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MATHEMATISCHE NACHRICHTEN

ORIGINAL ARTICLE

On fractional semidiscrete Dirac operators of Lévy–Leblond type

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Abstract

In this paper, we introduce a wide class of space-fractional and time-fractional semidiscrete Dirac operators of Lévy–Leblond type on the semidiscrete space-time lattice $h\mathbb{Z}^n \times [0,\infty)$ $(h > 0)$, resembling to fractional semidiscrete counterparts of the so-called parabolic Dirac operators. The methods adopted here are fairly operational, relying mostly on the algebraic manipulations involving Clifford algebras, discrete Fourier analysis techniques as well as standard properties of the analytic fractional semidiscrete semigroup $\left\{\exp(-te^{i\theta}(-\Delta_h)^\alpha)\right\}_{t\geq 0}$, carrying the parameter constraints $0 < \alpha \leq 1$ and $|\theta| \leq \frac{\alpha\pi}{2}$. The results obtained involve the study of Cauchy problems on $h\mathbb{Z}^n \times [0,\infty)$.

KEYWORDS

fractional semidiscrete Dirac operators, Riemann–Liouville fractional derivative, fractional discrete Laplacian

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1 | INTRODUCTION

1.1 | The state of art

The study of null solutions of Dirac-like operators is known as the heart of several function theories in the context of Clifford algebras. From the fact that Dirac-like operators factorize the Laplace operator and its analogs, it is of foremost importance to study them algebraically and analytically in order to obtain refinements of well-known results in harmonic analysis and to develop applications in the fields of mathematical physics and applied mathematics as well.

In 1967, Lévy–Leblond investigated the factorization of the Schrödinger operator with the aim of obtaining non-relativistic analogs of the Dirac operator in the $(1 + 3)$-dimensional space carrying any spin (cf. [35]). However, we had to wait for contemporary times to realize how the $(1 + n)$-dimensional generalization of the Lévy–Leblond-type picture, coined as parabolic Dirac-type operators, can be adopted to cover a wide range of applications on the crossroads of function theory and boundary value problems. Between the many notable results regarding this new class of operators, it is worth to stress the importance of the ground-breaking works of Cerejeiras, Kähler & Sommen [12] and Cerejeiras, Sommen & Vieira [13], who have proposed the matter of factorizing the heat operator and its relatives. Since then, their work have conducted many researchers to explore this type of approach to a wide variety of model problems, including higher-dimensional analogs of the nonlinear Schrödinger equation [7, 10].

Interestingly, the Lévy–Leblond picture had continue to receiving an increasing interest during the last decade. Mainly, on several physical-phase-space formulations of spinning particles through supersymmetric Lie algebraic representations of Partial Differential Equations (PDEs) (see [1, 2]) as well as on operational models involving parabolic Dirac operators and their fractional analogs (see, e.g., [5, 6, 26, 27], and references therein), to mention a few. Therewith, it is reasonable to say that the Lévy–Leblond framework is no longer just an emerging topic in the mathematics and physics communities, so that nowadays one can say that its foundations and applications are well understood by many scholars.

In parallel, the study of discrete Dirac operators has experienced a rapid increase in hypercomplex analysis during the last two decades, strongly influenced by the pioneering papers [9, 18, 23–25]. The literature toward this topic, whose physical roots may be found, for example, on the research papers of Kogut & Susskind [33] and Rabin [40] (see also [31, 44], and references therein) is very diverse. To the purpose of this paper, we will depict only an abridged overview of it to motivate our approach.

For the construction of a faithful discretization for the discrete Dirac operator on the lattice $h\mathbb{Z}^n$, say D_h, whose precise definition will be introduced afterward, the Clifford algebra of signature of (n,n) may be embody on the ladder structure of D_h, with the aid of the so-called Witt basis $\left\{\mathbf{e}_j^+, \mathbf{e}_j^- \, : \, j = 1, 2, \dots, n\right\}$, to seamlessly encode the canonical structure of the exterior algebra (cf. [29, Chap. 1] and [45, Chap. 2]).

Originally highlighted in [24] and on the series of papers [9, 18], such fact was only duly clarified in author's recent paper [19], through the canonical isomorphism $C\ell_{n,n} \cong \mathrm{End}(C\ell_{0,n})$ between the Clifford algebra of signature (n,n), $C\ell_{n,n}$, and the algebra of endomorphisms acting on the Clifford algebra of signature $(0,n)$, $C\ell_{0,n}$. This in turn allows us to establish a canonical correspondence between the discrete counterpart of the Dirac–Kähler operator, $d - d^*$, and the multivector approximation of Dirac operator, D_h, over $h\mathbb{Z}^n$ (see also [41]).

In a larger extent, starting from forward and backward discretizations of the Dirac operator $D = \sum_{j=1}^n \mathbf{e}_j \partial_{x_j}$, D_h^+ and D_h^- respectively, formally introduced in [23], one can make use of the wedge ($\wedge$) and the dot ($\bullet$) actions on $C\ell_{0,n}$ to establish the canonical one-to-one correspondences

$$d \overset{1\text{-}1}{\longleftrightarrow} D_h^- \wedge (\cdot) \quad \text{and} \quad d^* \overset{1\text{-}1}{\longleftrightarrow} D_h^+ \bullet (\cdot)$$

in a way that the resulting discrete Dirac operator D_h is nothing else than a $C\ell_{n,n}$-valued representation of the multivector counterpart of the Dirac–Kähler operator, $D_h^- \wedge (\cdot) - D_h^+ \bullet (\cdot)$ (cf. [19, Section 2.3]).

1.2 | Main targets

In this paper, we will focus our attention on the time-fractional and space-fractional Dirac operators, carrying the parameters $0 < \alpha \leq 1$, $\beta \geq 1$ and $|\theta| \leq \frac{\alpha\pi}{2}$. On the construction neatly summarized in Figure 1, the notation Δ_h stands for the discrete Laplacian on the lattice $h\mathbb{Z}^n$ considered in several author's contributions such as [19, 41].

In addition, the family of fractional discrete operators $(-\Delta_h)^\sigma$ $(0 < \sigma \leq 1)$ will be defined in terms of its Fourier symbol in the streamlines of [36, Section 6] (see also [21, Section 21.4.3]). For our main purposes, we will adopt the notation $\mathbb{D}_t^\beta$ for the so-called right-sided Riemann–Liouville fractional derivative (cf. [43, Chap. 2]). The operator $\mathbb{D}_t^\beta$, defined as follows:

$$\mathbb{D}_t^\beta \Psi(x,t) = \begin{cases} (-\partial_t)^k \displaystyle\int_t^{+\infty} g_{k-\beta}(s-t)\Psi(x,s)ds & \text{for} \quad k-1 < \beta < k \\ (-\partial_t)^k \Psi(x,t) & \text{for} \quad \beta = k, \end{cases} \tag{1.1}$$

where $k = \lfloor \beta \rfloor + 1$ ($\lfloor \beta \rfloor$ denotes the integer part of β), is an integro-differential operator for values of $\beta \neq k$, involving the higher order time-derivative $(-\partial_t)^k := (-1)^k (\partial_t)^k$ and an integral part, corresponding to the convolution between $\Psi(x,t)$ and the Gel'fand–Shilov function $g_\nu : \mathbb{R} \to [0,\infty)$, defined for $-\nu \notin \mathbb{N}_0$ by

$$g_\nu(p) = \begin{cases} \dfrac{p^{\nu-1}}{\Gamma(\nu)} & \text{for} \quad p > 0 \\ 0 & \text{for} \quad p \leq 0. \end{cases} \tag{1.2}$$

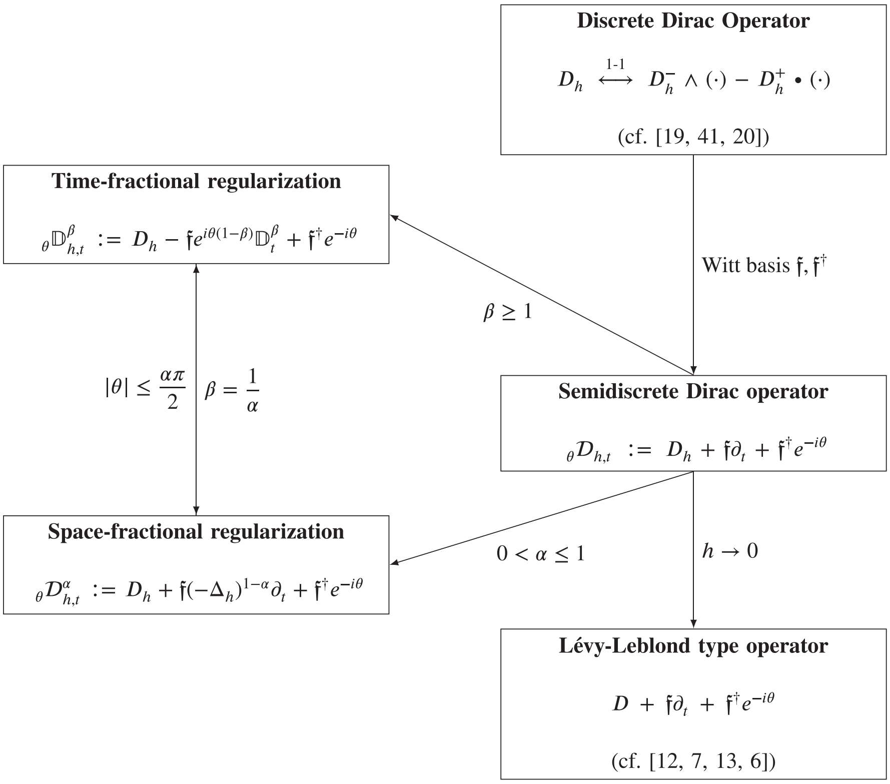

FIGURE 1 The Lévy–Leblond picture on the fractional semidiscrete case.

