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ORIGINAL PAPER

The Aboodh Transform Techniques to Ulam Type Stability of Linear Delay Differential Equation

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Abstract
This paper mainly focuses on solving the Ulam–Hyers and Ulam–Hyers–JRassias stability problem of the linear delay differential equation using Aboodh transform technique. In addition, the results are extended to investigate the Mittag–Leffler–Ulam–Hyers and Mittag–Leffler–Ulam–Hyers–JRassias stability of this differential equation. Further, appropriate examples are illustrated to justify the efficiency of the obtained theoretical results.

Keywords Aboodh transform · Linear delay differential equation · Mittag–Leffler–Ulam–Hyers and Mittag–Leffler–Ulam–Hyers–JRassias stability

Mathematics Subject Classification 34A40 · 39B82 · 26D10 · 34K20

Introduction

An intriguing and celebrated talk presented by Ulam [36] in 1940 triggered the study of solving stability problems pertaining to functional equations. Hyers [12] presented an affirmative partial answer to Ulam's question concerning the stability of functional equations. In 1950, Aoki [4] and in 1978 Rassias [27] excellently proved the generalization of the Hyers theorem involving unbounded Cauchy difference. Many investigations ensued to explore the Ulam stability problem of the different functional and differential equations (see [8, 23–25, 32–35]).

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The delayed dynamical systems have attracted increasing interest in recent years from various fields of science and engineering, including sociology, biology, physics, and chemistry, as well as control technology, industrial robotics, communication engineering, and so on [5, 6]. Circuits which include delayed elements have become more important due to the increase in performance of VLSI systems. The two types of circuits which include elements with delay are transmission lines [7] and partial element equivalent circuits [30].

Huang et al. [11], in their research paper, it is described the first-order delay differential equation with Lipschitz condition by successive approximation method. Zada et al. [37] discussed the nonlinear delay differential equation under fixed point technique. In [2] further described the nonlinear delay differential equation of fractional integrable impulses.

In [15], the authors examined the solution of Blasius differential equation with Adomian decomposition method using Mohand transform. Rezaei et al. [29] studied the Ulam-Hyers stability of linear differential equation for nth order via Laplace transform approach. In [3], the authors dealt with the generalized stability results of the linear differential equation for higher order. The various types of Ulam stabilities of linear differential equations obtained through the Laplace transform are available in [18].

Moreover, the solution of Ulam-Hyers stability of various differential equations through the Mahgoub transform was obtained by Jung et al. [14], Aruldass et al. [26], Deepa et al. [9] and Murali et al. [19]. In [22], the authors provided the stability problems of first-order linear differential equations using the method of Fourier transform and Rassias et al. [28], second order linear differential equations using the method of Fourier transform. Mohanapriya et al. [17] demonstrated the Ulam stability problems under the method of Fourier transform for the linear differential equation. Also, Mohanapriya et al. [16] obtained the stability results of second order differential equation through Fourier transform method. Some effective techniques have been developed, for example, integral transform [13, 31].

In [10], the authors investigated the linear differential equations using the Shehu transform and Ulam stability. In [20, 21] have recently focused on the stability result of first-order and second-order linear differential equations through Aboodh transform method.

There are no existing previous studies on the Ulam-Hyers stability of linear delay differential equation with the Aboodh transform. Motivated and inspired by [20], this work employs Aboodh transform to prove Ulam-Hyers and Ulam-Hyers-JRassias stability of linear delay differential equation as follows:

$$\nu^{'}(\ell) - \alpha\nu(\ell - 1) = \beta, \tag{1}$$

with the initial conditions

$$\nu(0) = 0.$$

where α and β are constants and $\nu : \mathcal{I} \to \mathcal{W}$ is a continuously differentiable function of exponential order, $\mathcal{I} = (0, \infty)$ denotes an open interval, $\mathcal{W}$ denotes either the real field $\mathcal{R}$ or the complex field $\mathcal{C}$. Moreover, we extend the results related to the Mittag-Leffler-Ulam-Hyers and Mittag-Leffler-Ulam-Hyers-JRassias stabilities of this differential equation. In addition, appropriate examples are presented to understand the efficiency of the obtained theoretical results.

Preparatory Discussions

Here, we refer to some basic concepts related with Aboodh transform that are useful to obtain the desired results in this study. Throughout this article, $\mathcal{W}$ denotes either the real field $\mathcal{R}$ or the complex field $\mathcal{C}$. A function $\nu : \mathcal{I} \to (0, \infty)$ is of exponential order if there exist constants $A, B \in \mathcal{R}$ such that $|\nu(\ell)| \leq Ae^{Bs}$ for all $\ell \geq 0$.

