Explicit criterion of uniform LP duality for linear problems of copositive optimization *

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Abstract

An uniform LP duality is an useful property of conic matrix systems. A consistent linear conic optimization problem yields uniform LP duality if for any linear cost function, for which the primal problem has finite optimal value, the corresponding Lagrange dual problem is attainable and the duality gap vanishes.

In this paper, we establish new necessary and sufficient conditions guaranteing the uniform LP duality for linear problems of Copositive Programming and formulate these conditions in different equivalent forms. The main results are obtained using an approach developed in previous papers of the authors and based on a concept of immobile indices that permits alternative representations of the set of feasible solutions.

Keywords: Copositive Programming, uniform LP duality, immobile indices, duality gap

1 Introduction

Conic optimization is a subfield of convex optimization devoted to problems of minimizing a convex function over an intersection of an affine subspace and a convex cone. Conic problems form a broad and important class of optimization problems since, according to [18], any convex optimization problem can be represented as a conic one. This class includes some of the most well-known types of convex problems, such as linear and semidefinite programming problems ((LP) and (SDP), respectively). Many problems of semi-infinite programming (SIP) consisting in optimization of a cost function w.r.t. an infinite number of functional constraints, can be also considered as conic optimization problems.

Copositive Programming (CoP) can be thought of as a special case of SIP and a generalization of SDP. In CoP, a linear function is optimized over a cone of matrices that are positive semidefinite in the non-negative ortant $\mathbb{R}^p_+$ (*copositive matrices*). Formally, problems of CoP are very similar to that of SDP, but CoP deals with more complex and less studied problems than SDP. Being

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a fairly new field of research, CoP has already gained popularity, as it has been proven to be very useful in modeling particularly complex problems of convex optimization, graph theory, algebra, and different applications (see, for example, [1, 6], and the references there).

Optimality conditions and the duality relations are among the most emerging optimization topics, and the importance of studying them has long been recognized (see e.g. [2, 5, 10, 17, 21], and the references therein). Duality plays a central role in testing optimality, identifying infeasibility, establishing lower-bounds of optimal objective value, design and analysis of iterative algorithms.

Traditionally, for a given (primal) convex problem, based on its initial data, the Lagrangian dual problem is constructed. The difference between optimal values of primal and dual cost functions is called the *duality gap*.

The primal and the (Lagrange) dual problems are closely related. The strength of this relationship depends on the initial problem data, which specify the constraints and the cost function. Roughly speaking, a pair of dual problems is said to satisfy (i) a *weak duality* if the duality gap is non-negative, (ii) a *strong duality* if, for a given cost function, the duality gap is zero, and (iii) an *uniform duality* if the duality gap is zero for any cost function.

It is well-known that the strong duality is guaranteed unconditionally only for the LP problems, while for most important classes of convex and conic problems this property is satisfied only under certain rather strong assumptions. Many works are devoted to study of these assumptions (see e.g. [3, 7, 9], and the references therein).

For a linear SIP problem, one of the known criteria of the uniform duality is the closedness of some cone built on the basis of the problem's data (see [5]). The criterion can be applied to any linear SIP problems, but it has an implicit form. That is why researchers often try to find more explicit conditions for the uniform duality, taking into account the specifics of the problems under consideration.

The concept of a *uniform LP duality* was considered in the work of Duffin et.al ([5]) for linear SIP problems, and in [8], it has been used for a wider class of convex SIP problems in the form of an *uniform convex duality*. In [17, 20, 22], the uniform duality property was studied for certain classes of linear conic, convex SIP and SDP problems and the conditions guaranteing that this property is satisfied, were deduced.

Not much literature is available for optimality and duality conditions for CoP problems. Moreover, the strong/uniform duality in CoP is not easy to establish due to intrinsic complexity of copositive problems connected with the fact that the cone of copositive matrices and the corresponding dual cone of *completely positive* matrices do not possess some "good" properties: they are neither self-dual, no facially exposed, no symmetric. At the same time, a more in-depth study of the duality issues and the description of explicit criteria for the fulfillment of the properties of strong/uniform duality is an extremely important challenge not only for the theory of the CoP, but also for the development of algorithms and numerical applications.

The aim of this paper is to establish new necessary and sufficient conditions guaranteing the uniform LP duality for linear CoP problems, and to formulate these conditions in different equivalent forms thus broadening their scope. The main results are obtained on the base of an approach developed in previous papers of the authors. This approach is based on a concept of immobile indices and first was described for SIP problems (see, for example, [11]), and then applied to various classes of convex conic problems in [10, 14], and others. The concept of immobile indices and the properties of the sets generated by these indices, make it possible to constructively represent for the CoP problem some important subcones used in conic optimization (the faces of the copositive cone, in particular, the minimal face containing the feasible set) to obtain new CQ-free strong dual formulations and optimality conditions. In this paper, the set of immobile indices is used to obtain new criteria of the uniform duality for linear CoP problems.

The remainder of this paper is organized as follows. The problem's statement and relevant research is overviewed in section 2. The main results of the paper, new necessary and sufficient conditions of uniform duality for linear copositive problems, are formulated and proved in section 3. Several equivalent formulations of the uniform duality conditions from section 3, are deduced in section 4. Section 5 contains examples that confirm that the conditions obtained in the paper are essential. Some comparison with known results is given. In section 6, we analyze the uniform duality conditions for SIP problems applied to CoP. We show that results obtained in this paper allow one to reformulate these conditions in a more explicit form. The final section 7 contains some conclusions, and several technical proofs are situated in Appendix.

2 Problem statement and preliminary results

Given a finite-dimensional vector space $\mathfrak{X}$, let's, first, recall some generally accepted definitions.

A set $C \subset \mathfrak{X}$ is *convex* if for any $x, y \in C$ and any $\alpha \in [0,1]$, it holds $\alpha x + (1-\alpha)y \in C$. A set $K \subset \mathfrak{X}$ is a *cone* if for any $x \in K$ and any $\alpha > 0$, it holds $\alpha x \in K$. Given a cone $K \subset \mathfrak{X}$, its *dual cone* K^* is given by

$$K^* := \{x \in \mathfrak{X} : \langle x, y\rangle \geq 0 \ \forall y \in K\}.$$

Given a set $\mathcal{B}\mathfrak{X}$, denote by conv$\mathcal{B}$ its *convex hull*, i.e., the minimal (by inclusion) convex set, containing this set, by span$(\mathcal{B})$ its *span*, i.e., the smallest linear subspace containing $\mathcal{B}$, and by cone$\mathcal{B}$ its *conic hull*, i.e the set of all conic combinations of the points of $\mathcal{B}$. In what follows, we will denote by cl $\mathcal{B}$ the *closure* of the set $\mathcal{B}$, by int $\mathcal{B}$ its *interior*, and by relint $\mathcal{B}$ its *relative interior*.

Given an integer $p > 1$, consider the vector space $\mathbb{R}^p$ with the standard orthogonal basis $\{\mathbf{e}_k,\ k = 1, 2, \dots, p\}$. Denote by $\mathbb{R}^p_+$ the set of all p - vectors with non-negative components, by $\mathcal{S}^p$ the space of real symmetric $p \times p$ matrices, and by $\mathcal{S}^p_+$ the cone of symmetric positive semidefinite $p \times p$ matrices. The space $\mathcal{S}^p$ is considered here as a vector space with the *trace inner product* $A \bullet B :=$ trace$\,(AB)$.

In this paper, we deal with special classes of cones, the elements of which are matrices, in particular, with the cones of *copositive* and *completely positive* matrices.

Let $\mathcal{COP}^p$ denote the cone of symmetric copositive $p \times p$ matrices:

$$\mathcal{COP}^p := \{D \in \mathcal{S}^p : \mathbf{t}^\top D\mathbf{t} \geq 0 \ \forall \mathbf{t} \in \mathbb{R}^p_+\}.$$

Consider a compact subset of $\mathbb{R}^p_+$ in the form of the simplex

$$T := \{\mathbf{t} \in \mathbb{R}^p_+ : \mathbf{e}^\top \mathbf{t} = 1\} \tag{1}$$

with $\mathbf{e} = (1, 1, ..., 1)^\top \in \mathbb{R}^p$. It is evident that the cone $\mathcal{COP}^p$ can be equivalently described as follows:

$$\mathcal{COP}^p = \{D \in \mathcal{S}^p : \ \mathbf{t}^\top D\mathbf{t} \geq 0 \ \forall \mathbf{t} \in T\}. \tag{2}$$

The dual cone to $\mathcal{COP}^p$ is the cone of completely positive matrices defined as

$$(\mathcal{COP}^p)^* = \mathcal{CP}^p : = \text{conv}\{\mathbf{t}\mathbf{t}^\top : \ \mathbf{t} \in \mathbb{R}^p_+\}.$$

The cones of copositive and completely positive matrices are known to be *proper* cones, which means that they are closed, convex, pointed, and full-dimensional.
Consider a linear copositive programming problem in the form

$$\mathbf{P}: \qquad \min_{\mathbf{x}\in\mathbb{R}^n} \ \mathbf{c}^\top \mathbf{x} \ \text{ s.t. } \mathcal{A}(\mathbf{x}) \in \mathcal{COP}^p,$$

where $\mathbf{x} = (x_1, ..., x_n)^\top$ is the vector of decision variables, the constraint matrix function $\mathcal{A}(\mathbf{x})$ is defined as $\mathcal{A}(\mathbf{x}) := \sum_{m=1}^{n} A_m x_m + A_0$; vector $\mathbf{c} \in \mathbb{R}^n$ and matrices $A_m \in \mathcal{S}^p$, $i = 0, 1, \ldots, n$ are given. Denote by X the set of feasible solutions of this problem:

$$X = \{\mathbf{x} \in \mathbb{R}^n : \mathcal{A}(\mathbf{x}) \in \mathcal{COP}^p\}.$$

For the problem ($\mathbf{P}$), the *Lagrange dual problem* takes the form:

$$\mathbf{D}: \qquad \max -U \bullet A_0, \text{ s.t. } U \bullet A_m = c_m \ \forall m = 1, 2, ..., n; \ U \in \mathcal{CP}^p.$$

In what follows, for an optimization problem ($\mathbf{Q}$), $Val(\mathbf{Q})$ denotes the optimal value of the objective function in the problem ($\mathbf{Q}$) (shortly, the optimal value of the problem ($\mathbf{Q}$)).

It is a known fact (see, for example, [13] and section 5 below) that for CoP problems, the optimal values $Val(\mathbf{P})$ and $Val(\mathbf{D})$ of the primal problem ($\mathbf{P}$) and the corresponding Lagrange dual problem ($\mathbf{D}$) are not necessarily equal, even if they exist and are finite. A situation where, assuming $Val(\mathbf{P}) > -\infty$, the problem ($\mathbf{D}$) has optimal solution and the so-called *duality gap*, the difference $Val(\mathbf{P}) - Val(\mathbf{D})$ equals to zero, is called a *strong duality*. The paper [13] is devoted to strong dual formulations for copositive problems that differ from the Lagrange dual problem ($\mathbf{D}$).
In this paper, for linear CoP problems, we consider a slightly different duality property of their feasible sets that guarantees vanishing of the duality gap for all cost vectors $\mathbf{c}$. Since this property is related to the constraint system $\mathcal{A}(\mathbf{x}) \in \mathcal{COP}^p$ of the problem ($\mathbf{P}$), we will refer to this property as to a property of this system.

Definition 1 *A consistent system $\mathcal{A}(\mathbf{x}) \in \mathcal{COP}^p$ yields the property of uniform LP duality if for all $\mathbf{c} \in \mathbb{R}^n$, such that the optimal value of the problem ($\mathbf{P}$) is finite ($Val(\mathbf{P}) > -\infty$), the corresponding Lagrange dual problem ($\mathbf{D}$) has an optimal solution, and it holds $Val(\mathbf{P}) = Val(\mathbf{D})$.*

It is known that under the Slater condition (the property that for some $\mathbf{x} \in \mathbb{R}^n$, it holds $\mathcal{A}(\mathbf{x}) \in \text{int}(\mathcal{COP}^p)$), the system $\mathcal{A}(\mathbf{x}) \in \mathcal{COP}^p$ yields the uniform LP duality property.

Given the set X of feasible solutions of the problem ($\mathbf{P}$), denote by T_{im} the set of *normalized immobile indices* of constraints in this problem:

$$T_{im} := \{\mathbf{t} \in T : \mathbf{t}^\top \mathcal{A}(\mathbf{x})\mathbf{t} = 0 \ \forall \mathbf{x} \in X\}.$$

Some properties of the set T_{im} in copositive problems were established in our previous works (see e.g. [10, 12, 15]). In particular, it is known that the set T_{im} is either empty or an union of a finite number of convex bounded polyhedra. Also, it was shown in [10] that the emptiness of the set T_{im} is equivalent to the fulfillment of the Slater condition.

