_n^{\mathsf{L}}(z), \qquad \widetilde{M}_n^{\mathsf{R}}(z) = z(1-z)M_n^{\mathsf{R}}(z),$$

then $\widetilde{M}_n^{\mathsf{L}}$ and $\widetilde{M}_n^{\mathsf{R}}$ are matrices of entire functions, *cf.* Theorem 3.14.

Theorem 3.15 (First order differential equation for the fundamental matrices)**.** *In the conditions of Theorem 3.14 we have that*

$$z(1-z)\big(Y_n^{\mathsf{L}}\big)'(z)+Y_n^{\mathsf{L}}(z)\begin{bmatrix}h^{\mathsf{L}}(z) & \mathbf{0}\\ \mathbf{0} & -h^{\mathsf{R}}(z)\end{bmatrix}=\widetilde{M}_n^{\mathsf{L}}(z)Y_n^{\mathsf{L}}(z) \tag{3.32}$$

$$z(1-z)\big(Y_n^{\mathsf{R}}\big)'(z)+\begin{bmatrix}h^{\mathsf{R}}(z) & \mathbf{0}\\ \mathbf{0} & -h^{\mathsf{L}}(z)\end{bmatrix}Y_n^{\mathsf{R}}(z)=Y_n^{\mathsf{R}}(z)\widetilde{M}_n^{\mathsf{R}}(z). \tag{3.33}$$

Proof. Equations (3.32) and (3.33) follows immediately from the definition of the matrices M_n^{L} and M_n^{R} in (3.3). □

Now, we introduce the $\mathcal{N}$ map, $\mathcal{N}(F(z))=F'(z)+\dfrac{F^2(z)}{z(1-z)}$.

Theorem 3.16 (Second order differential equation for the fundamental matrices)**.** *In the conditions of Theorem 3.14 we have that*

$$\begin{aligned}z(1-z)\big(Y_n^{\mathsf{L}}\big)''(z)+\big(Y_n^{\mathsf{L}}\big)'(z)&\begin{bmatrix}2h^{\mathsf{L}}(z)+(1-2z)\,\mathrm{I} & \mathbf{0}\\ \mathbf{0} & -2h^{\mathsf{R}}(z)+(1-2z)\,\mathrm{I}\end{bmatrix}\\ &+Y_n^{\mathsf{L}}(z)\begin{bmatrix}\mathcal{N}(h^{\mathsf{L}}(z)) & \mathbf{0}\\ \mathbf{0} & \mathcal{N}(-h^{\mathsf{R}}(z))\end{bmatrix}=\mathcal{N}(\widetilde{M}_n^{\mathsf{L}}(z))Y_n^{\mathsf{L}}(z),\end{aligned} \tag{3.34}$$

$$\begin{aligned}z(1-z)\big(Y_n^{\mathsf{R}}\big)''(z)+&\begin{bmatrix}2h^{\mathsf{R}}(z)+(1-2z)\,\mathrm{I} & \mathbf{0}\\ \mathbf{0} & -2h^{\mathsf{L}}(z)+(1-2z)\,\mathrm{I}\end{bmatrix}\big(Y_n^{\mathsf{R}}\big)'(z)\\ &+\begin{bmatrix}\mathcal{N}(h^{\mathsf{R}}(z)) & \mathbf{0}\\ \mathbf{0} & \mathcal{N}(-h^{\mathsf{L}}(z))\end{bmatrix}Y_n{}^{\mathsf{R}}(z)=Y_n^{\mathsf{R}}(z)\mathcal{N}(\widetilde{M}_n^{\mathsf{R}}(z)).\end{aligned} \tag{3.35}$$

Proof. Differentiating in (3.3) we get

$$\left(Z_n^{\mathsf{L}}\right)''\left(Z_n^{\mathsf{L}}\right)^{-1}=\frac{\left(\widetilde{M}_n^{\mathsf{L}}\right)'}{z(1-z)}-(1-2z)\frac{\widetilde{M}_n^{\mathsf{L}}}{z^2(1-z)^2}+\frac{\left(\widetilde{M}_n^{\mathsf{L}}\right)^2}{z^2(1-z)^2},$$

so that

$$z(1-z)\left(Z_n^{\mathsf{L}}\right)''\left(Z_n^{\mathsf{L}}\right)^{-1}+(1-2z)M_n^{\mathsf{L}}=\left(\widetilde{M}_n^{\mathsf{L}}\right)'+\frac{\left(\widetilde{M}_n^{\mathsf{L}}\right)^2}{z(1-z)}=\mathcal{N}(\widetilde{M}_n^{\mathsf{L}}).$$

Now let us see that

$$(1-2z)M_n^{\mathsf{L}}=z(1-z)\left(Y_n^{\mathsf{L}}\right)'\left(Y_n^{\mathsf{L}}\right)^{-1}+Y_n^{\mathsf{L}}\begin{bmatrix}h^{\mathsf{L}} & \mathbf{0}\\ \mathbf{0} & -h^{\mathsf{R}}\end{bmatrix}\left(Y_n^{\mathsf{L}}\right)^{-1}.$$

From (3.23) we have

$$z(1-z)\big(W^{\mathsf{L}}\big)''\big(W^{\mathsf{L}}\big)^{-1}=\frac{\left(h^{\mathsf{L}}\right)^2}{z(1-z)}-\frac{1-2z}{z(1-z)}h^{\mathsf{L}}+\left(h^{\mathsf{L}}\right)',$$

$$z(1-z)\big((W^{\mathsf{R}})^{-1}\big)''W^{\mathsf{R}}=\frac{\left(h^{\mathsf{R}}\right)^2}{z(1-z)}+\frac{1-2z}{z(1-z)}h^{\mathsf{R}}-\left(h^{\mathsf{R}}\right)'.$$

Since

$$z(1-z)\left(Z^{\mathsf{L}}_n\right)''\left(Z^{\mathsf{L}}_n\right)^{-1} = z(1-z)(Y^{\mathsf{L}}_n)''Y^{\mathsf{L}}_n + \left(\left(Y^{\mathsf{L}}_n\right)\right)'\begin{bmatrix} 2h^{\mathsf{L}} & \mathbf{0} \\ \mathbf{0} & -2h^{\mathsf{R}}\end{bmatrix}\left(Y^{\mathsf{L}}_n\right)^{-1}$$
$$+ Y^{\mathsf{L}}_n \begin{bmatrix} z(1-z)\left(W^{\mathsf{L}}\right)''\left(W^{\mathsf{L}}\right)^{-1} & \mathbf{0} \\ \mathbf{0} & z(1-z)\left((W^{\mathsf{R}})^{-1}\right)''W^{\mathsf{R}}\end{bmatrix}\left(Y^{\mathsf{L}}_n\right)^{-1},$$

we get the stated result (3.34). The equation (3.35) follows in a similar way from definition of M^{R}_n in (3.3). □

We introduce the following $\mathbb{C}^{2N\times 2N}$ valued functions

$$\mathsf{H}^{\mathsf{L}}_n = \begin{bmatrix} \mathsf{H}^{\mathsf{L}}_{1,1,n} & \mathsf{H}^{\mathsf{L}}_{1,2,n} \\ \mathsf{H}^{\mathsf{L}}_{2,1,n} & \mathsf{H}^{\mathsf{L}}_{2,2,n}\end{bmatrix} := \mathcal{N}(\widetilde{M}^{\mathsf{L}}_n), \qquad \mathsf{H}^{\mathsf{R}}_n = \begin{bmatrix} \mathsf{H}^{\mathsf{R}}_{1,1,n} & \mathsf{H}^{\mathsf{R}}_{1,2,n} \\ \mathsf{H}^{\mathsf{R}}_{2,1,n} & \mathsf{H}^{\mathsf{R}}_{2,2,n}\end{bmatrix} := \mathcal{N}(\widetilde{M}^{\mathsf{R}}_n).$$

It holds that the second order matrix differential equations (3.34) and (3.35) split in the following differential relations

$$z(1-z)\left(P^{\mathsf{L}}_n\right)'' + \left(P^{\mathsf{L}}_n\right)'\left(2h^{\mathsf{L}} + (1-2z)\,\mathrm{I}\right) + P^{\mathsf{L}}_n\mathcal{N}(h^{\mathsf{L}}) = \mathsf{H}^{\mathsf{L}}_{1,1,n}P^{\mathsf{L}}_n - \mathsf{H}^{\mathsf{L}}_{1,2,n}C_{n-1}P^{\mathsf{L}}_{n-1},$$
$$z(1-z)\left(Q^{\mathsf{L}}_n\right)'' - \left(Q^{\mathsf{L}}_n\right)'\left(2h^{\mathsf{R}} - (1-2z)\,\mathrm{I}\right) + Q^{\mathsf{L}}_n\mathcal{N}(-h^{\mathsf{R}}) = \mathsf{H}^{\mathsf{L}}_{1,1,n}Q^{\mathsf{L}}_n - \mathsf{H}^{\mathsf{L}}_{1,2,n}C_{n-1}Q^{\mathsf{L}}_{n-1},$$
$$z(1-z)\left(P^{\mathsf{R}}_n\right)'' + \left(2h^{\mathsf{R}} + (1-2z)\,\mathrm{I}\right)\left(P^{\mathsf{R}}_n\right)' + \mathcal{N}(h^{\mathsf{R}})P^{\mathsf{R}}_n = P^{\mathsf{R}}_n\mathsf{H}^{\mathsf{R}}_{1,1,n} - P^{\mathsf{R}}_{n-1}C_{n-1}\mathsf{H}^{\mathsf{R}}_{2,1,n},$$
$$z(1-z)\left(Q^{\mathsf{R}}_n\right)'' - \left(2h^{\mathsf{L}} - (1-2z)\,\mathrm{I}\right)\left(Q^{\mathsf{R}}_n\right)' + \mathcal{N}(-h^{\mathsf{L}})Q^{\mathsf{R}}_n = Q^{\mathsf{R}}_n\mathsf{H}^{\mathsf{R}}_{1,1,n} - Q^{\mathsf{R}}_{n-1}C_{n-1}\mathsf{H}^{\mathsf{R}}_{2,1,n}.$$

5. Bessel type weights

Within this section, we use the Riemann–Hilbert problem in the context of jump supported on a suitably chosen curve situated on the complex plane and possessing a finite end point at the origin. Our aim is to investigate the matrix biorthogonal polynomials associated with Bessel type matrices of weights, which are constructed in terms of matrix Pearson equation (3.1) with $\phi(z) = z^2$, *i.e.*

(3.36) $$z^2\left(W^{\mathsf{L}}\right)'(z) = h^{\mathsf{L}}(z)W^{\mathsf{L}}(z), \qquad z^2\left(W^{\mathsf{R}}\right)'(z) = W^{\mathsf{R}}(z)h^{\mathsf{R}}(z),$$

then the matrix Pearson type equation (3.2) transforms into

(3.37) $$z^2W'(z) = h^{\mathsf{L}}(z)W(z) + W(z)h^{\mathsf{R}}(z).$$

We deal with *regular weight matrix* W where its support, is a non self-intersecting smooth closed curve γ with beginning point at 0 and ending point at ∞, and such that it intersects the circles $|z| = R$, $R\in\mathbb{R}^+$, once and only once (*i.e.*, it can be taken as a determination curve for $\arg(z)$).

In the current paragraph, we focus on Bessel type examples.

Definition 3.4. *We say that a* $N\times N$ weight matrix W *with support* γ *is of* Bessel type *if the entries* $W^{(j,k)}$ *of the matrix weight* W *can be written as*

(3.38) $$W^{(j,k)}(z) = \sum_{m\in I_{j,k}} \varphi_m(z)(a+z)^{p_m}\log^{q_m}(a+z)\,\mathrm{e}^{\frac{-\lambda_m}{a+z}}, \qquad z\in\gamma,$$

where $I_{j,k}$ denotes a finite set of indexes, $\operatorname{Re}(\lambda_m) \geq 0$, $\operatorname{Re}(p_m) > -1$, $q_m \in \mathbb{N}$ *and* φ_m *is Hölder continuous, bounded and non-vanishing on* γ. *In the development of the theory, that the functions* φ_m *have an holomorphic extension to the whole complex plane.*

In this section, for the sake of simplicity, $\gamma = (0, +\infty)$ and the pole is taken at the origin, $a = 0$ with no loss of generality, as a similar arguments apply for $a \neq 0$.

5.1. Riemann–Hilbert problem for the Bessel type weights on $(0, +\infty)$**.** Now, we state a theorem on Riemann–Hilbert problem for the Bessel type weights.

Theorem 3.17. *Given a regular Bessel type weight matrix W with support on γ we have the matrix function Y_n^{L} and Y_n^{R}, defined by* (1.22) *and* (1.23) *is, for each $n \in \mathbb{N}$, the unique solution of the following Riemann–Hilbert problems, which consists, respectively, in the determination of a $2N \times 2N$ complex matrix function such that:*

(RH1): *Y_n^{L} and Y_n^{R} are holomorphic in $\mathbb{C} \setminus (0, +\infty)$.*

(RH2): *Satisfies the jump condition*

$$\big(Y_n^{\mathsf{L}}(z)\big)_+ = \big(Y_n^{\mathsf{L}}(z)\big)_- \begin{bmatrix} \mathrm{I} & W(z) \\ \mathbf{0} & \mathrm{I} \end{bmatrix}, \qquad \big(Y_n^{\mathsf{R}}(z)\big)_+ = \begin{bmatrix} \mathrm{I} & \mathbf{0} \\ W(z) & \mathrm{I} \end{bmatrix} \big(Y_n^{\mathsf{R}}(z)\big)_-, \qquad z \in \gamma.$$

(RH3): *Have the following asymptotic behavior, as $z \to \infty$*

$$Y_n^{\mathsf{L}}(z) = (\mathrm{I} + \mathrm{O}(1/z)) \begin{bmatrix} z^n \,\mathrm{I} & \mathbf{0} \\ \mathbf{0} & z^{-n}\,\mathrm{I} \end{bmatrix}, \qquad Y_n^{\mathsf{R}}(z) = \begin{bmatrix} \mathrm{I}\, z^n & \mathbf{0} \\ \mathbf{0} & \mathrm{I}\, z^{-n} \end{bmatrix} \Big(\mathrm{I} + \mathrm{O}(1/z)\Big).$$

(RH4): $Y_n^{\mathsf{L}}(z) = \begin{bmatrix} \mathrm{O}(1) & s_1^{\mathsf{L}}(z) \\ \mathrm{O}(1) & s_2^{\mathsf{L}}(z) \end{bmatrix}$, $Y_n^{\mathsf{R}}(z) = \begin{bmatrix} \mathrm{O}(1) & \mathrm{O}(1) \\ s_1^{\mathsf{R}}(z) & s_2^{\mathsf{R}}(z) \end{bmatrix}$, *as $z \to 0$, with*

$$\lim_{z\to 0} z s_j^{\mathsf{L}}(z) = \mathbf{0}, \qquad \lim_{z\to 0} z s_j^{\mathsf{R}}(z) = \mathbf{0}, \qquad j = 1, 2.$$

Proof. (RH1), (RH2) and (RH3) follows from Theorem 1.8. We will prove (RH4) and the unicity. The entries $W^{j,k}$ of the matrix weight W are given in (3.38). It holds (*cf.* [57]) that in a neighborhood of $z = 0$ the Cauchy transform

$$\phi_m(z) = \frac{1}{2\pi\,\mathrm{i}} \int_\gamma \frac{p(t)\varphi_m(t) t^{\alpha_m} \log^{p_m}(t)\, \mathrm{e}^{-\frac{\lambda_m}{t}}}{t - z}\, \mathrm{d}\, t,$$

where p denotes a polynomial, that satisfies $\lim_{z\to 0} z\phi_m(z) = 0$. Then, $(\mathrm{RH4})$ is fulfilled by the matrices Y_n^{L} and Y_n^{R}, respectively. To prove the unicity of both Riemann–Hilbert problems let us consider the matrix function

$$G(z) = Y_n^{\mathsf{L}}(z) \begin{bmatrix} \mathbf{0} & \mathrm{I} \\ -\mathrm{I} & \mathbf{0} \end{bmatrix} Y_n^{\mathsf{R}}(z) \begin{bmatrix} \mathbf{0} & -\mathrm{I} \\ \mathrm{I} & \mathbf{0} \end{bmatrix}.$$

The function G has no jump or discontinuity on the curve γ and its behavior at the end point 0 is given by

$$G(z) \sim \begin{bmatrix} s_1^{\mathsf{L}}(z) + s_2^{\mathsf{R}}(z) & s_1^{\mathsf{L}}(z) + s_1^{\mathsf{R}}(z) \\ s_2^{\mathsf{L}}(z) + s_2^{\mathsf{R}}(z) & s_2^{\mathsf{L}}(z) + s_1^{\mathsf{R}}(z) \end{bmatrix}, \qquad z \to 0,$$

so it holds that $\lim_{z\to 0} zG(z) = 0$ and we conclude that the end point 0 is a removable singularity of G. Now, from the behavior for $z \to \infty$,

$$G(z) \sim \begin{bmatrix} \mathrm{I}\, z^n & \mathbf{0} \\ \mathbf{0} & \mathrm{I}\, z^{-n} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathrm{I} \\ -\mathrm{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathrm{I}\, z^n & \mathbf{0} \\ \mathbf{0} & \mathrm{I}\, z^{-n} \end{bmatrix} \begin{bmatrix} \mathbf{0} & -\mathrm{I} \\ \mathrm{I} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathrm{I} & \mathbf{0} \\ \mathbf{0} & \mathrm{I} \end{bmatrix},$$

the Liouville Theorem implies that $G(z) = \mathrm{I}$. To prove the unicity of the solution, we consider another solution $\widetilde{Y}_n^{\mathsf{L}}$ of the left Riemann–Hilbert problem. Then

$$\widetilde{Y}_n^{\mathsf{L}}(z) = \left(\begin{bmatrix} \mathbf{0} & \mathrm{I} \\ -\mathrm{I} & \mathbf{0} \end{bmatrix} Y_n^{\mathsf{R}}(z) \begin{bmatrix} \mathbf{0} & -\mathrm{I} \\ \mathrm{I} & \mathbf{0} \end{bmatrix} \right)^{-1}.$$

Hence any solution of this left Riemann–Hilbert problem is equal to the inverse of a fixed matrix, and the uniqueness follows. We obtain the uniqueness of the solution of the right Riemann–Hilbert in a similar way. □

5.2. Constant jump fundamental matrices. The solution of (3.36) will have possibly branch point at 0, *cf.* [91]. This means that there exists constant matrices, $\mathsf{C}_j^{\mathsf{L}}$, $\mathsf{C}_j^{\mathsf{R}}$, with $j = 0, 1$, such that

(3.39) $(W^{\mathsf{L}}(z))_- = (W^{\mathsf{L}}(z))_+ \mathsf{C}_0^{\mathsf{L}}, \qquad (W^{\mathsf{R}}(z))_- = \mathsf{C}_0^{\mathsf{R}} (W^{\mathsf{R}}(z))_+, \qquad$ in $(0, +\infty)$

The constant jump fundamental matrices Z_n^{L} and Z_n^{R} satisfy, for each $n \in \mathbb{N}$, the following properties:

- Are holomorphic on $\mathbb{C} \setminus [0, +\infty)$.
- Present the following *constant jump condition* on $(0, +\infty)$

$$\big(Z_n^{\mathsf{L}}(z)\big)_+ = \big(Z_n^{\mathsf{L}}(z)\big)_- \begin{bmatrix} \mathsf{C}_0^{\mathsf{L}} & \mathsf{C}_0^{\mathsf{L}} \\ \mathbf{0} & \mathrm{I} \end{bmatrix}, \qquad \big(Z_n^{\mathsf{R}}(z)\big)_+ = \begin{bmatrix} \mathrm{I} & \mathbf{0} \\ \mathsf{C}_0^{\mathsf{R}} & \mathsf{C}_0^{\mathsf{R}} \end{bmatrix} \big(Z_n^{\mathsf{R}}(z)\big)_-.$$

Now, we discuss the holomorphic properties of the structure matrices (3.3).