Definition 1 [1] The Aboodh transform is defined, for a function $\nu(\ell)$ of exponential order, by

$$\mathcal{A}\{\nu(\ell)\} = \mathcal{V}(\omega) = \frac{1}{\omega}\int_0^\infty \nu(\ell)e^{-\omega\ell}d\ell,\ \forall\, \ell \geq 0,\ k_1 \leq \omega \leq k_2,$$

where the operator is $\mathcal{A}$ called the Aboodh transform operator, ω is a variable factor of ℓ and k_1, k_2 are finite or infinite. If $\mathcal{V}(\omega)$ is Aboodh transform of $\nu(\ell)$,then $\nu(\ell) = \mathcal{A}^{-1}\{\mathcal{V}(\omega)\}$ is the inverse of $\mathcal{A}^{-1}$ is the inverse Aboodh operator.

Definition 2 [1] If $\mathcal{A}\{\nu(\ell)\} = \mathcal{V}(\omega)$ and $\mathcal{A}\{g(\ell)\} = \mathcal{G}(\omega)$, then

$$\mathcal{A}\{\nu(\omega) * g(\omega)\} = \mathcal{A}\{\nu(\ell)\}\,\mathcal{A}\{g(\ell)\},$$
$$\mathcal{A}\{\nu(\omega) * g(\omega)\} = \omega\mathcal{V}(\omega)\mathcal{G}(\omega),$$

where $\nu(\omega) * g(\omega)$ is defined by

$$\nu(\omega) * g(\omega) = \int_0^\ell \nu(s)r(\ell - s)ds.$$

Lemma 1 [1] *If* $\mathcal{A}\{\nu(\ell)\} = \mathcal{V}(\omega)$, *then*

- $\mathcal{A}\{\nu^{'}(\omega)\} = \omega\mathcal{V}(\omega) - \frac{\nu(0)}{\omega}$,
- $\mathcal{A}\{\nu^{''}(\omega)\} = \omega^2\mathcal{V}(\omega) - \nu(0) - \frac{\nu'(0)}{\omega}$,
- $\mathcal{A}\{\nu^{(n)}(\omega)\} = \omega^{(n)}\mathcal{V}(\omega) - \sum_{k=0}^{n-1} \frac{\nu^k(0)}{\omega^{2-n+k}}$.

Definition 3 [20] The Mittag-Leffler function of one parameter, denoted by $E_\gamma(z)$, is defined as

$$E_\gamma(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma k + 1)},$$

where $\mathcal{R}e(\gamma) > 0$ and $z, \gamma \in \mathcal{C}$. If we put $\gamma = 1$, then the above equation becomes

$$E_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z.$$

The generalization of $E_\gamma(z)$ is defined as a function

$$E_{\gamma,\delta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma k + \delta)},$$

where $\mathcal{R}e(\gamma) > 0$, $\mathcal{R}e(\delta) > 0$ and $z, \gamma, \delta \in \mathcal{C}$.

Definition 4 [28] Let $\nu : \mathcal{I} \rightarrow \mathcal{W}$ be a continuously differentiable function. Suppose $\nu(\ell)$ satisfies

$$\left|\nu^{'}(\ell) - \alpha\nu(\ell-1) - \beta\right| \leq \varepsilon,\ \forall\, \ell \geq 0, \tag{2}$$

then equation (1) is said to have Ulam-Hyers stability and the delay differential equation (1) has a solution $\nu_a : \mathcal{I} \rightarrow \mathcal{W}$ with the condition that

$$|\nu(\ell) - \nu_a(\ell)| \leq \mathcal{Z}\varepsilon, \forall\, \ell \geq 0,$$

where $\varepsilon > 0$, $\mathcal{Z}$ is a non-negative and Ulam-Hyers stability constant.

Definition 5 [28] Let $\phi : \mathcal{I} \rightarrow (0, \infty)$ be an integrable function. Suppose $\nu : \mathcal{I} \rightarrow \mathcal{W}$ is a continuously differentiable function. If $\nu(\ell)$ satisfies

$$\left|\nu^{'}(\ell) - \alpha\nu(\ell-1) - \beta\right| \leq \phi(\ell)\varepsilon,\ \forall\, \ell \geq 0, \tag{3}$$

then equation (1) is Ulam-Hyers-JRassias stable and the delay differential equation (1) has a solution $\nu_a : \mathcal{I} \rightarrow \mathcal{W}$ with the condition that

$$|\nu(\ell) - \nu_a(\ell)| \leq \mathcal{Z}\phi(\ell)\varepsilon, \forall\, \ell \geq 0,$$

where $\varepsilon > 0$, $\mathcal{Z}$ is a non-negative and Ulam-Hyers-JRassias stability constant.