Suppose that the set T_{im} is not empty and denote by

$$\mathcal{T} := \{\boldsymbol{\tau}(j), j \in J\},\ |J| < \infty,$$

the set of vertices of its convex hull convT_{im}. It was shown in [10] that

$$X \subset Z := \{\mathbf{x} \in \mathbb{R}^n : \mathcal{A}(\mathbf{x})\boldsymbol{\tau}(j) \geq 0,\ j \in J\}. \tag{3}$$

Denote $P := \{1, ..., p\}$ and introduce the following sets:

$$\overline{M}(j) := \{k \in P : \mathbf{e}_k^\top \mathcal{A}(\mathbf{x})\boldsymbol{\tau}(j) = 0\ \forall \mathbf{x} \in Z\},\ j \in J, \tag{4}$$

$$M(j) = \{k \in P : \mathbf{e}_k^\top \mathcal{A}(\mathbf{x})\boldsymbol{\tau}(j) = 0\ \forall \mathbf{x} \in X\},\ j \in J, \tag{5}$$

$$N_*(j) := \{k \in P :\ \exists\, \mathbf{x}(k,j) \in X \text{ such that } \mathbf{e}_k^\top \mathcal{A}(\mathbf{x}(k,j))\boldsymbol{\tau}(j) = 0\},\ j \in J. \tag{6}$$

The following lemma is proved in [14].

Lemma 1 *Given a consistent system $\mathcal{A}(\mathbf{x}) \in \mathcal{COP}^p$ with the corresponding set of normalized immobile indices of constraints T_{im} and the sets $\overline{M}(j)$, $M(j)$, $j \in J$, defined in (4) and (5) respectively, it holds $\overline{M}(j) = M(j)$ $\forall j \in J$.*

It was shown in [12] that the problem ($\mathbf{P}$) is equivalent to the following problem:

$$\mathbf{P}_* : \qquad \min \mathbf{c}^\top \mathbf{x}$$
$$\text{s.t. } \mathbf{t}^\top \mathcal{A}(\mathbf{x})\mathbf{t} \geq 0\ \forall \mathbf{t} \in \Omega,\ \mathbf{e}_k^\top \mathcal{A}(\mathbf{x})\boldsymbol{\tau}(j) \geq 0\ \forall k \in N_*(j)\ \forall j \in J, \tag{7}$$

and there exists a *minimally active* feasible solution $\mathbf{x}^* \in X$ such that

$$\mathbf{t}^\top \mathcal{A}(\mathbf{x}^*)\mathbf{t} > 0\ \forall \mathbf{t} \in T \setminus T_{im}, \tag{8}$$
$$\mathbf{e}_k^\top \mathcal{A}(\mathbf{x}^*)\boldsymbol{\tau}(j) = 0,\ \forall k \in M(j);\ \mathbf{e}_k^\top \mathcal{A}(\mathbf{x}^*)\boldsymbol{\tau}(j) > 0\, \forall k \in P \setminus M(j), \forall j \in J.$$

Here and in what follows, we will use the set

$$\Omega := \{\mathbf{t} \in T : \rho(\mathbf{t}, \text{conv}T_{im}) \geq \sigma\} \subset T \setminus T_{im}, \tag{9}$$

where $\sigma := \min\{\tau_k(j), k \in P_+(\boldsymbol{\tau}(j)), j \in J\} > 0$, $P_+(\mathbf{t}) := \{k \in P : t_k > 0\}$ for $\mathbf{t} = (t_k, k \in P)^\top \in \mathbb{R}^p_+$, $\rho(\mathbf{t}, \mathcal{B}) = \min\limits_{\boldsymbol{\tau} \in \mathcal{B}} \sum\limits_{k \in P} |t_k - \tau_k|$ for some set $\mathcal{B} \subset \mathbb{R}^p$.

Remark 1 *In [12], it was formulated a similar problem equivalent to the problem* ($\mathbf{P}$). *This problem has additional constraints $\mathbf{e}_k^\top \mathcal{A}(\mathbf{x})\boldsymbol{\tau}(j) \geq 0$ for all $k \in P \setminus (N_*(j))$, $j \in J$. It worth mentioning that these constraints can be omitted since they are non-active for all $\mathbf{x} \in X$.*

For $\mathbf{t} \in T$ and $k \in P$, denote

$$\bar{\mathbf{b}}(k, \mathbf{t}) = \begin{pmatrix} \mathbf{e}_k^\top A_m \mathbf{t} \\ m = 0, 1, ..., n \end{pmatrix} \in \mathbb{R}^{n+1}, \mathbf{a}(\mathbf{t}) = \begin{pmatrix} \mathbf{t}^\top A_m \mathbf{t} \\ m = 0, 1, ..., n \end{pmatrix} = \sum_{k \in P_+(\mathbf{t})} t_k \bar{\mathbf{b}}(k, \mathbf{t}) \in \mathbb{R}^{n+1}, \tag{10}$$

$$\mathbf{b}(k, j) = \begin{pmatrix} \mathbf{e}_k^\top A_m \boldsymbol{\tau}(j) \\ m = 0, 1, ..., n \end{pmatrix} = \bar{\mathbf{b}}(k, \boldsymbol{\tau}(j)) \in \mathbb{R}^{n+1}. \tag{11}$$

Proposition 1 *In the terminology of Lemma 1, for any $j_0 \in J$ and $k_0 \in M(j_0)$, there exist numbers $\alpha_{kj} = \alpha_{kj}(k_0, j_0)$, $k \in M(j)$, $j \in J$, such that*

$$-\mathbf{b}(k_0, j_0) = \sum_{j\in J} \sum_{k\in M(j)} \alpha_{kj}\mathbf{b}(k, j), \ \alpha_{kj} \geq 0 \ \forall k \in M(j), \ j \in J, \tag{12}$$

where vectors $\mathbf{b}(k, j)$ are defined in (11).

Proof. Using the notation introduced above, the set Z defined in (3) can be represented as

$$Z = \{\mathbf{x} \in \mathbb{R}^n : (1, \mathbf{x}^\top)\mathbf{b}(k, j) \geq 0 \ \forall k \in P, \forall j \in J\}. \tag{13}$$

Consider the following LP problem:

$$\mathbf{LP}_* : \qquad \max(1, \mathbf{z}^\top)\mathbf{b}(k_0, j_0) \quad \text{s.t. } \mathbf{z} \in Z.$$

Due to Lemma 1, we have $k_0 \in M(j_0) = \overline{M}(j_0)$ and it follows from the definition of the set $\overline{M}(j_0)$ that $Val(\mathbf{LP}_*) = 0$. Hence a vector $\mathbf{x}^*$ satisfying (8) is an optimal solution of the problem $(\mathbf{LP}_*)$. Taking into account representation (13) of the set Z and relations (8) we see that relations (12) are necessary and sufficient optimality conditions for $\mathbf{x}^*$ in the problem $(\mathbf{LP}_*)$. □

Let us partition the set J into subsets $J(s)$, $s \in S$, as it was done in [15]. Then (see [15])

$$T_{im} = \bigcup_{s\in S} T_{im}(s), \ T_{im}(s) := \text{conv}\{\boldsymbol{\tau}(j), j \in J(s)\}, s \in S, \tag{14}$$

$$P_*(s) := \bigcup_{j\in J(s)} P_+(\boldsymbol{\tau}(j)) \subset M(j) \ \forall j \in J(s), \ \forall s \in S. \tag{15}$$

Proposition 2 *Consider a consistent system $\mathcal{A}(\mathbf{x}) \in \mathcal{COP}^p$ with the corresponding sets T_{im}, $\mathcal{T}$, $M(j)$, $j \in J$, and the vectors $\mathbf{a}(\mathbf{t}), \mathbf{b}(k, j), k \in M(j)$, $j \in J$ defined above. The following inclusions hold true:*

$$\mathbf{a}(\mathbf{t}) \in \text{cone}\{\mathbf{b}(k, j), \ k \in M(j), \ j \in J\} \ \ \forall \mathbf{t} \in T_{im}. \tag{16}$$

Proof. Consider any $\mathbf{t} \in T_{im}$. It follows from (14) that $\mathbf{t} \in T_{im}(s)$ with some $s \in S$. Hence

$$\mathbf{t} = \sum_{j\in J(s)} \alpha_j \boldsymbol{\tau}(j), \ \alpha_j \geq 0 \ \forall j \in J(s); \ \sum_{j\in J(s)} \alpha_j = 1. \tag{17}$$

Then we obtain

$$\mathbf{a}(\mathbf{t}) = \sum_{k\in P_+(\mathbf{t})} t_k \bar{\mathbf{b}}(k, \mathbf{t}) = \sum_{k\in P_+(\mathbf{t})} t_k \sum_{j\in J(s)} \alpha_j \mathbf{b}(k, j) = \sum_{j\in J(s)} \sum_{k\in M(j)} t_k \alpha_j \mathbf{b}(k, j). \tag{18}$$

Here we took into account that $P_+(\mathbf{t}) \subset \bigcup_{j\in J(s)} P_+(\boldsymbol{\tau}^*(j)) = P_*(s)$ and (15).

Since $t_k\alpha_j \geq 0$ for all $k \in M(j)$ and $j \in J(s) \subset J$, we conclude from (18) that inclusions (16) take place. □

3 Necessary and sufficient uniform LP duality conditions

In this section, we will prove two statements containing new necessary and sufficient uniform LP duality conditions for linear CoP systems.

Proposition 3 *A consistent linear CoP system $\mathcal{A}(\mathbf{x}) \in \mathcal{COP}^p$ yields the uniform LP duality iff the following relations hold:*

$$\mathbf{b}(k,j) \in \text{cone}\{\mathbf{a}(\mathbf{t}), \mathbf{t} \in T\} \ \forall k \in N_*(j), \quad \forall j \in J. \tag{19}$$

Proof. Notice that if $T_{im} = \emptyset$, then $J = \emptyset$, and we consider that conditions (19) are fulfilled.
$\Rightarrow$) Suppose that the consistent system $\mathcal{A}(\mathbf{x}) \in \mathcal{COP}^p$ yields the uniform LP duality. Then for any $\mathbf{c} \in \mathbb{R}^n$ for which $Val(\mathbf{P}) > -\infty$, there exists a matrix $U = U(\mathbf{c})$ in the form

$$U = \sum_{i \in I} \alpha_i \mathbf{t}(i)(\mathbf{t}(i))^\top, \ \alpha_i > 0, \ \mathbf{t}(i) \in T, \ i \in I, \ |I| < \infty, \tag{20}$$

such that

$$A_m \bullet U = c_m, \ m = 1, ..., n; \ A_0 \bullet U = -Val(\mathbf{P}). \tag{21}$$

For fixed $j \in J$ and $k \in N_*(j)$, consider the problem $(\mathbf{P})$ with

$$\mathbf{c}^\top = (c_m = \mathbf{e}_k^\top A_m \boldsymbol{\tau}(j), m = 1, ..., n).$$

It follows from (3) that

$$\mathbf{c}^\top \mathbf{x} = \sum_{m=1}^{n} \mathbf{e}_k^\top A_m x_m \boldsymbol{\tau}(j) \geq -\mathbf{e}_k^\top A_0 \boldsymbol{\tau}(j) \ \ \forall \mathbf{x} \in X,$$

and it follows from (6) that there exists $\mathbf{x}(k,j) \in X$ such $\mathbf{c}^\top \mathbf{x}(k,j) = -\mathbf{e}_k^\top A_0 \boldsymbol{\tau}(j)$. Thus we can conclude that $Val(\mathbf{P}) = -\mathbf{e}_k^\top A_0 \boldsymbol{\tau}(j) > -\infty$. Taking into account that the system $\mathcal{A}(\mathbf{x}) \in \mathcal{COP}^p$ yields the uniform LP duality, we conclude that there exists a matrix U in the form (20) such that

$$A_m \bullet U = \sum_{i \in I} \alpha_i (\mathbf{t}(i))^\top A_m \mathbf{t}(i) = \mathbf{e}_k^\top A_m \boldsymbol{\tau}(j), \ m = 1, ..., n;$$
$$A_0 \bullet U = \sum_{i \in I} \alpha_i (\mathbf{t}(i))^\top A_0 \mathbf{t}(i) = \mathbf{e}_k^\top A_0 \boldsymbol{\tau}(j).$$

It is easy to see that these equalities can be rewritten as

$$\mathbf{b}(k,j) = \begin{pmatrix} A_m \bullet U \\ m = 0, ..., n \end{pmatrix} = \sum_{i \in I} \alpha_i \mathbf{a}(\mathbf{t}(i)) \text{ with } \alpha_i > 0, \ \mathbf{t}(i) \in T, \ i \in I.$$

Thus we have shown that inclusions (19) hold true.

$\Leftarrow$) Now, having supposed that inclusions (19) hold true, let us show that the consistent system $\mathcal{A}(\mathbf{x}) \in \mathcal{COP}^p$ yields the uniform LP duality.
Consider any $\mathbf{c} \in \mathbb{R}^n$ such that $Val(\mathbf{P}) > -\infty$. It was stated in section 2 that the problem $(\mathbf{P})$ is equivalent to the problem $(\mathbf{P}_*)$ and there exists $\mathbf{x}^* \in \mathbb{R}^n$ satisfying (8).