Theorem 3.18. *Let W be a regular Bessel matrix weight that satisfies a Pearson type equation* (3.37) *with $\phi(z) = z^2$, that admits a factorization $W(z) = W^{\mathsf{L}}(z) W^{\mathsf{R}}(z)$, where W^{L} and W^{R} satisfies* (3.36)*. Then, the structure matrices M_n^{L} and M_n^{R} are, for each $n \in \mathbb{N}$, meromorphic on $\mathbb{C}$, with singularity located at $z = 0$, which happens to be a pole of degree at most two.*

Proof. Let us prove the statement for M_n^{L}. The matrix function M_n^{L} is holomorphic in $\mathbb{C} \setminus [0, +\infty)$ by definition, *cf.* (3.3). Due to the fact that Z_n^{L} has a constant jump on $(0, +\infty)$, *cf.* (3.39), the matrix function $\big(Z_n^{\mathsf{L}}\big)'$ has the same constant jump on $(0, +\infty)$, so that the matrix M_n^{L} has no jump on $(0,+\infty)$, and it follows that at $z=0$, M_n^{L} has an isolated singularity.

From (2.33) and (3.3) it holds

$$\begin{aligned} M_n^{\mathsf{L}}(z) &= \big(Z_n^{\mathsf{L}}\big)'(z)\big(Z_n^{\mathsf{L}}(z)\big)^{-1} \\ &= \big(Y_n^{\mathsf{L}}\big)'(z)\big(Y_n^{\mathsf{L}}(z)\big)^{-1} + \frac{1}{z^2} Y_n^{\mathsf{L}}(z) \begin{bmatrix} h^{\mathsf{L}}(z) & \mathbf{0} \\ \mathbf{0} & -h^{\mathsf{R}}(z) \end{bmatrix} \big(Y_n^{\mathsf{L}}(z)\big)^{-1}, \end{aligned}$$

where Y_n^{L} is given in (1.22). Each entry of the matrix Q_n^{L} is the Cauchy transform of certain function, f, of type

$$f(z) = \sum_{j\in I} \varphi_j(z) z^{p_j} \log^{q_j}(z)\,\mathrm{e}^{-\frac{\lambda}{z}},$$

where φ_j is, for each $j\in I$, an entire function with $\operatorname{Re}(\lambda_j)\geq 0$, $\operatorname{Re}(p_j)>-1$, $q_j\in\mathbb{N}$, and I is a finite set of indices. It's clear that $\lim_{z\to 0} zf(z)=0$. By [57, §8.3-8.6] and [83], we deduce that the Cauchy transform of f have the same properties:

$$\lim_{z\to 0} zg(z) = 0, \qquad \text{where} \qquad g(z) := \int_0^1 \frac{f(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}.$$

We can also see that $\lim_{z\to 0} z^3 g'(z)=0$. Indeed,

$$\begin{aligned} z^2 g'(z) &= \int_\gamma \frac{z^2 f(t)}{(t-z)^2}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} = \int_\gamma \frac{(z-t)(z+t)f(t)}{(t-z)^2}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} + \int_\gamma \frac{t^2 f(t)}{(t-z)^2}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}, \\ &= -\int_\gamma \frac{(z+t)f(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} - \frac{t^2 f(t)}{2\pi\,\mathrm{i}(t-z)}\bigg]_{\partial\gamma} + \int_\gamma \frac{(t^2 f(t))'}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} \\ &= -z\int_\gamma \frac{f(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} - \frac{tf(t)}{t-z}\bigg]_{\partial\gamma} + \int_\gamma \frac{t^2 f'(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} + \int_\gamma \frac{tf(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}}. \end{aligned}$$

From the boundary conditions, $\frac{tf(t)}{2\pi\,\mathrm{i}(t-z)}\Big]_{\partial\gamma}$ is zero and we get

$$z^2 g'(z) = -z\int_\gamma \frac{f(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} + \int_\gamma \frac{t^2 f'(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} + \int_\gamma \frac{tf(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}},$$

and from the definition of f we get that $t^2 f'(t)$ and $tf(t)$ are functions in the same class of f. Moreover, their Cauchy transforms verify,

$$\lim_{z\to 0} z\int_\gamma \frac{t^2 f'(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} = 0, \qquad \lim_{z\to 0} z\int_\gamma \frac{tf(t)}{t-z}\frac{\mathrm{d}\,t}{2\pi\,\mathrm{i}} = 0;$$

consequently, $\lim_{z\to 0} z^3 g'(z)=0$. From these considerations it follows,

$$\big(Y_n^{\mathsf{L}}\big)'(z) = \begin{bmatrix} \mathrm{O}(1) & r_1^{\mathsf{L}}(z) \\ \mathrm{O}(1) & r_2^{\mathsf{L}}(z) \end{bmatrix}, \qquad \big(Y_n^{\mathsf{L}}(z)\big)^{-1} = \begin{bmatrix} r_3^{\mathsf{L}}(z) & r_4^{\mathsf{L}}(z) \\ \mathrm{O}(1) & \mathrm{O}(1) \end{bmatrix}, \qquad z\to 0,$$

where $\lim_{z\to 0} z^3 r_i^{\mathsf{L}}(z) = \mathbf{0}$, for $i=1,2$, and $\lim_{z\to 0} z^3 r_i^{\mathsf{R}}(z) = \mathbf{0}$, for $i=3,4$, so it holds that

$$\lim_{z\to 0} z^3 \big(Y_n^{\mathsf{L}}\big)'(z)\big(Y_n^{\mathsf{L}}\big)^{-1} = \lim_{z\to 0} z^3 \begin{bmatrix} r_1^{\mathsf{L}}(z)+r_3^{\mathsf{L}}(z) & r_1^{\mathsf{L}}(z)+r_4^{\mathsf{L}}(z) \\ r_2^{\mathsf{L}}(z)+r_3^{\mathsf{L}}(z) & r_2^{\mathsf{L}}(z)+r_4^{\mathsf{L}}(z) \end{bmatrix} = \mathbf{0}.$$

Similar considerations leads us to

$$\lim_{z\to 0} z\, Y_n^{\mathsf{L}}(z) \begin{bmatrix} h^{\mathsf{L}}(z) & \mathbf{0} \\ \mathbf{0} & -h^{\mathsf{R}}(z) \end{bmatrix} \big(Y_n^{\mathsf{L}}(z)\big)^{-1} = \mathbf{0},$$

so we obtain that $\lim_{z\to 0} z^3 M_n^{\mathsf{L}}(z) = \mathbf{0}$, and hence the matrix function M_n^{L} has at most a pole of degree 2 at the point $z=0$. By analogous arguments we get the results for M_n^{R}. □

5.3. Differential relations from the Riemann–Hilbert problem. Our objective is to derive differential equations satisfied by the biorthogonal matrix polynomials associated to regular Bessel type matrices of weights.

Let us define a new matrix functions,

$$\widetilde{M}_n^{\mathsf{L}}(z) = z^2 M_n^{\mathsf{L}}(z), \qquad \widetilde{M}_n^{\mathsf{R}}(z) = z^2 M_n^{\mathsf{R}}(z),$$

then $\widetilde{M}_n^{\mathsf{L}}$ and $\widetilde{M}_n^{\mathsf{R}}$ are matrices of entire functions, *cf.* Theorem 3.18.

Theorem 3.19 (First order differential equation for the fundamental matrices)**.** *In the conditions of Theorem 3.18 we have that*

$$z^2\big(Y_n^{\mathsf{L}}\big)'(z) + Y_n^{\mathsf{L}}(z) \begin{bmatrix} h^{\mathsf{L}}(z) & \mathbf{0} \\ \mathbf{0} & -h^{\mathsf{R}}(z) \end{bmatrix} = \widetilde{M}_n^{\mathsf{L}}(z) Y_n^{\mathsf{L}}(z) \tag{3.40}$$

$$z^2\big(Y_n^{\mathsf{R}}\big)'(z) + \begin{bmatrix} h^{\mathsf{R}}(z) & \mathbf{0} \\ \mathbf{0} & -h^{\mathsf{L}}(z) \end{bmatrix} Y_n^{\mathsf{R}}(z) = Y_n^{\mathsf{R}}(z) \widetilde{M}_n^{\mathsf{R}}(z). \tag{3.41}$$

Proof. Equations (3.40) and (3.41) follows immediately from the definition of the matrices M_n^{L} and M_n^{R} in (3.3) and taking into account Theorem 3.18. □

Now, we introduce the $\mathcal{N}$ map, $\mathcal{N}(F(z)) = F'(z) + \dfrac{F^2(z)}{z^2}$.

Theorem 3.20 (Second order differential equation for the fundamental matrices)**.** *In the conditions of Theorem 3.18 we have that*

$$z^2\big(Y_n^{\mathsf{L}}\big)'' + \big(Y_n^{\mathsf{L}}\big)' \begin{bmatrix} 2h^{\mathsf{L}} + 2z\,\mathrm{I} & \mathbf{0} \\ \mathbf{0} & -2h^{\mathsf{R}} + 2z\,\mathrm{I} \end{bmatrix} + Y_n^{\mathsf{L}}(z) \begin{bmatrix} \mathcal{N}(h^{\mathsf{L}}) & \mathbf{0} \\ \mathbf{0} & \mathcal{N}(-h^{\mathsf{R}}) \end{bmatrix} = \mathcal{N}(\widetilde{M}_n^{\mathsf{L}}) Y_n^{\mathsf{L}},$$

$$z^2\big(Y_n^{\mathsf{R}}\big)'' + \begin{bmatrix} 2h^{\mathsf{R}} + 2z\,\mathrm{I} & \mathbf{0} \\ \mathbf{0} & -2h^{\mathsf{L}} + 2z\,\mathrm{I} \end{bmatrix} \big(Y_n^{\mathsf{R}}\big)' + \begin{bmatrix} \mathcal{N}(h^{\mathsf{R}}) & \mathbf{0} \\ \mathbf{0} & \mathcal{N}(-h^{\mathsf{L}}) \end{bmatrix} {Y_n}^{\mathsf{R}}(z) = Y_n^{\mathsf{R}} \mathcal{N}(\widetilde{M}_n^{\mathsf{R}}).$$

Proof. Very similar to the previous cases. □

CHAPTER 4

Structure matrices and Painlevé discrete matrix equations

1. Introduction

The study of equations for the recursion coefficients of orthogonal polynomials on the real line or on the unit circle constitutes a subject of current interest. The question of how the expression of the weight and its properties (for example if it satisfies a Pearson type equation), translate to the recursion coefficients has been treated in several places, for a review see [89].

In 1976, Freud [56] studied weights in $\mathbb{R}$ of exponential variation of type $w(x) = |x|^\rho \exp(-|x|^m)$, $\rho > -1$ and $m > 0$. For $m = 2, 4, 6$ he constructed relations among them as well as determined its asymptotic behavior. However, Freud did not find the role of the discrete Painlevé I, that was discovered later by Magnus [79].

For the unit circle and a weight of the form $w(\theta) = \exp\big(k\cos(\theta)\big)$, $k \in \mathbb{R}$, Periwal and Shevitz [84, 85], in the context of matrix models, found the discrete Painlevé II equation for the recursion relations of the corresponding orthogonal polynomials. This result was rediscovered later and connected with the Painlevé III equation [69].

In [6] the discrete Painlevé II was found using the Riemann–Hilbert problem given in [7], see also [88]. For a nice account of the relation of these discrete Painlevé equations and integrable systems see [32], and for a survey on the subject of differential and discrete Painlevé equations *cf.* [29]. We also mention the recent paper [30] where a discussion on the relationship between the recurrence coefficients of orthogonal polynomials with respect to a semiclassical Laguerre weight and classical solutions of the fourth Painlevé equation, can be found. Also, in [31] the solution of the discrete alternate Painlevé equations is presented in terms of the Airy function.

In [20] the Riemann–Hilbert problem for this matrix situation and the appearance of non-Abelian discrete versions of Painlevé I were explored, showing singularity confinement (see also, [22]). The singularity analysis for a matrix discrete version of the Painlevé I equation was performed. It was found that the singularity confinement holds generically, *i.e.* in the whole space of parameters except possibly for algebraic subvarieties. The situation was considered in [21] for the matrix extension of the Szegő polynomials in the unit circle and corresponding non-Abelian versions discrete Painlevé II equations.

In this chapter, we begin by reviewing some results for the classical families of orthogonal polynomials and their associated functions in a scalar setting, where h^{L}

and h^{R} are scalar polynomials of degree one. Next, we revisit previous results related to Hermite type weights from [15]. In that work, various nonlinear matrix relations were discovered for the recursion coefficients when the degree of h^{L} is one, two, or three, and the role of h^{R} is deleted for simplicity.

Moving on to weights of Laguerre [11], Jacobi [12], and Bessel types [14], we derive Painlevé equations for the three-term recurrence relation coefficients of the monic orthogonal polynomials, when h^{L} is a matrix polynomial of degree two. It is important to note that when the degree of h^{L} is one, the recursion coefficients satisfy a nonlinear equation.

2. Classical matrix orthogonal polynomials

Here we investigate the classical case when $\max\big\{\deg h^{\mathsf{L}}(z), \deg h^{\mathsf{R}}(z)\big\} = 1$ in full generality. A scalar second order differential equation is recovered for Hermite, Laguerre, Jacobi and Bessel polynomials. We take

$$h^{\mathsf{L}}(z) = A^{\mathsf{L}}z + B^{\mathsf{L}}, \qquad\qquad h^{\mathsf{R}}(z) = A^{\mathsf{R}}z + B^{\mathsf{R}}.$$

2.1. Hermite case.

The results in this subsection are taken from [15].

Theorem 4.1. *for arbitrary matrices $A^{\mathsf{L}}, B^{\mathsf{L}}, A^{\mathsf{R}}, B^{\mathsf{R}} \in \mathbb{C}^{N\times N}$, with $A^{\mathsf{L}}, A^{\mathsf{R}}$ definite negative matrices. Thus, the weight matrix W is a solution of the following Pearson equation (a Sylvester linear differential equation)*

$$W'(z) = (A^{\mathsf{L}}z + B^{\mathsf{L}})W(z) + W(z)(A^{\mathsf{R}}z + B^{\mathsf{R}}).$$

For simplicity we take $\gamma = \mathbb{R}$. Hence, the structure matrices have the following form

$$M_n^{\mathsf{L}}(z) = \begin{bmatrix} A^{\mathsf{L}}z + B^{\mathsf{L}} + \big[p^1_{\mathsf{L},n}, A^{\mathsf{L}}\big] & C_n^{-1}A^{\mathsf{R}} + A^{\mathsf{L}}C_n^{-1} \\ -C_{n-1}A^{\mathsf{L}} - A^{\mathsf{R}}C_{n-1} & -A^{\mathsf{R}}z - B^{\mathsf{R}} - \big[q^1_{\mathsf{L},n-1}, A^{\mathsf{R}}\big], \end{bmatrix}$$

Corollary 4.1. *The scalar Hermite second order differential equation is satisfied in such way*

$$P_n''(z) - 2zP_n'(z) = -2\gamma_n P_n(z)$$
$$Q_n''(z) - 2zQ_n'(z) = (2 - 2\gamma_n)Q_n(z)$$

Proof. For $W^{\mathsf{L}}(z) = W^{\mathsf{R}}(z) = \mathrm{e}^{-\frac{1}{2}z^2}$ we get $A^{\mathsf{L}} = A^{\mathsf{R}} = -1$ and $B^{\mathsf{L}} = B^{\mathsf{R}} = 0$. Using Theorem 4.1 then plot these data on (3.11). □

2.2. Laguerre case.

Theorem 4.2. *Let $h^{\mathsf{L}}(z) = A^{\mathsf{L}}z + B^{\mathsf{L}}$ and $h^{\mathsf{R}}(z) = A^{\mathsf{R}}z + B^{\mathsf{R}}$ be two first degree matrix polynomials. The left and right fundamental matrices are given respectively by,*

$$M_n^{\mathsf{L}}(z) = \frac{1}{z}\begin{bmatrix} A^{\mathsf{L}}z + [p^1_{\mathsf{L},n}, A^{\mathsf{L}}] + n\,\mathrm{I} + B^{\mathsf{L}} & A^{\mathsf{L}}C_n^{-1} + C_n^{-1}A^{\mathsf{R}} \\ -C_{n-1}A^{\mathsf{L}} - A^{\mathsf{R}}C_{n-1} & -A^{\mathsf{R}}z + [p^1_{\mathsf{R},n}, A^{\mathsf{R}}] - n\,\mathrm{I} - B^{\mathsf{R}} \end{bmatrix}, \tag{4.1}$$

$$M_n^{\mathsf{R}}(z)=\frac{1}{z}\begin{bmatrix}A^{\mathsf{R}}z-[p^1_{\mathsf{R},n},A^{\mathsf{R}}]+n\,\mathrm{I}+B^{\mathsf{R}} & -C_{n-1}A^{\mathsf{L}}-A^{\mathsf{R}}C_{n-1}\\ A^{\mathsf{L}}C_n^{-1}+C_n^{-1}A^{\mathsf{R}} & -A^{\mathsf{L}}z-[p^1_{\mathsf{L},n},A^{\mathsf{L}}]-n\,\mathrm{I}-B^{\mathsf{L}}\end{bmatrix}. \tag{4.2}$$

Proof. By considering (3.16) in the proof of Theorem 3.10, the asymptotic expansion at infinity of the fundamental matrix Y_n^{L} and Q_n^{L}, *cf.* Theorem 1.8, and using the identities $p^1_{\mathsf{R},n}=-q^1_{\mathsf{L},n-1}$ and $p^1_{\mathsf{L},n}=-q^1_{\mathsf{R},n-1}$ (4.1) follows. The relation (3.4) leads to (4.2). □

From now on let us concentrate in the following general matrix Laguerre weight

$$W(z)=\mathrm{e}^{A_1 z}\,z^{\alpha}\,\mathrm{e}^{A_2 z}, \qquad z\in\mathbb{C},$$

defined in $\mathbb{C}\setminus[0,+\infty)$ with support on $\gamma=[0,+\infty)$. Here α, A_1, $A_2\in\mathbb{C}^{N\times N}$ are matrices such that $[\alpha,A_1]=[\alpha,A_2]=\mathbf{0}$, with spectrum $\sigma(\alpha)$, $\operatorname{Re}(\sigma(\alpha))\subset(-1,+\infty)$. This class of weights contains in the Hermitian case some of the cases studied in the literature [19, 47, 48, 49, 50].

For this class of Laguerre weights, we get, using analytic arguments, an alternative formula for the residue matrix with the simple pole at $z=0$ of the left fundamental matrix. In a similar manner we could get the result for the right fundamental matrix. Notice that the fundamental matrix is completely determined in Theorem 4.2, where A^{L}, A^{R}, is substituted respectively by A_1, A_2, and B^{L}, B^{R} by $\frac{\alpha}{2}$. This alternative formula enables us to make an important simplification in the equation (3.19) previously obtained.