Definition 6 [28] Let $E_\gamma(\ell)$ denotes Mittag-Leffler function. Let $\nu : \mathcal{I} \rightarrow \mathcal{W}$ be a continuously differentiable function. If $\nu(\ell)$ satisfies

$$\left|\nu^{'}(\ell) - \alpha\nu(\ell-1) - \beta\right| \leq \varepsilon E_\gamma(\ell),\ \forall\, \ell \geq 0, \tag{4}$$

then equation (1) has Mittag-Leffler-Ulam-Hyers stability and the delay differential equation (1) has a solution $\nu_a : \mathcal{I} \rightarrow \mathcal{W}$ with the condition that

$$|\nu(\ell) - \nu_a(\ell)| \leq \mathcal{Z}\varepsilon E_\gamma(\ell), \forall\, \ell \geq 0,$$

where $\varepsilon > 0$, $\mathcal{Z}$ is a non-negative and Mittag-Leffler-Ulam-Hyers stability constant.

Definition 7 [28] Let $\phi : \mathcal{I} \rightarrow (0, \infty)$ is a integrable function and $E_\gamma(\ell)$ denotes Mittag-Leffler function. Let $\nu : \mathcal{I} \rightarrow \mathcal{W}$ be a continuously differentiable function. If $\nu(\ell)$ satisfies

$$\left|\nu^{'}(\ell) - \alpha\nu(\ell-1) - \beta\right| \leq \phi(\ell)\varepsilon E_\gamma(\ell),\ \forall\, \ell \geq 0, \tag{5}$$

then equation (1) is Mittag-Leffler-Ulam-Hyers-JRassias stable and the delay differential equation (1) has a solution $\nu_a : \mathcal{I} \rightarrow \mathcal{W}$ with the condition that

$$|\nu(\ell) - \nu_a(\ell)| \leq \mathcal{Z}\phi(\ell)\varepsilon E_\gamma(\ell), \forall\, \ell \geq 0,$$

where $\varepsilon > 0$, $\mathcal{Z}$ is a non-negative and Mittag-Leffler-Ulam-Hyers-JRassias stability constant.

Main Results

Now, we obtain various types of Ulam stabilities for the equation (1) via Aboodh transform technique.

Theorem 1 *The delay differential equation (1) has Ulam-Hyers stability.*

Proof Let $\nu : \mathcal{I} \to \mathcal{W}$ satisfies (2), for every $\varepsilon > 0$. Then, there exists a solution $\nu_a : \mathcal{I} \to \mathcal{W}$ of (1) with $|\nu(\ell) - \nu_a(\ell)| \leq \mathcal{Z}\varepsilon$, for any $\ell \geq 0$. Choosing a function $d(\ell)$ as follows:

$$d(\ell) = \nu^{'}(\ell) - \alpha\nu(\ell - 1) - \beta. \tag{6}$$

Now, applying the Aboodh integral transform on both sides of (6), we obtain

$$\begin{aligned} \mathcal{A}\{d(\ell)\} &= \mathcal{A}\left\{\nu^{'}(\ell) - \alpha\nu(\ell-1) - \beta\right\}, \\ \mathcal{D}(\omega) &= \omega\mathcal{V}(\omega) - \frac{\nu(0)}{\omega} - \alpha\,\frac{e^{-\omega}}{\omega}\,[\mathcal{V}(\omega)] - \frac{\beta}{\omega^2}, \\ &= \mathcal{V}(\omega)\left(\omega - \alpha\,\frac{e^{-\omega}}{\omega}\right) - \frac{\beta}{\omega^2}. \end{aligned}$$

Thus,

$$\mathcal{V}(\omega) = \frac{\mathcal{D}(\omega) + \frac{\beta}{\omega^2}}{\left(\omega - \alpha\,\frac{e^{-\omega}}{\omega}\right)}. \tag{7}$$

If we put $\nu_a(\ell) = e^{-\alpha\ell}\nu(0)$, then $\nu_a(0) = \nu(0)$ and applying Aboodh integral transform on $\nu_a : \mathcal{I} \to \mathcal{W}$, we have

$$\mathcal{V}_a(\omega) = \frac{\frac{\beta}{\omega^2}}{\left(\omega - \alpha\,\frac{e^{-\omega}}{\omega}\right)}. \tag{8}$$