Notice that $Val(\mathbf{P}_*) = Val(\mathbf{P})$ and system (7) in problem $(\mathbf{P}_*)$ can be rewritten as

$$(1, \mathbf{x}^\top)\mathbf{a}(\mathbf{t}) \geq 0 \ \forall \mathbf{t} \in \Omega, \ (1, \mathbf{x}^\top)\mathbf{b}(k, j) \geq 0 \ \forall k \in N_*(j), \ \forall j \in J. \tag{22}$$

Taking into account the inequalities in (8) (that can be considered as a *generalized Slater condition*), let us show that system (7) yields the uniform LP duality. In fact, it follows from Theorem 1 in [16] that under conditions (8), there exist vectors $\mathbf{t}(i) \in \Omega, \ i \in I, |I| \leq n$, such that a Linear Programming problem

$$\mathbf{LP}: \quad \min \mathbf{c}^\top \mathbf{x} \ \text{ s.t. } (1, \mathbf{x}^\top)\mathbf{a}(\mathbf{t}(i)) \geq 0 \ \forall i \in I, \ (1, \mathbf{x}^\top)\mathbf{b}(k, j) \geq 0 \ \forall k \in N_*(j), \ \forall j \in J,$$

has the same optimal value as the problem $(\mathbf{P}_*)$: $Val(\mathbf{P}_*) = Val(\mathbf{LP}) > -\infty$. The problem $(\mathbf{LP})$ is consistent since any $\mathbf{x} \in X$ is feasible in this problem. Hence the problem $(\mathbf{LP})$ has an optimal solution. Consequently, there exist numbers and vector

$$\alpha_i, \ \mathbf{t}(i) \in \Omega, \ i \in I, \ \lambda_k(j), \, k \in N_*(j), \ j \in J,$$

such that $\alpha_i \geq 0, i \in I$, $\lambda_k(j) \geq 0$, $k \in N_*(j), \ j \in J$, and

$$\sum_{i \in I} \alpha_i \mathbf{a}(\mathbf{t}(i)) + \sum_{j \in J} \sum_{k \in N_*(j)} \lambda_k(j)\mathbf{b}(k, j) = (-Val(\mathbf{P}), c_m, \ m = 1, ..., n)^\top. \tag{23}$$

From (19), one can conclude that for all indices $j \in J$, $k \in N_*(j)$ and any $\lambda_k(j) > 0$, the vector $\lambda_k(j)\mathbf{b}(k, j)$ admits a representation

$$\lambda_k(j)\mathbf{b}(k, j) = \sum_{i \in I(k,j)} \alpha_i(k, j)\mathbf{a}(\boldsymbol{\tau}(i, k, j))$$
$$\text{with } \alpha_i(k, j) > 0, \ \boldsymbol{\tau}(i, k, j) \in T, \ i \in I(k, j), \ |I(k, j)| < \infty.$$

It follows from the representations above and from (23) that

$$\sum_{i \in \bar{I}} \bar{\alpha}_i \mathbf{a}(\bar{\mathbf{t}}(i)) = (-Val(\mathbf{P}), c_m, m = 1, ..., n)^\top \ \text{ with some } \bar{\alpha}_i > 0, \, \bar{\mathbf{t}}(i) \in T, \, |\bar{I}| < \infty. \tag{24}$$

Denote $U := \sum_{i \in \bar{I}} \bar{\alpha}_i \bar{\mathbf{t}}(i)(\bar{\mathbf{t}}(i))^\top$. It is evident that $U \in \mathcal{CP}^p$ and relations (24) can be rewritten as (21). Hence we have shown that if inclusions (19) hold true, then the system $\mathcal{A}(\mathbf{x}) \in \mathcal{COP}^p$ yields the uniform LP duality. $\square$

For $j \in J$ and $k \in N_*(j)$, consider the following sets:

$$X(k, j) := \{\mathbf{x} \in X : \mathbf{e}_k^\top \mathcal{A}(\mathbf{x})\boldsymbol{\tau}(j) = 0\}, \ T_{im}(k, j) := \{\mathbf{t} \in T : \mathbf{t}^\top \mathcal{A}(\mathbf{x})\mathbf{t} = 0 \ \forall \mathbf{x} \in X(k, j)\}.$$

Denote $N(j) = N_*(j) \setminus M(j)$, $j \in J$. Notice that by construction,

$$\emptyset \neq X(k, j) \subset X \setminus \{\mathbf{x}^*\}, \ T_{im} \subset T_{im}(k, j) \ \forall k \in N(j), \ \forall j \in J;$$

$$X(k, j) = X, \quad T_{im} = T_{im}(k, j) \ \ \forall k \in M(j), \ \forall j \in J.$$

For $j \in J$ and $k \in N_*(j)$, denote by $\mathbf{x}^*(k, j)$ a vector such that $\mathbf{x}^*(k, j) \in X(k, j)$ and

$$\mathbf{e}_k^\top \mathcal{A}(\mathbf{x}^*(k, j))\boldsymbol{\tau}(j) = 0, \ \mathbf{t}^\top \mathcal{A}(\mathbf{x}^*(k, j))\mathbf{t} > 0 \ \forall \mathbf{t} \in T \setminus T_{im}(k, j). \tag{25}$$

Notice that such vectors exist for $j \in J, \ k \in N_*(j)$.

Theorem 1 *A consistent linear system $\mathcal{A}(\mathbf{x}) \in \mathcal{COP}^p$ with the corresponding sets T_{im}, $\mathcal{T}$, and other defined above, yields the uniform LP duality iff the following conditions hold:*

$$\textbf{I)} \qquad \mathbf{b}(k,j) \in \text{cone}\{\mathbf{a}(\mathbf{t}), \mathbf{t} \in T_{im}\} \ \forall k \in M(j), \ \forall j \in J, \tag{26}$$

$$\textbf{II)} \qquad \mathbf{b}(k,j) \in \text{cone}\{\mathbf{a}(\mathbf{t}), \mathbf{t} \in T_{im}(k,j)\} \ \forall k \in N(j), \ \forall j \in J. \tag{27}$$

Proof. It follows from Proposition 3 that to prove the theorem, it is enough to show that relations (19) are equivalent to relations (26) and (27).
Since $T_{im} \subset T$ and $T_{im}(k,j) \subset T \ \forall k \in N(j), \ j \in J$, it is evident that the relations (26) and (27) imply the inclusions (19).
Suppose that inclusions (19) take a place. Hence for any $j \in J$ and $k \in N_*(j)$, the equality

$$\mathbf{b}(k,j) = \sum_{i \in I} \alpha_i \mathbf{a}(\mathbf{t}(i)) \text{ with some } \alpha_i = \alpha_i(k,j) > 0, \ \mathbf{t}(i) = \mathbf{t}(i,k,j) \in T, \ i \in I, \tag{28}$$

holds true. Let's multiply the right and left parts of this equality by $(1, (\mathbf{x}^*(k,j))^\top)$. As a result, we get

$$\sum_{i \in I} \alpha_i (\mathbf{t}(i))^\top \mathcal{A}(\mathbf{x}^*(k,j)) \mathbf{t}(i) = \mathbf{e}_k^\top \mathcal{A}(\mathbf{x}^*(k,j)) \boldsymbol{\tau}(j) = 0 \ \text{ with some } \alpha_i > 0, \ \mathbf{t}(i) \in T, i \in I. \tag{29}$$

Here we took into account the equality in (25). It follows from (29) and the inequalities in (25) that in (28), the vectors $\mathbf{t}(i), i \in I$, should satisfy the conditions $\mathbf{t}(i) \in T_{im}(k,j), \ i \in I$. Consequently, we have shown that inclusions (19) imply the inclusions

$$\mathbf{b}(k,j) \in \text{cone}\{\mathbf{a}(\mathbf{t}), \mathbf{t} \in T_{im}(k,j)\}, k \in N_*(j) = M(j) \cup N(j), \ j \in J.$$

Taking into account that $T_{im}(k,j) = T_{im} \ \forall k \in M(j), \ \forall j \in J$, we conclude that inclusions (19) imply (26) and (27). $\square$

Notice that in (26) and (27) we have a *finite number* of inclusions.

4 Equivalent formulations of the condition I)

In this section, we will present several equivalent formulations of condition **I**), set forth in the previous section, which is one of the conditions that guarantee the uniform LP duality of the copositive system. This gives us the opportunity to analyze this condition from different points of view and create a theoretical basis for comparing our results with others known in the literature.

Proposition 4 *Given a linear system $\mathcal{A}(\mathbf{x}) \in \mathcal{COP}^p$ with the corresponding sets T_{im}, $\mathcal{T}$, $M(j), \ j \in J$, and the vectors $\mathbf{a}(\mathbf{t}), \mathbf{b}(k,j), k \in M(j), \ j \in J$ defined above, the following statements are equivalent:*

- *j) the condition* **I**) *is satisfied;*
- *jj) the cones* $\text{cone}\{\mathbf{b}(k,j), \ k \in M(j), \ j \in J\}$ *and* $\text{cone}\{\mathbf{a}(\mathbf{t}), \mathbf{t} \in T_{im}\}$ *coincide;*
- *jjj) the equality* $\text{span}\{\mathbf{b}(k,j), \ k \in M(j), \ j \in J\} = \text{cone}\{\mathbf{a}(\mathbf{t}), \mathbf{t} \in T_{im}\}$ *holds true.*

In the terminology from [5], condition *jj)* means that the sets $\{\mathbf{b}(k,j),\ k\in M(j),\ j\in J\}$ and $\{\mathbf{a}(\mathbf{t}),\mathbf{t}\in T_{im}\}$ are *positively equivalent*.

Proof. It is evident that the condition *jj)* implies the condition *j)*. Let us show that *j)* implies *jj)*. In fact, it follows from *j)* that $\text{cone}\{\mathbf{b}(k,j),\ k\in M(j),\ j\in J\}\subset\text{cone}\{\mathbf{a}(\mathbf{t}),\mathbf{t}\in T_{im}\}$. On the other hand, it follows from Proposition 2 that $\text{cone}\{\mathbf{a}(\mathbf{t}),\mathbf{t}\in T_{im}\}\subset\text{cone}\{\mathbf{b}(k,j),\ k\in M(j),\ j\in J\}$. Hence, $\text{cone}\{\mathbf{b}(k,j),\ k\in M(j),\ j\in J\}=\text{cone}\{\mathbf{a}(\mathbf{t}),\mathbf{t}\in T_{im}\}$, and we have shown that *j)* implies that *jj)*. Thus the equivalence of *j)* and *jj)* is proved.
To prove the equivalence of the conditions *jj)* and *jjj)*, it is enough to show that

$$\text{cone}\{\mathbf{b}(k,j),\ k\in M(j),\ j\in J\}=\text{span}\{\mathbf{b}(k,j),\ k\in M(j),\ j\in J\}. \tag{30}$$

From Proposition 1, it follows that $-\mathbf{b}(\bar{k},\bar{j})\in\text{cone}\{\mathbf{b}(k,j),\ k\in M(j),\ j\in J\}$ for all $\bar{k}\in M(\bar{j})$ and $\bar{j}\in J$. This implies equality (30) and hence the conditions *jj)* and *jjj)* are equivalent. $\square$

Let the set J be partitioned into subsets $J(s),\ s\in S$, as in (14). For $s\in S$, let us number the indices in $J(s)$ as follows: $J(s)=\{i_1,i_2,...,i_{k(s)}\},\ k(s)=|J(s)|$ and denote

$$V(s):=\{(i_k,i_q),\ k=1,...,k(s),\ q=k,...,k(s)\},\ \ V_0:=\bigcup_{s\in S}V(s). \tag{31}$$

Notice that the sets introduced above are finite.
For a given vector $\mathbf{z}=(z_0,z_1,...,z_n)^\top$, denote

$$\mathcal{B}(\mathbf{z})=\sum_{m=0}^{n}A_m z_m. \tag{32}$$

Proposition 5 *The condition* **I**) *of Theorem 1 (see (26)) is equivalent to the following two conditions:*

A1) *The set* $L:=\text{cone}\{\mathbf{a}(\mathbf{t}),\mathbf{t}\in T_{im}\}$ *is a subspace;*

B1) *for any* $\mathbf{z}\in\mathbb{R}^{n+1}$*, the equalities*

$$(\boldsymbol{\tau}(i))^\top\mathcal{B}(\mathbf{z})\boldsymbol{\tau}(j)=0\ \forall(i,j)\in V_0, \tag{33}$$

imply the equalities

$$\mathbf{e}_k^\top\mathcal{B}(\mathbf{z})\boldsymbol{\tau}(j)=0\ \forall k\in M(j),\ \forall j\in J. \tag{34}$$

Proof. Suppose that inclusions (26) hold true. Then it follows from Proposition 4 that L is a subspace.
Now let us prove that inclusions (26) imply the condition **B1**). First, notice that it follows from Proposition 9 (see Appendix) that equalities (33) are equivalent to the equalities

$$\mathbf{t}^\top\mathcal{B}(\mathbf{z})\mathbf{t}=0\ \forall\mathbf{t}\in T_{im}. \tag{35}$$

There equalities and equalities (34) can be written as follows:

$$\mathbf{z}^\top\mathbf{a}(\mathbf{t})=0\ \forall\mathbf{t}\in T_{im}, \tag{36}$$

$$\mathbf{z}^\top\mathbf{b}(k,j)=0\ \forall k\in M(j),\ \forall j\in J. \tag{37}$$

Then it is evident that under conditions (26), equalities (36) imply the equalities (37). Thus we have shown that the condition **B1**) follows from (26).