Accordingly, we choose

$$W^{\mathsf{L}}(z)=\mathrm{e}^{A_1 z}\,z^{\frac{\alpha}{2}}, \qquad W^{\mathsf{R}}(z)=z^{\frac{\alpha}{2}}\,\mathrm{e}^{A_2 z}.$$

Straightforward calculation shows that h^{L} and h^{R} appearing in (3.14) are given by,

$$h^{\mathsf{L}}(z)=A_1 z+\frac{\alpha}{2}, \qquad h^{\mathsf{R}}(z)=A_2 z+\frac{\alpha}{2}.$$

Theorem 4.3. *The structure matrix M_n^{L} defined in* (4.1) *has a simple pole given by the yielding expression,*

i) *If $\operatorname{Re}(\sigma(\alpha))\subset(-1,+\infty)$ and $\sigma(\alpha)\cap\mathbb{N}=\emptyset$, then*

$$M_n^{\mathsf{L}}(z)=\frac{1}{z}F_n^{\mathsf{L}}(0)\begin{bmatrix}\frac{\alpha}{2} & \mathbf{0}\\ \mathbf{0} & -\frac{\alpha}{2}\end{bmatrix}\left(F_n^{\mathsf{L}}(0)\right)^{-1}+\mathrm{O}(1), \qquad z\to 0,$$

where $F_n^{\mathsf{L}}(0)$ is defined as follows

$$F_n^{\mathsf{L}}(0)=\hat{Y}_n^{\mathsf{L}}(0)\begin{bmatrix}P & \mathbf{0}\\ \mathbf{0} & P\end{bmatrix}\begin{bmatrix}\begin{bmatrix}I_{N^+} & 0_{N^+\times N^-}\\ 0_{N^-\times N^+} & 0_{N^-}\end{bmatrix} & \begin{bmatrix}0_{N^+} & 0_{N^+\times N^-}\\ 0_{N^-\times N^+} & \mathrm{e}^{-\mathrm{i}\pi J^-}\end{bmatrix}\\ \begin{bmatrix}0_{N^+} & 0_{N^+\times N^-}\\ 0_{N^-\times N^+} & I_{N^-}-\mathrm{e}^{2\mathrm{i}\pi J^-}\end{bmatrix} & \begin{bmatrix}\mathrm{e}^{\mathrm{i}\pi J^+}-\mathrm{e}^{-\mathrm{i}\pi J^+} & 0_{N^+\times N^-}\\ 0_{N^-\times N^+} & 0_{N^-}\end{bmatrix}\end{bmatrix}\begin{bmatrix}P & \mathbf{0}\\ \mathbf{0} & P\end{bmatrix}^{-1}.$$

where α has the yielding canonical Jordan form, $\alpha = PJP^{-1}$ with

$$J = \begin{bmatrix} J^{+} & 0_{N^{+}\times N^{-}} \\ 0_{N^{-}\times N^{+}} & J^{-} \end{bmatrix},$$

and N^{+} (respectively, N^{-}) being the sum of the algebraic multiplicities associated with eigenvalues having positive (respectively, negative) real part and in J^{+} (respectively, J^{-}) we gather together the Jordan blocks of all eigenvalues with positive (respectively, negative) real part, and $\hat{Y}_n^{\mathsf{L}}(z)$ being given by

$$\hat{Y}_n^{\mathsf{L}}(z) := Y_n^{\mathsf{L}}(z) \begin{bmatrix} z^{-\alpha} & \mathbf{0} \\ \mathrm{I} - \mathrm{e}^{2\,\mathrm{i}\,\pi\alpha} & z^{\alpha} \end{bmatrix}^{-1}.$$

ii) *If $\alpha = m\,\mathrm{I}, m \in \mathbb{N}$*

$$M_n^{\mathsf{L}}(z) = \frac{1}{z} F_n^{\mathsf{L}}(0) \begin{bmatrix} \frac{m}{2}\,\mathrm{I} & -\frac{z^m}{2\pi\,\mathrm{i}}\,\mathrm{I} \\ \mathbf{0} & -\frac{m}{2}\,\mathrm{I} \end{bmatrix} \left(F_n^{\mathsf{L}}(0)\right)^{-1} + \mathrm{O}(1), \qquad z \to 0,$$

where $F_n^{\mathsf{L}}(0) = \hat{Y}_n^{\mathsf{L}}(0) \begin{bmatrix} \mathbf{0} & \frac{1}{2\pi\,\mathrm{i}}\,\mathrm{I} \\ -2\pi\,\mathrm{i}\,\mathrm{I} & \mathbf{0} \end{bmatrix}$, with

$$\hat{Y}_n^{\mathsf{L}}(z) := Y_n^{\mathsf{L}}(z) \begin{bmatrix} (\log(z))^{-1}\,\mathrm{I} & \mathbf{0} \\ -2\pi\,\mathrm{i}\,\mathrm{I} & \log(z)\,\mathrm{I} \end{bmatrix}^{-1}.$$

Remark 4.1. *In the first case, $F_n^{\mathsf{L}}(0)$ have a simpler form if $\operatorname{Re}(\sigma(\alpha))$ are all positive or all negative*

(1) *If $\operatorname{Re}(\sigma(\alpha)) \subset (0, +\infty)$, then $F_n^{\mathsf{L}}(0) = Y_n^{\mathsf{L}}(0) \begin{bmatrix} \mathrm{I} & \mathbf{0} \\ \mathbf{0} & \mathrm{e}^{\mathrm{i}\,\pi\alpha} - \mathrm{e}^{-\mathrm{i}\,\pi\alpha} \end{bmatrix}$.*

(2) *If $\operatorname{Re}(\sigma(\alpha)) \subset (-1, 0)$, then $F_n^{\mathsf{L}}(0) = \lim_{z\to 0} Y_n^{\mathsf{L}}(z) \begin{bmatrix} \mathbf{0} & z^{\alpha}\,\mathrm{e}^{-\mathrm{i}\,\pi\alpha} \\ z^{\alpha}\left(\mathrm{I} - \mathrm{e}^{2\,\mathrm{i}\,\pi\alpha}\right) & \mathrm{e}^{\mathrm{i}\,\pi\alpha} - \mathrm{e}^{-\mathrm{i}\,\pi\alpha} \end{bmatrix}$.*

Proof. It can be seen that the matrix function Z_n^{L} defined by

$$Z_n^{\mathsf{L}}(z) = Y_n^{\mathsf{L}}(z)\mathcal{C}(z), \qquad \text{where} \qquad \mathcal{C}(z) = \begin{bmatrix} W^{\mathsf{L}}(z) & 0 \\ 0 & (W^{\mathsf{R}}(z))^{-1} \end{bmatrix},$$

with $W^{\mathsf{L}}(z)W^{\mathsf{R}}(z) = W(z)$, satisfies

- Z_n^{L} is holomorphic in $\mathbb{C} \setminus [0, +\infty)$.
- $\left(Z_n^{\mathsf{L}}(z)\right)_{+} = \left(Z_n^{\mathsf{L}}(z)\right)_{-} \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\,\pi\alpha} & \mathrm{e}^{-\mathrm{i}\,\pi\alpha} \\ 0 & \mathrm{e}^{\mathrm{i}\,\pi\alpha} \end{bmatrix}$ over $(0, +\infty)$.

Let's start with the first case: $\operatorname{Re}(\sigma(\alpha)) \subset (-1, +\infty)$ and $\sigma(\alpha) \cap \mathbb{N} = \emptyset$. In this case the constant jump matrix $\begin{bmatrix} \mathrm{e}^{-\mathrm{i}\,\pi\alpha} & \mathrm{e}^{-\mathrm{i}\,\pi\alpha} \\ 0 & \mathrm{e}^{\mathrm{i}\,\pi\alpha} \end{bmatrix}$ can be block diagonalized. For that aim we consider the matrix

$$P = \begin{bmatrix} \mathrm{I} & \mathrm{e}^{-\mathrm{i}\pi\alpha} \\ 0 & \mathrm{e}^{\mathrm{i}\pi\alpha} - \mathrm{e}^{-\mathrm{i}\pi\alpha} \end{bmatrix} \qquad \text{such that} \qquad \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\pi\alpha} & \mathrm{e}^{-\mathrm{i}\pi\alpha} \\ 0 & \mathrm{e}^{\mathrm{i}\pi\alpha} \end{bmatrix} P = P \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\pi\alpha} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\pi\alpha} \end{bmatrix}.$$

So, over the interval $(0, +\infty)$, we have

$$\big(Z_n^{\mathsf{L}}(z)P\big)_+ = \big(Z_n^{\mathsf{L}}(z)P\big)_- \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\pi\alpha} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\pi\alpha} \end{bmatrix}.$$

For $z \in \mathbb{C} \setminus [0, +\infty)$, let us define the matrix

$$\psi(z) \coloneqq \begin{bmatrix} z^{\frac{\alpha}{2}} & 0 \\ 0 & z^{-\frac{\alpha}{2}} \end{bmatrix}, \tag{4.3}$$

which satisfies, over $(0, +\infty)$, the following jump condition

$$\big(\psi(z)\big)_+ = \big(\psi(z)\big)_- \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\pi\alpha} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\pi\alpha} \end{bmatrix}.$$

Consequently, the matrix

$$F_n^{\mathsf{L}}(z) \coloneqq Z_n^{\mathsf{L}}(z) P \psi^{-1}(z),$$

has no jump in the interval $(0, +\infty)$. The matrix function F_n^{L} has an isolated singularity at the origin which, as we will show now, is a removable singularity, *i.e.* $\lim_{z\to 0} z F_n^{\mathsf{L}}(z) = \mathbf{0}$. From its definition we have that

$$\begin{aligned} zF_n^{\mathsf{L}}(z) &= \begin{bmatrix} \mathrm{O}(z) & zs_1^{\mathsf{L}}(z) \\ \mathrm{O}(z) & zs_2^{\mathsf{L}}(z) \end{bmatrix} \begin{bmatrix} \mathrm{e}^{A_1 z} z^{\frac{\alpha}{2}} & \mathbf{0} \\ \mathbf{0} & \mathrm{e}^{-A_2 z} z^{-\frac{\alpha}{2}} \end{bmatrix} \begin{bmatrix} \mathrm{I} & \mathrm{e}^{-\mathrm{i}\pi\alpha} \\ \mathbf{0} & \mathrm{e}^{\mathrm{i}\pi\alpha} - \mathrm{e}^{-\mathrm{i}\pi\alpha} \end{bmatrix} \begin{bmatrix} z^{-\frac{\alpha}{2}} & 0 \\ 0 & z^{\frac{\alpha}{2}} \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{O}(z) & zs_1^{\mathsf{L}}(z) \\ \mathrm{O}(z) & zs_2^{\mathsf{L}}(z) \end{bmatrix} \begin{bmatrix} \mathrm{e}^{A_1 z} & \mathrm{e}^{A_1 z} \mathrm{e}^{-\mathrm{i}\pi\alpha} z^{\alpha} \\ \mathbf{0} & \mathrm{e}^{-A_2 z}(\mathrm{e}^{\mathrm{i}\pi\alpha} - \mathrm{e}^{-\mathrm{i}\pi\alpha}) \end{bmatrix}, \qquad z \to 0, \end{aligned}$$

and as zs_1^{L}, $zs_2^{\mathsf{L}} \to \mathbf{0}$ as $z \to 0$ and $\mathrm{O}(z)z^{\alpha} \to \mathbf{0}$, as $z \to 0$ (because the eigenvalues of α are bounded from below by -1) we conclude that $zF_n^{\mathsf{L}}(z) \to \mathbf{0}$, for $z \to 0$. Hence, $F_n^{\mathsf{L}}(z)$ is a matrix of entire functions.

Now, we want to compute $F_n^{\mathsf{L}}(0) = \lim_{z\to 0} F_n^{\mathsf{L}}(z)$. For this fact, we will discuss with respect to the sign of the real part of spectrum of α. Notice that,

$$F_n^{\mathsf{L}}(0) = \lim_{z\to 0} Y_n^{\mathsf{L}}(z) \begin{bmatrix} \mathrm{e}^{A_1 z} & \mathrm{e}^{A_1 z} \mathrm{e}^{-\mathrm{i}\pi\alpha} z^{\alpha} \\ \mathbf{0} & \mathrm{e}^{-A_2 z}(\mathrm{e}^{\mathrm{i}\pi\alpha} - \mathrm{e}^{-\mathrm{i}\pi\alpha}) \end{bmatrix},$$

where the limit of each factor do not need to exist.

We separately compute $F_n^{\mathsf{L}}(0)$ in the cases, when $\operatorname{Re}(\sigma(\alpha)) \subset (0, +\infty)$ and when $\operatorname{Re}(\sigma(\alpha)) \subset (-1, 0)$, and then we give $F_n^{\mathsf{L}}(0)$ in general.

Case $\operatorname{Re}(\sigma(\alpha)) \subset (0,+\infty)$ ***and*** $\operatorname{Re}(\sigma(\alpha)) \cap \mathbb{N} = \emptyset$. When the real part of all the eigenvalues of α are strictly positive then each limit exists and

$$F_n^{\mathsf{L}}(0) = Y_n^{\mathsf{L}}(0)\begin{bmatrix} \mathrm{I} & \mathbf{0} \\ \mathbf{0} & \mathrm{e}^{\mathrm{i}\pi\alpha} - \mathrm{e}^{-\mathrm{i}\pi\alpha} \end{bmatrix}.$$

Case $\operatorname{Re}(\sigma(\alpha)) \subset (-1,0)$ ***and*** $\sigma(\alpha) \cap \mathbb{N} = \emptyset$. We cannot proceed as before. However, as the limit exists, if we are able to rewrite

$$Y_n^{\mathsf{L}}(z)\begin{bmatrix} \mathrm{e}^{A_1 z} & \mathrm{e}^{A_1 z}\,\mathrm{e}^{-\mathrm{i}\pi\alpha}\, z^{\alpha} \\ \mathbf{0} & \mathrm{e}^{-A_2 z}\big(\mathrm{e}^{\mathrm{i}\pi\alpha} - \mathrm{e}^{-\mathrm{i}\pi\alpha}\big) \end{bmatrix} = \hat{Y}_n^{\mathsf{L}}(z) f(z),$$

in terms of two matrix factors $\hat{Y}_n^{\mathsf{L}}(z)$ and $f(z)$, a nonsingular matrix, with f having a well defined limit for $z \to 0$, also being a nonsingular matrix, we can ensure the existence of $\lim_{z\to 0}\hat{Y}_n^{\mathsf{L}}(z)$, and $F_n^{\mathsf{L}}(0) = \big(\lim_{z\to 0}\hat{Y}_n^{\mathsf{L}}(z)\big)\big(\lim_{z\to 0} f(z)\big)$. This can be achieved by considering

$$\begin{aligned}
\hat{Y}_n^{\mathsf{L}}(z) &:= Y_n^{\mathsf{L}}(z)\begin{bmatrix} z^{-\alpha} & \mathbf{0} \\ \mathrm{I} - \mathrm{e}^{2\mathrm{i}\pi\alpha} & z^{\alpha} \end{bmatrix}^{-1}, \\
f(z) &:= \begin{bmatrix} z^{-\alpha} & \mathbf{0} \\ \mathrm{I} - \mathrm{e}^{2\mathrm{i}\pi\alpha} & z^{\alpha} \end{bmatrix}\begin{bmatrix} \mathrm{e}^{A_1 z} & \mathrm{e}^{A_1 z}\,\mathrm{e}^{-\mathrm{i}\pi\alpha}\, z^{\alpha} \\ \mathbf{0} & \mathrm{e}^{-A_2 z}\big(\mathrm{e}^{\mathrm{i}\pi\alpha} - \mathrm{e}^{-\mathrm{i}\pi\alpha}\big) \end{bmatrix} \\
&= \begin{bmatrix} z^{-\alpha}\,\mathrm{e}^{A_1 z} & \mathrm{e}^{A_1 z}\,\mathrm{e}^{-\mathrm{i}\pi\alpha} \\ \big(\mathrm{I} - \mathrm{e}^{2\mathrm{i}\pi\alpha}\big)\,\mathrm{e}^{A_1 z} & \big(-\mathrm{e}^{A_1 z} + \mathrm{e}^{-A_2 z}\big)\big(\mathrm{e}^{\mathrm{i}\pi\alpha} - \mathrm{e}^{-\mathrm{i}\pi\alpha}\big) z^{\alpha} \end{bmatrix}.
\end{aligned}$$

So that,

$$\lim_{z\to 0} f(z) = \begin{bmatrix} \mathbf{0} & \mathrm{e}^{-\mathrm{i}\pi\alpha} \\ \mathrm{I} - \mathrm{e}^{2\mathrm{i}\pi\alpha} & \mathbf{0} \end{bmatrix}, \qquad F_n^{\mathsf{L}}(0) = \hat{Y}_n^{\mathsf{L}}(0)\begin{bmatrix} \mathbf{0} & \mathrm{e}^{-\mathrm{i}\pi\alpha} \\ \mathrm{I} - \mathrm{e}^{2\mathrm{i}\pi\alpha} & \mathbf{0} \end{bmatrix}.$$

General case $\operatorname{Re}(\sigma(\alpha)) \subset (-1,+\infty)$ ***and*** $\sigma(\alpha) \cap \mathbb{N} = \emptyset$. Recalling the canonical Jordan form, we can write $\alpha = PJP^{-1}$ with

$$J = \begin{bmatrix} J^{+} & 0_{N^{+}\times N^{-}} \\ 0_{N^{-}\times N^{+}} & J^{-} \end{bmatrix},$$

and N^{+} (respectively, N^{-}) being the sum of the algebraic multiplicities associated with positive (respectively, negative) eigenvalues and in J^{+} (respectively, J^{-}), we gather together the Jordan blocks of all positive (respectively, negative) eigenvalues. Hence,

$$\begin{bmatrix} \mathrm{e}^{A_1 z} & \mathrm{e}^{A_1 z}\,\mathrm{e}^{-\mathrm{i}\pi\alpha}\, z^{\alpha} \\ \mathbf{0} & \mathrm{e}^{-A_2 z}\big(\mathrm{e}^{\mathrm{i}\pi\alpha} - \mathrm{e}^{-\mathrm{i}\pi\alpha}\big) \end{bmatrix} = \begin{bmatrix} P & \mathbf{0} \\ \mathbf{0} & P \end{bmatrix}\begin{bmatrix} \mathrm{e}^{A_1 z} & \mathrm{e}^{\widetilde{A}_1 z}\,\mathrm{e}^{-\mathrm{i}\pi J}\, z^{J} \\ \mathbf{0} & \mathrm{e}^{-\widetilde{A}_2 z}\big(\mathrm{e}^{\mathrm{i}\pi J} - \mathrm{e}^{-\mathrm{i}\pi J}\big) \end{bmatrix}\begin{bmatrix} P & \mathbf{0} \\ \mathbf{0} & P \end{bmatrix}^{-1}$$

with $\widetilde{A}_j = P^{-1} A_j P$, $j = 1, 2$.