It should be noted that the Eulerian integral representation involving the Gamma function $\Gamma(\cdot)$ (cf. [39, formula 2.3.3. (1), p. 322]):

$$\int_0^\infty \mathrm{e}^{-p\lambda} p^{\nu-1} dp = \Gamma(\nu)\lambda^{-\nu}, \Re(\nu) > 0 \ \& \quad \Re(\lambda) > 0 \tag{1.3}$$

assures that for every $k-1 < \beta < k$ the function $p \mapsto g_{k-\beta}(p)$, appearing on the definition of $\mathbb{D}_t^\beta$, defines a *probability density function* converging to the Dirac delta function, that is,

$$\lim_{\beta \to k^-} g_{k-\beta}(p) = \delta(p).$$

The later set of properties allows us to say that $\mathbb{D}_t^\beta$ $(k-1 < \beta < k)$ provides us a regularization of $(-\partial_t)^k$ in the sense of the topology of the underlying space of tempered distributions.

The motivation to this paper comes from the series of contributions [16, 28, 36], focused on the study of Cauchy problems involving fractional discrete Laplacians, and from the interrelationship between space-fractional and time-fractional operators highlighted in [14] toward superdiffusion equations. Such link has played a crucial role in the theory of PDEs (cf. [32, 34]) and on stochastic calculus as well (cf. [4]). For a survey on both topics, we also refer to the monograph [38].

Thus, we are not only interested on fractional difference analogs for the parabolic operator of the heat type ($\alpha = \beta = 1$, $\theta = 0$ & $h \to 0$) and of the Schrödinger type ($\alpha = \beta = 1$, $\theta = \pm\frac{\pi}{2}$ & $h \to 0$), considered in the series of papers [1, 2, 6, 7, 12, 13], neither fractional counterparts of semidiscrete models involving the *semidiscrete heat operator* $\partial_t - \Delta_h = (D_h + \mathfrak{f}\partial_t + \mathfrak{f}^\dagger)^2$ (i.e., when $\alpha = \beta = 1$ & $\theta = 0$), already considered in [3], but also on the interface between space-fractional and time-fractional semidiscrete operators in a way that the *fractional discrete Laplacian* $-(-\Delta_h)^\alpha$, carrying the parameter $0 < \alpha \leq 1$, turns out be treated as a temporal regularization of order $\beta := \frac{1}{\alpha} \geq 1$, encoded on the time-fractional derivative $\mathbb{D}_t^\beta$.

Moreover, the parameter condition $|\theta| \leq \frac{\alpha\pi}{2}$ encoded on the [fractional semidiscrete] analytic semigroup $\left\{\exp(-t\mathrm{e}^{i\theta}(-\Delta_h)^\alpha)\right\}_{t\geq 0}$ is ubiquitous on space-fractional diffusion models (cf. [37]). Its incorporation on our model problem is seamlessly justified by the constraint $|\theta| = |\arg(te^{i\theta})| < \pi$, carrying the integral representation obtained in [21, p. 458, Equation (21.35)] for *modified Bessel functions of the first kind* (cf. [36, Sections 2.1 and 3.1]). This will be the main novelty of this paper in comparison with the semidiscrete heat semigroup representations considered in the series of author's contributions [20–22].

Up to author's knowledge, the overlap between space-fractional Dirac operators—such as the ones introduced by Bernstein in [8]—and time-fractional Dirac operators of Riemann–Liouville type—such as the ones considered, for example, by Ferreira & Vieira [26]—was not yet addressed so that the idea of connecting fractional order in time and space on this paper shall be seen eventually as another step further to pursue the goal of studying the mild solutions for time-fractional Navier–Stokes equations in the superdiffusive case, from a hypercomplex analysis perspective (see, e.g., [17, 46], and references therein for an overview on the subdiffusive case).

1.3 | Layout of the paper

This paper is organized as follows:

- In Section 2, we briefly provide some background on Clifford algebras and on *discrete Fourier analysis* required throughout the paper. From the point of view of the theory of pseudo-differential operators, that will allows us to shift all the well-known constructions in the space of square-integrable functions to the space of Clifford-valued distributions over the lattice $h\mathbb{Z}^n$ (see, for instance, [21, Section 21.2] and [21, Section 2.2]).
- In Section 3, we introduce a time-fractional and a space-fractional variant of the *semidiscrete Dirac operator* $D_h + \mathfrak{f}\partial_t + \mathfrak{f}^\dagger e^{-i\theta}$. Guided by the approaches of Cerejeiras, Kähler & Sommen [12] and Cerejeiras, Sommen & Vieira [13], we introduce the time-fractional regularization of $D_h + \mathfrak{f}\partial_t + \mathfrak{f}^\dagger e^{-i\theta}$ by replacing the time-derivative ∂_t by a time-fractional counterpart $-e^{i\theta(1-\beta)}\mathbb{D}_t^\beta$ $(\beta \geq 1)$, mixing the Riemann–Liouville derivative and a unitary term lying on the unit circle $\mathbb{S}^1$. For the space-fractional regularization, we consider the space-fractional counterpart $(-\Delta_h)^{\alpha-1}\partial_t$ of ∂_t, involving the fractional discrete operator $(-\Delta_h)^{\alpha-1}$ $(0 < \alpha \leq 1)$ instead of ∂_t (case of $\alpha = 1$). With the proof of Theorems 3.1 and Theorem 3.4, we will show that the formulations highlighted on Figure 1 retain all of the salient features of the null solutions of the *Lévy–Leblond-type operator* $D + \mathfrak{f}\partial_t + \mathfrak{f}^\dagger e^{-i\theta}$ (limit case $h \to 0$) considered by several authors (see also the papers of Bernstein [7], and Bao, Constales, De Bie & Mertens [6]).
- In Section 4, we will show that the null solutions of both fractional semidiscrete operators are indeed interrelated. Starting from the pseudo-differential representation of the fractional semidiscrete analytic semigroup $\left\{\exp(-te^{i\theta}(-\Delta_h)^\alpha)\right\}_{t\geq 0}$ in terms of its Fourier symbol we will provide first, with the proof of Theorem 4.1, a bridge result that establishes the correspondence between Cauchy problems of space-fractional and time-fractional type, encoded by the set of fractional operators, $e^{-i\theta}\partial_t + (-\Delta_h)^\alpha$ and $-e^{-i\theta\beta}\mathbb{D}_t^\beta - \Delta_h$, respectively. This theorem is essentially a wise reformulation of [14, Theorem 3] in terms of the right-sided Riemann–Liouville fractional derivative. Afterward, we will prove Theorem 4.3—the main contribution of this paper—by combining Theorems 3.1, 3.4, and 4.1.
- In Section 5, we will briefly discuss an alternative formulation for the *semidiscrete Dirac operator of the space-fractional type* that factorizes the space-fractional operator $e^{-i\theta}\partial_t + (-\Delta_h)^\alpha$. In the end, we will also comment on the key ingredients considered to obtain our main results and depict further directions of research on the crossroads of function spaces and Helmholtz–Leray type decompositions.

2 | DEFINITIONS

2.1 | Clifford algebra setup

Let $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n, \mathbf{e}_{n+1}, \mathbf{e}_{n+2} \dots, \mathbf{e}_{2n}, \mathbf{e}_{2n+1}$ be the generators of the Clifford algebra of signature $(n+1, n+1)$, $C\ell_{n+1,n+1}$, satisfying

$$\begin{aligned} \mathbf{e}_j\mathbf{e}_k + \mathbf{e}_k\mathbf{e}_j &= -2\delta_{jk}, && 0 \leq j,k \leq n \\ \mathbf{e}_j\mathbf{e}_{n+k} + \mathbf{e}_{n+k}\mathbf{e}_j &= 0, && 0 \leq j \leq n \;\&\; 1 \leq k \leq n+1 \\ \mathbf{e}_{n+j}\mathbf{e}_{n+k} + \mathbf{e}_{n+k}\mathbf{e}_{n+j} &= 2\delta_{jk}, && 1 \leq j,k \leq n+1. \end{aligned} \tag{2.1}$$

As it is well-known from the literature (cf. [45, Chap. 3]), $C\ell_{n+1,n+1}$ is a universal algebra of dimension 2^{2n+2} linear isomorphic to the exterior algebra $\Lambda^*\left(\mathbb{R}^{n+1,n+1}\right)$, containing the field of real numbers $\mathbb{R}$, the n-dimensional Euclidean space $\mathbb{R}^n$ and the Minkowski space $\mathbb{R}^{n+1,n+1}$ of signature $(n+1,n+1)$ as proper subspaces.

In particular, the ladder structure of $C\ell_{n+1,n+1}$ allows us to represent the space-time tuple $(x_1,x_2,\dots,x_n,t)$ of $\mathbb{R}^{n+1}$ through the paravector representation $t+x=t+\sum_{j=1}^n x_j\mathbf{e}_j$ of $\mathbb{R}\oplus\mathbb{R}^n$.

Here and elsewhere, the 1-vector representations $x=\sum_{j=1}^n x_j\mathbf{e}_j$ and $x\pm h\mathbf{e}_j$ of $\mathbb{R}^n$ will be adopted to describe the lattice point $(x_1,x_2,\dots,x_n)$ of $h\mathbb{Z}^n$ and the forward/backward shifts $(x_1,x_2,\dots,x_j\pm h,\dots,x_n)$ over $h\mathbb{Z}^n$, respectively. For a sake of readibility, one will use throughout the paper the tuple notations $(x+h\mathbf{e}_j,t)$ and $(x-h\mathbf{e}_j,t)$ to define shifts over the semidiscrete space-time lattice

$$h\mathbb{Z}^n\times[0,\infty):=\left\{(x,t)\in\mathbb{R}^{n+1}\,:\,\frac{x}{h}\in\mathbb{Z}^n\wedge t\geq 0\right\}.$$

We note also that the Clifford algebras $C\ell_{0,n}$, $C\ell_{1,1}$, and $C\ell_{1,n+1}$, considered, that is, on the series of papers [6, 7, 12, 26, 27], are subalgebras of $C\ell_{n+1,n+1}$. For our purposes, we assume that $C\ell_{1,1}$ is generated by the nilpotents

$$\mathfrak{f}=\frac{1}{2}(\mathbf{e}_{2n+1}+\mathbf{e}_0)\quad\text{and}\quad\mathfrak{f}^\dagger=\frac{1}{2}(\mathbf{e}_{2n+1}-\mathbf{e}_0).\tag{2.2}$$

Thereby, the graded anti-commuting relations (2.1) are equivalent to

$$\begin{array}{lll}\mathbf{e}_j\mathfrak{f}+\mathfrak{f}\mathbf{e}_j=0, & \mathbf{e}_{n+j}\mathfrak{f}+\mathfrak{f}\mathbf{e}_{n+j}=0, & (1\leq j\leq n)\\ \mathbf{e}_j\mathfrak{f}^\dagger+\mathfrak{f}^\dagger\mathbf{e}_j=0, & \mathbf{e}_{n+j}\mathfrak{f}^\dagger+\mathfrak{f}^\dagger\mathbf{e}_{n+j}=0, & (1\leq j\leq n)\\ (\mathfrak{f})^2=(\mathfrak{f}^\dagger)^2=0, & \mathfrak{f}\mathfrak{f}^\dagger+\mathfrak{f}^\dagger\mathfrak{f}=1. & \end{array}\tag{2.3}$$

Furthermore, based on the decomposition $C\ell_{n+1,n+1}=C\ell_{1,1}\otimes C\ell_{n,n}$, we will represent the Clifford-vector-valued functions $(x,t)\mapsto\Psi(x,t)$ with membership in the *complexified Clifford algebra* $\mathbb{C}\otimes C\ell_{n+1,n+1}$, through the ansatz

$$\Psi(x,t)=\Psi^{[0]}(x,t)+\mathfrak{f}\Psi^{[1]}(x,t)+\mathfrak{f}^\dagger\Psi^{[2]}(x,t)+\mathfrak{f}\mathfrak{f}^\dagger\Psi^{[3]}(x,t),$$

whereby $(x,t)\mapsto\Psi^{[m]}(x,t)$ $(m=0,1,2,3)$ are Clifford-vector-valued functions with membership in $\mathbb{C}\otimes C\ell_{n,n}$.