Hence, we get

$$\mathcal{A}\left\{\nu_a^{'}(\ell) - \alpha\nu_a(\ell-1) - \beta\right\} = \mathcal{V}(\omega)\left(\omega - \alpha\,\frac{e^{-\omega}}{\omega}\right) - \frac{\beta}{\omega^2}.$$

Then, by using (6), we obtain

$$\mathcal{A}\left\{\nu_a^{'}(\ell) - \alpha\nu_a(\ell-1) - \beta\right\} = 0.$$

Since $\mathcal{A}$ is one-to-one operator, we have

$$\nu_a^{'}(\ell) - \alpha\nu_a(\ell-1) - \beta = 0.$$

Hence $\nu_a(\ell)$ is a solution of (1). By (7) and (8), we obtain

$$\mathcal{V}(\omega) - \mathcal{V}_a(\omega) = \frac{\mathcal{D}(\omega)}{\left(\omega - \alpha\,\frac{e^{-\omega}}{\omega}\right)},$$

where $R(\omega) = \frac{1}{\omega\left(\omega - \alpha\,\frac{e^{-\omega}}{\omega}\right)}$ which given $r(\ell) \Rightarrow \mathcal{A}^{-1}\left\{\frac{1}{(\omega^2 - \alpha e^{-\omega})}\right\} = e^{-\alpha\ell}$. These equalities show that

$$\nu(\ell) - \nu_a(\ell) = d(\ell) * r(\ell).$$

Now, applying modulus on both sides, we have

$$|\nu(\ell) - \nu_a(\ell)| = \left|\int_0^{\ell} d(\ell) r(\ell - s) ds\right|.$$

By the inequality (2), $|d(\ell)| \leq \varepsilon$, we have

$$\begin{aligned} \Big|\nu(\ell) - \nu_a(\ell)\Big| &\leq \int_0^\ell \Big|d(\ell)\Big| \Big|r(\ell - s)ds\Big| \\ &\leq \mathcal{Z}\varepsilon, \ \forall \ \ell \geq 0, \ \mathcal{Z} = \int_0^\ell \Big|r(\ell - s)ds\Big|. \end{aligned}$$

Therefore, the equation (1) has Ulam-Hyers stability. □

Theorem 2 *The delay differential equation (1) has Mittag-Leffler-Ulam-Hyers stability.*

Proof Let $\nu : \mathcal{I} \to \mathcal{W}$ satisfies (4), for every $\varepsilon > 0$. Then, there exists a solution $\nu_a : \mathcal{I} \to \mathcal{W}$ of (1) with $|\nu(\ell) - \nu_a(\ell)| \leq \mathcal{Z}\varepsilon E_\gamma(\ell)$, for any $\ell \geq 0$. Let us consider a function $d : \mathcal{I} \to \mathcal{W}$ as follows:

$$d(\ell) = \nu^{'}(\ell) - \alpha\nu(\ell - 1) - \beta. \tag{9}$$

Now, applying the Aboodh integral transform on both sides of (9), we obtain

$$\mathcal{V}(\omega) = \frac{\mathcal{D}(\omega) + \frac{\beta}{\omega^2}}{\left(\omega - \alpha\ \frac{e^{-\omega}}{\omega}\right)}. \tag{10}$$

If we put $\nu_a(\ell) = e^{-\alpha\ell}\nu(0)$, then $\nu_a(0) = \nu(0)$ and applying Aboodh integral transform on $\nu_a : \mathcal{I} \to \mathcal{W}$, we obtain

$$\mathcal{V}_a(\omega) = \frac{\frac{\beta}{\omega^2}}{\left(\omega - \alpha\ \frac{e^{-\omega}}{\omega}\right)}. \tag{11}$$

Hence, we get

$$\mathcal{A}\Big\{\nu_a^{'}(\ell) - \alpha\nu_a(\ell - 1) - \beta\Big\} = \mathcal{V}(\omega)\left(\omega - \alpha\ \frac{e^{-\omega}}{\omega}\right) - \frac{\beta}{\omega^2}.$$