Now we will show that the conditions **A1**) and **B1**) imply inclusions (26).
Notice that under the condition **A1**) a vector $\mathbf{z}$ satisfies (36) iff $\mathbf{z} \in L^{\perp}$ where $L^{\perp}$ is the orthogonal complement to L in $\mathbb{R}^{n+1}$. Hence it follows from the condition **A1**) and Proposition 9 that the condition **B1**) can be reformulated as follows:

$$\mathbf{e}_k^{\top}\mathcal{B}(\mathbf{z})\boldsymbol{\tau}(j) = 0 \ \forall k \in M(j), \ \forall j \in J, \ \forall \mathbf{z} \in L^{\perp},$$

or equivalently

$$\mathbf{z}^{\top}\mathbf{b}(k,j) = 0 \ \forall k \in M(j), \ \forall j \in J, \ \forall \mathbf{z} \in L^{\perp}, \tag{38}$$

Given $k \in M(j)$ and $j \in J$, the vector $\mathbf{b}(k,j)$ admits the representation

$$\mathbf{b}(k,j) = \mathbf{b}^1(k,j) + \mathbf{b}^2(k,j) \ \text{ with } \mathbf{b}^1(k,j) \in L \text{ and } \mathbf{b}^2(k,j) \in L^{\perp}.$$

It follows from (38) that $\mathbf{b}^2(k,j) = 0$ and $\mathbf{b}(k,j) = \mathbf{b}^1(k,j) \in L$ in the representation above. Hence the conditions **A1**) and **B1**) imply the inclusions (26). $\square$

Remark 2 *It is easy to see that under the condition* **A1**)*, the condition* **B1**) *can be reformulated as follows: the equalities (34) hold true for any* $\mathbf{z} \in L^{\perp}$.

Remark 3 *Let us introduce* $n+1$ *vectors* $\mathbf{a}(i,j) := ((\boldsymbol{\tau}(i))^{\top} A_m \boldsymbol{\tau}(j), \ m = 0, ..., n)^{\top}$, $(i,j) \in V_0$, *and consider matrices with finite dimensions*

$$\mathbb{A} := (\mathbf{a}(i,j), \ (i,j) \in V_0), \ \mathbb{B} := (\mathbf{b}(k,j), \ k \in M(j), \ j \in J).$$

Then the condition **B1**) *can be formulated as* $\text{rank}\mathbb{A} = \text{rank}(\mathbb{A}, \mathbb{B})$.

Proposition 6 *The condition* **A1**) *is equivalent to the following one:*

A2) *There exists a matrix* $U^* \in \mathcal{CP}^p$ *in the form*

$$U^* = \sum_{i \in I} \alpha_i \mathbf{t}(i)(\mathbf{t}(i))^{\top} + \tfrac{1}{4} \sum_{(l,q) \in V_0} (\boldsymbol{\tau}(l) + \boldsymbol{\tau}(q))(\boldsymbol{\tau}(l) + \boldsymbol{\tau}(q))^{\top} \tag{39}$$

$$\text{with } \alpha_i > 0, \ \mathbf{t}(i) \in T_{im}, \ i \in I, \ |I| < \infty,$$

such that

$$A_m \bullet U^* = 0 \ \forall m = 0, 1, ..., n. \tag{40}$$

Proof. Suppose that the condition **A1**) holds true. Hence it follows from Proposition 10 (see Appendix) that there exist numbers and vectors $\bar{\alpha}_i > 0, \ \mathbf{t}(i) \in T_{im}, i \in I, \ |I| < \infty$, such that

$$\sum_{i \in I} \bar{\alpha}_i \mathbf{a}(\mathbf{t}(i)) = 0, \ \text{rank}(\mathbf{a}(\mathbf{t}(i)), i \in I) = r_{im},$$

where $r_{im} := \text{rank}(\mathbf{a}(\mathbf{t}), \mathbf{t} \in T_{im})$. Notice that $\frac{1}{2}(\boldsymbol{\tau}(l) + \boldsymbol{\tau}(q)) \in T_{im}$ for all $(l,q) \in V_0$. Then, according to Proposition 11 (see the Appendix), there exist numbers $\alpha_i > 0, i \in I$, such that

$$\sum_{i \in I} \alpha_i \mathbf{a}(\mathbf{t}(i)) + \frac{1}{4} \sum_{(l,q) \in V_0} \mathbf{a}(\boldsymbol{\tau}(l) + \boldsymbol{\tau}(q)) = 0. \tag{41}$$

Let U^* be a matrix in the form (39). Then (41) can be rewritten in the form (40). Thus we have proved that the condition **A1**) implies the condition **A2**).
Now suppose that the condition **A2**) holds true and rewrite equalities (40) in the form (41). It follows from Proposition 12 (see Appendix) that

$$\text{rank}(\mathbf{a}(\mathbf{t}(i)), i \in I, \ \mathbf{a}(\boldsymbol{\tau}(l) + \boldsymbol{\tau}(q)), \ (l, q) \in V_0) = r_{im}.$$

Taking into account this equality, equality (41), and Proposition 10, we conclude that $L := \text{cone}\{\mathbf{a}(\mathbf{t}), \mathbf{t} \in T_{im}\}$ is a subspace. Thus we have shown that the condition **A2**) implies the condition **A1**). $\square$

Remark 4 *The condition* **A2**) *can be reformulated as follows: there exists a matrix* $U^* \in \mathcal{CP}^p$ *in the form*

$$U^* = \sum_{i \in I} \alpha_i \mathbf{t}(i)(\mathbf{t}(i))^\top \ \textit{with } \alpha_i > 0, \mathbf{t}(i) \in T_{im}, i \in I,$$

such that $\text{rank}(\mathbf{a}(\mathbf{t}(i)), \ i \in I) = r_{im}$ *and equalities (40) hold true.*

To formulate the next propositions and lemmas, we need the following notation and definitions (see [20]). Given a matrix $Y \in \mathcal{S}^p$, denote

$$\text{dir}(Y, \mathcal{COP}^p) := \{D \in \mathcal{S}^p : \ Y + \varepsilon D \in \mathcal{COP}^p \text{ for some } \varepsilon > 0\},$$

$$\text{ldir}(Y, \mathcal{COP}^p) := \text{dir}(Y, \mathcal{COP}^p) \cap -\text{dir}(Y, \mathcal{COP}^p),$$
$$\tan(Y, \mathcal{COP}^p) := \text{cl}\,(\text{dir}(Y, \mathcal{COP}^p)) \cap -\text{cl}\,(\text{dir}(Y, \mathcal{COP}^p)).$$

For matrices $Y \in \mathcal{COP}^p$ and $U \in \mathcal{CP}^p$, we say that U is *strictly complementary* to Y if $U \in \text{relint}(\mathcal{CP}^p \cap Y^\perp)$.

Definition 2 *A matrix* Y *is called a maximum slack in the system* $\mathcal{A}(\mathbf{x}) \in \mathcal{COP}^p$ *if* $Y \in \text{relint}\,\mathcal{D}$*, where* $\mathcal{D} := \{D = \mathcal{A}(\mathbf{x}), \, \mathbf{x} \in X\}$.

Denote

$$\mathcal{R}(\mathcal{B}) := \{D = \mathcal{B}(\mathbf{z}), \, \mathbf{z} \in \mathbb{R}^{n+1}\}, \ \mathcal{N}(\mathcal{B}^*) := \{U \in \mathcal{S}^p : A_m \bullet U = 0 \ \forall m = 0, 1, ..., n\}$$

where $\mathcal{B}(\mathbf{z})$ is defined in (32).

Lemma 2 *A matrix* Y^* *is a maximum slack in the system* $\mathcal{A}(\mathbf{x}) \in \mathcal{COP}^p$ *iff there exists* $\mathbf{y}^* \in X$ *such that* $Y^* = \mathcal{A}(\mathbf{y}^*)$ *and*

$$\mathbf{t}^\top \mathcal{A}(\mathbf{y}^*)\mathbf{t} > 0 \ \forall \mathbf{t} \in T \setminus T_{im}, \ \mathbf{e}_k^\top \mathcal{A}(\mathbf{y}^*)\boldsymbol{\tau}(j) > 0 \ \forall k \in P \setminus M(j), j \in J. \tag{42}$$

Proof. Suppose that relations (42) hold true for some $\mathbf{y}^* \in X$. Let us show that $Y^* := \mathcal{A}(\mathbf{y}^*) \in \text{relint}\,\mathcal{D}$. Here, as above, $\mathcal{D} := \{D = \mathcal{A}(\mathbf{x}), \, \mathbf{x} \in X\}$. Since the set $\mathcal{D}$ is convex, we can state that the following equivalence takes a place:

$$\mathcal{A}(\mathbf{y}^*) \in \text{relint}\,\mathcal{D} \iff \forall \mathbf{x} \in X \ \exists \varepsilon > 0 \text{ such that } D(\varepsilon, \mathbf{x}) \in \mathcal{COP}^p, \tag{43}$$
$$D(\varepsilon, \mathbf{x}) := (1 + \varepsilon)\mathcal{A}(\mathbf{y}^*) - \varepsilon \mathcal{A}(\mathbf{x}).$$

Notice that due to Theorem 1 from [12], we have

$$D(\varepsilon, \mathbf{x}) \in \mathcal{COP}^p \iff \tag{44}$$
$$\mathbf{t}^\top D(\varepsilon, \mathbf{x})\mathbf{t} \geq 0 \; \forall \, \mathbf{t} \in \Omega; \;\; D(\varepsilon, \mathbf{x})\boldsymbol{\tau}(j) \geq 0 \; \forall j \in J. \tag{45}$$

where the set Ω is defined in (9). Denote

$$\xi_1 := \min_{\mathbf{t} \in \Omega} t^\top \mathcal{A}(\mathbf{y}^*)t, \;\; \xi_2 := \min\{\mathbf{e}_k^\top \mathcal{A}(\mathbf{y}^*)\boldsymbol{\tau}(j), \; k \in P \setminus M(j), \; j \in J\}.$$

It follows from (42) that $\xi_1 > 0$ and $\xi_2 > 0$.
For a fixed $\mathbf{x} \in X$, set

$$\eta_1(\mathbf{x}) := \max_{\mathbf{t} \in \Omega} \mathbf{t}^\top \mathcal{A}(\mathbf{x})\mathbf{t}, \;\; \eta_2(\mathbf{x}) = \max\{\mathbf{e}_k^\top \mathcal{A}(\mathbf{x})\boldsymbol{\tau}(j), \; k \in P \setminus M(j), \; j \in J\},$$

and calculate

$$\begin{gathered}\mathbf{t}^\top D(\varepsilon, \mathbf{x})\mathbf{t} = (1+\varepsilon)\mathbf{t}^\top \mathcal{A}(\mathbf{y}^*)\mathbf{t} - \varepsilon \mathbf{t}^\top \mathcal{A}(\mathbf{x})\mathbf{t} \geq (1+\varepsilon)\xi_1 - \varepsilon\eta_1(\mathbf{x}) \; \forall t \in \Omega, \\ \mathbf{e}_k^\top D(\varepsilon, \mathbf{x})\boldsymbol{\tau}(j) = (1+\varepsilon)\mathbf{e}_k^\top \mathcal{A}(\mathbf{y}^*)\boldsymbol{\tau}(j) - \varepsilon \mathbf{e}_k^\top \mathcal{A}(\mathbf{x})\boldsymbol{\tau}(j) \geq (1+\varepsilon)\xi_2 - \varepsilon\eta_2(\mathbf{x}) \\ \forall \, k \in P \setminus M(j), \; \forall j \in J.\end{gathered} \tag{46}$$

For $s = 1, 2$, set $\varepsilon_s(\mathbf{x}) = \infty$ if $\xi_s - \eta_s(\mathbf{x}) \geq 0$; $\varepsilon_s(\mathbf{x}) = \xi_s/(\eta_s(\mathbf{x}) - \xi_s) > 0$ if $\xi_s - \eta_s(\mathbf{x}) < 0$; $\varepsilon_0(\mathbf{x}) := \min\{\varepsilon_1, \varepsilon_2\} > 0$. It follows from (46) that for any $\mathbf{x} \in X$, there exists $\varepsilon_0(\mathbf{x}) > 0$ such that inequalities (45) hold true. Hence we have shown that inequalities (42) imply the condition $\mathcal{A}(\mathbf{y}^*) \in \operatorname{relint} \mathcal{D}$.
Now suppose that Y^* is a maximum slack in the system $\mathcal{A}(\mathbf{x}) \in \mathcal{COP}^p$. Then there exists $\mathbf{y}^* \in X$ such that $Y^* = \mathcal{A}(\mathbf{y}^*)$, $\mathcal{A}(\mathbf{y}^*) \in \operatorname{relint} \mathcal{D}$, and due to (43) we have

$$\forall \, \mathbf{x} \in X \; \exists \; \varepsilon > 0 \;\; \text{such that } D(\varepsilon, \mathbf{x}) \in \mathcal{COP}^p. \tag{47}$$

Suppose that this vector $\mathbf{y}^*$ does not satisfy (42). Consequently one of the following situations should have place:

a) there exists $\bar{\mathbf{t}} \in T \setminus T_{im}$ such that $\bar{\mathbf{t}}^\top \mathcal{A}(\mathbf{y}^*)\bar{\mathbf{t}} = 0$;

b) $\mathbf{t}^\top \mathcal{A}(\mathbf{y}^*)\mathbf{t}$ for all $\mathbf{t} \in T \setminus T_{im}$ and there exist $j_0 \in J$ and $k_0 \in P \setminus M(j_0)$ such that $\mathbf{e}_{k_0}^\top \mathcal{A}(\mathbf{y}^*)\boldsymbol{\tau}(j_0) = 0$.