To introduce in the following function spaces and operators underlying to the $\mathbb{C}\otimes C\ell_{n+1,n+1}$-valued functions $(x,t)\mapsto\Psi(x,t)$ and to its $\mathbb{C}\otimes C\ell_{n,n}$-valued components $(x,t)\mapsto\Psi^{[m]}(x,t)$ $(m=0,1,2,3)$, as well as $\mathbb{C}\otimes C\ell_{n,n}$-valued representations of the discrete Dirac operator D_h in terms of its Fourier multipliers (cf. [21, Section 21.2.2] and [22, Section 2.3]), one has to consider the $\dagger$-*conjugation* operation $\mathbf{a}\mapsto\mathbf{a}^\dagger$ on the *complexified Clifford algebra* $\mathbb{C}\otimes C\ell_{n+1,n+1}$, defined recursively as follows:

$$\begin{array}{l}(\mathbf{ab})^\dagger=\mathbf{b}^\dagger\mathbf{a}^\dagger\\ (a_J\mathbf{e}_J)^\dagger=\overline{a_J}\mathbf{e}_{j_r}^\dagger\cdots\mathbf{e}_{j_2}^\dagger\mathbf{e}_{j_1}^\dagger\quad(0\leq j_1<j_2<\cdots<j_r\leq 2n+1)\\ \mathbf{e}_j^\dagger=-\mathbf{e}_j\text{and}\mathbf{e}_{n+1+j}^\dagger=\mathbf{e}_{n+1+j}\quad(0\leq j\leq n).\end{array}\tag{2.4}$$

From Equation (2.4), the $\dagger$-conjugation identities

$$(\mathfrak{f}^\dagger)^\dagger=\mathfrak{f}\text{and}(\mathfrak{f}\mathfrak{f}^\dagger)^\dagger=\mathfrak{f}\mathfrak{f}^\dagger,$$

involving the nilpotents $\mathfrak{f}$ and $\mathfrak{f}^\dagger$ defined viz. Equation (2.2), are then immediate. Also, from Equation (2.4) one readily has that

$$\mathbf{a}\mapsto\|\mathbf{a}\|:=\sqrt{\mathbf{a}^\dagger\mathbf{a}}$$

defines a $\|\cdot\|$-norm endowed by the *complexified Clifford algebra* structure of $\mathbb{C}\otimes C\ell_{n+1,n+1}$, since $\mathbf{a}^\dagger\mathbf{a}=\mathbf{a}\mathbf{a}^\dagger$ is a non-negative real number. In case where $\mathbf{a}$ belongs to $\mathbb{C}\otimes\mathbb{R}^{n+1,n+1}$, the quantity $\|\mathbf{a}\|$ equals to the standard norm of $\mathbf{a}$ on $\mathbb{C}^{2n+2}$.

2.2 The discrete Fourier analysis background

Let us define by $\mathcal{S}(h\mathbb{Z}^n;\mathbb{C}\otimes C\ell_{n,n}) := \mathcal{S}(h\mathbb{Z}^n)\otimes\left(\mathbb{C}\otimes C\ell_{n,n}\right)$ the Schwartz class of $\mathbb{C}\otimes C\ell_{n,n}$-valued functions on the lattice $h\mathbb{Z}^n$, consisting on *rapidly decaying functions* $x\mapsto \Phi(x,t)$ $(t\in[0,\infty))$ defined for any $M\in[0,\infty)$ by the semi-norm condition

$$\sup_{x\in h\mathbb{Z}^n}\left(1+\|x\|^2\right)^M\|\Phi(x,t)\|<\infty,$$

and by $\ell_2(h\mathbb{Z}^n;\mathbb{C}\otimes C\ell_{n,n}) := \ell_2(h\mathbb{Z}^n)\otimes\left(\mathbb{C}\otimes C\ell_{n,n}\right)$ the *right Hilbert module* endowed by the Clifford-valued sesquilinear form

$$\langle\Phi(\cdot,t),\Psi(\cdot,t)\rangle_h=\sum_{x\in h\mathbb{Z}^n}h^n\Phi(x,t)^\dagger\Psi(x,t). \tag{2.5}$$

By exploiting [42, Exercise 3.1.7] to the Clifford-valued setting, it is easy to check that the seminorm condition

$$\sup_{x\in h\mathbb{Z}^n}(1+\|x\|^2)^{-M}\|\Phi(x,t)\|<\infty$$

induces the set of all *continuous linear functionals* with membership in the Schwarz class $\mathcal{S}(h\mathbb{Z}^n;\mathbb{C}\otimes C\ell_{n,n})$, induced by the mapping

$$\Phi(\cdot,t)\mapsto\langle\Phi(\cdot,t),\Psi(\cdot,t)\rangle_h,$$

whereby the family of functions $\Psi(\cdot,t) : h\mathbb{Z}^n\to\mathbb{C}\otimes C\ell_{n,n}$ $(t\in[0,\infty))$ belong to the space of $\mathbb{C}\otimes C\ell_{n,n}$-valued tempered distributions on the lattice $h\mathbb{Z}^n$, denoted as

$$\mathcal{S}'(h\mathbb{Z}^n;\mathbb{C}\otimes C\ell_{n,n}) := \mathcal{S}'(h\mathbb{Z}^n)\otimes\left(\mathbb{C}\otimes C\ell_{n,n}\right).$$

In particular, the mapping property $\Phi(\cdot,t)\mapsto\langle\Phi(\cdot,t),\Psi(\cdot,t)\rangle_h$ together with density arguments allows us to define, for every $x\mapsto\Phi(x,t)$ with membership in $\ell_2(h\mathbb{Z}^n;\mathbb{C}\otimes C\ell_{n,n})$, a distribution $\Phi(\cdot,t)\mapsto\langle\Phi(\cdot,t),\Psi(\cdot,t)\rangle_h$ lying to $\mathcal{S}'(h\mathbb{Z}^n;\mathbb{C}\otimes C\ell_{n,n})$.

Next, we denote by $\left(-\frac{\pi}{h},\frac{\pi}{h}\right]^n$ the n-dimensional *Brioullin zone* representation for the n-torus $\mathbb{R}^n/\frac{2\pi}{h}\mathbb{Z}^n$ (cf. [40, p. 324]), by

$$L_2\left(\left(-\frac{\pi}{h},\frac{\pi}{h}\right]^n;\mathbb{C}\otimes C\ell_{n,n}\right) := L_2\left(\left(-\frac{\pi}{h},\frac{\pi}{h}\right]^n\right)\otimes\left(\mathbb{C}\otimes C\ell_{n,n}\right)$$

the $\mathbb{C}\otimes C\ell_{n,n}$-*Hilbert module* endowed by the sesquilinear form

$$\langle\mathbf{f}(\cdot,t),\mathbf{g}(\cdot,t)\rangle_{\left(-\frac{\pi}{h},\frac{\pi}{h}\right]^n}=\int_{\left(-\frac{\pi}{h},\frac{\pi}{h}\right]^n}\mathbf{f}(\xi,t)^\dagger\mathbf{g}(\xi,t)d\xi, \tag{2.6}$$

and by $C^\infty\left(\left(-\frac{\pi}{h},\frac{\pi}{h}\right]^n;\mathbb{C}\otimes C\ell_{n,n}\right)$ the space of $\mathbb{C}\otimes C\ell_{n,n}$-valued test functions. The *discrete Fourier transform*, defined by

$$(\mathcal{F}_h\Phi)(\xi,t)=\begin{cases}\dfrac{h^n}{(2\pi)^{\frac{n}{2}}}\displaystyle\sum_{x\in h\mathbb{Z}^n}\Phi(x,t)e^{ix\cdot\xi}, & \xi\in\left(-\frac{\pi}{h},\frac{\pi}{h}\right]^n\\ 0 & \xi\in\mathbb{R}^n\setminus\left(-\frac{\pi}{h},\frac{\pi}{h}\right]^n\end{cases} \tag{2.7}$$

yields the isometric isomorphism

$$\mathcal{F}_h : \ell_2(h\mathbb{Z}^n;\mathbb{C}\otimes C\ell_{n,n}) \to L_2\left(\left(-\frac{\pi}{h},\frac{\pi}{h}\right]^n;\mathbb{C}\otimes C\ell_{n,n}\right),$$

whose inverse $(\mathcal{F}_h^{-1}\mathbf{g})(x,t) = \widehat{\mathbf{g}}_h(x,t)$ is provided by the Fourier coefficients

$$\widehat{\mathbf{g}}_h(x,t) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\left(-\frac{\pi}{h},\frac{\pi}{h}\right]^n}(\mathcal{F}_h\mathbf{g})(\xi,t)e^{-ix\cdot\xi}d\xi. \tag{2.8}$$