Now, as we did in the previous case, with negative eigenvalues only, we left multiply by the following nonsingular matrix

$$S(z) := \begin{bmatrix} P & \mathbf{0} \\ \mathbf{0} & P \end{bmatrix} \begin{bmatrix} \begin{bmatrix} I_{N^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & z^{-J^-} \end{bmatrix} & \mathbf{0} \\ \begin{bmatrix} 0_{N^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & I_{N^-} - \mathrm{e}^{2\,\mathrm{i}\,\pi J^-} \end{bmatrix} & \begin{bmatrix} I_{N^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & z^{J^-} \end{bmatrix} \end{bmatrix} \begin{bmatrix} P & \mathbf{0} \\ \mathbf{0} & P \end{bmatrix}^{-1},$$

to get

$$\left[\begin{smallmatrix} P & \mathbf{0} \\ \mathbf{0} & P \end{smallmatrix}\right] \begin{bmatrix} \begin{bmatrix} I_{N^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & z^{-J^-} \end{bmatrix} \mathrm{e}^{\widetilde{A}_1 z} & \begin{bmatrix} I_{N^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & z^{-J^-} \end{bmatrix} \mathrm{e}^{\widetilde{A}_1 z} \begin{bmatrix} \mathrm{e}^{-\,\mathrm{i}\,\pi J^+} z^{J^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & \mathrm{e}^{-\,\mathrm{i}\,\pi J^-} z^{J^-} \end{bmatrix} \\ \begin{bmatrix} 0_{N^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & I_{N^-} - \mathrm{e}^{2\,\mathrm{i}\,\pi J^-} \end{bmatrix} \mathrm{e}^{\widetilde{A}_1 z} & \begin{matrix} \begin{bmatrix} 0_{N^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & I_{N^-} - \mathrm{e}^{2\,\mathrm{i}\,\pi J^-} \end{bmatrix} \mathrm{e}^{\widetilde{A}_1 z} \begin{bmatrix} \mathrm{e}^{-\,\mathrm{i}\,\pi J^+} z^{J^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & \mathrm{e}^{-\,\mathrm{i}\,\pi J^-} z^{J^-} \end{bmatrix} \\ + \begin{bmatrix} I_{N^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & z^{J^-} \end{bmatrix} \mathrm{e}^{-\widetilde{A}_2 z} \begin{bmatrix} \mathrm{e}^{\mathrm{i}\,\pi J^+} - \mathrm{e}^{-\,\mathrm{i}\,\pi J^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & \mathrm{e}^{\mathrm{i}\,\pi J^-} - \mathrm{e}^{-\,\mathrm{i}\,\pi J_-} \end{bmatrix} \end{matrix} \end{bmatrix} \left[\begin{smallmatrix} P & \mathbf{0} \\ \mathbf{0} & P \end{smallmatrix}\right]^{-1}$$

which for $z \to 0$ has a well defined limit, being a nonsingular matrix, given by

$$\begin{bmatrix} P & \mathbf{0} \\ \mathbf{0} & P \end{bmatrix} \begin{bmatrix} \begin{bmatrix} I_{N^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & 0_{N^-} \end{bmatrix} & \begin{bmatrix} 0_{N^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & \mathrm{e}^{-\,\mathrm{i}\,\pi J^-} \end{bmatrix} \\ \begin{bmatrix} 0_{N^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & I_{N^-} - \mathrm{e}^{2\,\mathrm{i}\,\pi J^-} \end{bmatrix} & \begin{bmatrix} \mathrm{e}^{\mathrm{i}\,\pi J^+} - \mathrm{e}^{-\,\mathrm{i}\,\pi J^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & 0_{N^-} \end{bmatrix} \end{bmatrix} \begin{bmatrix} P & \mathbf{0} \\ \mathbf{0} & P \end{bmatrix}^{-1}.$$

Thus,

$$F_n^{\mathsf{L}}(0) = \hat{Y}_n^{\mathsf{L}}(0) \begin{bmatrix} P & \mathbf{0} \\ \mathbf{0} & P \end{bmatrix} \begin{bmatrix} \begin{bmatrix} I_{N^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & 0_{N^-} \end{bmatrix} & \begin{bmatrix} 0_{N^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & \mathrm{e}^{-\,\mathrm{i}\,\pi J^-} \end{bmatrix} \\ \begin{bmatrix} 0_{N^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & I_{N^-} - \mathrm{e}^{2\,\mathrm{i}\,\pi J^-} \end{bmatrix} & \begin{bmatrix} \mathrm{e}^{\mathrm{i}\,\pi J^+} - \mathrm{e}^{-\,\mathrm{i}\,\pi J^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & 0_{N^-} \end{bmatrix} \end{bmatrix} \begin{bmatrix} P & \mathbf{0} \\ \mathbf{0} & P \end{bmatrix}^{-1}.$$

By definition,

$$M_n^{\mathsf{L}} = \big(Z_n^{\mathsf{L}}\big)'\big(Z_n^{\mathsf{L}}\big)^{-1} = \big(F_n^{\mathsf{L}}\big)'\big(F_n^{\mathsf{L}}\big)^{-1} + F_n^{\mathsf{L}}\psi'\psi^{-1}\big(F_n^{\mathsf{L}}\big)^{-1},$$

as $\det F_n^{\mathsf{L}}(z) \neq 0$, we know that $\big(F_n^{\mathsf{L}}\big)'\big(F_n^{\mathsf{L}}\big)^{-1}$ has no singularities, while

$$F_n^{\mathsf{L}}\psi'\psi^{-1}\big(F_n^{\mathsf{L}}\big)^{-1} = \frac{1}{z}F_n^{\mathsf{L}} \begin{bmatrix} \frac{\alpha}{2} & \mathbf{0} \\ \mathbf{0} & -\frac{\alpha}{2} \end{bmatrix} \big(F_n^{\mathsf{L}}\big)^{-1}.$$

Consequently, M_n^{L} has a simple pole at the origin with

$$M_n^{\mathsf{L}}(z) = \frac{1}{z}F_n^{\mathsf{L}}(0) \begin{bmatrix} \frac{\alpha}{2} & \mathbf{0} \\ \mathbf{0} & -\frac{\alpha}{2} \end{bmatrix} \big(F_n^{\mathsf{L}}(0)\big)^{-1} + \mathrm{O}(1), \qquad z \to 0.$$

Let us move to the proof of the second case, *i.e.* $\alpha = m\,\mathrm{I}$, $m \in \mathbb{N}$.

It can be seen that the matrix function Z_n^{L} satisfies over $(0, +\infty)$ the following jump condition

$$\big(Z_n^{\mathsf{L}}(z)\big)_+ = \big(Z_n^{\mathsf{L}}(z)\big)_- \begin{bmatrix} (-1)^m\,\mathrm{I} & (-1)^m\,\mathrm{I} \\ 0 & (-1)^m\,\mathrm{I} \end{bmatrix}.$$

For $z \in \mathbb{C} \setminus [0, +\infty)$, instead of (4.3), let us define the matrix

$$\psi(z) := \begin{bmatrix} z^{\frac{m}{2}}\,\mathrm{I} & -\frac{1}{2\pi\,\mathrm{i}} z^{\frac{m}{2}} \log(z)\,\mathrm{I} \\ 0 & z^{-\frac{m}{2}}\,\mathrm{I} \end{bmatrix},$$

where we take the branch of the logarithmic function defined in $\mathbb{C} \setminus [0, +\infty)$, which satisfies, over $(0, +\infty)$, the same jump condition

$$\big(\psi(z)\big)_+ = \big(\psi(z)\big)_- \begin{bmatrix} (-1)^m\,\mathrm{I} & (-1)^m\,\mathrm{I} \\ 0 & (-1)^m\,\mathrm{I} \end{bmatrix}.$$

Consequently, the matrix

$$F_n^{\mathsf{L}}(z) := Z_n^{\mathsf{L}}(z)\psi^{-1}(z)$$

has no jump in the interval $(0, +\infty)$. The matrix function F_n^{L} has an isolated singularity at the origin which, as we will show now, is a removable one, *i.e.*

$$\begin{aligned} zF_n^{\mathsf{L}}(z) &= \begin{bmatrix} \mathrm{O}(z) & zs_1^{\mathsf{L}}(z) \\ \mathrm{O}(z) & zs_2^{\mathsf{L}}(z) \end{bmatrix} \begin{bmatrix} \mathrm{O}(1) & \mathbf{0} \\ \mathbf{0} & \mathrm{O}(1) \end{bmatrix} \begin{bmatrix} \mathrm{O}(1) & \mathrm{O}(\log(z)) \\ \mathrm{O}(1) & \mathrm{O}(1) \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{O}(z) + zs_1^{\mathsf{L}}(z) & \mathrm{O}(z\log(z)) + zs_1^{\mathsf{L}}(z) \\ \mathrm{O}(z) + zs_2^{\mathsf{L}}(z) & \mathrm{O}(z\log(z)) + zs_2^{\mathsf{L}}(z) \end{bmatrix}, \qquad z \to 0, \end{aligned}$$

and as zs_1^{L}, $zs_2^{\mathsf{L}} \to \mathbf{0}$ as $z \to 0$, we conclude that $zF_n^{\mathsf{L}}(z) \to \mathbf{0}$, as $z \to 0$. Hence, $F_n^{\mathsf{L}}(z)$ is a matrix of entire functions. To compute $F_n^{\mathsf{L}}(0)$ we notice that,

$$F_n^{\mathsf{L}}(0) = \lim_{z\to 0} Y_n^{\mathsf{L}}(z) \begin{bmatrix} \mathrm{e}^{A_1 z} & \frac{1}{2\pi\,\mathrm{i}} z^m \log(z)\,\mathrm{e}^{A_1 z} \\ \mathbf{0} & \mathrm{e}^{-A_2 z} \end{bmatrix}.$$

For $m = 1, 2, \dots$ it holds that $F_n^{\mathsf{L}}(0) = Y_n^{\mathsf{L}}(0)$. For $m = 0$ the limit of each factor inside the limit does not need to exist. As the limit exists, let us write

$$Y_n^{\mathsf{L}}(z) \begin{bmatrix} \mathrm{e}^{A_1 z} & \frac{1}{2\pi\,\mathrm{i}} \log(z)\,\mathrm{e}^{A_1 z} \\ \mathbf{0} & \mathrm{e}^{-A_2 z} \end{bmatrix} = \hat{Y}_n^{\mathsf{L}}(z) f(z),$$

with

$$\begin{aligned} \hat{Y}_n^{\mathsf{L}}(z) &:= Y_n^{\mathsf{L}}(z) \begin{bmatrix} (\log(z))^{-1}\,\mathrm{I} & \mathbf{0} \\ -2\pi\,\mathrm{i}\,\mathrm{I} & \log(z)\,\mathrm{I} \end{bmatrix}^{-1}, \\ f(z) &:= \begin{bmatrix} (\log(z))^{-1}\,\mathrm{I} & \mathbf{0} \\ -2\pi\,\mathrm{i}\,\mathrm{I} & \log(z)\,\mathrm{I} \end{bmatrix} \begin{bmatrix} \mathrm{e}^{A_1 z} & \frac{1}{2\pi\,\mathrm{i}} \log(z)\,\mathrm{e}^{A_1 z} \\ \mathbf{0} & \mathrm{e}^{-A_2 z} \end{bmatrix} \\ &= \begin{bmatrix} (\log(z))^{-1}\,\mathrm{e}^{A_1 z} & \frac{1}{2\pi\,\mathrm{i}}\,\mathrm{e}^{A_1 z} \\ -2\pi\,\mathrm{i}\,\mathrm{e}^{A_1 z} & -\log(z)(\mathrm{e}^{A_1 z} - \mathrm{e}^{-A_2 z}) \end{bmatrix}. \end{aligned}$$

So that,

$$\lim_{z\to 0} f(z) = \begin{bmatrix} \mathbf{0} & \frac{1}{2\pi\,\mathrm{i}}\,\mathrm{I} \\ -2\pi\,\mathrm{i}\,\mathrm{I} & \mathbf{0} \end{bmatrix}, \qquad F_n^{\mathsf{L}}(0) = \hat{Y}_n^{\mathsf{L}}(0) \begin{bmatrix} \mathbf{0} & \frac{1}{2\pi\,\mathrm{i}}\,\mathrm{I} \\ -2\pi\,\mathrm{i}\,\mathrm{I} & \mathbf{0} \end{bmatrix}.$$

Using the same kind of reasoning as above we get that, M_n^{L} has a simple pole at the origin with

$$M_n^{\mathsf{L}}(z) = \frac{1}{z} F_n^{\mathsf{L}}(0) \begin{bmatrix} \frac{m}{2}\,\mathrm{I} & -\frac{z^m}{2\pi\,\mathrm{i}}\,\mathrm{I} \\ \mathbf{0} & -\frac{m}{2}\,\mathrm{I} \end{bmatrix} \big(F_n^{\mathsf{L}}(0)\big)^{-1} + \mathrm{O}(1), \qquad z \to 0,$$

which ends the proof. □

Theorem 4.4. *The structure matrix M_n^{L} has the yielding expression*

$$M_n^{\mathsf{L}}(z) = \frac{1}{z} \begin{bmatrix} A_1 z + [p^1_{\mathsf{L},n}, A_1] + n\,\mathrm{I} + \frac{\alpha}{2} & A_1 C_n^{-1} + C_n^{-1} A_2 \\ -C_{n-1} A_1 - A_2 C_{n-1} & -A_2 z + [p^1_{\mathsf{R},n}, A_2] - n\,\mathrm{I} - \frac{\alpha}{2} \end{bmatrix}.$$

Proof. Substituting A^{L}, A^{R}, respectively by A_1, A_2, and B^{L}, B^{R} by $\frac{\alpha}{2}$ in (4.1) and (4.2) we get the result. □

Theorem 4.5. *Let α, A_1 and A_2, such that $[\alpha, A_1] = [\alpha, A_2] = \mathbf{0}$, and the real part of spectrum of α, $\sigma(\alpha)$, is contained on $(-1, +\infty)$ with $\sigma(\alpha) \cap \{\mathbb{N}\} = \emptyset$. If there exists $\lambda \in (0, +\infty)$ such that $\alpha^2 = \lambda\,\mathrm{I}$, or $\alpha = m\,\mathrm{I}$, for some $m \in \{0, 1, 2, \dots\}$, then the second order differential equation is simplified to*

$$z\big(Y_n^{\mathsf{L}}\big)'' + \big(Y_n^{\mathsf{L}}\big)' \begin{bmatrix} \alpha + \mathrm{I} + 2A_1 z & \mathbf{0} \\ \mathbf{0} & \mathrm{I} - \alpha - 2A_2 z \end{bmatrix} + Y_n^{\mathsf{L}} \begin{bmatrix} A_1 + A_1\alpha + {A_1}^2 z & \mathbf{0} \\ \mathbf{0} & -A_2 + A_2\alpha + {A_2}^2 z \end{bmatrix}$$
$$= \begin{bmatrix} A_1 + [p^1_{\mathsf{L},n}, A_1^2] + (n\,\mathrm{I} + \alpha) A_1 + A_1^2 z & A_1^2 C_n^{-1} - C_n^{-1} A_2^2 \\ -C_{n-1} A_1^2 + A_2^2 C_{n-1} & -A_2 - [p^1_{\mathsf{R},n}, A_2^2] + (n\,\mathrm{I} + \alpha) A_2 + A_2^2 z \end{bmatrix} Y_n^{\mathsf{L}}(z).$$

Proof. If we take into account that $\widetilde{M}_n^{\mathsf{L}}(z) = \widetilde{M}_n^{\mathsf{L}}(0) + z(\widetilde{M}_n^{\mathsf{L}})'(0)$ and that

$$\mathcal{N}(\widetilde{M}_n^{\mathsf{L}}(z)) = (\widetilde{M}_n^{\mathsf{L}})'(0) + (\widetilde{M}_n^{\mathsf{L}}(0))^2 \frac{1}{z} + (\widetilde{M}_n^{\mathsf{L}})'(0)\widetilde{M}_n^{\mathsf{L}}(0) + \widetilde{M}_n^{\mathsf{L}}(0)(\widetilde{M}_n^{\mathsf{L}})'(0) + \big((\widetilde{M}_n^{\mathsf{L}})'(0)\big)^2 z,$$

we get that (3.11), the second order differential equation that the fundamental matrix satisfies, can be written as

$$z\big(Y_n^{\mathsf{L}}\big)'' + \big(Y_n^{\mathsf{L}}\big)' \begin{bmatrix} \alpha + \mathrm{I} + 2A_1 z & \mathbf{0} \\ \mathbf{0} & \mathrm{I} - \alpha - 2A_2 z \end{bmatrix}$$
$$+ Y_n^{\mathsf{L}} \begin{bmatrix} A_1 + \frac{1}{2}A_1\alpha + \frac{1}{2}\alpha A_1 + z{A_1}^2 & \mathbf{0} \\ \mathbf{0} & -A_2 + \frac{1}{2}A_2\alpha + \frac{1}{2}\alpha A_2 + z{A_2}^2 \end{bmatrix} + \frac{1}{z} Y_n^{\mathsf{L}} \begin{bmatrix} (\frac{\alpha}{2})^2 & \mathbf{0} \\ \mathbf{0} & (\frac{\alpha}{2})^2 \end{bmatrix}$$
$$= \Big((\widetilde{M}_n^{\mathsf{L}})'(0) + (\widetilde{M}_n^{\mathsf{L}}(0))^2 \frac{1}{z} + (\widetilde{M}_n^{\mathsf{L}})'(0)\widetilde{M}_n^{\mathsf{L}}(0) + \widetilde{M}_n^{\mathsf{L}}(0)(\widetilde{M}_n^{\mathsf{L}})'(0) + \big((\widetilde{M}_n^{\mathsf{L}})'(0)\big)^2 z\Big) Y_n^{\mathsf{L}}(z).$$

Under the restriction that the real part of the spectrum of α is contained on $(-1, +\infty)$ and $\sigma(\alpha) \cap \{\mathbb{N}\} = \emptyset$ the matrix $M_n^{\mathsf{L}} = \big(Z_n^{\mathsf{L}}\big)'\big(Z_n^{\mathsf{L}}\big)^{-1}$ has a pole of order 1 at $z = 0$, with residue given by

$$\widetilde{M}_n^{\mathsf{L}}(0) = F_n^{\mathsf{L}}(0) \begin{bmatrix} \frac{\alpha}{2} & \mathbf{0} \\ \mathbf{0} & -\frac{\alpha}{2} \end{bmatrix} \big(F_n^{\mathsf{L}}(0)\big)^{-1}.$$

If we now also assume on the matrix α that $\alpha^2 = \lambda\,\mathrm{I}$, we get

$$(\widetilde{M}^{\mathsf{L}}_n(0))^2 = F^{\mathsf{L}}_n(0)\begin{bmatrix}\left(\frac{\alpha}{2}\right)^2 & \mathbf{0}\\ \mathbf{0} & \left(\frac{\alpha}{2}\right)^2\end{bmatrix}\big(F^{\mathsf{L}}_n(0)\big)^{-1} = \frac{\lambda}{4}\,\mathrm{I}\,.$$

In the case that $\alpha = m\,\mathrm{I}$, for some $m\in\mathbb{N}$, we get that

$$(\widetilde{M}^{\mathsf{L}}_n(0))^2 = F^{\mathsf{L}}_n(0)\begin{bmatrix}\left(\frac{m}{2}\right)^2 & \mathbf{0}\\ \mathbf{0} & \left(\frac{m}{2}\right)^2\end{bmatrix}\big(F^{\mathsf{L}}_n(0)\big)^{-1} = \frac{m^2}{4}\,\mathrm{I}$$

In both cases we have

$$\begin{aligned} z\big(Y^{\mathsf{L}}_n\big)'' + \big(Y^{\mathsf{L}}_n\big)'&\begin{bmatrix}\alpha+\mathrm{I}+2A_1z & \mathbf{0}\\ \mathbf{0} & \mathrm{I}-\alpha-2A_2z\end{bmatrix}\\ &+Y^{\mathsf{L}}_n\begin{bmatrix}A_1+A_1\alpha+{A_1}^2z & \mathbf{0}\\ \mathbf{0} & -A_2+A_2\alpha+{A_2}^2z\end{bmatrix}\\ &=\Big((\widetilde{M}^{\mathsf{L}}_n)'(0)+(\widetilde{M}^{\mathsf{L}}_n)'(0)\widetilde{M}^{\mathsf{L}}_n(0)+\widetilde{M}^{\mathsf{L}}_n(0)(\widetilde{M}^{\mathsf{L}}_n)'(0)+\big((\widetilde{M}^{\mathsf{L}}_n)'(0)\big)^2z\Big)Y^{\mathsf{L}}_n(z). \end{aligned}$$

and substituting

$$\widetilde{M}^{\mathsf{L}}_n(z) = \begin{bmatrix}A_1z+[p^1_{\mathsf{L},n},A_1]+n\,\mathrm{I}+\frac{\alpha}{2} & A_1C_n^{-1}+C_n^{-1}A_2\\ -C_{n-1}A_1-A_2C_{n-1} & -A_2z+[p^1_{\mathsf{R},n},A_2]-n\,\mathrm{I}-\frac{\alpha}{2}\end{bmatrix}.$$

the result follows. □

Remark 4.2. *We remark that if the spectrum of α is contained in $(-1,+\infty)\setminus\mathbb{Z}_+$ when $|\lambda|<1$ the $\pm\lambda$ are admissible eigenvalues for α, and when $|\lambda|>1$ only positive and bigger than 1 eigenvalues are admissible for α, and then $\alpha=\lambda\,\mathrm{I}$.*

Corollary 4.2. *Let us consider $N=1$ (i.e the scalar case). If $A_1=A_2=-\frac{1}{2}$, and $\alpha>-1$ then the second order equation for $\big\{P^{\mathsf{L}}_n\big\}_{n\in\mathbb{N}}$ and $\big\{Q^{\mathsf{L}}_n\big\}_{n\in\mathbb{N}}$ is given by*

$$\begin{aligned} zP_n''(z)-(z-\alpha-1)P_n'(z)&=-nP_n(z),\\ zQ_n''(z)+(z-\alpha+1)Q_n'(z)&=-(n+1)Q_n(z). \end{aligned}$$