Let $\mathbf{x}^*$ be a vector in X satisfying (8). We will show that for $\mathbf{x} = \mathbf{x}^*$, inclusion (44) does not hold true with any $\varepsilon > 0$.
In fact, in the situation **a**), we have that $\bar{\mathbf{t}}^\top \mathcal{A}(\mathbf{y}^*)\bar{\mathbf{t}} = 0$ and $\bar{\mathbf{t}}^\top \mathcal{A}(\mathbf{x}^*)\bar{\mathbf{t}} > 0$, and hence

$$\bar{\mathbf{t}}^\top((1+\varepsilon)\mathcal{A}(\mathbf{y}^*) - \varepsilon\mathcal{A}(\mathbf{x}^*))\bar{\mathbf{t}} < 0 \text{ with any } \varepsilon > 0.$$

It follows from these relations and the condition $\bar{\mathbf{t}} \in T$ that

$$(1+\varepsilon)\mathcal{A}(\mathbf{y}^*) - \varepsilon\mathcal{A}(\mathbf{x}^*) = D(\varepsilon, \mathbf{x}^*) \notin \mathcal{COP}^p \; \forall \varepsilon > 0. \tag{48}$$

In the situation **b**), for $\theta \geq 0$ and $\mathbf{t}(\theta) := (\boldsymbol{\tau}(j_0) + \theta\mathbf{e}_{k_0}) \in \mathbb{R}^p_+$, let us calculate

$$\mathbf{t}^\top(\theta)D(\varepsilon, \mathbf{x}^*)\mathbf{t}(\theta) = [2\theta\mathbf{e}_{k_0}^\top D(\varepsilon, \mathbf{x}^*)\boldsymbol{\tau}(j_0) + \theta^2\mathbf{e}_{k_0}^\top D(\varepsilon, \mathbf{x}^*)\mathbf{e}_{k_0}]$$

$$= \theta[-2\varepsilon \mathbf{e}_{k_0}^\top \mathcal{A}(\mathbf{x}^*)\boldsymbol{\tau}(j_0) + \theta \mathbf{e}_{k_0}^\top((1+\varepsilon)\mathcal{A}(\mathbf{y}^*) - \varepsilon\mathcal{A}(\mathbf{x}^*))\mathbf{e}_{k_0}]$$
$$\leq \theta[-2\varepsilon \mathbf{e}_{k_0}^\top \mathcal{A}(\mathbf{x}^*)\boldsymbol{\tau}(j_0) + \theta(1+\varepsilon)\mathbf{e}_{k_0}^\top \mathcal{A}(\mathbf{y}^*)\mathbf{e}_{k_0}]. \quad (49)$$

For any $\varepsilon > 0$, let us set $\theta(\varepsilon) = \varepsilon \mathbf{e}_{k_0}^\top \mathcal{A}(\mathbf{x}^*))\boldsymbol{\tau}(j_0)/(1+\varepsilon)\mathbf{e}_{k_0}^\top \mathcal{A}(\mathbf{y}^*)\mathbf{e}_{k_0} > 0$ if $\mathbf{e}_{k_0}^\top \mathcal{A}(\mathbf{y}^*)\mathbf{e}_{k_0} > 0$ and $\theta(\varepsilon) = 1$ if $\mathbf{e}_{k_0}^\top \mathcal{A}(\mathbf{y}^*)\mathbf{e}_{k_0} = 0$. Then taking into account (49), for $\mathbf{t}(\theta(\varepsilon)) \in \mathbb{R}_+^p$ we obtain

$$\mathbf{t}^\top(\theta(\varepsilon))D(\varepsilon, \mathbf{x}^*)\mathbf{t}(\theta(\varepsilon)) \leq -\varepsilon\theta(\varepsilon)\mathbf{e}_{k_0}^\top \mathcal{A}(\mathbf{x}^*)\boldsymbol{\tau}(j_0) < 0 \ \forall \varepsilon > 0.$$

This implies that relations (48) take place. But these relations contradict (47). Hence we have shown that (47) implies (42). □

Proposition 7 *The condition* **A1)** *is equivalent to the following one:*

A3) *For a maximum slack Y^* in the system $\mathcal{A}(\mathbf{x}) \in \mathcal{COP}^p$, there is $U^* \in \mathcal{N}(\mathcal{B}^*) \cap \mathcal{CP}^p$ strictly complementary to Y^*.*

Proof. Suppose that the condition **A3**) holds true. Since the set $\mathcal{CP}^p \cap (Y^*)^\perp$ is convex, then

$$U^* \in \text{relint}(\mathcal{CP}^p \cap (Y^*)^\perp) \iff$$
$$\forall U \in \mathcal{CP}^p \cap (Y^*)^\perp \ \ \exists\, \varepsilon > 0 \text{ such that } (1+\varepsilon)U^* - \varepsilon U \in \mathcal{CP}^p \cap (Y^*)^\perp. \quad (50)$$

It follows from Lemma 2 that $U \in \mathcal{CP}^p \cap (Y^*)^\perp$ iff U admits a representation

$$U = \sum_{i \in \bar{I}} \bar{\alpha}_i \bar{\mathbf{t}}(i)(\bar{\mathbf{t}}(i))^\top \text{ with some } \bar{\alpha}_i > 0, \ \bar{\mathbf{t}}(i) \in T_{im}, \ i \in \bar{I}. \quad (51)$$

Hence U^* takes the form

$$U^* = \sum_{i \in I} \alpha_i \mathbf{t}(i)(\mathbf{t}(i))^\top \text{ with some } \alpha_i > 0, \ \mathbf{t}(i) \in T_{im}, \ i \in I. \quad (52)$$

Since $U^* \in \mathcal{N}(\mathcal{B}^*)$ then the equalities (40) hold true.
For any $\mathbf{t} \in T_{im}$, consider the matrix $U = \mathbf{t}\mathbf{t}^\top \in \mathcal{CP}^p \cap (Y^*)^\perp$. It follows from (50), (51) that there exist $\varepsilon > 0$ and $\bar{\alpha}_i > 0$, $\bar{\mathbf{t}}(i) \in T_{im}$, $i \in \bar{I}$, such that

$$(1+\varepsilon)U^* - \varepsilon \mathbf{t}\mathbf{t}^\top = \sum_{i \in \bar{I}} \bar{\alpha}_i \bar{\mathbf{t}}(i)(\bar{\mathbf{t}}(i))^\top \iff (1+\varepsilon)U^* = \varepsilon \mathbf{t}\mathbf{t}^\top + \sum_{i \in \bar{I}} \bar{\alpha}_i \bar{\mathbf{t}}(i)(\bar{\mathbf{t}}(i))^\top.$$

Taking into account the latter equality and (40), we obtain the equalities

$$A_m \bullet (\varepsilon \mathbf{t}\mathbf{t}^\top + \sum_{i \in \bar{I}} \bar{\alpha}_i \bar{\mathbf{t}}(i)(\bar{\mathbf{t}}(i))^\top) = 0 \ \forall m = 0, 1, ..., n,$$

that can be rewritten in the form

$$\varepsilon \mathbf{a}(\mathbf{t}) + \sum_{i \in \bar{I}} \bar{\alpha}_i \mathbf{a}(\bar{\mathbf{t}}(i)) = 0 \text{ where } \varepsilon > 0, \ \bar{\alpha}_i > 0, \ \bar{\mathbf{t}}(i) \in T_{im} \ \forall i \in \bar{I}.$$

It follows from these relations that $-\mathbf{a}(\mathbf{t}) \in L$ for any $\mathbf{a}(\mathbf{t}) \in L$ and hence, L is a subspace.

Now suppose that the condition **A1**) holds true. It follows from Proposition 6 that the condition **A1**) is equivalent to the condition **A2**). According to this condition, there exists a matrix

$U^* \in \mathcal{CP}^p$ in the form (39) satisfying equalities (40). By construction, $U^* \in \mathcal{CP}^p \cap (Y^*)^\perp$ and $U^* \in \mathcal{N}(\mathcal{B}^*) \cap \mathcal{CP}^p$. Let us show that relations (50) hold true.

Consider any matrix $U \in \mathcal{CP}^p \cap (Y^*)^\perp$. It follows from Lemma 2 that this matrix admits representation (51). For a fixed $i \in \bar{I}$, consider the corresponding $\bar{\mathbf{t}}(i) \in T_{im}$. Then it follows from Proposition 13 (see Appendix) that the matrix $\bar{\mathbf{t}}(i)(\bar{\mathbf{t}}(i))^\top$ can be presented in the form

$$\bar{\mathbf{t}}(i)(\bar{\mathbf{t}}(i))^\top = \sum_{(l,q)\in V_0} \beta_{l,q}(i)(\boldsymbol{\tau}(l)+\boldsymbol{\tau}(q))(\boldsymbol{\tau}(l)+\boldsymbol{\tau}(q))^\top.$$

Consequently,

$$\begin{aligned}(1+\varepsilon)U^* - \varepsilon U =&(1+\varepsilon)\Big(\sum_{i\in I}\alpha_i \mathbf{t}(i)(\mathbf{t}(i))^\top + \frac{1}{4}\sum_{(l,q)\in V_0}(\boldsymbol{\tau}(l)+\boldsymbol{\tau}(q))(\boldsymbol{\tau}(l)+\boldsymbol{\tau}(q))^\top\Big)\\ &-\varepsilon\sum_{i\in\bar{I}}\bar{\alpha}_i\Big(\sum_{(l,q)\in V_0}\beta_{l,q}(i)(\boldsymbol{\tau}(l)+\boldsymbol{\tau}(q))(\boldsymbol{\tau}(l)+\boldsymbol{\tau}(q))^\top\Big)\\ =&(1+\varepsilon)\sum_{i\in I}\alpha_i\mathbf{t}(i)(\mathbf{t}(i))^\top + \sum_{(l,q)\in V_0}\bar{\beta}_{l,q}(\boldsymbol{\tau}(l)+\boldsymbol{\tau}(q))(\boldsymbol{\tau}(l)+\boldsymbol{\tau}(q))^\top,\end{aligned}$$

where $\alpha_i > 0$, $\mathbf{t}(i) \in T_{im}\, \forall i \in I$, $0.5(\boldsymbol{\tau}(l)+\boldsymbol{\tau}(q)) \in T_{im}$, $\forall (l,q) \in V_0$, and where for a sufficiently small $\varepsilon > 0$, it holds $\bar{\beta}_{l,q} := (1+\varepsilon)/4 - \varepsilon\sum\limits_{i\in\bar{I}}\bar{\alpha}_i\beta_{l,q}(i) > 0\ \forall (l,q) \in V_0$.

Then, evidently, $(1+\varepsilon)U^* - \varepsilon U \in \mathcal{CP}^p \cap (Y^*)^\perp$ for any $U \in \mathcal{CP}^p \cap (Y^*)^\perp$ and for a some sufficiently small $\varepsilon > 0$, and, consequently $U^* \in \operatorname{relint}(\mathcal{CP}^p \cap (Y^*)^\perp)$. □

Proposition 8 *The condition* **B1**) *is equivalent to the following one:*

B2) *For a maximum slack Y^* in the system $\mathcal{A}(\mathbf{x}) \in \mathcal{COP}^p$, the set*

$$\mathcal{R}(\mathcal{B}) \cap (\tan(Y^*, \mathcal{COP}^p) \setminus \operatorname{ldir}(Y^*, \mathcal{COP}^p))$$

is empty.

Proof. In [4] (see Theorems 6, 13), for $A \in \mathcal{COP}^p$, it is shown that

$$\begin{aligned}\operatorname{dir}(A, \mathcal{COP}^p) = \quad &\{B \in \mathcal{S}^p : \mathbf{t}^\top B\mathbf{t} \geq 0\ \forall \mathbf{t} \in \mathcal{V}^A;\\ &\mathbf{e}_k^\top B\mathbf{t} \geq 0\ \forall \mathbf{t} \in \mathcal{V}^A \cap \mathcal{V}^B,\ \forall k \in \{s \in P : \mathbf{e}_s^\top A\mathbf{t} = 0\}\},\end{aligned} \tag{53}$$

$$\begin{aligned}\tan(A, \mathcal{COP}^p) = \quad &\{B \in \mathcal{S}^p : \mathbf{t}^\top B\mathbf{t} = 0\ \forall \mathbf{t} \in \mathcal{V}^A\} =\\ &\{B \in \mathcal{S}^p : \mathbf{t}^\top B\boldsymbol{\tau} = 0\ \forall \{\mathbf{t}, \boldsymbol{\tau}\} \subset \mathcal{V}^A_{min} \text{ s.t. } \mathbf{t}^\top A\boldsymbol{\tau} = 0\},\end{aligned} \tag{54}$$

where $\mathcal{V}^A := \{\mathbf{t} \in T : \mathbf{t}^\top A\mathbf{t} = 0\}$ is the set of zeros of A and $\mathcal{V}^A_{min} \subset \mathcal{V}^A$ is the set of minimal zeros of A. For definitions see [4].