Moreover, by noticing that the sesquilinear form (2.6) allows us to define a mapping that identifies $C^\infty\left(\left(-\frac{\pi}{h},\frac{\pi}{h}\right]^n;\mathbb{C}\otimes C\ell_{n,n}\right)$ with the dual space $C^\infty\left(\left(-\frac{\pi}{h},\frac{\pi}{h}\right]^n;\mathbb{C}\otimes C\ell_{n,n}\right)'$, the so-called space of $\mathbb{C}\otimes C\ell_{n,n}$-valued distributions over $\left(-\frac{\pi}{h},\frac{\pi}{h}\right]^n$ (cf. [42, Exercise 3.1.15.] and [42, Definition 3.1.25]), we immediately get that the *Parseval -type relation*

$$\langle\mathcal{F}_h\Phi(\cdot,t),\mathbf{g}(\cdot,t)\rangle_{\left(-\frac{\pi}{h},\frac{\pi}{h}\right]^n} = \langle\Phi(\cdot,t),\widehat{\mathbf{g}}_h(\cdot,t)\rangle_h, \tag{2.9}$$

involving the sesquilinear forms (2.5) and (2.6) (cf. [42, Definition 3.1.27]), allows us to extend properly $\mathcal{F}_h$ (see, Equation (2.7)) to the setting of distributions, through the mapping property (cf. [42, Definitions 3.1.27 and 3.1.28])

$$\mathcal{F}_h : \mathcal{S}(h\mathbb{Z}^n;\mathbb{C}\otimes C\ell_{n,n}) \to C^\infty\left(\left(-\frac{\pi}{h},\frac{\pi}{h}\right]^n;\mathbb{C}\otimes C\ell_{n,n}\right),$$

in a way that the Fourier coefficients $x\mapsto\widehat{\mathbf{g}}_h(x,t)$, defined *viz* Equation (2.8), belong to $\mathcal{S}'(h\mathbb{Z}^n;\mathbb{C}\otimes C\ell_{n,n})$.

In the same order of ideas of [21, Section 21.1.3] and [22, Section 2.2] (see also [15, Section 6]), one can also define the *discrete convolution operation* $\star_h$ between the *discrete distribution* $\Psi(\cdot,t)\in\mathcal{S}'(h\mathbb{Z}^n;\mathbb{C}\otimes C\ell_{n,n})$, and the *discrete function* $\Phi(x)\in\mathcal{S}(h\mathbb{Z}^n;\mathbb{C}\otimes C\ell_{n,n})$ as follows:

$$(\Psi(\cdot,t)\star_h\Phi)(x) = \sum_{y\in h\mathbb{Z}^n} h^n\Phi(y)\Psi(x-y,t). \tag{2.10}$$

Indeed, from the duality condition

$$\langle\Psi(\cdot,t)\star_h\Phi,\mathbf{g}(\cdot,t)\rangle_h = \langle\Psi(\cdot,t),\widetilde{\Phi}\star_h\mathbf{g}(\cdot,t)\rangle_h,\text{with} \qquad \widetilde{\Phi}(x) = [\Phi(-x)]^\dagger$$

that yields straightforwardly from the following sequence of identities:

$$\begin{aligned}\langle\mathcal{F}_h[\Psi(\cdot,t)\star_h\Phi],\mathbf{g}(\cdot,t)\rangle_{\left(-\frac{\pi}{h},\frac{\pi}{h}\right]^n} &= \langle\Psi(\cdot,t)\star_h\Phi,\mathcal{F}_h^{-1}[\mathbf{g}(\cdot,t)]\rangle_h\\ &= \langle\Psi(\cdot,t),\widetilde{\Phi}\star_h\mathcal{F}_h^{-1}[\mathbf{g}(\cdot,t)]\rangle_h\\ &= \left\langle\Psi(\cdot,t),\mathcal{F}_h^{-1}\left(\mathcal{F}_h\widetilde{\Phi}\mathbf{g}(\cdot,t)\right)\right\rangle_h\\ &= \left\langle\mathcal{F}_h\Psi(\cdot,t),\mathcal{F}_h\widetilde{\Phi}\mathbf{g}(\cdot,t)\right\rangle_h\\ &= \langle(\mathcal{F}_h\Psi(\cdot,t))(\mathcal{F}_h\Phi),\mathbf{g}(\cdot,t)\rangle_{\left(-\frac{\pi}{h},\frac{\pi}{h}\right]^n},\end{aligned}$$

one can say the *discrete convolution operation* (2.10) is well-defined.

3 | FRACTIONAL SEMIDISCRETE DIRAC OPERATORS OF LÉVY–LEBLOND TYPE

3.1 | The time-fractional case

Let us now recall the basic setup and results from the series of papers [19–21, 41] to discuss further aspects of our construction. In the following, we will use the nilpotents $\mathfrak{f}$ and $\mathfrak{f}^\dagger$ of $C\ell_{1,1}$ to introduce firstly on $\mathbb{C}\otimes C\ell_{n+1,n+1}$, with $C\ell_{n+1,n+1}=C\ell_{1,1}\otimes C\ell_{n,n}$, the semidiscrete Dirac-type operator

$$ {}_\theta\mathcal{D}_{h,t} := D_h + \mathfrak{f}\partial_t + \mathfrak{f}^\dagger e^{-i\theta}, \tag{3.1}$$

carrying the discrete Dirac operator

$$\begin{aligned} D_h\Psi(x,t) = &\sum_{j=1}^n \mathbf{e}_j \frac{\Psi(x+h\mathbf{e}_j,t)-\Psi(x-h\mathbf{e}_j,t)}{2h} + \\ &+\sum_{j=1}^n \mathbf{e}_{n+j}\frac{2\Psi(x,t)-\Psi(x+h\mathbf{e}_j,t)-\Psi(x-h\mathbf{e}_j,t)}{2h}, \end{aligned}\tag{3.2}$$

the time derivative ∂_t and the unitary group parameter $\theta\mapsto e^{-i\theta}$.

From the factorization property $(D_h)^2=-\Delta_h$ involving the *discrete Laplacian* (cf. [19, Proposition 2.1])

$$\Delta_h\Psi(x,t)=\sum_{j=1}^n \frac{\Psi(x+h\mathbf{e}_j,t)+\Psi(x-h\mathbf{e}_j,t)-2\Psi(x,t)}{h^2}, \tag{3.3}$$

and from the graded anti-commuting relations (2.3) we can easily prove that

$$\left({}_\theta\mathcal{D}_{h,t}\right)^2 = e^{-i\theta}\partial_t - \Delta_h.$$

Here, we notice that $\left({}_\theta\mathcal{D}_{h,t}\right)^2$ is, up to the unitary term $e^{-i\theta}$, the semidiscrete heat operator considered in [3].

Accordingly to [5, 26, 27], a time-fractional counterpart of Equation (3.1) can be straightforwardly obtained by replacing the time-derivative by a fractional analog. In particular, based on the Riemann–Liouville derivative (1.1) we introduce the time-fractional regularization of ${}_\theta\mathcal{D}_{h,t}$, defined for $\beta\geq 1$ by the *time-fractional semidiscrete* operator

$$ {}_\theta\mathbb{D}^\beta_{h,t} := D_h - \mathfrak{f}e^{i\theta(1-\beta)}\mathbb{D}^\beta_t + \mathfrak{f}^\dagger e^{-i\theta}. \tag{3.4}$$

In addition, from the set of graded anti-commuting relations (2.3) one can also obtain the factorization property

$$\left({}_\theta\mathbb{D}^\beta_{h,t}\right)^2 = -e^{-i\theta\beta}\mathbb{D}^\beta_t - \Delta_h$$

for Equation (3.4) in terms of the *time-fractional semidiscrete* operator $-e^{-i\theta\beta}\mathbb{D}^\beta_t-\Delta_h$.

3.2 | Time-fractional case vs. space-fractional case

By taking into account the *discrete Fourier transform* introduced viz. Equation (2.7), one can obtain an equivalent formulation of ${}_\theta\mathbb{D}^\beta_{h,t}$ resp. $-e^{-i\theta\beta}\partial_t-\Delta_h$ in terms of the Fourier multipliers (cf. [20, Section 3.3])

$$\begin{aligned} d_h(\xi)^2 &= \sum_{j=1}^n \frac{4}{h^2}\sin^2\left(\frac{h\xi_j}{2}\right) \\ \mathbf{z}_h(\xi) &= \sum_{j=1}^n -i\mathbf{e}_j\frac{\sin(h\xi_j)}{h} + \sum_{j=1}^n \mathbf{e}_{n+j}\frac{1-\cos(h\xi_j)}{h} \end{aligned}\tag{3.5}$$

of $\mathcal{F}_h \circ (-\Delta_h) \circ \mathcal{F}_h^{-1}$ and $\mathcal{F}_h \circ D_h \circ \mathcal{F}_h^{-1}$. Here, we emphasize that the Fourier multiplier formulation lead to the following mapping properties (cf. [21, Section 21.2.2])

$$\begin{aligned} D_h &: \mathcal{S}(h\mathbb{Z}^n; \mathbb{C} \otimes C\ell_{n,n}) \to \mathcal{S}(h\mathbb{Z}^n; \mathbb{C} \otimes C\ell_{n,n}), \\ -\Delta_h &: \mathcal{S}(h\mathbb{Z}^n; \mathbb{C} \otimes C\ell_{n,n}) \to \mathcal{S}(h\mathbb{Z}^n; \mathbb{C} \otimes C\ell_{n,n}) \end{aligned}$$

in a way that

$$\begin{aligned} \mathcal{F}_h\left[\left(-\mathrm{e}^{-i\theta\beta}\mathbb{D}_t^\beta - \Delta_h\right)\Psi(x,t)\right] &= \left(-\mathrm{e}^{-i\theta\beta}\mathbb{D}_t^\beta + d_h(\xi)^2\right)\mathcal{F}_h\Psi(\xi,t), \\ \mathcal{F}_h\left[{}_\theta\mathbb{D}_{h,t}^\beta\Psi(x,t)\right] &= \left(\mathbf{z}_h(\xi) - \mathfrak{f}\mathrm{e}^{i\theta(1-\beta)}\mathbb{D}_t^\beta + \mathfrak{f}^\dagger\mathrm{e}^{-i\theta}\right)\mathcal{F}_h\Psi(\xi,t). \end{aligned}$$

Thereby, by mimicking [13, Theorem 4.1 and Corollary 4.2] we obtain the following theorem.