Proof. In the scalar case this equation reduces to

$$\begin{aligned} z\big(Y^{\mathsf{L}}_n\big)''+\big(Y^{\mathsf{L}}_n\big)'\begin{bmatrix}\alpha+1+2A_1z & 0\\ 0 & 1-\alpha-2A_1z\end{bmatrix}+Y^{\mathsf{L}}_n\begin{bmatrix}A_1+A_1\alpha+{A_1}^2z & 0\\ 0 & -A_1+A_1\alpha+{A_1}^2z\end{bmatrix}\\ =\begin{bmatrix}A_1+(n+\alpha)A_1+A_1^2z & 0\\ 0 & -A_1+(n+\alpha)A_1+A_1^2z\end{bmatrix}Y^{\mathsf{L}}_n(z), \end{aligned}$$

as $A_1^2C_n^{-1}=C_n^{-1}A_1^2$ and $A_1=A_2=-\frac{1}{2}$, and so

$$z\big(Y^{\mathsf{L}}_n\big)''+\big(Y^{\mathsf{L}}_n\big)'\begin{bmatrix}\alpha+1-z & 0\\ 0 & 1-\alpha+z\end{bmatrix}+Y^{\mathsf{L}}_n\begin{bmatrix}-\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{bmatrix}=\begin{bmatrix}-\frac{n+1}{2} & 0\\ 0 & -\frac{n-1}{2}\end{bmatrix}Y^{\mathsf{L}}_n(z),$$

now, considering the $(1,1)$ and the $(1,2)$ entry of this differential matrix equation the result follows. □

2.3. Jacobi case.

Theorem 4.6. *In the conditions of Theorem 3.14. If* $h^{\mathsf{L}}(z)=A^{\mathsf{L}}z+B^{\mathsf{L}}$ *and* $h^{\mathsf{R}}(z)=A^{\mathsf{R}}z+B^{\mathsf{R}}$, *then the left and right fundamental matrices are given respectively by,*

$$\widetilde{M}^{\mathsf{L}}_n(z)=\begin{bmatrix}\big(A^{\mathsf{L}}-n\,\mathrm{I}\big)\,z+[p^1_{\mathsf{L},n},A^{\mathsf{L}}]+p^1_{\mathsf{L},n}+n\,\mathrm{I}+B^{\mathsf{L}} & A^{\mathsf{L}}C_n^{-1}+C_n^{-1}A^{\mathsf{R}}-(2n+1)C_n^{-1}\\ -C_{n-1}A^{\mathsf{L}}-A^{\mathsf{R}}C_{n-1}+(2n-1)C_{n-1} & \big(n\,\mathrm{I}-A^{\mathsf{R}}\big)\,z+[p^1_{\mathsf{R},n},A^{\mathsf{R}}]-p^1_{\mathsf{R},n}-n\,\mathrm{I}-B^{\mathsf{R}}\end{bmatrix},\tag{4.4}$$

$$\widetilde{M}^{\mathsf{R}}_n(z)=\begin{bmatrix}\big(A^{\mathsf{R}}-n\,\mathrm{I}\big)\,z-[p^1_{\mathsf{R},n},A^{\mathsf{R}}]+p^1_{\mathsf{R},n}+n\,\mathrm{I}+B^{\mathsf{R}} & -C_{n-1}A^{\mathsf{L}}-A^{\mathsf{R}}C_{n-1}+(2n-1)C_{n-1}\\ A^{\mathsf{L}}C_n^{-1}+C_n^{-1}A^{\mathsf{R}}-(2n+1)C_n^{-1} & \big(n\,\mathrm{I}-A^{\mathsf{L}}\big)\,z-[p^1_{\mathsf{L},n},A^{\mathsf{L}}]-p^1_{\mathsf{L},n}-n\,\mathrm{I}-B^{\mathsf{L}}\end{bmatrix}.\tag{4.5}$$

Proof. Taking $|z|\to+\infty$ in (3.29) we have that

$$Y^{\mathsf{L}}_n=\begin{bmatrix}\mathrm{I}\,z^n+p^1_{\mathsf{L},n}z^{n-1}+\cdots & -C_n^{-1}\big(\mathrm{I}\,z^{-n-1}+q^1_{\mathsf{L},n}z^{-n-2}+\cdots\big)\\ -C_{n-1}\big(\mathrm{I}\,z^{n-1}+p^1_{\mathsf{L},n-1}z^{n-2}+\cdots\big) & \mathrm{I}\,z^{-n-2}+q^1_{\mathsf{L},n-1}z^{-n-3}+\cdots\end{bmatrix},$$

$$\big(Y^{\mathsf{L}}_n\big)^{-1}=\begin{bmatrix}\mathrm{I}\,z^{-n-2}+q^1_{\mathsf{R},n-1}z^{-n-3}+\cdots & \big(\mathrm{I}\,z^{-n-1}+q^1_{\mathsf{R},n}z^{-n-2}+\cdots\big)\,C_n^{-1}\\ \big(\mathrm{I}\,z^{n-1}+p^1_{\mathsf{R},n-1}z^{n-2}+p^2_{\mathsf{R},n-1}z^{n-3}+\cdots\big)\,C_{n-1} & \mathrm{I}\,z^n+p^1_{\mathsf{R},n}z^{n-1}+p^2_{\mathsf{R},n}z^{n-2}+\cdots\end{bmatrix}.$$

Hence, as $|z|\to+\infty$

$$\begin{aligned}&z(1-z)\big(Y^{\mathsf{L}}_n\big)'\big(Y^{\mathsf{L}}_n\big)^{-1}\\&=\begin{bmatrix}-n\,\mathrm{I}\,z+n\,\mathrm{I}-(nq^1_{\mathsf{R},n-1}+(n-1)p^1_{\mathsf{L},n}) & -(2n+1)C_n^{-1}\\ (2n-1)C_{n-1} & n\,\mathrm{I}\,z-n\,\mathrm{I}+np^1_{\mathsf{R},n}+(n+1)q^1_{\mathsf{L},n-1}\end{bmatrix}+\mathrm{O}(1/z),\\&Y^{\mathsf{L}}_n(z)\begin{bmatrix}h^{\mathsf{L}}(z) & \mathbf{0}\\ \mathbf{0} & -h^{\mathsf{R}}(z)\end{bmatrix}\big(Y^{\mathsf{L}}_n(z)\big)^{-1}\\&=\begin{bmatrix}A^{\mathsf{L}}z+A^{\mathsf{L}}q^1_{\mathsf{R},n-1}+p^1_{\mathsf{L},n}A^{\mathsf{L}}+B^{\mathsf{L}} & A^{\mathsf{L}}C_n^{-1}+C_n^{-1}A^{\mathsf{R}}\\ C_{n-1}A^{\mathsf{L}}+A^{\mathsf{R}}C_{n-1} & -A^{\mathsf{R}}z-A^{\mathsf{R}}p^1_{\mathsf{R},n}-q^1_{\mathsf{L},n-1}A^{\mathsf{R}}-B^{\mathsf{R}}\end{bmatrix}+\mathrm{O}(1/z).\end{aligned}$$

Using the Liouville Theorem for $\widetilde{M}^{\mathsf{L}}_n$, and by considering the identities $p^1_{\mathsf{R},n}=-q^1_{\mathsf{L},n-1}$ and $p^1_{\mathsf{L},n}=-q^1_{\mathsf{R},n-1}$, then (4.4) follows. The relation (3.4) leads to (4.5). □

Using the calculation made in (4.4) We want to recover here some known formulas in the scalar case.

Example 4.1. *Let us consider the weight* $W(z)=z^\alpha(1-z)^\beta$, *with* α, β *scalars in* $(-1,\infty)$. *Then, the scalar second order equation for* $\{P^{\mathsf{L}}_n\}_{n\in\mathbb{N}}$ *and* $\{Q^{\mathsf{L}}_n\}_{n\in\mathbb{N}}$ (cf. *for example [87]) is given by*

$$z(1-z)P''_n(z)+\big(1+\alpha-(\alpha+\beta+2)z\big)P'_n(z)+n(\alpha+\beta+n+1)P_n(z)=0,\tag{4.6}$$

$$z(1-z)Q''_n(z)+\big(1-\alpha+(\alpha+\beta-2)z\big)Q'_n(z)+(n+1)(\alpha+\beta+n)Q_n(z)=0.\tag{4.7}$$

In fact, from (4.4)

$$\widetilde{M}^{\mathsf{L}}_n(z)=\begin{bmatrix}-(\frac{\alpha+\beta}{2}+n)z+p^1_n+n+\frac{\alpha}{2} & -C_n^{-1}(\alpha+\beta+2n+1)\\ C_{n-1}(\alpha+\beta+2n-1) & (\frac{\alpha+\beta}{2}+n)z-p^1_n-n-\frac{\alpha}{2}\end{bmatrix},$$

it is easy to see that

$$\big(\widetilde{M}_n^{\mathsf{L}}(z)\big)^2 = \left(-(\frac{\alpha+\beta}{2}+n)z + p_n^1 + n + \frac{\alpha}{2}\right)^2 - \gamma_n\left((\alpha+\beta+2n)^2-1\right)\mathrm{I},$$

and also, $\big(\widetilde{M}_n^{\mathsf{L}}(z)\big)' = \begin{bmatrix} -\frac{\alpha+\beta}{2}+n & 0 \\ 0 & \frac{\alpha+\beta}{2}-n \end{bmatrix}$. Using now Theorem 3.16 we get

$$\begin{aligned} &z(1-z)P_n''(z) + \big(1+\alpha-(\alpha+\beta+2)z\big)P_n'(z) - n(\alpha+\beta+n+1)P_n(z) \\ &\quad = \left(\frac{n(\alpha+n)+p_n^1(\alpha+\beta+2n)}{1-z} + \frac{\frac{\alpha^2}{4} - \big(p_n^1+\frac{\alpha}{2}+n\big)^2 + \gamma_n\left((\alpha+\beta+2n)^2-1\right)}{z(1-z)}\right)P_n(z). \end{aligned}$$

By equalizing poles between left and right hand side on 0 then on 1 we have

$$\left(\frac{\alpha^2}{4} - \left(p_n^1+\frac{\alpha}{2}+n\right)^2 + \gamma_n\left((\alpha+\beta+2n)^2-1\right)\right)P_n(0) = 0$$

$$\big(n(\alpha+n)+p_n^1(\alpha+\beta+2n)\big)P_n(1) = 0$$

which, taking into account $P_n(0)$, $P_n(1) \neq 0$, leads to the representation of p_n^1 and γ_n, as well as (4.6). The equation (4.7) for the $\{Q_n\}_{n\in\mathbb{N}}$ follows from the above considerations.

2.4. Bessel case. Here we want to give explicitly the structure matrix when a Pearson equation is satisfied with polynomial coefficients of degree one. Inside this class we can consider $W(z) = W^{\mathsf{L}}(z)W^{\mathsf{R}}(z)$, with

$$W^{\mathsf{L}}(z) = Z^{\alpha^{\mathsf{L}}}\,\mathrm{e}^{-\frac{\beta^{\mathsf{L}}}{z}} \qquad \text{and} \qquad W^{\mathsf{R}}(z) = Z^{\alpha^{\mathsf{R}}}\,\mathrm{e}^{-\frac{\beta^{\mathsf{R}}}{z}},$$

where $[\alpha^{\mathsf{L}},\beta^{\mathsf{L}}] = \mathbf{0}$, $[\alpha^{\mathsf{R}},\beta^{\mathsf{R}}] = \mathbf{0}$, $\operatorname{Re}\left(\sigma\left(\alpha^{\mathsf{L,R}}\right)\right) > -1$ and $\operatorname{Re}\left(\sigma\left(\beta^{\mathsf{L,R}}\right)\right) > 0$.

Theorem 4.7. *If $h^{\mathsf{L}}(z) = A^{\mathsf{L}}z + B^{\mathsf{L}}$ and $h^{\mathsf{R}}(z) = A^{\mathsf{R}}z + B^{\mathsf{R}}$, then the left and right fundamental matrices are given respectively by,*

$$\widetilde{M}_n^{\mathsf{L}}(z) = \begin{bmatrix} \left(A^{\mathsf{L}}+n\,\mathrm{I}\right)z - p_{\mathsf{L},n}^1 + B^{\mathsf{L}} & A^{\mathsf{L}}C_n^{-1} + C_n^{-1}A^{\mathsf{R}} + (2n+1)C_n^{-1} \\ -C_{n-1}A^{\mathsf{L}} - A^{\mathsf{R}}C_{n-1} - (2n-1)C_{n-1} & -\left(n\,\mathrm{I}+A^{\mathsf{R}}\right)z + p_{\mathsf{R},n}^1 - B^{\mathsf{R}} \end{bmatrix}, \tag{4.8}$$

$$\widetilde{M}_n^{\mathsf{R}}(z) = \begin{bmatrix} \left(n\,\mathrm{I}+A^{\mathsf{R}}\right)z - p_{\mathsf{R},n}^1 + B^{\mathsf{R}} & -C_{n-1}A^{\mathsf{L}} - A^{\mathsf{R}}C_{n-1} - (2n-1)C_{n-1} \\ A^{\mathsf{L}}C_n^{-1} + C_n^{-1}A^{\mathsf{R}} + (2n+1)C_n^{-1} & -\left(A^{\mathsf{L}}+n\,\mathrm{I}\right)z + p_{\mathsf{L},n}^1 - B^{\mathsf{L}} \end{bmatrix}.$$

Proof. Very similar to the proof of (4.4) and (4.5). □

We aim to utilize the previously performed computation (4.8) to derive familiar formulas in the context of scalar scenarios.

Example 4.2. *Let us consider the weight $W(z) = z^{a-2}\,\mathrm{e}^{\frac{-b}{z}}$, with a is not a negative integer or zero and b is not zero. Then, the scalar second order equation for $\left\{P_n^{\mathsf{L}}\right\}_{n\in\mathbb{N}}$ (cf. for example [73]) and $\left\{Q_n^{\mathsf{L}}\right\}_{n\in\mathbb{N}}$ is given by*

$$z^2P_n''(z) + \big(az+b\big)P_n'(z) - n(a+n-1)P_n(z) = 0, \tag{4.9}$$

$$z^2Q_n''(z)+\big((4-a)z-b\big)Q_n'(z)-\big(n(a+n-1)+2-a\big)Q_n(z)=0. \tag{4.10}$$

Proof. If we consider $W^{\mathsf L}=W^{\mathsf R}=z^{\frac a2-1}\,\mathrm e^{\frac{-b}{2z}}$ then $W(z)=W^{\mathsf L}W^{\mathsf R}=z^{a-2}\,\mathrm e^{\frac{-b}{z}}$. In fact, from (4.8)

$$\widetilde{M}_n^{\mathsf L}(z)=\begin{bmatrix}(\frac{a-2}{2}+n)z-p_n^1+\frac b2 & C_n^{-1}(a+2n-1)\\ -C_{n-1}(a+2n-3) & -(\frac{a-2}{2}+n)z+p_n^1-\frac b2\end{bmatrix},$$

it is easy to see that

$$\big(\widetilde{M}_n^{\mathsf L}(z)\big)^2=\left((\frac{a-2}{2}+n)z-p_n^1+\frac b2\right)^2-\gamma_n\left((a+2n-2)^2-1\right)\mathrm I,$$

and also, $\big(\widetilde{M}_n^{\mathsf L}\big)'(z)=\begin{bmatrix}\frac{a-2}{2}+n & 0\\ 0 & -\frac{a-2}{2}+n\end{bmatrix}$. Using now Theorem 3.20 we get

$$\begin{aligned}&z^2P_n''(z)+\big(az+b\big)P_n'(z)-n(a+n-1)P_n(z)\\&\quad=\left(\frac{nb-p_n^1(a+2n-2)}{z}+\frac{p_n^1\left(p_n^1-b\right)-\gamma_n\left((a+2n-2)^2-1\right)}{z^2}\right)P_n(z).\end{aligned}$$

By equalizing poles between left and right hand side on 0 we obtain

$$\Big(p_n^1\left(p_n^1-b\right)-\gamma_n\left((a+2n-2)^2-1\right)\Big)P_n(0)=0$$

which, taking into account $P_n(0)\neq 0$, then

$$\left(nb-p_n^1(a+2n-2)\right)P_n(0)=0$$

leads to the representation of p_n^1 and γ_n, as well as (4.9). The equation (4.10) for the $\{Q_n\}_{n\in\mathbb N}$ follows from the above considerations. □

3. Nonlinear difference equations for the recursion coefficients

Using the Riemann–Hilbert approach we will derive in this section nonlinear matrix difference equations fulfilled by the recursion coefficients, some of them are identified as a non-abelian extension of Painlevé scalar equations.

3.1. Hermite case. The results in this subsection are taken from [15]. We now explore the simplest case when $\max\big\{\deg h^{\mathsf L}(z),\ \deg h^{\mathsf R}(z)\big\}=1$ in full generality. We take

$$h^{\mathsf L}(z)=A^{\mathsf L}z+B^{\mathsf L}, \qquad h^{\mathsf R}(z)=A^{\mathsf R}z+B^{\mathsf R},$$

for arbitrary matrices $A^{\mathsf L},B^{\mathsf L},A^{\mathsf R},B^{\mathsf R}\in\mathbb C^{N\times N}$, with $A^{\mathsf L},A^{\mathsf R}$ definite negative matrices. Thus, the weight matrix W is a solution of the following Pearson equation

$$W'(z)=(A^{\mathsf L}z+B^{\mathsf L})W(z)+W(z)(A^{\mathsf R}z+B^{\mathsf R}).$$

For simplicity we take $\gamma=\mathbb R$. Hence, the structure matrices have the following form

$$M_n^{\mathsf L}(z)=\mathcal A^{\mathsf L}z+\mathcal K_n^{\mathsf L},\qquad \mathcal A^{\mathsf L}=\begin{bmatrix}A^{\mathsf L} & \mathbf 0\\ \mathbf 0 & -A^{\mathsf R}\end{bmatrix},\qquad \mathcal K_n^{\mathsf L}=\begin{bmatrix}B^{\mathsf L}+\left[p_{\mathsf L,n}^1,A^{\mathsf L}\right] & C_n^{-1}A^{\mathsf R}+A^{\mathsf L}C_n^{-1}\\ -C_{n-1}A^{\mathsf L}-A^{\mathsf R}C_{n-1} & -B^{\mathsf R}-\left[q_{\mathsf L,n-1}^1,A^{\mathsf R}\right]\end{bmatrix},$$