Let us present the condition **B2**) in another form. It follows from Lemma 2 that if Y^* is a maximum slack in the system $\mathcal{A}(\mathbf{x}) \in \mathcal{COP}^p$, then there exists a vector $\mathbf{y}^* \in X$ such that conditions (42) hold true. Taking into account the relations (42), (53), and (54), it is easy to see that

$$\tan(\mathcal{A}(\mathbf{y}^*), \mathcal{COP}^p) \setminus \operatorname{ldir}(\mathcal{A}(\mathbf{y}^*), \mathcal{COP}^p) = \{B \in \mathcal{S}^p : (\boldsymbol{\tau}(i))^\top B\boldsymbol{\tau}(j) = 0\,\forall (i,j) \in V_0;$$

$$\exists\,\bar{\mathbf{t}} \in T_{im} \cap \mathcal{V}^B \text{ and } \exists\,\bar{k} \in P \text{ such that } \mathbf{e}_{\bar{k}}^\top \mathcal{A}(\mathbf{y}^*)\bar{\mathbf{t}} = 0,\ \mathbf{e}_{\bar{k}}^\top B\bar{\mathbf{t}} \neq 0\}.$$

Consequently, the condition **B2)** can be reformulated as follows:

B2*): *for any* $\mathbf{z} \in \mathbb{R}^{n+1}$, *the equalities* $(\boldsymbol{\tau}(i))^\top \mathcal{B}(\mathbf{z})\boldsymbol{\tau}(j) = 0\ \forall (i,j) \in V_0$ *imply the equalities*

$$\mathbf{e}_k^\top \mathcal{B}(\mathbf{z})\mathbf{t} = 0\ \forall\, k \in \{q \in P : \mathbf{e}_q^\top \mathcal{A}(\mathbf{y}^*)\mathbf{t} = 0\}\ \forall \mathbf{t} \in T_{im} \cap \mathcal{V}^{\mathcal{B}(\mathbf{z})} = T_{im}. \tag{55}$$

Suppose that that the condition **B2*)** holds true. For $j \in J$, consider the corresponding vector $\boldsymbol{\tau}(j) \in T_{im}$. By construction (see (42)), we have $\{q \in P : \mathbf{e}_q^\top \mathcal{A}(\mathbf{y}^*)\boldsymbol{\tau}(j) = 0\} = M(j)$. Consequently, it follows from conditions (55) that $\mathbf{e}_k^\top \mathcal{B}(\mathbf{z})\boldsymbol{\tau}(j) = 0$ for all $k \in M(j)$. Hence we have shown that the condition **B2*)** implies the condition **B1)**.
Now suppose that the condition **B1**) holds true. Consider any $\mathbf{t} \in T_{im}$. It follows from (14) that $\mathbf{t} \in T_{im}(s)$ with some $s \in S$ and consequently, $\mathbf{t}$ admits a representation

$$\mathbf{t} = \sum_{j \in \Delta J(s)} \alpha_j \boldsymbol{\tau}(j),\ \alpha_j > 0,\ j \in \Delta J(s) \subset J(s);\ \sum_{j \in \Delta J(s)} \alpha_j = 1. \tag{56}$$

Suppose that $k \in \{q \in P : \mathbf{e}_q^\top \mathcal{A}(\mathbf{y}^*)\mathbf{t} = 0\}$. Hence

$$0 = \mathbf{e}_k^\top \mathcal{A}(\mathbf{y}^*)\mathbf{t} = \sum_{j \in \Delta J(s)} \alpha_j \mathbf{e}_k^\top \mathcal{A}(\mathbf{y}^*)\boldsymbol{\tau}(j).$$

Taking into account this equality and the inequalities $\alpha_j > 0$, $j \in \Delta J(s)$, and $\mathbf{e}_q^\top \mathcal{A}(\mathbf{y}^*)\boldsymbol{\tau}(j) > 0$ $\forall q \in P \setminus M(j)\ \forall j \in J(s)$, we conclude that

$$k \in M(j)\ \forall j \in \Delta J(s). \tag{57}$$

Now, for the same $\mathbf{t} \in T_{im}(s)$ and any $\mathbf{z} \in \mathbb{R}^{n+1}$ satisfying (33) calculate $\mathbf{e}_k^\top \mathcal{B}(\mathbf{z})\mathbf{t}$ taking into account conditions (34), (56), and (57):

$$\mathbf{e}_k^\top \mathcal{B}(\mathbf{z})\mathbf{t} = \sum_{j \in \Delta J(s)} \alpha_j \mathbf{e}_k^\top \mathcal{B}(\mathbf{z})\boldsymbol{\tau}(j) = 0.$$

Thus we have shown that the condition **B1**) implies **B2***). □

The above considerations can be formulated as follows.

Lemma 3 *For any* $k \in \{1,2,3\}$ *and any* $m \in \{1,2\}$, *the conditions i) and ii) below are equivalent to each other, and are necessary for the consistent system* $\mathcal{A}(\mathbf{x}) \in \mathcal{COP}^p$ *to yield the uniform LP duality property:*
i) the condition **I**) *holds true;*
ii) the conditions **A**k) *and* **B**m) *hold true.*

The condition *ii*) with $k = 3$ and $m = 2$ (i.e., the conditions **A3**) and **B2**)) is a necessary condition for the linear conic system to yield the uniform LP duality formulated and proved in [20] and applied to the copositive cone $\mathcal{COP}^p$.
It was shown in [20] that when $\mathcal{K}$ is a nice cone, the conditions **A3**) and **B2**) are necessary and sufficient for the linear conic consistent system $\mathcal{A}(\mathbf{x}) \in \mathcal{K}$ to yield the uniform LP duality.
In general, the conditions formulated in Lemma 3 are only necessary, but not sufficient for the system $\mathcal{A}(\mathbf{x}) \in \mathcal{COP}^p$ to yield the uniform LP duality. It is illustrated by a simple example presented in the next section.

5 Examples

In this section, we consider several examples that illustrate our results and help us to compare our results with ones known in the literature.

Example 1. Consider the system $\mathcal{A}(\mathbf{x}) \in \mathcal{COP}^p$ with the following data:

$$n=2,\ p=3,\ A_0=\begin{pmatrix} a & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, A_1=\begin{pmatrix} 0 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}, A_2=\begin{pmatrix} -1 & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & 0\end{pmatrix},\ a>0. \quad (58)$$

For $\mathbf{t}^*=(0,0,1)^\top$, we have $(\mathbf{t}^*)^\top\mathcal{A}(\mathbf{x})\mathbf{t}^*=0,\ \mathbf{e}_1^\top\mathcal{A}(\mathbf{x})\mathbf{t}^*=0,\ \mathbf{e}_3^\top\mathcal{A}(\mathbf{x})\mathbf{t}^*=0$ for all $\mathbf{x}\in\mathbb{R}^2$. It is easy to check that for the vector $\mathbf{x}^*=(-1,-1)^\top$ and the data in (58) we have $\mathbf{t}^\top\mathcal{A}(\mathbf{x}^*)\mathbf{t}>0$ for all $\mathbf{t}\in\mathbb{R}^3_+\setminus\{\mathbf{t}^*\}$ and $\mathbf{e}_2^\top\mathcal{A}(\mathbf{x}^*)\mathbf{t}^*>0$. Hence, for the system under consideration, $\mathcal{A}(\mathbf{x}^*)$ is a maximum slack, $T_{im}=\{\boldsymbol{\tau}(1)=\mathbf{t}^*\}$, $M(1)=\{1,3\}$, $J=\{1\}$, $\mathbf{a}(\mathbf{t}^*)=(0,\,0,\,0)^\top$, $\mathbf{b}(1,1)=\mathbf{b}(3,1)=(0,\,0,\,0)^\top$, $\mathbf{b}(2,1)=(0,\,-1,\,0)^\top$. Hence $L:=\text{cone}\{\mathbf{a}(\mathbf{t}),\mathbf{t}\in T_{im}\}=\{\mathbf{a}(\mathbf{t}^*)\}$ and $\mathbf{b}(k,j)\in L$ for all $k\in M(j)$ and all $j\in J$.
Thus we see that for this system, the condition **I**) is satisfied, and it follows from Lemma 3 that the conditions **A**k) for $k=1,2,3$ and the conditions **B**m) for $m=1,2$ are satisfied as well. (The fulfillment of the conditions **A**k) for $k=1,2,3$ and the conditions **B**m) for $m=1,2$ can be checked directly.)
But the system under consideration does not yield the uniform LP duality. In fact, it was shown in [10] that for the primal problem (**P**) with the cost vector $c^\top=(0,\,-1)$ and the corresponding dual problem (**D**), there is a positive duality gap: $Val(\mathbf{P})-Val(\mathbf{D})=a>0$.
The reason for not complying with the uniform duality is that for the system with the data (58), the condition **II**) is not satisfied. Indeed, for the vector $\bar{\mathbf{x}}=(-1,\,0)^\top$, we have $\mathbf{t}^\top\mathcal{A}(\bar{\mathbf{x}})\mathbf{t}>0$ $\forall\mathbf{t}\in T\setminus\{\mathbf{t}^*\}$ and $\mathbf{e}_k^\top\mathcal{A}(\bar{\mathbf{x}})\mathbf{t}^*=0$ for $k=1,2,3$. Hence for the system under consideration, we have $N(1)=\{2\}$, $T_{im}(k=2,j=1)=T_{im}=\{\mathbf{t}^*\}$ and the condition **II**), for $j=1\in J$ and $k=2\in N(j)$, takes the form

$$\mathbf{b}(2,1)=(0,\,-1,\,0)^\top\in\text{cone}\{\mathbf{a}(\mathbf{t}),\mathbf{t}\in T_{im}(k,j)\}=\{\mathbf{a}(\mathbf{t}^*)\}=\{(0,\,0,\,0)^\top\}.$$

It is evident that this condition does not hold true.

This example shows that the condition **II**) is essential and can not be omitted. The example also shows that for the cone $\mathcal{COP}^p$, the conditions formulated in [20] are not sufficient unlike the case with the cone of positive semi-definite matrices $\mathcal{S}^p_+$ for which these conditions are necessary and sufficient.

Example 2. Let us now consider an example where the condition **I**) is violated. Consider the system $\mathcal{A}(\mathbf{x})\in\mathcal{COP}^p$ with the following data:

$$n=1,\ p=3,\ A_0=\begin{pmatrix} a & 0 & -a\\ 0 & 0 & 0\\ -a & 0 & a\end{pmatrix},\ A_1=\begin{pmatrix} 1 & -1 & 2\\ -1 & 0 & 1\\ 2 & 1 & -5\end{pmatrix},\ a>0. \quad (59)$$

This system admits a unique feasible solution $\mathbf{x}=x_1=0$. Hence $X=\{\mathbf{0}\}$ and it is easy to check that $T_{im}=\{\mathbf{t}\in T: t_1=t_3\}$, where $T=\{\mathbf{t}\in\mathbb{R}^3_+: t_1+t_2+t_3=1\}$. The vertices of the set convT_{im} are $\boldsymbol{\tau}(1)=0.5(1,\,0,\,1)^\top$, $\boldsymbol{\tau}(2)=(0,\,1,\,0)^\top$, and the sets $M(j)$, $N(j)=N_*(j)\setminus M(j)$, defined in (5) and (6) take the form $M(j)=\{1,2,3\}$, $N(j)=\emptyset$ for $j\in J=\{1,2\}$.

It is easy to see that $\mathbf{t}^\top A_0\mathbf{t} = 0$, $\mathbf{t}^\top A_1\mathbf{t} = 0$ for all $\mathbf{t} \in T_{im}$, and $\mathbf{e}_1^\top A_0\boldsymbol{\tau}(1) = 0$, $\mathbf{e}_1^\top A_1\boldsymbol{\tau}(1) = 1.5$. Hence, $\mathbf{a}(\mathbf{t}) = \mathbf{0}$ $\forall \mathbf{t} \in T_{im}$ and $\mathbf{b}(k_0, j_0) = (0,\, 1.5)^\top$ for $k_0 = 1,\ j_0 = 1,\ k_0 \in M(j_0)$. Thus we obtain

$$\mathbf{b}(k_0, j_0) \notin \text{cone}\{\mathbf{a}(\mathbf{t}), \mathbf{t} \in T_{im}\},$$

wherefrom we conclude that condition **I**) does not hold true and, consequently, the system under consideration does not yield the uniform LP duality.
Let show this directly. Since the system $\mathcal{A}(\mathbf{x}) \in \mathcal{COP}^p$ with data (59) has a unique feasible solution $\mathbf{x} = x_1 = 0$, then the corresponding primal problem (**P**) has the optimal solution $\mathbf{x}^* = x_1^* = 0$ with $Val(\mathbf{P}) = 0$ for any objective function $\mathbf{c}^\top\mathbf{x}^* = c_1x_1^*,\ c_1 \in \mathbb{R}$. The corresponding dual problem (**P**) takes the form

$$\max(-A_0 \bullet U) \ \text{ s.t. } A_1 \bullet U = c_1,\ U \in \mathcal{CP}^p. \tag{60}$$

Suppose that the system $\mathcal{A}(\mathbf{x}) \in \mathcal{COP}^p$ yields the uniform LP duality. Hence the dual problem should have an optimal solution U^0 such that

$$U^0 = \sum_{i \in I} \alpha_i \mathbf{t}(i)(\mathbf{t}(i))^\top \ \text{ with } \alpha_i > 0,\ \mathbf{t}(i) \in T,\ i \in I,\ -A_0 \bullet U^0 = 0,\ A_1 \bullet U^0 = c_1. \tag{61}$$

Since $\mathbf{t}^\top A_0\mathbf{t} = a(t_1 - t_3)^2$ for all $\mathbf{t} \in \mathbb{R}^3$, we conclude that the equality $-A_0 \bullet U^0 = 0$ implies the equalities $t_1(i) = t_3(i)$ for all $i \in I$ and hence $(\mathbf{t}(i))^\top A_1\mathbf{t}(i) = 0$ for all $i \in I$. Then $A_1 \bullet U^0 = 0$. Thus we have shown that for any $c_1 \neq 0$, the dual problem has no solutions satisfying relations (61) which permits to conclude that the system $\mathcal{A}(\mathbf{x}) \in \mathcal{COP}^p$ with the data defined in (59) does not yield the uniform LP duality.