Theorem 3.1. *For every $t \in [0,\infty)$, let us assume that the components $\Psi^{[m]}(x,t)$ of the $\mathbb{C} \otimes C\ell_{n+1,n+1}$-valued function*

$$\Psi(x,t) = \Psi^{[0]}(x,t) + \mathfrak{f}\Psi^{[1]}(x,t) + \mathfrak{f}^\dagger\Psi^{[2]}(x,t) + \mathfrak{f}\mathfrak{f}^\dagger\Psi^{[3]}(x,t)$$

satisfy the set of conditions ($m = 0,1,2,3$)

$$\Psi^{[m]}(\cdot,t) \in \mathcal{S}(h\mathbb{Z}^n; \mathbb{C} \otimes C\ell_{n,n}) \ \&\ \mathbb{D}_t^\beta\Psi^{[m]}(\cdot,t) \in \mathcal{S}(h\mathbb{Z}^n; \mathbb{C} \otimes C\ell_{n,n}).$$

Then, for every $\beta \geq 1$ the function $\Psi(x,t)$ is a null solution of ${}_\theta\mathbb{D}_{h,t}^\beta$ (see Equation (3.4)) if, and only if

$$\begin{cases} \mathbb{D}_t^\beta\Psi^{[m]}(x,t) = -e^{i\theta\beta}\Delta_h\Psi^{[m]}(x,t) & \textit{for} \quad m = 0,2 \\ \Psi^{[1]}(x,t) = -e^{i\theta}D_h\Psi^{[0]}(x,t) \\ \Psi^{[3]}(x,t) = e^{i\theta}D_h\Psi^{[2]}(x,t) - \Psi^{[0]}(x,t). \end{cases} \tag{3.6}$$

Remark 3.2. Under the conditions of Theorem 3.1, one can say that the $\mathbb{C} \otimes C\ell_{n,n}$-valued components $\Psi^{[0]}(x,t)$ and $\Psi^{[2]}(x,t)$ of $\Psi(x,t)$ are null solutions of the *time-fractional semidiscrete operator* $-e^{-i\theta\beta}\mathbb{D}_t^\beta - \Delta_h$.

In the space-fractional case, one has to consider the *discrete Fourier transform* $\mathcal{F}_h$ and its inverse $\mathcal{F}_h^{-1}$, defined viz. Equation (2.8), to introduce a space-fractional regularization of the semidiscrete Dirac operator ${}_\theta\mathcal{D}_{h,t}$. Namely, with the aid of the fractional discrete operator $(-\Delta_h)^\sigma = \mathcal{F}_h^{-1} \circ \left(d_h(\xi)^2\right)^\sigma \circ \mathcal{F}_h$:

$$(-\Delta_h)^\sigma\Psi(x,t) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\left(-\frac{\pi}{h},\frac{\pi}{h}\right]^n} \left(d_h(\xi)^2\right)^\sigma \mathcal{F}_h\Psi(\xi,t)\mathrm{e}^{-ix\cdot\xi}d\xi, \tag{3.7}$$

one can define

$${}_\theta\mathcal{D}_{h,t}^\alpha := D_h + \mathfrak{f}(-\Delta_h)^{1-\alpha}\partial_t + \mathrm{e}^{-i\theta}\mathfrak{f}^\dagger \tag{3.8}$$

as the space-fractional regularization underlying to Equation (3.1).

We note already that the factorization property

$$({}_\theta\mathcal{D}_{h,t}^\alpha)^2 = \mathrm{e}^{-i\theta}(-\Delta_h)^{1-\alpha}\partial_t - \Delta_h$$

yields from the set of identities involving the Fourier multipliers of

$$\mathcal{F}_h \circ {}_\theta\mathcal{D}_{h,t}^\alpha \circ \mathcal{F}_h^{-1} \ \&\ \mathcal{F}_h \circ \left(\mathrm{e}^{-i\theta}(-\Delta_h)^{1-\alpha}\partial_t - \Delta_h\right) \circ \mathcal{F}_h^{-1}.$$

Indeed, from Equation (2.3) we obtain

$$\mathbf{z}_h(\xi)\mathfrak{f} + \mathfrak{f}\mathbf{z}_h(\xi) = \mathbf{z}_h(\xi)\mathfrak{f}^\dagger + \mathfrak{f}^\dagger\mathbf{z}_h(\xi) = 0$$

$$\begin{aligned}\left(\mathfrak{f}(d_h(\xi)^2)^{1-\alpha}\partial_t + \mathfrak{f}^\dagger \mathrm{e}^{-i\theta}\right)^2 &= \mathfrak{f}\mathfrak{f}^\dagger \mathrm{e}^{-i\theta}(d_h(\xi)^2)^{1-\alpha}\partial_t + \mathfrak{f}^\dagger\mathfrak{f}\mathrm{e}^{-i\theta}(d_h(\xi)^2)^{1-\alpha}\partial_t \\ &= \mathrm{e}^{-i\theta}(d_h(\xi)^2)^{1-\alpha}\partial_t\end{aligned}$$

so that

$$\begin{aligned}\left(\mathbf{z}_h(\xi) + \mathfrak{f}(d_h(\xi)^2)^{1-\alpha}\partial_t + \mathfrak{f}^\dagger \mathrm{e}^{-i\theta}\right)^2 &= \mathbf{z}_h(\xi)^2 + \left(\mathfrak{f}(d_h(\xi)^2)^{1-\alpha}\partial_t + \mathfrak{f}^\dagger \mathrm{e}^{-i\theta}\right)^2 \\ &\quad + \mathbf{z}_h(\xi)\left(\mathfrak{f}(d_h(\xi)^2)^{1-\alpha}\partial_t + \mathfrak{f}^\dagger \mathrm{e}^{-i\theta}\right) + \left(\mathfrak{f}(d_h(\xi)^2)^{1-\alpha}\partial_t + \mathfrak{f}^\dagger \mathrm{e}^{-i\theta}\right)\mathbf{z}_h(\xi) \\ &= \mathrm{e}^{-i\theta}(d_h(\xi)^2)^{1-\alpha}\partial_t + d_h(\xi)^2.\end{aligned}$$

Regarding the definition of $(-\Delta_h)^\sigma$, we would like to stress the tangible connection between Equation (3.7) and the Bochner's definition (cf. [15, Section 6]). Essentially, from the fact that the Fourier symbol $\mathrm{e}^{-td_h(\xi)^2}$ belongs to $C^\infty\left(\left(-\frac{\pi}{h}, \frac{\pi}{h}\right]^n ; \mathbb{C}\otimes C\ell_{n,n}\right)$ one can moreover show that $(-\Delta_h)^\sigma$ admits, for every of $0<\sigma<1$, a *semidiscrete heat semigroup representation* in terms of $\left\{\mathrm{e}^{s\Delta_h}\right\}_{s\geq 0}$. Namely, a wise adaptation of the proof of [36, Lemma 6.5.] (see also [16, Section 2.]):

$$\left(d_h(\xi)^2\right)^\sigma = \int_0^\infty g_{-\sigma}(s)\left(\mathrm{e}^{-sd_h(\xi)^2} - 1\right)ds,\ g_{-\sigma}(s) = \frac{s^{-1-\sigma}}{\Gamma(-\sigma)}(\text{see Equation}(1.2))$$

allows us to guarantee that $(d_h(\xi)^2)^\sigma$ also belongs to $C^\infty\left(\left(-\frac{\pi}{h}, \frac{\pi}{h}\right]^n ; \mathbb{C}\otimes C\ell_{n,n}\right)$ and moreover that $(-\Delta_h)^\sigma$, represented as follows:

$$(-\Delta_h)^\sigma = \begin{cases}\displaystyle\int_0^\infty g_{-\sigma}(s)\left(\mathrm{e}^{s\Delta_h} - I\right)ds & ,0<\sigma<1 \\ -\Delta_h & ,\sigma = 1\end{cases}$$

satisfies the mapping property

$$(-\Delta_h)^\sigma : \mathcal{S}(h\mathbb{Z}^n;\mathbb{C}\otimes C\ell_{n,n}) \to \mathcal{S}(h\mathbb{Z}^n;\mathbb{C}\otimes C\ell_{n,n}).$$

Thus, the space-fractional regularization ${}_\theta\mathcal{D}_{h,t}^\alpha$ of ${}_\theta\mathcal{D}_{h,t}$, defined viz. Equation (3.8), is thus well defined. The foregoing lemma permits us to derive the following integro-differential-difference representations, involving the *space-fractional semidiscrete* operators ${}_\theta\mathcal{D}_{h,t}^\alpha$ and $\mathrm{e}^{-i\theta}\partial_t(-\Delta_h)^{1-\alpha} - \Delta_h$, respectively.

Lemma 3.3. *For every $t\in[0,\infty)$, let us assume that the components $\Psi^{[m]}(x,t)$ of the $\mathbb{C}\otimes C\ell_{n+1,n+1}$-valued function*

$$\Psi(x,t) = \Psi^{[0]}(x,t) + \mathfrak{f}\Psi^{[1]}(x,t) + \mathfrak{f}^\dagger\Psi^{[2]}(x,t) + \mathfrak{f}\mathfrak{f}^\dagger\Psi^{[3]}(x,t)$$

satisfy the set of conditions ($m = 0,1,2,3$)

$$\Psi^{[m]}(\cdot,t)\in\mathcal{S}(h\mathbb{Z}^n;\mathbb{C}\otimes C\ell_{n,n})\ \&\ \partial_t\Psi^{[m]}(\cdot,t)\in\mathcal{S}(h\mathbb{Z}^n;\mathbb{C}\otimes C\ell_{n,n}).$$