The Sylvester differential system (3.8) for the left fundamental matrix is

$$\big(Y^{\mathsf{L}}_n(z)\big)' + \Big[Y^{\mathsf{L}}_n(z), \Big[\begin{smallmatrix} A^{\mathsf{L}}z+B^{\mathsf{L}} & \mathbf{0}\\ \mathbf{0} & -A^{\mathsf{R}}z-B^{\mathsf{R}}\end{smallmatrix}\Big]\Big] = \begin{bmatrix} \big[p^1_{\mathsf{L},n},A^{\mathsf{L}}\big] & C_n^{-1}A^{\mathsf{R}}+A^{\mathsf{L}}C_n^{-1}\\ -C_{n-1}A^{\mathsf{L}}-A^{\mathsf{R}}C_{n-1} & -\big[q^1_{\mathsf{L},n-1},A^{\mathsf{R}}\big]\end{bmatrix} Y_n(z), \quad n\in\mathbb{Z}_+,$$

that is, for all $n \in \mathbb{Z}_+$,

$$(P^{\mathsf{L}}_n)' + \Big[P^{\mathsf{L}}_n, A^{\mathsf{L}}z + B^{\mathsf{L}}\Big] = \big[p^1_{\mathsf{L},n}, A^{\mathsf{L}}\big]P^{\mathsf{L}}_n - \big(C_n^{-1}A^{\mathsf{R}} + A^{\mathsf{L}}C_n^{-1}\big)C_{n-1}P^{\mathsf{L}}_{n-1}, \tag{4.11}$$

$$\begin{aligned} C_{n-1}(Q^{\mathsf{L}}_{n-1})' - \Big[C_{n-1}Q^{\mathsf{L}}_{n-1}, A^{\mathsf{R}}z + B^{\mathsf{R}}\Big]& \\ = \big(C_{n-1}A^{\mathsf{L}} + A^{\mathsf{R}}C_{n-1}\big)Q^{\mathsf{L}}_n &- \big[q^1_{n-1}, A^{\mathsf{R}}\big]C_{n-1}Q^{\mathsf{L}}_{n-1},\end{aligned} \tag{4.12}$$

$$\begin{aligned} C_{n-1}(P^{\mathsf{L}}_{n-1})' + C_{n-1}P_{n-1}\big(A^{\mathsf{L}}z + B^{\mathsf{L}}\big) + \big(A^{\mathsf{R}}z + B^{\mathsf{R}}\big)\, C_{n-1}P^{\mathsf{L}}_{n-1}& \\ = \big(C_{n-1}A^{\mathsf{L}} + A^{\mathsf{R}}C_{n-1}\big)P^{\mathsf{L}}_n &- \big[q^1_{\mathsf{L},n-1}, A^{\mathsf{R}}\big]C_{n-1}P^{\mathsf{L}}_{n-1},\end{aligned} \tag{4.13}$$

$$\begin{aligned} (Q^{\mathsf{L}}_n)' - Q^{\mathsf{L}}_n\big(A^{\mathsf{R}}z + B^{\mathsf{R}}\big) - \big(A^{\mathsf{L}}z + B^{\mathsf{L}}\big)Q^{\mathsf{L}}_n& \\ = \big[p^1_n, A^{\mathsf{L}}\big]Q^{\mathsf{L}}_n &- \big(C_n^{-1}A^{\mathsf{R}} + A^{\mathsf{L}}C_n^{-1}\big)C_{n-1}Q^{\mathsf{L}}_{n-1}.\end{aligned} \tag{4.14}$$

Taking the $(n-1)$-th z power of the (4.11), the $-n$-th of (4.12), the $-(n-1)$-th of (4.13) and the $-(n+1)$-th of (4.14) we get, for all $n \in \mathbb{Z}_+$,

$$\begin{aligned} n\,\mathrm{I} + \big[p^1_{\mathsf{L},n}, B^{\mathsf{L}}\big] + \big[p^2_{\mathsf{L},n}, A^{\mathsf{L}}\big] &= \big[p^1_{\mathsf{L},n}, A^{\mathsf{L}}\big]\, p^1_n - \big(C_n^{-1}A^{\mathsf{R}} + A^{\mathsf{L}}C_n^{-1}\big)C_{n-1},\\ n\,\mathrm{I} + \big[q^1_{n-1}, B^{\mathsf{R}}\big] + \big[q^2_{\mathsf{L},n-1}, A^{\mathsf{R}}\big] &= -\big(C_{n-1}A^{\mathsf{L}} + A^{\mathsf{R}}C_{n-1}\big)C_n^{-1} + \big[q^1_{\mathsf{L},n-1}, A^{\mathsf{R}}\big]q^1_{\mathsf{L},n-1},\\ C_{n-1}B^{\mathsf{L}} + B^{\mathsf{R}}C_{n-1} + C_{n-1}\big[p^1_{\mathsf{L},n-1}, A^{\mathsf{L}}\big] &= -\big(C_{n-1}A^{\mathsf{L}} + A^{\mathsf{R}}C_{n-1}\big)\beta^{\mathsf{L}}_{n-1} - \big[q^1_{\mathsf{L},n-1}, A^{\mathsf{R}}\big]C_{n-1},\\ B^{\mathsf{R}}C_n + C_nB^{\mathsf{L}} + \big[q^1_{\mathsf{L},n}, A^{\mathsf{R}}\big]C_n &= -C_n\big[p^1_{\mathsf{L},n}, A^{\mathsf{L}}\big] - \big(A^{\mathsf{R}}C_n + C_nA^{\mathsf{L}}\big)\beta^{\mathsf{L}}_n.\end{aligned}$$

After some cleaning we reckon that the system is, for all $n \in \mathbb{Z}_+$, equivalent to

$$\begin{cases} I - \left[\beta^{\mathsf{L}}_n, B^{\mathsf{L}} - \Big[\displaystyle\sum_{k=0}^{n-1}\beta^{\mathsf{L}}_k, A^{\mathsf{L}}\Big] + A^{\mathsf{L}}\beta^{\mathsf{L}}_n\right] \\ \qquad\qquad = C_n^{-1}C_{n-1}A^{\mathsf{L}} - C_{n+1}^{-1}A^{\mathsf{R}}C_n - A^{\mathsf{L}}C_{n+1}^{-1}C_n + C_n^{-1}A^{\mathsf{R}}\,C_{n-1},\\ C_{n-1}B^{\mathsf{L}} + B^{\mathsf{R}}C_{n-1} - C_{n-1}\Big[\displaystyle\sum_{k=0}^{n-2}\beta^{\mathsf{L}}_k, A^{\mathsf{L}}\Big] \\ \qquad\qquad = -\big(C_{n-1}A^{\mathsf{L}} + A^{\mathsf{R}}C_{n-1}\big)\beta^{\mathsf{L}}_{n-1} - \Big[\displaystyle\sum_{k=0}^{n-1}C_k\beta^{\mathsf{L}}_k(C_k)^{-1}, A^{\mathsf{R}}\Big]C_{n-1}.\end{cases}$$

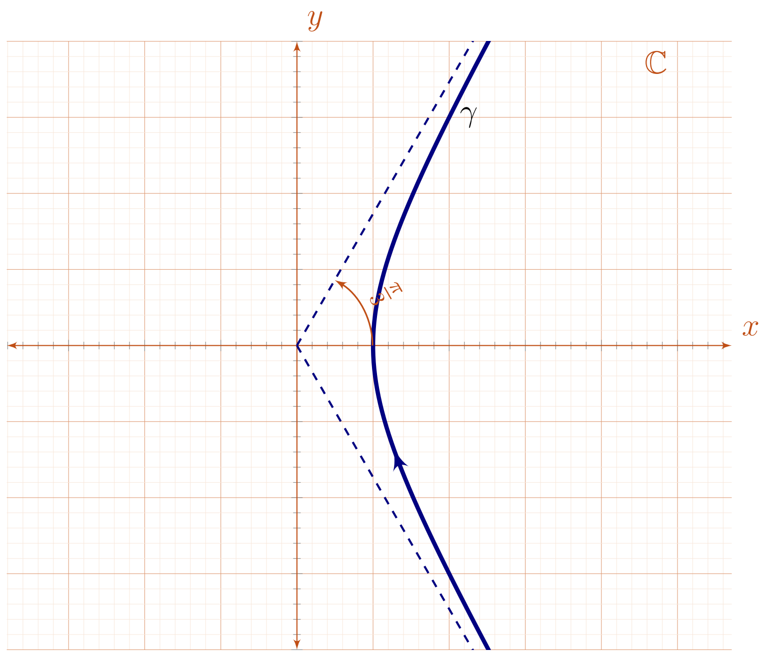
3.1.1. *A matrix extension of the alt-dPI.* We now discuss the case

$$\max\big\{h^{\mathsf{L}}(z), h^{\mathsf{R}}(z)\big\} = 2,$$

but we perform a strong simplification as we take $h^{\mathsf{R}} = \mathbf{0}$ and $h^{\mathsf{L}} = \lambda + \mu z + \nu z^2$, with $\lambda, \mu, \nu \in \mathbb{C}^{N\times N}$ arbitrary matrices but for ν being negative definite nonsingular matrix. Thus, the Pearson equation will be

$$W'(z) = (\lambda + \mu z + \nu z^2)W(z). \tag{4.15}$$

We obviously drop off the notation that distinguish left and right polynomials and only describe the results for the left case. The integrals are taken along γ, a smooth curve for which we have a *simple* Riemann–Hilbert problem as depicted in the following diagram (taken from [15] with the permission of the authors):



Branch of the hyperbola $3x^2 - y^2 = 3$

The structure matrix, *cf.* (3.3), is a second order polynomial $M_n(z) = M_n^0 z^2 + M_n^1 z + M_n^2$ with

$$M_n^0 = \begin{bmatrix} \nu & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \qquad M_n^1 = \begin{bmatrix} \mu - \big[\nu, p_n^1\big] & \nu C_n^{-1} \\ -C_{n-1}\nu & \underline{0} \end{bmatrix},$$

$$M_n^2 = \begin{bmatrix} \lambda - \big[\beta, p_n^1\big] - \big[\nu, p_n^2\big] + \nu\big(p_n^1\big)^2 - p_n^1 \nu\, p_n^1 + \nu C_n^{-1} C_{n-1} & \big(\mu - \big[\nu, p_n^1\big] + \gamma\beta_n\big) C_n^{-1} \\ -C_{n-1}\left(\mu + p_{n-1}^1 \nu - \nu p_n^1\right) & -C_{n-1}\nu\, C_n^{-1} \end{bmatrix}.$$

Theorem 4.8 (Matrix alt-dPI system)**.** *The recursion coefficients* β_n, γ_n *of the matrix orthogonal polynomials with weight matrix a solution of the Pearson equation* (4.15) *are subject to the following system of equations, for all* $n \in \mathbb{Z}_+$*,*

$$\left(\mu + \Big[\nu, \sum_{k=0}^{n-1} \beta_k\Big] + \gamma(\beta_n + \beta_{n+1})\right)\gamma_{n+1} = -(n+1)\,\mathrm{I}, \tag{4.16}$$

$$\begin{aligned} \lambda + \gamma\big(\gamma_n + \gamma_{n+1} + \beta_n^2\big) - \mu\beta_n + \Big[\mu, \sum_{k=0}^{n-1} \beta_k\Big]\big(\,\mathrm{I} + \beta_n\big) \\ + \Big[\nu, \sum_{m=1}^{n-1} \gamma_m - \sum_{0 \le k < m \le n-1} \beta_m \beta_k\Big] + \Big[\nu, \sum_{k=0}^{n-1} \beta_k\Big] \sum_{k=0}^{n-1} \beta_k = \mathbf{0}. \end{aligned} \tag{4.17}$$

Proof. Given the asymptotics about ∞,

$$-C_nQ_n(z)=\mathrm{I}\,z^{-n-1}+q_n^1z^{-n-2}+\cdots,$$

we read the coefficient of z^{-n-1} coming from

$$C_{n-1}Q'_{n-1}(z)=-M^n_{2,1}(z)Q_n(z)+M^n_{2,2}(z)\,C_{n-1}Q_{n-1}(z),$$

with $M^n_{2,1}=-C_{n-1}\nu z-C_{n-1}\big(\mu+p^1_{n-1}\nu-\nu p^1_n\big),\quad M^n_{2,2}=-C_{n-1}\nu C_n^{-1}$, we get (4.16); and from

$$Q'_n(z)=M^n_{1,1}\,Q_n(z)-M^n_{1,2}(z)\,C_{n-1}\,Q_{n-1}(z),$$

with

$$M^n_{1,1}=\nu z^2+\big(\mu-\big[\nu,p^1_n\big]\big)z+\big(\lambda-\big[\mu,p^1_n\big]-\big[\nu,p^2_n\big]+\nu\big(p^1_n\big)^2+\nu C_n^{-1}C_{n-1}-p^1_n\nu\,p^1_n\big)$$
$$M^n_{1,2}=\nu C_n^{-1}z+\big(\mu-\big[\nu,p^1_n\big]+\nu\beta_n\big)C_n^{-1};$$

we deduce (4.17) from the z^{-n-1}-coefficient. □

Another form of writing this result is

Theorem 4.9 (Matrix alt-dPI system)**.** *Given matrix orthogonal polynomials with weight matrix W supported on γ, solution of the Pearson equation* (4.15)*, the recursion coefficients γ_n can be expressed directly in terms of the recursion coefficients β_n, for all $n\in\mathbb{Z}_+$,*

$$\gamma_{n+1}=-(n+1)\Big(\beta+\Big[\gamma,\sum_{k=0}^{n-1}\beta_k\Big]+\gamma(\beta_n+\beta_{n+1})\Big)^{-1}.$$

The coefficients β_n fulfill, for all $n\in\mathbb{Z}_+$, the following non-Abelian alt-dPI,

$$\lambda+\nu\big(\gamma_n+\gamma_{n+1}+\beta_n^2\big)-\mu\beta_n+\Big[\beta,\sum_{k=0}^{n-1}\beta_k\Big]\big(\mathrm{I}+\beta_n\big)$$
$$+\Big[\nu,\sum_{m=1}^{n-1}\gamma_m-\sum_{0\le k<m\le n-1}\beta_m\beta_k\Big]+\Big[\nu,\sum_{k=0}^{n-1}\beta_k\Big]\sum_{k=0}^{n-1}\beta_k=\mathbf{0}.$$

Proof. From (4.16) we get the γ_n in terms of β_n, plugged this relation into the second one gives the following nonlinear equation for the matrices β_n. □

If we assume that $\nu=-\mathrm{I}$ as expected strong simplifications occur. In the first place we find that

$$\gamma_{n+1}=-(n+1)(\mu-\beta_n-\beta_{n+1})^{-1},$$

and, secondly, we derive the following simplified version of a non-Abelian alt-dPI equation

$$\lambda-\beta_n^2+n(\beta-\beta_{n-1}+\beta_n)^{-1}+(n+1)(\mu-\beta_n-\beta_{n+1})^{-1}-\mu\beta_n=-\Big[\mu,\sum_{k=0}^{n-1}\beta_k\Big]\big(\mathrm{I}+\beta_n\big).$$

Moreover, when we choose $\nu = -I$ and $\mu = \mathbf{0}$ the non local terms disappear and the equation simplifies further to

$$-n(\beta_{n-1}+\beta_n)^{-1} - (n+1)(\beta_n+\beta_{n+1})^{-1} + \beta_n^2 = \lambda.$$

Let us remind the reader how the alt-dPI equation appeared for the first time. Going back to the scalar context, in Magnus' work [78], associated with the weight functions solution of the Pearson equation $W'(z) = \big(z^2+t\big)W(z)$, we can find the following scalar alternate discrete Painlevé I system

$$\begin{aligned}\gamma_n + \gamma_{n+1} + \beta_n^2 + t &= 0,\\ n + \gamma_n\left(\beta_n + \beta_{n-1}\right) &= 0,\end{aligned}$$

which can be written as

$$-\frac{n}{\beta_n+\beta_{n-1}} - \frac{n+1}{\beta_n+\beta_{n+1}} + \beta_n^2 + t = 0.$$

3.1.2. *The matrix dPI system.* We now increase further the degree of the polynomials appearing in the Pearson equations. We consider the case with

$$\max\left\{h^{\mathsf{L}}(z), h^{\mathsf{R}}(z)\right\} = 3,$$

but we perform a strong simplification we take $h^{\mathsf{R}} = \mathbf{0}$ and $h^{\mathsf{L}} = \mu z + \nu z^3$, with $\mu, \nu \in \mathbb{C}^{N\times N}$ arbitrary matrices but for ν being negative definite nonsingular matrix. Now we take $\gamma = \mathbb{R}$. Observe that we have now taken the more general possible polynomial of degree three, but an odd one, with well defined parity on z, this simplifies widely the computations.

The associated Pearson type equation for a weight matrix of Freud type:

$$W'(z) = (\mu z + \nu z^3)W(z) \tag{4.18}$$

The structure matrix, *cf.* (3.3), is a third order polynomial, that we write as follows

$$M_n(z) = M_n^0 z^3 + M_n^1 z^2 + M_n^2 z + M_n^3$$

with

$$M_n^0 = \begin{bmatrix}\nu & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}, \qquad M_n^1 = \begin{bmatrix}\mathbf{0} & \mu\, C_n^{-1}\\ -C_{n-1}\mu & \mathbf{0}\end{bmatrix},$$

$$M_n^2 = \begin{bmatrix}\nu + [p_n^2,\nu] + \mu C_n^{-1}C_{n-1} & \mathbf{0}\\ \mathbf{0} & -C_{n-1}\nu C_n^{-1}\end{bmatrix}, \qquad M_n^3 = \begin{bmatrix}\mathbf{0} & \xi_n C_n^{-1}\\ -C_{n-1}\xi_{n-1} & \mathbf{0}\end{bmatrix},$$

where $\xi_n = \mu + [p_n^2,\nu] + \nu(C_n^{-1}C_{n-1} + C_{n+1}^{-1}C_n)$, $n \in \mathbb{Z}_+$.

With this at hand we find.

Theorem 4.10 (Matrix dPI equation)**.** *The recursion coefficients γ_n of the matrix orthogonal polynomials with weight matrix satisfying the Pearson equation* (4.18) *fulfill the following non-Abelian dPI equation*

$$\Big(\mu + \nu(\gamma_{n+2}+\gamma_{n+1}+\gamma_n) + \big[\nu, \sum_{k=1}^{n-1}\gamma_k\big]\Big)\gamma_{n+1} = -(n+1)\,\mathrm{I}, \qquad n \in \mathbb{Z}_+.$$

Proof. Compare the coefficients of z^{-n-1} in the ODE for the second kind functions we get directly (without additional computations) the MdPI equations for the three term relation coefficients of $\{P_n(z)\}_{n\in\mathbb{Z}_+}$. □

Notice the appearance again of non local terms, that disappear if we take $\nu = -\mathrm{I}$ and the matrix dPI reads

$$\gamma_{n+1} = n\gamma_n^{-1} - \gamma_n - \gamma_{n-1} - \mu, \qquad n \in \mathbb{Z}_+,$$

which was derived in the matrix context for the first time in [20] and the confinement of singularities for this relation was proven in [22, 20], see also [65]. In 1995, Alphonse P. Magnus [78] for the Freud weight satisfying the Pearson equation $W'(z) = -\big(z^3 + 2tz\big)\, W(z)$ presented the following scalar discrete Painlevé I equation

$$\gamma_n\big(\gamma_{n-1} + \gamma_n + \gamma_{n+1}\big) + 2t\gamma_n = n.$$

3.2. Laguerre case.

3.2.1. *Matrix discrete Painlevé IV.* We can consider, using the notation introduced before, the matrix weight measure $W = W_{\mathsf{L}}W_{\mathsf{R}}$ such that

$$z(W^{\mathsf{L}})'(z) = (h_0^{\mathsf{L}} + h_1^{\mathsf{L}}z + h_2^{\mathsf{L}}z^2)W^{\mathsf{L}}(z), \qquad z(W^{\mathsf{R}})'(z) = W^{\mathsf{R}}(z)(h_0^{\mathsf{R}} + h_1^{\mathsf{R}}z + h_2^{\mathsf{R}}z^2).$$