Then, by using (9), we obtain

$$\mathcal{A}\Big\{\nu_a^{'}(\ell) - \alpha\nu_a(\ell - 1) - \beta\Big\} = 0.$$

Since $\mathcal{A}$ is one-to-one operator, then

$$\nu_a^{'}(\ell) - \alpha\nu_a(\ell - 1) - \beta = 0.$$

Hence $\nu_a(\ell)$ is a solution of (1). By (10) and (11), we obtain

$$\mathcal{V}(\omega) - \mathcal{V}_a(\omega) = \frac{\mathcal{D}(\omega)}{\left(\omega - \alpha\ \frac{e^{-\omega}}{\omega}\right)},$$

where $R(\omega) = \frac{1}{\omega\left(\omega - \alpha\ \frac{e^{-\omega}}{\omega}\right)}$ which given $r(\ell) \Rightarrow \mathcal{A}^{-1}\left\{\frac{1}{(\omega^2 - \alpha e^{-\omega})}\right\} = e^{-\alpha\ell}$. These equalities show that

$$\nu(\ell) - \nu_a(\ell) = d(\ell) * r(\ell).$$

Now, applying modulus on both sides, then we have

$$|\nu(\ell) - \nu_a(\ell)| = \left|\int_0^\ell d(\ell)r(\ell - s)ds\right|.$$

By the inequality (4), $|d(\ell)| \leq \varepsilon E_\gamma(\ell)$, we have

$$\begin{aligned}\Big|\nu(\ell) - \nu_a(\ell)\Big| &\leq \int_0^\ell \Big|d(\ell)\Big| \Big|r(\ell - s)ds\Big| \\ &\leq \mathcal{Z}\varepsilon E_\gamma(\ell), \ \forall\, \ell \geq 0, \ \mathcal{Z} = \int_0^\ell \Big|r(\ell - s)ds\Big|.\end{aligned}$$

Therefore, the equation (1) has Mittag-Leffler-Ulam-Hyers stability. □

Corollary 1 *The delay differential equation (1) has Ulam-Hyers-JRassias stability.*

Proof Let $\phi : \mathcal{I} \to (0, \infty)$ be an integrable function and $\nu : \mathcal{I} \to \mathcal{W}$ is satisfies (3), for every $\varepsilon > 0$. Then, there exists a solution $\nu_a : \mathcal{I} \to \mathcal{W}$ of (1) with $|\nu(\ell) - \nu_a(\ell)| \leq \mathcal{Z}\phi(\ell)\varepsilon$, for any $\ell \geq 0$. Define a function $d : \mathcal{I} \to \mathcal{W}$ as follows:

$$d(\ell) = \nu^{'}(\ell) - \alpha\nu(\ell - 1) - \beta. \tag{12}$$

Now, applying the Aboodh integral transform on both sides of (12), we obtain

$$\mathcal{V}(\omega) = \frac{\mathcal{D}(\omega) + \frac{\beta}{\omega^2}}{\left(\omega - \alpha\,\frac{e^{-\omega}}{\omega}\right)}. \tag{13}$$

Thus, applying Aboodh integral transform on $\nu_a : \mathcal{I} \to \mathcal{W}$ leads to

$$\mathcal{V}_a(\omega) = \frac{\frac{\beta}{\omega^2}}{\left(\omega - \alpha\,\frac{e^{-\omega}}{\omega}\right)}. \tag{14}$$

In view of (3), we have $|d(\ell)| \leq \phi(\ell)\varepsilon$. By using the same technique as in Theorem 1, we can easily obtain the rest of the proof. Also, one can prove that

$$\begin{aligned}\Big|\nu(\ell) - \nu_a(\ell)\Big| &\leq \int_0^\ell \Big|d(\ell)\Big| \Big|r(\ell - s)ds\Big| \\ &\leq \mathcal{Z}\phi(\ell)\varepsilon, \ \forall\, \ell \geq 0, \mathcal{Z} = \int_0^\ell \Big|r(\ell - s)ds\Big|.\end{aligned}$$

Therefore, the equation (1) has Ulam-Hyers-JRassias stability. □

Corollary 2 *The delay differential equation (1) has Mittag-Leffler-Ulam-Hyers-JRassias stability.*

Proof Let $\phi : \mathcal{I} \to (0, \infty)$ be an integrable function and $\nu : \mathcal{I} \to \mathcal{W}$ satisfies (5), for every $\varepsilon > 0$. Then, there exists a solution $\nu_a : \mathcal{I} \to \mathcal{W}$ of (1) with $|\nu(\ell) - \nu_a(\ell)| \leq \mathcal{Z}\phi(\ell)\varepsilon E_\gamma(\ell)$, for any $\ell \geq 0$. Now, consider a function $d : \mathcal{I} \to \mathcal{W}$ defined as follows:

$$d(\ell) = \nu^{'}(\ell) - \alpha\nu(\ell - 1) - \beta, \tag{15}$$

In view of (5), we have $|d(\ell)| \leq \phi(\ell)\varepsilon E_\gamma(\ell)$. By utilizing the same technique as in Theorem 2, we can obtain remaining part of the proof. Also, one can easily prove that

$$\begin{aligned}\Big|\nu(\ell) - \nu_a(\ell)\Big| &\leq \int_0^\ell \Big|d(\ell)\Big| \Big|r(\ell - s)ds\Big| \\ &\leq \mathcal{Z}\phi(\ell)\varepsilon E_\gamma(\ell), \ \forall\, \ell \geq 0, \ \mathcal{Z} = \int_0^\ell \Big|r(\ell - s)ds\Big|.\end{aligned}$$

Therefore, the equation (1) has Mittag-Leffler-Ulam-Hyers-JRassias stability. □

Examples

Here, we illustrate appropriate examples to prove the Ulam-Hyers stability of equation (1) to justify our main results.