Then, for every $0<\alpha\leq 1$ and $|\theta|\leq\frac{\alpha\pi}{2}$, the componentwise action

$${}_\theta\mathcal{D}_{h,t}^\alpha\Psi(x,t) = D_h\Psi(x,t) + \mathfrak{f}(-\Delta_h)^{1-\alpha}\partial_t\Psi(x,t) + \mathfrak{f}^\dagger e^{-i\theta}\Psi(x,t)$$

admits the following integro-differential-difference representation

$$\begin{cases} D_h\Psi(x,t) + \int_0^\infty g_{\alpha-1}(s)\left(e^{s\Delta_h} - I\right)\mathfrak{f}\partial_t\Psi(x,t)ds + \mathfrak{f}^\dagger e^{-i\theta}\Psi(x,t), & 0<\alpha<1 \\ D_h\Psi(x,t) + \mathfrak{f}\partial_t\Psi(x,t) + \mathfrak{f}^\dagger e^{-i\theta}\partial_t\Psi(x,t), & \alpha = 1, \end{cases}$$

whereas

$$\begin{cases} \int_0^\infty g_{\alpha-1}(s)\left(e^{s\Delta_h} - I\right)e^{-i\theta}\partial_t\Psi(x,t)ds - \Delta_h\Psi(x,t), & 0<\alpha<1 \\ e^{-i\theta}\partial_t\Psi(x,t) - \Delta_h\Psi(x,t), & \alpha = 1 \end{cases}$$

stands for the integro-differential-difference representation of

$$e^{-i\theta}(-\Delta_h)^{1-\alpha}\partial_t\Psi(x,t) - \Delta_h\Psi(x,t) = {}_\theta\mathcal{D}^\alpha_{h,t}\left({}_\theta\mathcal{D}^\alpha_{h,t}\Psi(x,t)\right).$$

Similarly to Theorem 3.1, one can also obtain the following characterization for the *space-fractional semidiscrete operator*${}_\theta\mathcal{D}^\alpha_{h,t}$ defined through Equation (3.8). Although the definition of ${}_\theta\mathcal{D}^\alpha_{h,t}$ relies essentially on the replacement

$$-\mathrm{e}^{i\theta(1-\beta)}\mathbb{D}^\beta_t \longrightarrow (-\Delta_h)^{1-\alpha}\partial_t$$

on the right-hand side of Equation (3.4), its proof involves a lot of technicalities far beyond the Schwarz class $\mathcal{S}(h\mathbb{Z}^n;\mathbb{C}\otimes C\ell_{n,n})$.

Theorem 3.4. *For every $t\in[0,\infty)$, let us assume that the components $\Psi^{[m]}(x,t)$ of the $\mathbb{C}\otimes C\ell_{n+1,n+1}$-valued function*

$$\Psi(x,t) = \Psi^{[0]}(x,t) + \mathfrak{f}\Psi^{[1]}(x,t) + \mathfrak{f}^\dagger\Psi^{[2]}(x,t) + \mathfrak{f}\mathfrak{f}^\dagger\Psi^{[3]}(x,t),$$

satisfy the set of conditions ($m=0,1,2,3$)

$$\Psi^{[m]}(\cdot,t)\in\mathcal{S}(h\mathbb{Z}^n;\mathbb{C}\otimes C\ell_{n,n})\ \&\ \partial_t\Psi^{[m]}(\cdot,t)\in\mathcal{S}(h\mathbb{Z}^n;\mathbb{C}\otimes C\ell_{n,n}).$$

Then, for every $0<\alpha\le 1$ and $|\theta|\le\frac{\alpha\pi}{2}$ the function $\Psi(x,t)$ is a null solution of ${}_\theta\mathcal{D}^\alpha_{h,t}$ (see Equation (3.8)) if, and only if

$$\begin{cases} \partial_t\Psi^{[m]}(x,t) = -e^{i\theta}(-\Delta_h)^\alpha\Psi^{[m]}(x,t), & m=0,2 \\ \Psi^{[1]}(x,t) = -e^{i\theta}D_h\Psi^{[0]}(x,t) \\ \Psi^{[3]}(x,t) = e^{i\theta}D_h\Psi^{[2]}(x,t) - \Psi^{[0]}(x,t). \end{cases} \tag{3.9}$$

Proof. First, we recall that from Equations (3.2) and (2.2) it readily follows that the discrete Dirac operator D_h and the Witt basis $\mathfrak{f},\mathfrak{f}^\dagger$ of $C\ell_{1,1}$ satisfy the set of relations

$$\mathfrak{f}D_h = -D_h\mathfrak{f},\ \mathfrak{f}^\dagger D_h = -D_h\mathfrak{f}^\dagger\ \& D_h\mathfrak{f}\mathfrak{f}^\dagger = \mathfrak{f}\mathfrak{f}^\dagger D_h.$$

Then, by letting act the operator (3.8) on $\Psi(x,t)$ one gets, by a straightforwardly computation based on the aforementioned relations, that

$$\begin{aligned} {}_\theta\mathcal{D}^\alpha_{h,t}\Psi(x,t) &= \left(D_h + \mathfrak{f}(-\Delta_h)^{1-\alpha}\partial_t + \mathfrak{f}^\dagger\mathrm{e}^{-i\theta}\right)\Psi(x,t) \\ &= D_h\Psi^{[0]}(x,t) - \mathfrak{f}D_h\Psi^{[1]}(x,t) \\ &\quad -\mathfrak{f}^\dagger D_h\Psi^{[2]}(x,t) + \mathfrak{f}\mathfrak{f}^\dagger D_h\Psi^{[3]}(x,t) \end{aligned}$$

$$
\begin{aligned}
&+\mathfrak{f}(-\Delta_h)^{1-\alpha}\partial_t\Psi^{[0]}(x,t)\\
&+\mathfrak{f}\mathfrak{f}^\dagger(-\Delta_h)^{1-\alpha}\partial_t\Psi^{[2]}(x,t)\\
&+\mathfrak{f}^\dagger\left(\mathrm{e}^{-i\theta}\Psi^{[0]}(x,t)+\mathrm{e}^{-i\theta}\Psi^{[3]}(x,t)\right)\\
&+(1-\mathfrak{f}\mathfrak{f}^\dagger)\mathrm{e}^{-i\theta}\Psi^{[1]}(x,t).
\end{aligned}
$$

By rearranging now the previous identity, we obtain the above ansatz, written as a linear combination in terms of $1, \mathfrak{f}, \mathfrak{f}^\dagger$ and $\mathfrak{f}\mathfrak{f}^\dagger$. Namely, one has

$$
\begin{aligned}
{}_\theta\mathcal{D}^\alpha_{h,t}\Psi(x,t) = &\left(D_h\Psi^{[0]}(x,t)+\mathrm{e}^{-i\theta}\Psi^{[1]}(x,t)\right)\\
&+\mathfrak{f}\left((-\Delta_h)^{1-\alpha}\partial_t\Psi^{[0]}(x,t)-D_h\Psi^{[1]}(x,t)\right)\\
&+\mathfrak{f}^\dagger\left(-D_h\Psi^{[2]}(x,t)+\mathrm{e}^{-i\theta}\Psi^{[0]}(x,t)+\mathrm{e}^{-i\theta}\Psi^{[3]}(x,t)\right)\\
&+\mathfrak{f}\mathfrak{f}^\dagger\left(\partial_t(-\Delta_h)^{1-\alpha}\Psi^{[2]}(x,t)+D_h\Psi^{[3]}(x,t)-\mathrm{e}^{-i\theta}\Psi^{[1]}(x,t)\right).
\end{aligned}
$$

Then, simply the observation that

$$
(-\Delta_h)^{\alpha-1}=\mathcal{F}_h^{-1}\circ(d_h(\xi)^2)^{\alpha-1}\circ\mathcal{F}_h
$$

stands for the inverse of $(-\Delta_h)^{1-\alpha}=\mathcal{F}_h^{-1}\circ(d_h(\xi)^2)^{1-\alpha}\circ\mathcal{F}_h$ (see Equation (3.7)), one can thus show that $\Psi(x,t)$ solves the equation ${}_\theta\mathcal{D}^\alpha_{h,t}\Psi(x,t)=0$ if, and only if

$$
\begin{cases}
\Psi^{[1]}(x,t)=-\mathrm{e}^{i\theta}D_h\Psi^{[0]}(x,t) & (\text{Equation}(1))\\
\partial_t\Psi^{[0]}(x,t)=(-\Delta_h)^{\alpha-1}D_h\Psi^{[1]}(x,t) & (\text{Equation}(\mathfrak{f}))\\
\Psi^{[3]}(x,t)=\mathrm{e}^{i\theta}D_h\Psi^{[2]}(x,t)-\Psi^{[0]}(x,t) & (\text{Equation}(\mathfrak{f}^\dagger))\\
\partial_t\Psi^{[2]}(x,t)=-(-\Delta_h)^{\alpha-1}D_h\Psi^{[3]}(x,t)+\mathrm{e}^{-i\theta}(-\Delta_h)^{\alpha-1}\Psi^{[1]}(x,t) & (\text{Equation}(\mathfrak{f}\mathfrak{f}^\dagger)).
\end{cases}
\tag{3.10}
$$

Next, using the fact that the discrete Dirac operator D_h (see Equation (3.2)) satisfies $(D_h)^2=-\Delta_h$ (cf. [19, Proposition 2.1]), we immediately get

$$
\begin{aligned}
\partial_t\Psi^{[0]}(x,t)&=-\mathrm{e}^{i\theta}(-\Delta_h)^{\alpha-1}(D_h)^2\Psi^{[0]}(x,t)\\
&=-\mathrm{e}^{i\theta}(-\Delta_h)^{\alpha}\Psi^{[0]}(x,t),
\end{aligned}
$$

after substituting (Equation(1)) on the right-hand side of (Equation($\mathfrak{f}$)).

In the same order of ideas, by substituting (Equation(1)) and (Equation($\mathfrak{f}^\dagger$)) on the right-hand side of (Equation($\mathfrak{f}\mathfrak{f}^\dagger$)), we end up with

$$
\begin{aligned}
\partial_t\Psi^{[2]}(x,t)&=-(-\Delta_h)^{\alpha-1}D_h\left(\mathrm{e}^{i\theta}D_h\Psi^{[2]}(x,t)-\Psi^{[0]}(x,t)\right)+\mathrm{e}^{-i\theta}(-\Delta_h)^{\alpha-1}\left(-\mathrm{e}^{i\theta}D_h\Psi^{[0]}(x,t)\right)\\
&=-\mathrm{e}^{i\theta}(-\Delta_h)^{\alpha-1}(D_h)^2\Psi^{[2]}(x,t)\\
&=-\mathrm{e}^{i\theta}(-\Delta_h)^{\alpha}\Psi^{[2]}(x,t).
\end{aligned}
$$

Thus, we have shown that the coupled system of Equations (3.10) is equivalent to Equation (3.9). □

Remark 3.5. Under the conditions of Theorem 3.4, one can say that the $\mathbb{C}\otimes C\ell_{n,n}$-valued components $\Psi^{[0]}(x,t)$ and $\Psi^{[2]}(x,t)$ of $\Psi(x,t)$ are null solutions of the *space-fractional semidiscrete operator* $\mathrm{e}^{-i\theta}\partial_t+(-\Delta_h)^\alpha$.