From Theorem 3.10 we get that the matrix

$$\widetilde{M}_n = zM_n^{\mathsf{L}}$$

is given explicitly by

$$\begin{aligned}
\big(\widetilde{M}_n^{\mathsf{L}}\big)_{11} &= C_n^{-1}h_2^{\mathsf{R}}C_{n-1} + (h_0^{\mathsf{L}} + h_1^{\mathsf{L}}z + h_2^{\mathsf{L}}z^2) + h_1^{\mathsf{L}}q^1_{\mathsf{R},n-1} + p^1_{\mathsf{L},n}h_1^{\mathsf{L}}\\
&\qquad + z(h_2^{\mathsf{L}}q^1_{\mathsf{R},n-1} + p^1_{\mathsf{L},n}h_2^{\mathsf{L}}) + h_2^{\mathsf{L}}q^2_{\mathsf{R},n-1} + p^2_{\mathsf{L},n}h_2^{\mathsf{L}} + p^1_{\mathsf{L},n}h_2^{\mathsf{L}}q^1_{\mathsf{R},n-1} + n\,\mathrm{I},\\
\big(\widetilde{M}_n^{\mathsf{L}}\big)_{12} &= (h_1^{\mathsf{L}} + h_2^{\mathsf{L}}z + h_2^{\mathsf{L}}q^1_{\mathsf{R},n} + p^1_{\mathsf{L},n}h_2^{\mathsf{L}})C_n^{-1} + C_n^{-1}(h_1^{\mathsf{R}} + h_2^{\mathsf{R}}z + h_2^{\mathsf{R}}p^1_{\mathsf{R},n} + q^1_{\mathsf{L},n}h_2^{\mathsf{R}}),\\
\big(\widetilde{M}_n^{\mathsf{L}}\big)_{21} &= -C_{n-1}(h_1^{\mathsf{L}} + h_2^{\mathsf{L}}z + h_2^{\mathsf{L}}q^1_{\mathsf{R},n-1} + p^1_{\mathsf{L},n-1}C_{\mathsf{L}})\\
&\qquad - (h_1^{\mathsf{R}} + h_2^{\mathsf{R}}z + h_2^{\mathsf{R}}p^1_{\mathsf{R},n-1} + q^1_{\mathsf{L},n-1}h_2^{\mathsf{R}})C_{n-1},\\
\big(\widetilde{M}_n^{\mathsf{L}}\big)_{22} &= -C_{n-1}h_2^{\mathsf{L}}C_n^{-1} - (h_0^{\mathsf{R}} + h_1^{\mathsf{R}}z + h_2^{\mathsf{R}}z^2) - h_1^{\mathsf{R}}p^1_{\mathsf{R},n} - q^1_{\mathsf{L},n-1}h_1^{\mathsf{R}}\\
&\qquad - z(h_2^{\mathsf{R}}p^1_{\mathsf{R},n} + q^1_{\mathsf{L},n-1}h_2^{\mathsf{R}}) - h_2^{\mathsf{R}}p^2_{\mathsf{R},n} - q^2_{\mathsf{L},n-1}h_2^{\mathsf{R}} - q^1_{\mathsf{L},n-1}h_2^{\mathsf{R}}p^1_{\mathsf{R},n} - n\,\mathrm{I}\,.
\end{aligned}$$

From the three term recurrence relation for $\{P_n^{\mathsf{L}}\}_{n\in\mathbb{N}}$ we get that $p^1_{\mathsf{L},n} - p^1_{\mathsf{L},n+1} = \beta_n^{\mathsf{L}}$ and $p^2_{\mathsf{L},n} - p^2_{\mathsf{L},n+1} = \beta_n^{\mathsf{L}}p^1_{\mathsf{L},n} + \gamma_n^{\mathsf{L}}$ where $\gamma_n^{\mathsf{L}} = C_n^{-1}C_{n-1}$. Consequently,

$$p^1_{\mathsf{L},n} = -\sum_{k=0}^{n-1}\beta_k^{\mathsf{L}}, \qquad p^2_{\mathsf{L},n} = \sum_{i,j=0}^{n-1}\beta_i^{\mathsf{L}}\beta_j^{\mathsf{L}} - \sum_{k=0}^{n-1}\gamma_k^{\mathsf{L}}.$$

In the same manner, from the three term recurrence relation for $\{Q_n^{\mathsf{L}}\}_{n\in\mathbb{N}}$ we deduce that $q^1_{\mathsf{L},n} - q^1_{\mathsf{L},n-1} = \beta_n^{\mathsf{R}} := C_n\beta_n^{\mathsf{L}}C_n^{-1}$ and $q^2_{\mathsf{L},n} - q^2_{\mathsf{L},n-1} = \beta_n^{\mathsf{R}}q^1_{\mathsf{L},n} + \gamma_n^{\mathsf{R}}$, where $\gamma_n^{\mathsf{R}} = C_nC_{n+1}^{-1}$.

If we consider that $W = W^{\mathsf{L}}$ and $W^{\mathsf{R}} = \mathrm{I}$, and use the representation for $\{P^{\mathsf{L}}_n\}_{n\in\mathbb{N}}$ and $\{Q^{\mathsf{L}}_n\}_{n\in\mathbb{N}}$ in z powers, the $(1,2)$ and $(2,2)$ entries in (3.17) read

$$
\begin{aligned}
&(2n+1)\,\mathrm{I} + h^{\mathsf{L}}_0 + h^{\mathsf{L}}_2(\gamma^{\mathsf{L}}_{n+1} + \gamma^{\mathsf{L}}_n + (\beta^{\mathsf{L}}_n)^2) + h^{\mathsf{L}}_1\beta^{\mathsf{L}}_n \\
&\qquad\qquad = [p^1_{\mathsf{L},n}, h^{\mathsf{L}}_2]p^1_{\mathsf{L},n+1} - [p^2_{\mathsf{L},n}, h^{\mathsf{L}}_2] - [p^1_{\mathsf{L},n}, h^{\mathsf{L}}_1],\\
&\beta^{\mathsf{L}}_n = \gamma^{\mathsf{L}}_n\big(h^{\mathsf{L}}_2(\beta^{\mathsf{L}}_n + \beta^{\mathsf{L}}_{n-1}) + [p^1_{\mathsf{L},n-1}, h^{\mathsf{L}}_2] + h^{\mathsf{L}}_1\big) - \big(h^{\mathsf{L}}_2(\beta^{\mathsf{L}}_n + \beta^{\mathsf{L}}_{n+1}) + [p^1_{\mathsf{L},n}, h^{\mathsf{L}}_2] + h^{\mathsf{L}}_1\big)\gamma^{\mathsf{L}}_{n+1}.
\end{aligned}
$$

We can write these equations as follows

$$
\begin{aligned}
&(2n+1)\,\mathrm{I} + h^{\mathsf{L}}_0 + h^{\mathsf{L}}_2(\gamma^{\mathsf{L}}_{n+1} + \gamma^{\mathsf{L}}_n)) + (h^{\mathsf{L}}_2\beta^{\mathsf{L}}_n + h^{\mathsf{L}}_1)\beta^{\mathsf{L}}_n \\
&\qquad = \Big[\sum_{k=0}^{n-1}\beta^{\mathsf{L}}_k, h^{\mathsf{L}}_2\Big]\sum_{k=0}^{n}\beta^{\mathsf{L}}_k - \Big[\sum_{i,j=0}^{n-1}\beta^{\mathsf{L}}_i\beta^{\mathsf{L}}_j - \sum_{k=0}^{n-1}\gamma^{\mathsf{L}}_k, h^{\mathsf{L}}_2\Big] - \Big[\sum_{k=0}^{n-1}\beta^{\mathsf{L}}_k, h^{\mathsf{L}}_1\Big],
\end{aligned}
\tag{4.19}
$$

$$
\begin{aligned}
&\beta^{\mathsf{L}}_n - \gamma^{\mathsf{L}}_n\big(h^{\mathsf{L}}_2(\beta^{\mathsf{L}}_n + \beta^{\mathsf{L}}_{n-1}) + h^{\mathsf{L}}_1\big) + \big(h^{\mathsf{L}}_2(\beta^{\mathsf{L}}_n + \beta^{\mathsf{L}}_{n+1}) + h^{\mathsf{L}}_1\big)\gamma^{\mathsf{L}}_{n+1} \\
&\qquad = -\gamma^{\mathsf{L}}_n\Big[\sum_{k=0}^{n-1}\beta^{\mathsf{L}}_k, h^{\mathsf{L}}_2\Big] + \Big[-\sum_{k=0}^{n-1}\beta^{\mathsf{L}}_k, h^{\mathsf{L}}_2\Big]\gamma^{\mathsf{L}}_{n+1}.
\end{aligned}
\tag{4.20}
$$

We will show now that this system contains a noncommutative version of an instance of discrete Painlevé IV equation, as happens in the analogous case for the scalar scenario.

We see, on the *r.h.s.* of the nonlinear discrete equations (4.19) and (4.20) nonlocal terms (sums) in the recursion coefficients β^{L}_n and γ^{L}_n, all of them inside commutators. These nonlocal terms vanish whenever the three matrices $\{h^{\mathsf{L}}_0, h^{\mathsf{L}}_1, h^{\mathsf{L}}_2\}$ conform an Abelian set. Moreover, $\{h^{\mathsf{L}}_0, h^{\mathsf{L}}_1, h^{\mathsf{L}}_2, \beta^{\mathsf{L}}_n, \gamma^{\mathsf{L}}_n\}$ is also an Abelian set. In this commutative setting we have

$$
\begin{aligned}
(2n+1)\,\mathrm{I} + h^{\mathsf{L}}_0 + h^{\mathsf{L}}_2(\gamma^{\mathsf{L}}_{n+1} + \gamma^{\mathsf{L}}_n)) + (h^{\mathsf{L}}_2\beta^{\mathsf{L}}_n + h^{\mathsf{L}}_1)\beta^{\mathsf{L}}_n &= \mathbf{0},\\
\beta^{\mathsf{L}}_n - \gamma^{\mathsf{L}}_n\big(h^{\mathsf{L}}_2(\beta^{\mathsf{L}}_n + \beta^{\mathsf{L}}_{n-1}) + h^{\mathsf{L}}_1\big) + \big(h^{\mathsf{L}}_2(\beta^{\mathsf{L}}_n + \beta^{\mathsf{L}}_{n+1}) + h^{\mathsf{L}}_1\big)\gamma^{\mathsf{L}}_{n+1} &= \mathbf{0}.
\end{aligned}
$$

In terms of $\xi_n := \dfrac{h^{\mathsf{L}}_0}{2} + n\,\mathrm{I} + h^{\mathsf{L}}_2\gamma_n$ and $\mu_n := h^{\mathsf{L}}_2\beta^{\mathsf{L}}_n + h^{\mathsf{L}}_1$ the above equations are

$$
\beta^{\mathsf{L}}_n\mu_n = -(\xi_n + \xi_{n+1}), \qquad\qquad \beta^{\mathsf{L}}_n(\xi_n - \xi_{n+1}) = -\gamma_n\mu_{n-1} + \gamma_{n+1}\mu_{n+1}.
$$

Now, we multiply the second equation by μ_n and taking into account the first one we arrive to

$$
-(\xi_n + \xi_{n+1})(\xi_n - \xi_{n+1}) = -\gamma_n\mu_{n-1}\mu_n + \gamma_{n+1}\mu_n\mu_{n+1}
$$

and so

$$
\xi^2_{n+1} - \xi^2_n = \gamma_{n+1}\mu_n\mu_{n+1} - \gamma_n\mu_{n-1}\mu_n.
$$

Hence,

$$
\xi^2_{n+1} - \xi^2_0 = \gamma_{n+1}\mu_n\mu_{n+1} \qquad\text{and}\qquad \beta^{\mathsf{L}}_n\mu_n = -(\xi_n + \xi_{n+1})
\tag{4.21}
$$

coincide to the ones presented in [9] as discrete Painlevé IV (dPIV) equation. In fact, taking $\nu_n = \mu_n^{-1}$ we finally arrive to

$$\nu_n \nu_{n+1} = \frac{h_2^{\mathsf{L}}\big(\xi_{n+1} - h_0^{\mathsf{L}}/2 - n\,\mathrm{I}\big)}{\xi_{n+1}^2 - \xi_0^2} \quad \text{and} \quad \xi_n + \xi_{n+1} = \Big(\big(h_2^{\mathsf{L}}\big)^{-1} h_1^{\mathsf{L}} - \big(h_2^{\mathsf{L}}\big)^{-1} \nu_n^{-1}\Big)\,\nu_n^{-1}.$$

If we take $h_1^{\mathsf{L}} = 0$ in (4.21) then $\mu_n = h_2^{\mathsf{L}}\beta_n^{\mathsf{L}}$, and so

$$(\beta_n^{\mathsf{L}})^2 h_2^{\mathsf{L}} = -(\xi_n + \xi_{n+1}).$$

Now, taking square in the first equation in (4.21) we get

$$\big(\xi_n + \xi_{n+1}\big)\big(\xi_{n+1} + \xi_{n+2}\big) = \Big(\big(\xi_{n+1} - \frac{h_0^{\mathsf{L}}}{2} - n\,\mathrm{I}\big)^{-1}\big(\xi_{n+1}^2 - \xi_0^2\big)\Big)^2,$$

which is an instance of dPIV by Grammaticos, Hietarinta, and Ramani (*cf.* [60]).

Thus, (4.19) and (4.20) for $B_{\mathsf{L}} = \mathbf{0}$ may be considered as non-Abelian extension of this instance of dPIV.

We have just seen that,

Theorem 4.11 (Non-Abelian extension of the dPIV). *When $B_{\mathsf{L}} = \mathbf{0}$, the following nonlocal nonlinear non-Abelian system for the recursion coefficients is fulfilled*

$$\begin{aligned}
(2n+1)\,\mathrm{I} + h_0^{\mathsf{L}} + h_2^{\mathsf{L}}(\gamma_{n+1}^{\mathsf{L}} + \gamma_n^{\mathsf{L}})) + h_2^{\mathsf{L}}(\beta_n^{\mathsf{L}})^2 \\
= \Big[\sum_{k=0}^{n-1}\beta_k^{\mathsf{L}}, h_2^{\mathsf{L}}\Big]\sum_{k=0}^{n}\beta_k^{\mathsf{L}} - \Big[\sum_{i,j=0}^{n-1}\beta_i^{\mathsf{L}}\beta_j^{\mathsf{L}} - \sum_{k=0}^{n-1}\gamma_k^{\mathsf{L}}, h_2^{\mathsf{L}}\Big], \\
\beta_n^{\mathsf{L}} - \gamma_n^{\mathsf{L}}\big(h_2^{\mathsf{L}}(\beta_n^{\mathsf{L}} + \beta_{n-1}^{\mathsf{L}})\big) + \big(h_2^{\mathsf{L}}(\beta_n^{\mathsf{L}} + \beta_{n+1}^{\mathsf{L}})\big)\gamma_{n+1}^{\mathsf{L}} \\
= -\gamma_n^{\mathsf{L}}\Big[\sum_{k=0}^{n-1}\beta_k^{\mathsf{L}}, h_2^{\mathsf{L}}\Big] + \Big[-\sum_{k=0}^{n-1}\beta_k^{\mathsf{L}}, h_2^{\mathsf{L}}\Big]\gamma_{n+1}^{\mathsf{L}}.
\end{aligned}$$

Moreover, this system reduces in the commutative context to the standard dPIV equation.

3.3. Jacobi case. We can consider, using the notation introduced before, the matrix weight measure $W(z) = W_{\mathsf{L}}(z)W_{\mathsf{R}}(z)$ such that

$$\begin{aligned}
z(1-z)(W^{\mathsf{L}})'(z) &= (h_0^{\mathsf{L}} + h_1^{\mathsf{L}}z + h_2^{\mathsf{L}}z^2)W^{\mathsf{L}}(z), \\
z(1-z)(W^{\mathsf{R}})'(z) &= W^{\mathsf{R}}(z)(h_0^{\mathsf{R}} + h_1^{\mathsf{R}}z + h_2^{\mathsf{R}}z^2).
\end{aligned}$$

From Theorem 3.14 we get the matrix $\widetilde{M}_n = z(1-z)M_n^{\mathsf{L}}$ is given explicitly by

$$\begin{cases} (\widetilde{M}_n^{\mathsf{L}})_{11} = C_n^{-1}h_2^{\mathsf{R}}C_{n-1} + (h_0^{\mathsf{L}} + h_1^{\mathsf{L}}z + h_2^{\mathsf{L}}z^2) + h_1^{\mathsf{L}}q_{\mathsf{R},n-1}^1 + p_{\mathsf{L},n}^1 h_1^{\mathsf{L}} \\ \quad + z(h_2^{\mathsf{L}}q_{\mathsf{R},n-1}^1 + p_{\mathsf{L},n}^1 h_2^{\mathsf{L}}) + h_2^{\mathsf{L}}q_{\mathsf{R},n-1}^2 + p_{\mathsf{L},n}^2 h_2^{\mathsf{L}} + p_{\mathsf{L},n}^1 h_2^{\mathsf{L}} q_{\mathsf{R},n-1}^1 + n\,\mathrm{I} - zn\,\mathrm{I} + p_{\mathsf{L},n}^1, \\ (\widetilde{M}_n^{\mathsf{L}})_{12} = (h_1^{\mathsf{L}} + h_2^{\mathsf{L}}z + h_2^{\mathsf{L}}q_{\mathsf{R},n}^1 + p_{\mathsf{L},n}^1 h_2^{\mathsf{L}})C_n^{-1} + C_n^{-1}(h_1^{\mathsf{R}} + h_2^{\mathsf{R}}z + h_2^{\mathsf{R}}p_{\mathsf{R},n}^1 + q_{\mathsf{L},n}^1 h_2^{\mathsf{R}}) \\ \qquad\qquad - (2n+1)C_n^{-1}, \\ (\widetilde{M}_n^{\mathsf{L}})_{21} = -C_{n-1}(h_1^{\mathsf{L}} + h_2^{\mathsf{L}}z + h_2^{\mathsf{L}}q_{\mathsf{R},n-1}^1 + p_{\mathsf{L},n-1}^1 h_2^{\mathsf{L}}) \\ \qquad - (h_1^{\mathsf{R}} + h_2^{\mathsf{R}}z + h_2^{\mathsf{R}}p_{\mathsf{R},n-1}^1 + q_{\mathsf{L},n-1}^1 h_2^{\mathsf{R}})C_{n-1} + (2n-1)C_{n-1}, \\ (\widetilde{M}_n^{\mathsf{L}})_{22} = -C_{n-1}h_2^{\mathsf{L}}C_n^{-1} - (h_0^{\mathsf{R}} + h_1^{\mathsf{R}}z + h_2^{\mathsf{R}}z^2) - h_1^{\mathsf{R}}p_{\mathsf{R},n}^1 - q_{\mathsf{L},n-1}^1 h_1^{\mathsf{R}} \\ -z(h_2^{\mathsf{R}}p_{\mathsf{R},n}^1 + q_{\mathsf{L},n-1}^1 h_2^{\mathsf{R}}) - h_2^{\mathsf{R}}p_{\mathsf{R},n}^2 - q_{\mathsf{L},n-1}^2 h_2^{\mathsf{R}} - q_{\mathsf{L},n-1}^1 h_2^{\mathsf{R}} p_{\mathsf{R},n}^1 - n\,\mathrm{I} + zn\,\mathrm{I} - p_{\mathsf{R},n}^1. \end{cases}$$

Using the three term recurrence relation for $\{P_n^{\mathsf{L}}\}_{n\in\mathbb{N}}$ we get that $p_{\mathsf{L},n}^1 - p_{\mathsf{L},n+1}^1 = \beta_n^{\mathsf{L}}$ and $p_{\mathsf{L},n}^2 - p_{\mathsf{L},n+1}^2 = \beta_n^{\mathsf{L}}p_{\mathsf{L},n}^1 + \gamma_n^{\mathsf{L}}$ where $\gamma_n^{\mathsf{L}} = C_n^{-1}C_{n-1}$. Consequently,

$$p_{\mathsf{L},n}^1 = -\sum_{k=0}^{n-1}\beta_k^{\mathsf{L}}, \qquad\qquad p_{\mathsf{L},n}^2 = \sum_{i,j=0}^{n-1}\beta_i^{\mathsf{L}}\beta_j^{\mathsf{L}} - \sum_{k=0}^{n-1}\gamma_k^{\mathsf{L}}.$$

In the same manner, from the three term recurrence relation for $\{Q_n^{\mathsf{L}}\}_{n\in\mathbb{N}}$ we deduce that $q_{\mathsf{L},n}^1 - q_{\mathsf{L},n-1}^1 = \beta_n^{\mathsf{R}} := C_n\beta_n^{\mathsf{L}}C_n^{-1}$ and $q_{\mathsf{L},n}^2 - q_{\mathsf{L},n-1}^2 = \beta_n^{\mathsf{R}}q_{\mathsf{L},n}^1 + \gamma_n^{\mathsf{R}}$, where $\gamma_n^{\mathsf{R}} = C_nC_{n+1}^{-1}$.