Example 1 Consider the following linear delay differential equation

$$\nu^{'}(\ell) + \nu(\ell - 1) = 0, \tag{16}$$

with initial condition $\nu(0) = 0$, $\alpha = 1$ and $\beta = 0$.

Letting $d(\ell) = \nu^{'}(\ell) + \nu(\ell - 1)$ in Theorem 1. Applying the Aboodh transform, we obtain

$$\begin{aligned}
\mathcal{A}\{d(\ell)\} &= \mathcal{A}\left\{\nu^{'}(\ell) + \nu(\ell - 1)\right\}, \\
\mathcal{D}(\omega) &= \omega\mathcal{V}(\omega) - \frac{\nu(0)}{\omega} - \frac{e^{-\omega}}{\omega}\left[\mathcal{V}(\omega) - \nu(0) - \nu(\omega)\right], \\
&= \mathcal{V}(\omega)\left(\omega - \frac{e^{-\omega}}{\omega}\right).
\end{aligned}$$

Thus,

$$\mathcal{V}(\omega) = \frac{\mathcal{D}(\omega)}{\left(\omega - \frac{e^{-\omega}}{\omega}\right)}. \tag{17}$$

If we put $\nu_a(\ell) = e^{-\ell}\nu(0)$, then $\nu_a(0) = \nu(0)$ and applying Aboodh integral transform on $\nu_a : \mathcal{I} \to \mathcal{W}$ as follows:

$$\mathcal{V}_a(\omega) = \frac{1}{\left(\omega - \frac{e^{-\omega}}{\omega}\right)}. \tag{18}$$

Hence $\nu_a(\ell)$ is a solution of equation (1). By (7) and (8) that

$$\mathcal{V}(\omega) - \mathcal{V}_a(\omega) = \frac{\mathcal{D}(\omega)}{\left(\omega - \frac{e^{-\omega}}{\omega}\right)}.$$

Now, applying modulus on both sides, we have

$$|\nu(\ell) - \nu_a(\ell)| = \left|\int_0^{\ell} d(\ell) r(\ell - s) ds\right|.$$

Let $|d(\ell)| \leq \varepsilon$, we have

$$\begin{aligned}
\left|\nu(\ell) - \nu_a(\ell)\right| &\leq \int_0^{\ell} \left|d(\ell)\right| \left|r(\ell - s) ds\right| \\
&\leq \mathcal{Z}\varepsilon,\ \forall\, \ell \geq 0,\ \mathcal{Z} = \int_0^{\ell} \left|r(\ell - s) ds\right|.
\end{aligned}$$

Therefore, the equation (16) has Ulam-Hyers stability.

Example 2 Consider the following linear delay differential equation

$$\nu^{'}(\ell) + \nu(\ell - 1) = 2, \tag{19}$$

with initial condition $\nu(0) = 0$, $\alpha = 1$ and $\beta = 2$.

Taking $d(\ell) = \nu'(\ell) + \nu(\ell-1) - 2$ in Theorem 1 and applying Aboodh transform, we obtain

$$\mathcal{A}\{d(\ell)\} = \mathcal{A}\left\{\nu'(\ell) + \nu(\ell-1) - 2\right\},$$
$$\mathcal{D}(\omega) = \omega\mathcal{V}(\omega) - \frac{\nu(0)}{\omega} - \frac{e^{-\omega}}{\omega}\left[\mathcal{V}(\omega) - \nu(0) - \nu(\omega)\right] - \frac{2}{\omega^2},$$
$$= \mathcal{V}(\omega)\left(\omega - \frac{e^{-\omega}}{\omega}\right) - \frac{2}{\omega^2}.$$

Thus,

$$\mathcal{V}(\omega) = \frac{\mathcal{D}(\omega) + \frac{2}{\omega^2}}{\left(\omega - \frac{e^{-\omega}}{\omega}\right)}. \tag{20}$$