Note that in [19] (see page 3), it is stated that for the SDP systems $\mathcal{A}(\mathbf{x}) \in \mathcal{S}_+^p$ with $n = 1$, the uniform LP duality property is always satisfied. In our example, for $n = 1$ we present the CoP system $\mathcal{A}(\mathbf{x}) \in \mathcal{COP}^p$ that does not yield the uniform LP duality. This further confirms the fact that CoP systems are much more complex (more *pathological*) than SDP systems.

Example 3. Consider a system $\mathcal{A}(\mathbf{x}) \in \mathcal{COP}^p$ with the following data:

$$n = 1,\ p = 3,\ A_0 = \begin{pmatrix} a & 0 & -a \\ 0 & 0 & 0 \\ -a & 0 & a \end{pmatrix},\ A_1 = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} \text{ with } a > 0. \tag{62}$$

This system admits feasible solutions $\mathbf{x} = x_1 \geq 0$. It is easy to check that the set $T_{im} = \{\boldsymbol{\tau}(j), j \in J\}$ consists of two vectors $\boldsymbol{\tau}(1) = 0.5(1,\, 0,\, 1)^\top$, $\boldsymbol{\tau}(2) = (0,\, 1,\, 0)^\top$, and that here $J = \{1, 2\}$. Since $\mathcal{A}(\mathbf{x})\boldsymbol{\tau}(1) = (0,\, x_1,\, 0)^\top$ and $\mathcal{A}(\mathbf{x})\boldsymbol{\tau}(2) = (x_1,\, 0,\, x_1)^\top$, we conclude that the sets $M(j)$, $N(j)$, $j \in J = \{1, 2\}$, take the form $M(1) = \{1,\, 3\}$, $N(1) = \{2\}$, $M(2) = \{2\}$, $N(2) = \{1,\, 3\}$, and the vectors $\mathbf{x}^*(k, j) \in X$ and sets $T_{im}(k, j)$, $k \in N(j)$, $j \in J$, satisfying (25) are as follows:

$$\mathbf{x}^*(k, j) = 0,\ T_{im}(k, j) = \{\mathbf{t} \in T : t_1 = t_3\}\ \forall k \in N(j),\ j \in J.$$

It is evident that the system with data (62) does not satisfy the Slater condition.
It follows from the equalities $A_0\boldsymbol{\tau}(1) = (0,\, 0,\, 0)^\top$, $A_1\boldsymbol{\tau}(1) = (0,\, 1,\, 0)^\top$, $A_0\boldsymbol{\tau}(2) = (0,\, 0,\, 0)^\top$ and $A_1\boldsymbol{\tau}(2) = (1,\, 0,\, 1)^\top$, that

$$\mathbf{b}(k, j) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\ \forall k \in M(j),\ j \in J;\ \mathbf{b}(k, j) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \forall k \in N(j),\ j \in J.$$

Let us check the conditions I) and II). Here we have

$$\mathbf{b}(k,j) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \text{cone}\{\mathbf{a}(\mathbf{t}),\, \mathbf{t} \in T_{im}\} = \text{cone}\{\mathbf{a}(\boldsymbol{\tau}(j)) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\, j \in J\}\ \forall k \in M(j), j \in J,$$

$$\mathbf{b}(k,j) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \text{cone}\Big\{ \begin{pmatrix} 0 \\ 4t_2t_3 \end{pmatrix}, \mathbf{t} \in T_{im}(k,j) \Big\}\ \forall k \in N(j),\, j \in J.$$

Thus, we can conclude that the conditions I) and II) hold true and, consequently, the system under consideration yields the uniform LP duality despite it does not satisfy the Slater condition.
Let us check the uniform LP duality directly. Since the system $\mathcal{A}(\mathbf{x}) \in \mathcal{COP}^p$ with the data defined in (62), has the set of feasible solutions in the form $X = \{x_1 \geq 0\}$, then the corresponding primal problem ($\mathbf{P}$) has the finite optimal value $Val(\mathbf{P}) = 0$ only for the objective function c_1x_1 with $c_1 \geq 0$. The corresponding dual problem ($\mathbf{D}$) takes the form (60).
For a given $c_1 \geq 0$, set $U^0(c_1) = \frac{c_1}{4}\bar{\mathbf{t}}\,\bar{\mathbf{t}}^\top$ with $\bar{\mathbf{t}} = (1,\,1,\,1)^\top$. It is easy to check that

$$U^0(c_1) \in \mathcal{CP}^p,\ -A_0 \bullet U^0(c_1) = 0,\ A_1 \bullet U^0(c_1) = c_1.$$

Hence, for any $c_1 \geq 0$, $U^0(c_1)$ is a feasible solution of the corresponding dual problem ($\mathbf{D}$) and $Val(\mathbf{P}) = Val(\mathbf{D})$. Thus, in fact, system $\mathcal{A}(\mathbf{x}) \in \mathcal{COP}^p$ with data (62) yields the uniform LP duality.

6 On a relationship of the obtained results with the uniform duality for SIP

Consider a general linear SIP problem in the form

$$\mathbf{P}^*_{SIP}: \qquad \min_{x \in \mathbb{R}^n} \mathbf{c}^\top\mathbf{x} \quad \text{s.t. } (1, \mathbf{x}^\top)\mathbf{a}(\mathbf{t}) \geq 0\ \forall \mathbf{t} \in T,$$

where T is an index set and $\mathbf{a}(\mathbf{t}) = (a_m(\mathbf{t}), m = 0, 1, ..., n)^\top$, $\mathbf{t} \in T$. Denote

$$G := \left\{\mathbf{a}(\mathbf{t}), \mathbf{t} \in T; (1, \mathbf{0}_n^\top)^\top\right\}.$$

The following theorem is proved in [5] (see Theorem 3.2, conditions (ii) and (iv)).

Theorem 2 *The consistent constraint system of the problem ($\mathbf{P}^*_{SIP}$) yields the uniform LP duality iff*

$$\text{cone}(G) = \text{cone}(F \cup W) \tag{63}$$

with some $F \subset \mathbb{R}^{n+1}$ and $W \subset \mathbb{R}^{n+1}$ satisfying the following conditions:

- *F is finite and* cone(F) *is a linear space which is also contained in* cone G,
- *W is compact, and there exists a vector $\bar{\mathbf{x}} \in \mathbb{R}^n$ such that*

$$\begin{aligned} s_0 + \mathbf{s}^\top\bar{\mathbf{x}} = 0\ \forall (s_0, \mathbf{s}^\top)^\top \in F,\ s_0 \in \mathbb{R}, \\ t_0 + \mathbf{t}^\top\bar{\mathbf{x}} > 0\ \forall (t_0, \mathbf{t}^\top)^\top \in W,\ t_0 \in \mathbb{R}. \end{aligned} \tag{64}$$

Moreover, whenever $(s_0, \mathbf{s}^\top)^\top \in F$*, the equality* $\mathbf{s}^\top \mathbf{x} = -s_0$ *is implied by the constraint system of the problem* $(\mathbf{P}^*_{SIP})$ *and the equality* $\mathbf{s}^\top \mathbf{x} = 0$ *is implied by the homogeneous system* $(0, \mathbf{x}^\top)\mathbf{a}(\mathbf{t}) \geq 0, \mathbf{t} \in T$*. Furthermore, it holds:*

$$\mathbf{s}^\top \mathbf{t} + s_0 t_0 = 0 \text{ if } (s_0, \mathbf{s}^\top)^\top \in F \text{ and } (t_0, \mathbf{t}^\top)^\top \in W. \quad (65)$$

It follows from the equivalent description (2) of the cone $\mathcal{COP}^p$ that the problem $(\mathbf{P})$ is equivalent to the linear SIP problem $(\mathbf{P}^*_{SIP})$ where the set T and the vector $\mathbf{a}(\mathbf{t})$ are defined in (1) and (10), respectively. Let us denote this special SIP problem by $(\mathbf{P}_{SIP})$.
Since the problem $(\mathbf{P}_{SIP})$ is a special case of a general linear SIP problem, the statements of Theorem 2 should be satisfied for the problem $(\mathbf{P}_{SIP})$ as well.
In Theorem 2, nothing is said about how to build the sets F and W mentioned in the theorem. Obviously, it is interesting to know how to find these sets for our CoP problem. The following theorem gives an answer to this question for the CoP problem under consideration.

Theorem 3 *Given the problem* $(\mathbf{P}_{SIP})$*, the sets* F *and* W*, mentioned in Theorem 2, can be chosen as follows:*

$$F := \{\mathbf{b}(k, j), k \in M(j), j \in J\}, \ W := \mathrm{Pr}(\widetilde{W}, L^\perp), \quad (66)$$

$$\widetilde{W} := \{\mathbf{a}(\mathbf{t}), \mathbf{t} \in \Omega\} \cup \{\mathbf{b}(k, j), k \in N(j), j \in J\} \cup (1, \mathbf{0}_n^\top)^\top, \quad (67)$$

where $L := \mathrm{span}\, F$*,* $L^\perp$ *is the orthogonal complement to* L*, the set* Ω *is defined in (9), and* $\mathrm{Pr}(\widetilde{W}, L^\perp)$ *is the projection of the set* $\widetilde{W}$ *onto* $L^\perp$*.*

Proof. First, let us show that the consistent constraints system of the problem $(\mathbf{P}_{SIP})$ yields the uniform LP duality iff

$$\mathrm{cone}(F \cup \widetilde{W}) = \mathrm{cone}(G) \quad (68)$$

with the sets F and $\widetilde{W}$ defined in (66) and (67).
It was shown above that the problem $(\mathbf{P}_{SIP})$ is equivalent to the problem $(\mathbf{P}_*)$ (see (7)) that can be rewritten in the form

$$\mathbf{P}_* : \ \min_{\mathbf{x} \in \mathbb{R}^n} \ \mathbf{c}^\top \mathbf{x} \ \text{ s.t. } (1, \mathbf{x}^\top)\mathbf{a}(\mathbf{t}) \geq 0 \ \forall \mathbf{t} \in \Omega, \ (1, \mathbf{x}^\top)\mathbf{b}(k, j) \geq 0, k \in N_*(j), j \in J.$$

Notice that the problem $(\mathbf{P}_*)$ yields the uniform LP duality, the problems $(\mathbf{P}_{SIP})$ and $(\mathbf{P}_*)$ have the same feasible sets and, consequently, the same optimal values of the cost functions.
For a fixed $\mathbf{t} \in T$, consider the consistent problem $(\mathbf{P}_*)$ with $\mathbf{c}^\top = (c_m = \mathbf{t}^\top A_m \mathbf{t}, \ m = 1, ..., n)$. Then for all $\mathbf{x} \in X$,

$$\mathbf{c}^\top \mathbf{x} = \sum_{m=1}^{n} \mathbf{t}^\top A_m \mathbf{t} x_m \geq -\mathbf{t}^\top A_0 \mathbf{t} > -\infty.$$

Consequently $Val(\mathbf{P}_{SIP}) = \beta - \mathbf{t}^\top A_0 \mathbf{t}$ with some $\beta \geq 0$. Taking into account that the problem $(\mathbf{P}_*)$ yields the uniform LP duality, we conclude that there exist numbers and vector

$$\alpha_i, \ \mathbf{t}(i) \in \Omega, \ i \in I, \ \lambda_k(j), \ k \in N_*(j), \ j \in J,$$

such that $\alpha_i > 0, i \in I, |I| < \infty, \lambda_k(j) \geq 0, \ k \in N_*(j), \ j \in J$, and equality (23) holds true with $\mathbf{c} = (c_m = \mathbf{e}_m^\top \mathbf{a}(\mathbf{t}), \ m = 1, ..., n)$ and $Val(\mathbf{P}_{SIP}) = \beta - \mathbf{t}^\top A_0 \mathbf{t}$. This implies that

$$\mathbf{a}(\mathbf{t}) = \sum_{i \in I} \alpha_i \mathbf{a}(\mathbf{t}(i)) + \sum_{j \in J} \sum_{k \in N_*(j)} \lambda_k(j) \mathbf{b}(k, j) + (1, \mathbf{0}_n^\top)^\top \beta \in \mathrm{cone}(F \cup \widetilde{W}).$$

Since the inclusion above is satisfied for all $\mathbf{t} \in T$, we conclude that for the consistent problem $(\mathbf{P}_{SIP})$, it holds:

$$\text{cone}(G) \subset \text{cone}(F \cup \widetilde{W}). \tag{69}$$

On the other hand, due to Proposition 3, the consistent system of the problem $(\mathbf{P}_{SIP})$ yields the uniform LP duality iff

$$\mathbf{b}(k, j) \in \text{cone}\{\mathbf{a}(\mathbf{t}), \mathbf{t} \in T\} \subset \text{cone}(G) \; \forall \, k \in N_*(j), \; \forall \, j \in J.$$

Taking into account these inclusions and the definitions of the sets G, F and $\widetilde{W}$, we obtain that the consistent system of the problem $(\mathbf{P}_{SIP})$ yields the uniform LP duality iff

$$\text{cone}(F \cup \widetilde{W}) \subset \text{cone}(G).$$

Now taking into account inclusion (69), we conclude that the consistent system of the problem $(\mathbf{P}_{SIP})$ yields the uniform LP duality iff condition (68) is satisfied with the sets F and $\widetilde{W}$ defined in (66) and (67).