Now, we consider that $W = W^{\mathsf{L}}$ and $W^{\mathsf{R}} = \mathrm{I}$, and then use the representation for $\{P_n^{\mathsf{L}}\}_{n\in\mathbb{N}}$ and $\{Q_n^{\mathsf{L}}\}_{n\in\mathbb{N}}$ in z powers, the $(1,2)$ and $(2,2)$ entries in (3.32) read

$$\begin{aligned} &(2n+1)(\mathrm{I} - \beta_n^{\mathsf{L}}) + h_0^{\mathsf{L}} + h_2^{\mathsf{L}}(\gamma_{n+1}^{\mathsf{L}} + \gamma_n^{\mathsf{L}} + (\beta_n^{\mathsf{L}})^2) + h_1^{\mathsf{L}}\beta_n^{\mathsf{L}} \\ &\qquad = [p_{\mathsf{L},n}^1, h_2^{\mathsf{L}}]p_{\mathsf{L},n+1}^1 - [p_{\mathsf{L},n}^2, h_2^{\mathsf{L}}] - [p_{\mathsf{L},n}^1, h_1^{\mathsf{L}}] - p_{\mathsf{L},n}^1 - C_n^{-1}p_{\mathsf{L},n+1}^1 C_n, \\ &\beta_n^{\mathsf{L}} - (\beta_n^{\mathsf{L}})^2 = \gamma_n^{\mathsf{L}}\big(h_2^{\mathsf{L}}(\beta_n^{\mathsf{L}} + \beta_{n-1}^{\mathsf{L}}) + [p_{\mathsf{L},n-1}^1, h_2^{\mathsf{L}}] + h_1^{\mathsf{L}} - (2n-1)\,\mathrm{I}\big) \\ &\qquad - \big(h_2^{\mathsf{L}}(\beta_n^{\mathsf{L}} + \beta_{n+1}^{\mathsf{L}}) + [p_{\mathsf{L},n}^1, h_2^{\mathsf{L}}] + h_1^{\mathsf{L}} - (2n+3)\,\mathrm{I}\big)\gamma_{n+1}^{\mathsf{L}} - [p_{\mathsf{L},n}^1, p_{\mathsf{L},n+1}^1]. \end{aligned}$$

We can write these equations as follows

$$\begin{aligned} &(2n+1)\,\mathrm{I} + h_0^{\mathsf{L}} + h_2^{\mathsf{L}}(\gamma_{n+1}^{\mathsf{L}} + \gamma_n^{\mathsf{L}}) + \big(h_2^{\mathsf{L}}\beta_n^{\mathsf{L}} + h_1^{\mathsf{L}} - (2n+1)\,\mathrm{I}\big)\,\beta_n^{\mathsf{L}} + \sum_{k=0}^{n-1}\beta_k^{\mathsf{L}} \\ &\quad + C_n^{-1}\sum_{k=0}^{n}\beta_k^{\mathsf{L}}C_n = \Big[\sum_{k=0}^{n-1}\beta_k^{\mathsf{L}}, h_2^{\mathsf{L}}\Big]\sum_{k=0}^{n}\beta_k^{\mathsf{L}} - \Big[\sum_{i,j=0}^{n-1}\beta_i^{\mathsf{L}}\beta_j^{\mathsf{L}} - \sum_{k=0}^{n-1}\gamma_k^{\mathsf{L}}, h_2^{\mathsf{L}}\Big] - \Big[\sum_{k=0}^{n-1}\beta_k^{\mathsf{L}}, h_1^{\mathsf{L}}\Big], \end{aligned} \tag{4.22}$$

$$\begin{aligned} &\beta_n^{\mathsf{L}} - (\beta_n^{\mathsf{L}})^2 - \gamma_n^{\mathsf{L}}\big(h_2^{\mathsf{L}}(\beta_n^{\mathsf{L}} + \beta_{n-1}^{\mathsf{L}}) + h_1^{\mathsf{L}} - (2n-1)\,\mathrm{I}\big) + \big(h_2^{\mathsf{L}}(\beta_n^{\mathsf{L}} + \beta_{n+1}^{\mathsf{L}}) + h_1^{\mathsf{L}} \\ &\quad - (2n+3)\,\mathrm{I}\big)\gamma_{n+1}^{\mathsf{L}} = \gamma_n^{\mathsf{L}}\Big[\sum_{k=0}^{n-2}\beta_k^{\mathsf{L}}, h_2^{\mathsf{L}}\Big] - \Big[\sum_{k=0}^{n-1}\beta_k^{\mathsf{L}}, h_2^{\mathsf{L}}\Big]\gamma_{n+1}^{\mathsf{L}} - \Big[\sum_{k=0}^{n-1}\beta_k^{\mathsf{L}}, \sum_{k=0}^{n}\beta_k^{\mathsf{L}}\Big]. \end{aligned} \tag{4.23}$$

We will show now that this system contains a noncommutative version of an instance of discrete Painlevé IV equation.

We see, on the *r.h.s.* of the nonlinear discrete equations (4.22) and (4.23) nonlocal terms (sums) in the recursion coefficients β_n^{L} and γ_n^{L}, all of them inside commutators. These nonlocal terms vanish whenever the three matrices $\{h_0^{\mathsf{L}}, h_1^{\mathsf{L}}, h_2^{\mathsf{L}}\}$ conform an

Abelian set, so that $\{h_0^{\mathsf{L}}, h_1^{\mathsf{L}}, h_2^{\mathsf{L}}, \beta_n^{\mathsf{L}}, \gamma_n^{\mathsf{L}}\}$ is also an Abelian set. In this commutative setting we have

$$\begin{aligned}
&(2n+1)\,\mathrm{I} + h_0^{\mathsf{L}} + h_2^{\mathsf{L}}(\gamma_{n+1}^{\mathsf{L}} + \gamma_n^{\mathsf{L}})) + \big(h_2^{\mathsf{L}}\beta_n^{\mathsf{L}} + h_1^{\mathsf{L}} - (2n+1)\,\mathrm{I}\big)\,\beta_n^{\mathsf{L}} + p_{\mathsf{L},n}^1 + p_{\mathsf{L},n+1}^1 = \mathbf{0},\\
&\beta_n^{\mathsf{L}} - (\beta_n^{\mathsf{L}})^2 - \gamma_n^{\mathsf{L}}\big(h_2^{\mathsf{L}}(\beta_n^{\mathsf{L}} + \beta_{n-1}^{\mathsf{L}}) + h_1^{\mathsf{L}} - (2n-1)\,\mathrm{I}\,\big) + \big(h_2^{\mathsf{L}}(\beta_n^{\mathsf{L}} + \beta_{n+1}^{\mathsf{L}})\\
&\qquad\qquad\qquad\qquad + h_1^{\mathsf{L}} - (2n+3)\,\mathrm{I}\,\big)\gamma_{n+1}^{\mathsf{L}} = \mathbf{0}.
\end{aligned}$$

In terms of

$$\xi_n := \frac{h_0^{\mathsf{L}}}{2} + n\,\mathrm{I} + h_2^{\mathsf{L}}\gamma_n + p_{\mathsf{L},n}^1 \qquad \text{and} \qquad \mu_n := h_2^{\mathsf{L}}\beta_n^{\mathsf{L}} + h_1^{\mathsf{L}} - (2n+1)\,\mathrm{I},$$

the above equations reads as

$$-\mu_n\beta_n^{\mathsf{L}} = \xi_n + \xi_{n+1} \qquad \text{and} \qquad \beta_n^{\mathsf{L}}(\xi_n - \xi_{n+1}) = \mu_{n+1}\gamma_{n+1} - \gamma_n\mu_{n-1}.$$

Now, we multiply the second equation by μ_n and taking into account the first one we arrive to

$$-(\xi_n + \xi_{n+1})(\xi_n - \xi_{n+1}) = -\gamma_n\mu_{n-1}\mu_n + \gamma_{n+1}\mu_n\mu_{n+1},$$

and so

$$\xi_{n+1}^2 - \xi_n^2 = \gamma_{n+1}\mu_n\mu_{n+1} - \gamma_n\mu_{n-1}\mu_n.$$

Hence,

$$\xi_{n+1}^2 - \xi_0^2 = \gamma_{n+1}\mu_n\mu_{n+1} \qquad \text{and} \qquad \beta_n^{\mathsf{L}}\mu_n = -(\xi_n + \xi_{n+1})$$

coincide to the ones presented in [9] as discrete Painlevé IV (dPIV) equation. In fact, taking $\nu_n = \mu_n^{-1}$ we finally arrive to

$$\begin{aligned}
\nu_n\nu_{n+1} &= \frac{h_2^{\mathsf{L}}\big(\xi_{n+1} - h_0^{\mathsf{L}}/2 - n\,\mathrm{I} - p_{\mathsf{L},n}^1\big)}{\xi_{n+1}^2 - \xi_0^2},\\
\xi_n + \xi_{n+1} &= \Big(\big(h_2^{\mathsf{L}}\big)^{-1} h_1^{\mathsf{L}} - \big(h_2^{\mathsf{L}}\big)^{-1}\nu_n^{-1} - (2n+1)\,\big(h_2^{\mathsf{L}}\big)^{-1}\Big)\,\nu_n^{-1}.
\end{aligned}$$

Now, we are able to state that,

Theorem 4.12 (**Non-Abelian extension of the dPIV**). *Equations* (4.22) *and* (4.23) *defines a nonlocal nonlinear non-Abelian system for the recursion coefficients.*

3.4. Bessel case. We can consider, using the notation introduced before, the matrix weight measure $W(z) = W_{\mathsf{L}}(z)W_{\mathsf{R}}(z)$ such that

$$z^2(W^{\mathsf{L}})'(z) = (h_0^{\mathsf{L}} + h_1^{\mathsf{L}}z + h_2^{\mathsf{L}}z^2)W^{\mathsf{L}}(z), \qquad z^2(W^{\mathsf{R}})'(z) = W^{\mathsf{R}}(z)(h_0^{\mathsf{R}} + h_1^{\mathsf{R}}z + h_2^{\mathsf{R}}z^2).$$

The matrix $\widetilde{M}_n = z^2 M^{\mathsf{L}}_n$ is given explicitly by

$$
\begin{cases}
(\widetilde{M}^{\mathsf{L}}_n)_{11} = C_n^{-1} h^{\mathsf{R}}_2 C_{n-1} + (h^{\mathsf{L}}_0 + h^{\mathsf{L}}_1 z + h^{\mathsf{L}}_2 z^2) + \left[p^1_{\mathsf{L},n}, h^{\mathsf{L}}_1\right] + z\left(n\,\mathrm{I} + \left[p^1_{\mathsf{L},n}, h^{\mathsf{L}}_2\right]\right) \\
\qquad\qquad + \left[p^2_{\mathsf{L},n}, h^{\mathsf{L}}_2\right] - p^1_{\mathsf{L},n} h^{\mathsf{L}}_2 p^1_{\mathsf{L},n} - p^1_{\mathsf{L},n}, \\
(\widetilde{M}^{\mathsf{L}}_n)_{12} = (h^{\mathsf{L}}_1 + h^{\mathsf{L}}_2 z - h^{\mathsf{L}}_2 p^1_{\mathsf{L},n+1} + p^1_{\mathsf{L},n} h^{\mathsf{L}}_2) C_n^{-1} \\
\qquad\qquad + C_n^{-1}(h^{\mathsf{R}}_1 + h^{\mathsf{R}}_2 z + h^{\mathsf{R}}_2 p^1_{\mathsf{R},n} - p^1_{\mathsf{R},n+1} h^{\mathsf{R}}_2) + (2n+1) C_n^{-1}, \\
(\widetilde{M}^{\mathsf{L}}_n)_{21} = -C_{n-1}(h^{\mathsf{L}}_1 + h^{\mathsf{L}}_2 z - h^{\mathsf{L}}_2 p^1_{\mathsf{L},n} + p^1_{\mathsf{L},n-1} h^{\mathsf{L}}_2) \\
\qquad\qquad - (h^{\mathsf{R}}_1 + h^{\mathsf{R}}_2 z + h^{\mathsf{R}}_2 p^1_{\mathsf{R},n-1} - p^1_{\mathsf{R},n} h^{\mathsf{R}}_2) C_{n-1} - (2n-1) C_{n-1}, \\
(\widetilde{M}^{\mathsf{L}}_n)_{22} = -C_{n-1} h^{\mathsf{L}}_2 C_n^{-1} - (h^{\mathsf{R}}_0 + h^{\mathsf{R}}_1 z + h^{\mathsf{R}}_2 z^2) + \left[p^1_{\mathsf{R},n}, h^{\mathsf{R}}_1\right] - z\left(n\,\mathrm{I} + \left[h^{\mathsf{R}}_2, p^1_{\mathsf{R},n}\right]\right) \\
\qquad\qquad + \left[p^2_{\mathsf{R},n}, h^{\mathsf{R}}_2\right] + p^1_{\mathsf{R},n} h^{\mathsf{R}}_2 p^1_{\mathsf{R},n} + p^1_{\mathsf{R},n}.
\end{cases}
$$

Taking $h^{\mathsf{R}} = 0$

$$
\begin{cases}
(\widetilde{M}^{\mathsf{L}}_n)_{11} = (h^{\mathsf{L}}_0 + h^{\mathsf{L}}_1 z + h^{\mathsf{L}}_2 z^2) + \left[p^1_{\mathsf{L},n}, h^{\mathsf{L}}_1\right] + z\left(n\,\mathrm{I} + \left[p^1_{\mathsf{L},n}, h^{\mathsf{L}}_2\right]\right) \\
\qquad\qquad + \left[p^2_{\mathsf{L},n}, h^{\mathsf{L}}_2\right] - p^1_{\mathsf{L},n} h^{\mathsf{L}}_2 p^1_{\mathsf{L},n} - p^1_{\mathsf{L},n}, \\
(\widetilde{M}^{\mathsf{L}}_n)_{12} = (h^{\mathsf{L}}_1 + h^{\mathsf{L}}_2 z - h^{\mathsf{L}}_2 p^1_{\mathsf{L},n+1} + p^1_{\mathsf{L},n} h^{\mathsf{L}}_2) C_n^{-1} + (2n+1) C_n^{-1}, \\
(\widetilde{M}^{\mathsf{L}}_n)_{21} = -C_{n-1}(h^{\mathsf{L}}_1 + h^{\mathsf{L}}_2 z - h^{\mathsf{L}}_2 p^1_{\mathsf{L},n} + p^1_{\mathsf{L},n-1} h^{\mathsf{L}}_2) - (2n-1) C_{n-1}, \\
(\widetilde{M}^{\mathsf{L}}_n)_{22} = -C_{n-1} h^{\mathsf{L}}_2 C_n^{-1} - zn\,\mathrm{I} + p^1_{\mathsf{R},n}.
\end{cases}
$$

If we consider that $W = W^{\mathsf{L}}$ and $W^{\mathsf{R}} = \mathrm{I}$, and use the representation for $\left\{P^{\mathsf{L}}_n\right\}_{n\in\mathbb{N}}$ and $\{Q^{\mathsf{L}}_n\}_{n\in\mathbb{N}}$ in z powers, the $(1,2)$ and $(2,2)$ entries read

$$
\begin{aligned}
&(2n+1)\beta^{\mathsf{L}}_n + h^{\mathsf{L}}_0 + h^{\mathsf{L}}_2(\gamma^{\mathsf{L}}_{n+1} + \gamma^{\mathsf{L}}_n + (\beta^{\mathsf{L}}_n)^2) + h^{\mathsf{L}}_1 \beta^{\mathsf{L}}_n \\
&\qquad = [p^1_{\mathsf{L},n}, h^{\mathsf{L}}_2] p^1_{\mathsf{L},n+1} - [p^2_{\mathsf{L},n}, h^{\mathsf{L}}_2] - [p^1_{\mathsf{L},n}, h^{\mathsf{L}}_1] + p^1_{\mathsf{L},n} + C_n^{-1} p^1_{\mathsf{L},n+1} C_n, \\
&\left(\beta^{\mathsf{L}}_n\right)^2 = \gamma^{\mathsf{L}}_n\left(2n - 1 + h^{\mathsf{L}}_2(\beta^{\mathsf{L}}_n + \beta^{\mathsf{L}}_{n-1}) + [p^1_{\mathsf{L},n-1}, h^{\mathsf{L}}_2] + h^{\mathsf{L}}_1\right) \\
&\qquad - \left(2n + 3 + h^{\mathsf{L}}_2(\beta^{\mathsf{L}}_{n+1} + \beta^{\mathsf{L}}_n) + [p^1_{\mathsf{L},n}, h^{\mathsf{L}}_2] + h^{\mathsf{L}}_1\right)\gamma^{\mathsf{L}}_{n+1}.
\end{aligned}
$$

In this commutative setting we have

$$
\begin{aligned}
&-\left(h^{\mathsf{L}}_2 \beta^{\mathsf{L}}_n + h^{\mathsf{L}}_1 + (2n+1)\,\mathrm{I}\right)\beta^{\mathsf{L}}_n = h^{\mathsf{L}}_0 + h^{\mathsf{L}}_2\left(\gamma^{\mathsf{L}}_{n+1} + \gamma^{\mathsf{L}}_n\right) - p^1_{\mathsf{L},n} - p^1_{\mathsf{L},n+1}, \\
&(\beta^{\mathsf{L}}_n)^2 + \left(h^{\mathsf{L}}_2(\beta^{\mathsf{L}}_n + \beta^{\mathsf{L}}_{n+1}) + h^{\mathsf{L}}_1 + (2n+3)\,\mathrm{I}\right)\gamma^{\mathsf{L}}_{n+1} \\
&\qquad - \gamma^{\mathsf{L}}_n\left(h^{\mathsf{L}}_2(\beta^{\mathsf{L}}_n + \beta^{\mathsf{L}}_{n-1}) + h^{\mathsf{L}}_1 + (2n-1)\,\mathrm{I}\right) = \mathbf{0}.
\end{aligned}
$$

In terms of

$$
y_n := \frac{h^{\mathsf{L}}_0}{2} + h^{\mathsf{L}}_2 \gamma_n - p^1_{\mathsf{L},n} \qquad \text{and} \qquad \mu_n := h^{\mathsf{L}}_2 \beta^{\mathsf{L}}_n + h^{\mathsf{L}}_1 + (2n+1)\,\mathrm{I},
$$

the above equations reads as

$$
-\mu_n \beta^{\mathsf{L}}_n = y_n + y_{n+1} \qquad \text{and} \qquad \beta^{\mathsf{L}}_n(y_n - y_{n+1}) = \gamma_{n+1}\mu_{n+1} - \gamma_n \mu_{n-1}.
$$

Now, we multiply the second equation by μ_n and taking into account the first one we arrive to

$$
-(y_n + y_{n+1})(y_n - y_{n+1}) = \gamma_{n+1}\mu_n\mu_{n+1} - \gamma_n\mu_{n-1}\mu_n,
$$

and so

$$y_{n+1}^2 - y_n^2 = \gamma_{n+1}\mu_n\mu_{n+1} - \gamma_n\mu_n\mu_{n-1}.$$

Hence,

$$y_{n+1}^2 - y_0^2 = \gamma_{n+1}\mu_n\mu_{n+1} \qquad \text{and} \qquad \beta_n^{\mathsf{L}}\mu_n = -(y_n + y_{n+1})$$

coincide to the ones presented in [9] as discrete Painlevé IV (dPIV) equation. In fact, taking $x_n = \mu_n^{-1}$ we finally arrive to

$$x_n x_{n+1} = \frac{h_2^{\mathsf{L}}\big(y_{n+1} - h_0^{\mathsf{L}}/2 + p_{\mathsf{L},n+1}^1\big)}{y_{n+1}^2 - y_0^2},$$

$$y_n + y_{n+1} = \frac{\big(h_2^{\mathsf{L}}\big)^{-1}}{x_n}\left(h_1^{\mathsf{L}} + (2n+1) - \frac{1}{x_n}\right).$$

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