Applying Aboodh integral transform on $\nu_a : \mathcal{I} \to \mathcal{W}$, we have

$$\mathcal{V}_a(\omega) = \frac{1}{\left(\omega - \frac{e^{-\omega}}{\omega}\right)}. \tag{21}$$

By utilizing the same technique as in Theorem 1, we can obtain the rest of the proof. Also, one can easily prove that

$$\left|\nu(\ell) - \nu_a(\ell)\right| \le \int_0^\ell \left|d(\ell)\right| \left|r(\ell - s)ds\right|$$
$$\le \mathcal{Z}\varepsilon,\ \forall \ell \ge 0,\ \mathcal{Z} = \int_0^\ell \left|r(\ell - s)ds\right|.$$

Therefore, the equation (19) has Ulam-Hyers stability.

Conclusion

In this paper, we have studied the Ulam-Hyers stability of linear delay differential equations through Aboodh transform technique. Further, we developed Mittag-Leffler-Ulam-Hyers stability for the proposed problem. We have illustrated suitable examples to show the efficiency of the obtained theoretical results. Delay differential equations are commonly used to model various engineering systems with time delays. They allow for the analysis of stability results and overall system performance in the presence of delays. This work could recommend investigating stability of delay differential equations via various integral transforms in future and propose several physical phenomena through this novel method.

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Data Availability No data were used to support this work.

Declarations

Conflict of interest The authors declare that they have no competing interests.

Ethical Approval This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. Aggarwal, S., Chauhan, R.: A comparative study of Mohand and Aboodh transforms. Int. J. Res. Advent Technol. **7**(1), 520–529 (2019)
2. Akbar, Z., Shah, S.O.: Hyers–Ulam stability of first-order non-linear delay differential equations with fractional integrable impulses. Hacettepe J. Math. Stat. **47**(5), 1196–1205 (2018)
3. Alqifiary, Q.H., Jung, S.M.: Laplace transform and generalized Hyers–Ulam stability of linear differential equations. Electron. J. Differ. Equ. **2014**(80), 1–11 (2014)
4. Aoki, T.: On the stability of the linear transformation in Banach spaces. J. Math. Soc. Jpn. **2**(1–2), 64–66 (1950)
5. Baker, C.T., Paul, C.A., Willé, D.R.: Issues in the numerical solution of evolutionary delay differential equations. Adv. Comput. Math. **3**, 171–196 (1995)
6. Bellen, A., Guglielmi, N., Ruehli, A.E.: Methods for linear systems of circuit delay differential equations of neutral type. IEEE Trans. Circuits Syst. I Fundam. Theory Appl. **46**(1), 212–215 (1999)
7. Brayton, R.K.: Small-signal stability criterion for electrical networks containing lossless transmission lines. IBM J. Res. Dev. **12**(6), 431–440 (1968)
8. Cădariu, L., Găvruţa, L., Găvruţa, P.: Fixed points and generalized Hyers–Ulam stability. Abstract Appl. Anal. **2012**, 712743 (2012)
9. Deepa, S., Bowmiya, S., Ganesh, A., Govindan, V., Park, C., Lee, J.R.: Mahgoub transform and Hyers–Ulam stability of n^{th} order linear differential equations. AIMS Math. **7**(4), 4992–5014 (2022)
10. Govindan, V., Noeiaghdam, S., Fernandez-Gamiz, U., Sankeshwari, S.N., Arulprakasam, R., Li, B.Z.: Shehu integral transform and Hyers–Ulam stability of nth order linear differential equations. Sci. Afr. **18**(e01427), 1–22 (2022)
11. Huang, J., Li, Y.: Hyers–Ulam stability of delay differential equations of first order. Math. Nachr. **289**(1), 60–66 (2016)
12. Hyers, D.H.: On the stability of the linear functional equation. Proc. Natl. Acad. Sci. **27**(4), 222–224 (1941)
13. Jafari, H., Manjarekar, S.: A modification on the new general integral transform. Adv. Math. Models Appl. **7**(3), 253–263 (2022)
14. Jung, S.M., Ponmana Selvan, A., Murali, R.: Mahgoub transform and Hyers–Ulam stability of first-order linear differential equations. J. Math. Inequal. **15**(3), 1201–1218 (2021)
15. Khandelwal, R., Khandelwal, Y.: Solution of Blasius equation concerning with Mohand transform. Int. J. Appl. Comput. Math. **6**(5), 1–9 (2020)
16. Mohanapriya, A., Ganesh, A., Gunasekaran, N.: The Fourier transform approach to Hyers–Ulam stability of differential equation of second order. J. Phys. Conf. Ser. **1597**(1), 012027 (2020)
17. Mohanapriya, A., Park, C., Ganesh, A., Govindan, V.: Mittag–Leffler–Hyers–Ulam stability of differential equation using Fourier transform. Adv. Differ. Equ. **2020**(389), 1–16 (2020)
18. Murali, R., Selvan, A.P.: Mittag–Leffler–Hyers–Ulam stability of a linear differential equations of first order using Laplace transforms. Can. J. Appl. Math. **2**(2), 47–59 (2020)
19. Murali, R., Ponmana Selvan, A., Baskaran, S.: Stability of linear differential equation of higher order using Mahgoub transforms. J. Math. Comput. Sci. **30**, 1–9 (2023)
20. Murali, R., Ponmana Selvan, A., Baskaran, S., Park, C., Lee, J.R.: Hyers–Ulam stability of first-order linear differential equations using Aboodh transform. J. Inequal. Appl. **2021**(133), 1–18 (2021)
21. Murali, R., Ponmana Selvan, A., Park, C., Lee, J.R.: Aboodh transform and the stability of second order linear differential equations. Adv. Differ. Equ. **2021**(296), 1–18 (2021)
22. Murali, R., Ponmana Selvan, A., Park, C.: Ulam stability of linear differential equations using Fourier transform. AIMS Math. **5**(2), 766–780 (2020)
23. Ponmana Selvan, A., Sabarinathan, S., Selvam, A.: Approximate solution of the special type differential equation of higher order using Taylor's series. J. Math. Comput. Sci. **27**(2), 131–141 (2022)