4 MAIN RESULTS

4.1 The analytic semigroup $\left\{\exp\left(-te^{i\theta}(-\Delta_h)^\alpha\right)\right\}_{t\geq 0}$

In this section, we will show that the action of the [fractional semidiscrete] analytic semigroup $\left\{\exp\left(-te^{i\theta}(-\Delta_h)^\alpha\right)\right\}_{t\geq 0}$, carrying the parameters $0<\alpha\leq 1$ and $|\theta|\leq \frac{\alpha\pi}{2}$, allows us to establish a correspondence between the solution representation of two seemingly distinct Cauchy problems, involving the space-fractional and time-fractional semidiscrete Dirac operators studied previously in Section 3.

Before we proceed, it is important to recall some facts regarding the formulation of $\left\{\exp(-te^{i\theta}(-\Delta_h)^\alpha)\right\}_{t\geq 0}$ as an analytic semigroup encoded by the product of *modified Bessel functions of the first kind* along the same lines of [21, Section 21.4.2]. First, notice that in case of $\alpha=1$, $\left\{\exp\left(te^{i\theta}\Delta_h\right)\right\}_{t\geq 0}$ corresponds to an analytic extension of the *semidiscrete heat semigroup*, already treated in [36, Section 6] (see Remark 13). In concrete, from the *discrete convolution formula* (2.10) there holds

$$\exp\left(te^{i\theta}\Delta_h\right)\Phi(x)=\sum_{y\in h\mathbb{Z}^n} h^n\Phi(y)K(x-y,te^{i\theta}), \tag{4.1}$$

with

$$K(x,te^{i\theta})=\frac{1}{(2\pi)^n}\int_{Q_h} e^{-te^{i\theta}d_h(\xi)^2}e^{-ix\cdot\xi}d\xi. \tag{4.2}$$

Moreover, the closed formula[1] (cf. [21, p. 458, Equation (21.35)])

$$K(x,te^{i\theta})=\frac{1}{h^n}e^{-\frac{2nte^{i\theta}}{h^2}}I_{\frac{x_1}{h}}\left(\frac{2te^{i\theta}}{h^2}\right)I_{\frac{x_2}{h}}\left(\frac{2te^{i\theta}}{h^2}\right)\cdots I_{\frac{x_n}{h}}\left(\frac{2te^{i\theta}}{h^2}\right),$$

involving the product of *modified Bessel functions of the first kind* (cf. [39, p. 456, 2.5.40 (3)])

$$I_k(z)=\frac{1}{\pi}\int_0^\pi e^{z\cos(\omega)}e^{-ik\omega}d\omega,\ |\arg(z)|<\pi. \tag{4.3}$$

results from the identity associated with the Fourier multipliers $d_h(\xi)^2$ (see Equation (3.5)):

$$d_h(\xi)^2=\sum_{j=1}^n\frac{2}{h^2}\left(1-\cos(h\xi_j)\right),$$

and from the change of variables $\xi_j=\frac{\omega_j}{h}$ $(-\pi<\omega_j\leq\pi)$ on Equation (4.2).

Because of $|\arg(z)|=|\theta|$ for $z=te^{i\theta}$ one can moreover say that $K(x,te^{i\theta})$, described as above, is well defined for every $|\theta|\leq\frac{\alpha\pi}{2}$ $(\alpha=1)$. This conclusion is then immediate from the integral representation (4.3).

We can moreover extend the above analysis to $\left\{\exp\left(-te^{i\theta}(-\Delta_h)^\alpha\right)\right\}_{t\geq 0}$ in case of $0<\alpha<1$ and $|\theta|\leq\frac{\alpha\pi}{2}$. At this stage, we would like to stress that that the Laplace identity

$$e^{-s^\alpha}=\int_0^\infty e^{-rs}L_\alpha^{-\alpha}(r)dr,\ \Re(s)>0,\ 0<\alpha<1 \tag{4.4}$$

involving the *Lévy stable distribution* $L_\alpha^{-\alpha}(r)$ (cf. [37, Section 4]) provides us to establish a bridge between the semigroups

$$\left\{\exp\left(-te^{i\theta}(-\Delta_h)^\alpha\right)\right\}_{t\geq 0}\text{ and }\left\{\exp\left(te^{\frac{i\theta}{\alpha}}\Delta_h\right)\right\}_{t\geq 0},$$

respectively.

Indeed, for $s = te^{i\theta}(d_h(\xi)^2)^\alpha$ one can see from the change of variable $r \to rt^{-\frac{1}{\alpha}}$ on the right-hand side of Equation (4.4) that the underlying Fourier symbols $e^{-te^{i\theta}(d_h(\xi)^2)^\alpha}$ and $e^{-te^{\frac{i\theta}{\alpha}}d_h(\xi)^2}$, respectively, are linked by the integral formula

$$e^{-te^{i\theta}(d_h(\xi)^2)^\alpha} = \int_0^\infty e^{-re^{\frac{i\theta}{\alpha}}d_h(\xi)^2} f_{\alpha,\theta}(r)du, f_{\alpha,\theta}(r) = t^{-\frac{1}{\alpha}} L_\alpha^{-\alpha}\left(rt^{-\frac{1}{\alpha}}\right),$$

whereby $t > 0$ is assumed to ensure that the constraint $\mathrm{Re}(s) > 0$, appearing on Equation (4.4), is always fulfilled.

The above representation is useful in several applications on the crossroads of fractional calculus and stochastics, but we will not need to apply it in concrete throughout this paper. For the interested reader, we refer, for example, to [38, Chapters 3 and 6].

4.2 | Space-fractional vs. time-fractional Cauchy problems

For the remainder part of this paper, we will restrict ourselves to the analysis of $\left\{\exp\left(-te^{i\theta}(-\Delta_h)^\alpha\right)\right\}_{t\geq 0}$ in terms of its symbol $e^{-te^{i\theta}(d_h(\xi)^2)^\alpha}$. First of all, we will start to show that the technique used in [14] to prove Theorem 3 can be generalized to the fractional semidiscrete analytic semigroup $\left\{\exp\left(-te^{i\theta}(-\Delta_h)^\alpha\right)\right\}_{t\geq 0}$.

Theorem 4.1. *Let $\Phi_0 \in \mathcal{S}(h\mathbb{Z}^n; \mathbb{C} \otimes C\ell_{n,n})$ be given. Then, for every $0 < \alpha \leq 1$ and $|\theta| \leq \frac{\alpha\pi}{2}$ the function*

$$\Phi(x,t) = \exp(-te^{i\theta}(-\Delta_h)^\alpha)\Phi_0(x)$$

solves the following two Cauchy problems:

$$\begin{cases} \partial_t\Phi(x,t) = -e^{i\theta}(-\Delta_h)^\alpha\Phi(x,t) & \textit{for} \quad (x,t) \in h\mathbb{Z}^n \times [0,\infty) \\ \Phi(x,0) = \Phi_0(x) & \textit{for} \quad x \in h\mathbb{Z}^n, \end{cases} \tag{4.5}$$

and

$$\begin{cases} \mathbb{D}_t^{\frac{1}{\alpha}}\Phi(x,t) = -e^{\frac{i\theta}{\alpha}}\Delta_h\Phi(x,t) & \textit{for} \quad (x,t) \in h\mathbb{Z}^n \times [0,\infty) \\ \Phi(x,0) = \Phi_0(x) & \textit{for} \quad x \in h\mathbb{Z}^n. \end{cases} \tag{4.6}$$

Proof. By letting act the discrete Fourier transform $\mathcal{F}_h$, we obtain an equivalent formulation for the Cauchy problem (4.5) on the momentum space $Q_h \times [0,\infty)$:

$$\begin{cases} \partial_t[\mathcal{F}_h\Phi(\xi,t)] = -e^{i\theta}(d_h(\xi)^2)^\alpha\mathcal{F}_h\Phi(\xi,t) & \text{for} \quad (\xi,t) \in Q_h \times [0,\infty) \\ \mathcal{F}_h\Phi(\xi,0) = \mathcal{F}_h\Phi_0(\xi) & \text{for} \quad \xi \in Q_h \end{cases} \tag{4.7}$$

so that

$$\mathcal{F}_h\Phi(\xi,t) = e^{-te^{i\theta}\left(d_h(\xi)^2\right)^\alpha}\mathcal{F}_h\Phi_0(\xi)\ , (\xi,t) \in Q_h \times [0,\infty) \tag{4.8}$$

solves Equation (4.7), and whence

$$\begin{aligned} \Phi(x,t) &= \exp(-te^{i\theta}(-\Delta_h)^\alpha)\Phi_0(x) \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{Q_h} e^{-te^{i\theta}\left(d_h(\xi)^2\right)^\alpha}\mathcal{F}_h\Phi_0(\xi)e^{-ix\cdot\xi}d\xi \end{aligned}$$

solves Equation (4.5).

Thus, in order to show that $\Phi(x,t)$ is also a solution of the Cauchy problem (4.6), it suffices to show that Equation (4.8) solves

$$\begin{cases} \mathbb{D}_t^{\frac{1}{\alpha}}[\mathcal{F}_h\Phi(\xi,t)] = e^{\frac{i\theta}{\alpha}} d_h(\xi)^2\mathcal{F}_h\Phi(\xi,t) & \text{for} \quad (\xi,t)\in Q_h\times[0,\infty) \\ \mathcal{F}_h\Phi(\xi,0) = \mathcal{F}_h\Phi_0(\xi) & \text{for} \quad \xi\in Q_h, \end{cases} \tag{4.9}$$

or equivalently, that

$$\mathbb{D}_t^{\frac{1}{\alpha}}\left[e^{-te^{i\theta}\left(d_h(\xi)^2\right)^\alpha}\right] = e^{\frac{i\theta}{\alpha}} d_h(\xi)^2 e^{-te^{i\theta}\left(d_h(\xi)^2\right)^\alpha},$$

holds for every $0<\alpha\le 1$ and $|\theta|\le\frac{\alpha\pi}{2}$.