By construction, the set F is finite and the set $\widetilde{W}$ is compact. Relations (64) hold true with $\bar{\mathbf{x}} = \mathbf{x}^*$ where $\mathbf{x}^*$ is defined in (8). It follows from Proposition 1 that $\text{cone}\, F = \text{span}\, F$ and then it is evident that the set $\mathcal{L} := \text{cone}\, F$ is a subspace.
Hence we have shown that the sets F and $\widetilde{W}$ defined in (66) and (67) satisfy all statements of Theorem 2 except for the condition (65). Taking into account that the set $\mathcal{L}$ introduced above is a subspace, it is easy to see that to satisfy the condition (65) it is enough to replace the set $\widetilde{W}$ by its projection onto $\mathcal{L}^{\perp}$. The theorem is proved. □

7 Conclusions

The main result of the paper is to establish the necessary and sufficient conditions that guarantee the uniform LP duality for linear CoP problems. These conditions are obtained using the concept of immobile indices and the sets generated by them and are formulated in various equivalent forms thereby expanding the scope of their application. The examples illustrate how the conditions obtained can be applied to confirm or deny the uniform LP duality of a CoP system. The relationship between the uniform LP duality properties for the related problems of CoP and SIP is shown.

A Appendix

Proposition 9 *The equalities (33) and (35) are equivalent.*

Proof. Consider any $\mathbf{t} \in T_{im}$. From (14), it follows that $\mathbf{t} \in T_{im}(s)$ for some $s \in S$ and $\mathbf{t}$ admits a representation (17). Consequently,

$$\mathbf{t}^\top \mathcal{B}(\mathbf{z})\mathbf{t} = \sum_{i \in J(s)} \sum_{j \in J(s)} \alpha_i \alpha_j (\boldsymbol{\tau}(i))^\top \mathcal{B}(\mathbf{z}) \boldsymbol{\tau}(j).$$

Taking into account this equality, we conclude that equalities (33) imply equalities (35).

Now we suppose that equalities (35) hold true. Since for any $s \in S$ and for all $i \in J(s)$ and all $j \in J(s)$, we have $0.5(\boldsymbol{\tau}(i) + \boldsymbol{\tau}(j)) \in T_{im}$, it follows from equalities (35) that

$$(\boldsymbol{\tau}(i) + \boldsymbol{\tau}(j))^\top \mathcal{B}(\mathbf{z})(\boldsymbol{\tau}(i) + \boldsymbol{\tau}(j)) = 0 \ \forall i \in J(s), \ \forall j \in J(s).$$

It is evident that these equalities imply (33). $\square$

Proposition 10 *Consider a set $L := \text{cone}\{\mathbf{a}(i), i \in \mathcal{I}\}$ where $\mathbf{a}(i) \in \mathbb{R}^s$, $\mathcal{I}$ is a set of indices (it is possible that $|\mathcal{I}| = \infty$). The set L is a subspace iff there exist a finite subset $\{\mathbf{a}(j), j \in J\} \subset \{\mathbf{a}(i), i \in \mathcal{I}\}$, $|J| < \infty$, and numbers α_j, $j \in J$, such that*

$$\alpha_j > 0 \ \forall j \in J, \ \sum_{j \in J} \alpha_j \mathbf{a}(j) = 0, \ \text{rank}(\mathbf{a}(j), j \in J) = r_* := \text{rank}(\mathbf{a}(i), i \in \mathcal{I}). \tag{70}$$

Proof. Suppose that there exist numbers and vectors α_j, $\mathbf{a}(j)$, $j \in J$, such that (70) holds true. To show that L is a subspace, we have to show that $-\mathbf{d} \in L$ for any $\mathbf{d} \in L$.
Since $\mathbf{d} \in L$ and $\text{rank}(\mathbf{a}(j), j \in J) = r_*$, then $\mathbf{d} = \sum\limits_{j \in J} \beta_j \mathbf{a}(j)$ with some β_j, $j \in J$. Hence $-\mathbf{d} = -\sum\limits_{j \in J} \beta_j \mathbf{a}(j)$ and taking into account (70) we get

$$-\mathbf{d} = -\sum_{j \in J} \beta_j \mathbf{a}(j) = \sum_{j \in J} (M\alpha_j - \beta_j)\mathbf{a}(j) = \sum_{j \in J} \bar{\alpha}_j \mathbf{a}(j),$$

where $M := \max\{\beta_j/\alpha_j, j \in J, 0\}$, $\bar{\alpha}_j := M\alpha_j - \beta_j \geq 0 \ \forall \, j \in J$. Then $-\mathbf{d} \in L$. Thus we have shown that if relations (70) hold true then L is a subspace.
Now, assuming that L is a subspace, let us show that relations (70) hold true.
It is evident that there exists a subset $\mathcal{I}_* \subset \mathcal{I}$ such that

$$\text{rank}(\mathbf{a}(i), i \in \mathcal{I}_*) = |\mathcal{I}_*| = r_*.$$

Since L is a subspace then $-\mathbf{a}(i) \in L$ for all $i \in \mathcal{I}_*$. Hence, for any $i \in \mathcal{I}_*$, there exist a set $\mathcal{I}(i) \subset \mathcal{I}$ and numbers $\alpha_{ij}, j \in \mathcal{I}(i)$, such that

$$-\mathbf{a}(i) = \sum_{j \in \mathcal{I}(i)} \alpha_{ij} \mathbf{a}(j), \ \alpha_{ij} > 0, j \in \mathcal{I}(i), \ |\mathcal{I}(i)| \leq r_*.$$

Consider the following set of vectors:

$$\{\mathbf{a}(i), \ \mathbf{a}(j), \ j \in \mathcal{I}(i); \ i \in \mathcal{I}_*\}. \tag{71}$$

This set consists in a finite number of elements and, by construction,

$$\text{rank}(\mathbf{a}(i), \ \mathbf{a}(j), \ j \in \mathcal{I}(i); \ i \in \mathcal{I}_*) = r_*,$$

$$\sum_{j \in \mathcal{I}_*} (\mathbf{a}(i) + \sum_{j \in \mathcal{I}(i)} \alpha_{ij} \mathbf{a}(j)) = \sum_{j \in \mathcal{I}_*} (\mathbf{a}(i) - \mathbf{a}(i)) = 0.$$

Thus we get that relations (70) hold true with the finite set of vectors (71). $\square$

Proposition 11 *Consider a set* $\{\mathbf{a}(i), i \in \mathcal{I}\}$ *where* $\mathbf{a}(i) \in \mathbb{R}^s$, $\mathcal{I}$ *is a set of indices (it is possible that* $|\mathcal{I}| = \infty$*). Suppose that there exist a finite subset* $\{\mathbf{a}(j), j \in J\} \subset \{\mathbf{a}(i), i \in \mathcal{I}\}$, $|J| < \infty$, *and numbers* α_j, $j \in J$, *such that relations (70) are satisfied. Then for any* $\Delta J \subset \mathcal{I}$, $|\Delta J| < \infty$, *there exist numbers* $\bar{\alpha}_j$, $j \in J$, *such that*

$$\bar{\alpha}_j > 0 \ \forall j \in J, \ \sum_{j \in J} \bar{\alpha}_j \mathbf{a}(j) + \sum_{q \in \Delta J} \mathbf{a}(q) = 0. \tag{72}$$

Proof. It follows from (70) that for all $q \in \Delta J$, there exist numbers $\beta_j(q)$, $j \in J$, such that

$$\mathbf{a}(q) = \sum_{j \in J} \beta_j(q) \mathbf{a}(j).$$

Then, taking into account the equality in (70), we conclude that for any $M \in \mathbb{R}$, the equality

$$\sum_{j \in J} (M\alpha_j - \sum_{q \in \Delta J} \beta_j(q)) \mathbf{a}(j) + \sum_{q \in \Delta J} \mathbf{a}(q) = 0 \tag{73}$$

holds true. Let us denote $\Delta\alpha(j) = \sum\limits_{q \in \Delta J} \beta_j(q)$, $j \in J$, and choose $M := \max\{\Delta\alpha(j)/\alpha_j, j \in J, 0\} + 1$. Then (73) implies (72) with $\bar{\alpha}_j = M\alpha_j - \Delta\alpha(j) \geq \alpha_j > 0, j \in J$. □

Proposition 12 *The following equality holds true:*

$$r_{im} := \text{rank}(\mathbf{a}(\mathbf{t}), \mathbf{t} \in T_{im}) = \text{rank}(\mathbf{a}(\boldsymbol{\tau}(l) + \boldsymbol{\tau}(q)), (l, q) \in V_0) =: \bar{r},$$

where V_0 *is defined in (31).*

Proof. Since $\frac{1}{2}(\boldsymbol{\tau}(l) + \boldsymbol{\tau}(q)) \in T_{im}$ for all $(l, q) \in V_0$, it is evident that $\bar{r} \leq r_{im}$.
Suppose that $\bar{r} < r_{im}$. Then there exists $\mathbf{z} \in \mathbb{R}^{n+1}$ and $\mathbf{t} \in T_{im}$ such that $\mathbf{z}^\top \mathbf{a}(\mathbf{t}) \neq 0$ and

$$\mathbf{z}^\top \mathbf{a}(\boldsymbol{\tau}(l) + \boldsymbol{\tau}(q)) = 0 \ \forall (l, q) \in V_0.$$

Notice that the latter equalities imply the equalities

$$\sum_{m=0}^{n} z_m (\boldsymbol{\tau}(l))^\top A_m \boldsymbol{\tau}(q) = 0 \ \ \forall (l, q) \in V_0. \tag{74}$$

Since $\mathbf{t} \in T_{im}$, then $\mathbf{t} \in T_{im}(s)$ with some $s \in S$ and $\mathbf{t}$ admits a representation (17). Therefore, with respect to (74), we conclude that

$$\begin{aligned} 0 \neq \mathbf{z}^\top \mathbf{a}(\mathbf{t}) &= \sum_{m=0}^{n} z_m \mathbf{t}^\top A_m \mathbf{t} = \sum_{m=0}^{n} z_m \Big(\sum_{l \in J(s)} \alpha_l \boldsymbol{\tau}(l) \Big)^\top A_m \Big(\sum_{q \in J(s)} \alpha_q \boldsymbol{\tau}(q) \Big) \\ &= \sum_{l \in J(s)} \sum_{q \in J(s)} \alpha_l \alpha_q \sum_{m=0}^{n} z_m (\boldsymbol{\tau}(l))^\top A_m \boldsymbol{\tau}(q) = 0. \end{aligned}$$

The contradiction obtained shows that $\bar{r} = r_{im}$. □

Proposition 13 *For any* $\mathbf{t} \in T_{im}$ *there exist numbers* $\beta_{lq} \in \mathbb{R}$, *where* $(l, q) \in V_0$, *such that*

$$\mathbf{t}\mathbf{t}^\top = \sum_{(l,q) \in V_0} \beta_{lq} (\boldsymbol{\tau}(l) + \boldsymbol{\tau}(q))(\boldsymbol{\tau}(l) + \boldsymbol{\tau}(q))^\top. \tag{75}$$

Proof. Consider any $\mathbf{t} \in T_{im}$. Then $\mathbf{t} \in T_{im}(s)$ with some $s \in S$ and $\mathbf{t}$ admits a representation (17). Consequently

$$\begin{aligned}\mathbf{t}\mathbf{t}^\top &= \Big(\sum_{l\in J(s)} \alpha_l \boldsymbol{\tau}(l)\Big)\Big(\sum_{q\in J(s)} \alpha_q \boldsymbol{\tau}(q)\Big)^\top = \sum_{l\in J(s)}\sum_{q\in J(s)} \alpha_l\alpha_q \boldsymbol{\tau}(l)\boldsymbol{\tau}^\top(q) \\ &= \sum_{l\in J(s)} \alpha_l^2 \boldsymbol{\tau}(l)\boldsymbol{\tau}^\top(l) + \sum_{(l,q)\in V(s), l\neq q} \alpha_l\alpha_q[\boldsymbol{\tau}(l)\boldsymbol{\tau}^\top(q) + \boldsymbol{\tau}(q)\boldsymbol{\tau}^\top(l)].\end{aligned} \tag{76}$$

It is evident that for any $(l,q) \in V(s)$, $l \neq q$, we have

$$\boldsymbol{\tau}(l)\boldsymbol{\tau}^\top(q) + \boldsymbol{\tau}(q)\boldsymbol{\tau}^\top(l) = (\boldsymbol{\tau}(l) + \boldsymbol{\tau}(q))(\boldsymbol{\tau}(l) + \boldsymbol{\tau}(q))^\top - \boldsymbol{\tau}(l)\boldsymbol{\tau}^\top(l) - \boldsymbol{\tau}(q)\boldsymbol{\tau}^\top(q).$$

These equalities together with (76) imply the equality (75). $\square$

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