24. Qarawani, M.N.: Hyers–Ulam stability of linear and nonlinear differential equation of second order. Int. J. Appl. Math. **1**(4), 422–432 (2012)
25. Qarawani, M.N.: Hyers–Ulam stability of a generalized second order nonlinear differential equation. Appl. Math. **3**(12), 1857–1861 (2012)
26. Raj Aruldass, A., Pachaiyappan, D., Park, C.: Hyers–Ulam stability of second-order differential equations using Mahgoub transform. Adv. Differ. Equ. **2021**, 1–10 (2021)
27. Rassias, Th.M.: On the stability of the linear mappings in Banach Spaces. Proc. Am. Math. Soc. **72**(2), 297–300 (1978)
28. Rassias, J.M., Murali, R., Selvan, A.P.: Mittag–Leffler–Hyers–Ulam stability of linear differential equations using Fourier transforms. J. Comput. Anal. Appl. **29**(1), 68–85 (2021)
29. Rezaei, H., Jung, S.M., Rassias, T.M.: Laplace transform and Hyers–Ulam stability of linear differential equations. J. Math. Anal. Appl. **403**(1), 244–251 (2013)
30. Ruehli, A.E.: Equivalent circuit models for three-dimensional multiconductor systems. IEEE Trans. Microw. Theory Tech. **22**(3), 216–221 (1974)
31. Sayevand, K., Jafari, H.: A promising coupling of Daftardar–Jafari method and He's fractional derivation to approximate solitary wave solution of nonlinear fractional KDV equation. Adv. Math. Models Appl. **7**(2), 121–129 (2022)
32. Selvam, A., Sabarinathan, S., Noeiaghdam, S., Govindan, V.: Fractional Fourier transform and Ulam stability of fractional differential equation with fractional Caputo-type derivative. J. Funct. Spaces **2022**, 3777566 (2022)
33. Selvam, A., Sabarinathan, S., Senthil Kumar, B.V., Byeon, H., Guedri, K., Eldin, S.M., Khan, M.I., Govindan, V.: Ulam–Hyers stability of tuberculosis and COVID-19 co-infection model under Atangana–Baleanu fractal-fractional operator. Sci. Rep. **13**(1), 9012 (2023)
34. Sivashankar, M., Sabarinathan, S., Govindan, V., Fernandez-Gamiz, U., Noeiaghdam, S.: Stability analysis of COVID-19 outbreak using Caputo-Fabrizio fractional differential equation. AIMS Math. **8**(2), 2720–2735 (2023)
35. Sivashankar, M., Sabarinathan, S., Sooppy Nisar, K., Ravichandran, C., Senthil Kumar, B.V.: Some properties and stability of Helmholtz model involved with nonlinear fractional difference equations and its relevance with quadcopter, Chaos. Solitons Fractals **168**, 113161 (2023)
36. Ulam, S.M.: Problems in Modern Mathematics. Wiley, New York (1964)
37. Zada, A., Faisal, S., Li, Y.: On the Hyers–Ulam stability of first-order impulsive delay differential equations. J. Funct. Spaces **2016**, 8164978 (2016)

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