To do so, recall that for every $k\in\mathbb{N}$ one has the derivation rule

$$(-\partial_t)^k\left[e^{-te^{i\theta}\left(d_h(\xi)^2\right)^\alpha}\right] = e^{i\theta k}\left(d_h(\xi)^2\right)^{\alpha k} e^{-te^{i\theta}\left(d_h(\xi)^2\right)^\alpha}. \tag{4.10}$$

If $\frac{1}{\alpha}=k\in\mathbb{N}$, it readily follows that $\mathbb{D}_t^{\frac{1}{\alpha}}$ equals to $(-\partial_t)^k$ so that

$$\mathbb{D}_t^{\frac{1}{\alpha}}\left[e^{-te^{i\theta}\left(d_h(\xi)^2\right)^\alpha}\right] = e^{\frac{i\theta}{\alpha}} d_h(\xi)^2 e^{-te^{i\theta}\left(d_h(\xi)^2\right)^\alpha}.$$

Otherwise, set $k=\left\lfloor\frac{1}{\alpha}\right\rfloor+1$. Note that, for values of $0<\alpha<1$, the constraint $|\theta|\le\frac{\alpha\pi}{2}$ assures that the constant $\lambda = e^{i\theta}\left(d_h(\xi)^2\right)^\alpha$ satisfies the condition

$$\Re(\lambda) = \cos(\theta)\left(d_h(\xi)^2\right)^\alpha > 0.$$

Then, from the Laplace identity (1.3) we get that

$$\int_0^\infty g_{k-\frac{1}{\alpha}}(p)e^{-pe^{i\theta}\left(d_h(\xi)^2\right)^\alpha}dp = e^{-i\theta\left(k-\frac{1}{\alpha}\right)}\left(d_h(\xi)^2\right)^{-\alpha\left(k-\frac{1}{\alpha}\right)},$$

whereby $g_{k-\frac{1}{\alpha}}(p)$ stands for the Gel'fand–Shilov function (see Equation (1.2)).

Hence, Equation (4.10) together with the previous integral formula gives rise to the sequence of identities

$$\begin{aligned} \mathbb{D}_t^{\frac{1}{\alpha}}\left[e^{-te^{i\theta}\left(d_h(\xi)^2\right)^\alpha}\right] &= (-\partial_t)^k\int_t^\infty g_{k-\frac{1}{\alpha}}(s-t)e^{-se^{i\theta}\left(d_h(\xi)^2\right)^\alpha}ds \\ &= (-\partial_t)^k\int_0^\infty g_{k-\frac{1}{\alpha}}(p)e^{-(t+p)e^{i\theta}\left(d_h(\xi)^2\right)^\alpha}dp \\ &= (-\partial_t)^k\left[e^{-te^{i\theta}\left(d_h(\xi)^2\right)^\alpha}\right]\times\int_0^\infty g_{k-\frac{1}{\alpha}}(p)e^{-pe^{i\theta}\left(d_h(\xi)^2\right)^\alpha}dp \\ &= e^{i\theta k}\left(d_h(\xi)^2\right)^{\alpha k}e^{-te^{i\theta}\left(d_h(\xi)^2\right)^\alpha}\times e^{-i\theta\left(k-\frac{1}{\alpha}\right)}\left(d_h(\xi)^2\right)^{-\alpha(k-\frac{1}{\alpha})} \\ &= e^{\frac{i\theta}{\alpha}}d_h(\xi)^2e^{-te^{i\theta}\left(d_h(\xi)^2\right)^\alpha}, \end{aligned}$$

concluding the proof of Equation (4.9). □

Remark 4.2. Essentially, we have shown in Theorem 4.1 that the analytic semigroup $\left\{\exp\left(-te^{i\theta}(-\Delta_h)^\alpha\right)\right\}_{t\geq 0}$, carrying the parameters $0<\alpha\leq 1$ and $|\theta|\leq \frac{\alpha\pi}{2}$, generates simultaneously solutions for Cauchy problems induced by the fractional semidiscrete operators $e^{-i\theta}\partial_t+(-\Delta_h)^\alpha$ and $-e^{-\frac{i\theta}{\alpha}}\mathbb{D}_t^{\frac{1}{\alpha}}-\Delta_h$, respectively.

4.3 | Cauchy problems of Lévy–Leblond type

After the meaningful construction obtained previously in Theorem 4.1, we have now gathered the main ingredients to prove the main result of this section, whose starting point relies heavily on Theorems 3.1 and 3.4.

Theorem 4.3. *Let* $\Phi_0^{[0]},\Phi_0^{[2]}\in\mathcal{S}(h\mathbb{Z}^n;\mathbb{C}\otimes C\ell_{n,n})$ *and set*

$$\Phi_0(x)=\mathfrak{f}^\dagger\mathfrak{f}\Phi_0^{[0]}(x)+\mathfrak{f}^\dagger\Phi_0^{[2]}(x),$$
$$\Phi(x,t)=\exp(-te^{i\theta}(-\Delta_h)^\alpha)\Phi_0(x).$$

Then, for every $0<\alpha\leq 1$ *and* $|\theta|\leq\frac{\alpha\pi}{2}$ *the function*

$$\Psi(x,t)=\Phi(x,t)-\mathfrak{f}e^{i\theta}D_h\Phi(x,t) \tag{4.11}$$

solves the following two Cauchy problems of Lévy–Leblond type:

$$\begin{cases}\left(D_h+\mathfrak{f}(-\Delta_h)^{1-\alpha}\partial_t+\mathfrak{f}^\dagger e^{-i\theta}\right)\Psi(x,t)=0 & \textit{for}\quad (x,t)\in h\mathbb{Z}^n\times[0,\infty)\\ \Psi(x,0)=\Phi_0(x)-\mathfrak{f}e^{i\theta}D_h\Phi_0(x) & \textit{for}\quad x\in h\mathbb{Z}^n,\end{cases} \tag{4.12}$$

and

$$\begin{cases}\left(D_h-\mathfrak{f}e^{i\theta\left(1-\frac{1}{\alpha}\right)}\mathbb{D}_t^{\frac{1}{\alpha}}+\mathfrak{f}^\dagger e^{-i\theta}\right)\Psi(x,t)=0 & \textit{for}\quad (x,t)\in h\mathbb{Z}^n\times[0,\infty)\\ \Psi(x,0)=\Phi_0(x)-\mathfrak{f}e^{i\theta}D_h\Phi_0(x) & \textit{for}\quad x\in h\mathbb{Z}^n.\end{cases} \tag{4.13}$$

Proof. Under the assumption that $\Phi_0^{[0]},\Phi_0^{[2]}\in\mathcal{S}(h\mathbb{Z}^n;\mathbb{C}\otimes C\ell_{n,n})$, Theorem 4.1 asserts that the components

$$\Psi^{[m]}(x,t)=\exp\left(-te^{i\theta}(-\Delta_h)^\alpha\right)\Phi_0^{[m]}(x)\ (m=0,2)$$

of

$$\Phi(x,t)=\exp\left(-te^{i\theta}(-\Delta_h)^\alpha\right)\Phi_0(x),\text{ with }\Phi_0(x)=\mathfrak{f}^\dagger\mathfrak{f}\Phi_0^{[0]}(x)+\mathfrak{f}^\dagger\Phi_0^{[2]}(x)$$

are solutions of the following set of Cauchy problems ($m=0,2$) for values of $0<\alpha\leq 1$ and $|\theta|\leq\frac{\alpha\pi}{2}$:

$$\begin{cases}\partial_t\Psi^{[m]}(x,t)=-e^{i\theta}(-\Delta_h)^\alpha\Psi^{[m]}(x,t) & \text{for}\quad (x,t)\in h\mathbb{Z}^n\times[0,\infty)\\ \Psi_0^{[m]}(x,0)=\Phi_0^{[m]}(x) & \text{for}\quad x\in h\mathbb{Z}^n,\end{cases} \tag{4.14}$$

and

$$\begin{cases}\mathbb{D}_t^{\frac{1}{\alpha}}\Psi^{[m]}(x,t)=-e^{\frac{i\theta}{\alpha}}\Delta_h\Psi^{[m]}(x,t) & \text{for}\quad (x,t)\in h\mathbb{Z}^n\times[0,\infty)\\ \Psi_0^{[m]}(x,0)=\Phi_0^{[m]}(x) & \text{for}\quad x\in h\mathbb{Z}^n.\end{cases} \tag{4.15}$$

Then, from Theorems 3.4 and 3.1 one has that the function

$$\Psi(x,t) = \Psi^{[0]}(x,t) + \mathfrak{f}\Psi^{[1]}(x,t) + \mathfrak{f}^\dagger\Psi^{[2]}(x,t) + \mathfrak{f}\mathfrak{f}^\dagger\Psi^{[3]}(x,t),$$

solve simultaneously the Cauchy problems (4.12) and (4.13) if, and only if, the components $\Psi^{[m]}(x,t)$ $(m=1,3)$ of $\Psi(x,t)$ are uniquely determined by

$$\Psi^{[1]}(x,t) = -e^{i\theta}D_h\Psi^{[0]}(x,t),$$
$$\Psi^{[3]}(x,t) = e^{i\theta}D_h\Psi^{[2]}(x,t) - \Psi^{[0]}(x,t).$$

Hence, from the set of properties

$$\mathfrak{f}D_h = -D_h\mathfrak{f},\ \mathfrak{f}^\dagger D_h = -D_h\mathfrak{f}^\dagger \ \text{and} D_h\mathfrak{f}\mathfrak{f}^\dagger = \mathfrak{f}\mathfrak{f}^\dagger D_h$$

that yield from the combination of Equations (3.2) and (2.2), it readily follows that

$$\begin{aligned}\Psi(x,t) &= \Psi^{[0]}(x,t) + \mathfrak{f}^\dagger\Psi^{[2]}(x,t)\\ &\quad + \mathfrak{f}(-e^{i\theta}D_h\Psi^{[0]}(x,t)) + \mathfrak{f}\mathfrak{f}^\dagger\left(e^{i\theta}D_h\Psi^{[2]}(x,t) - \Psi^{[0]}(x,t)\right)\\ &= (1-\mathfrak{f}\mathfrak{f}^\dagger)\Psi^{[0]}(x,t) + \mathfrak{f}^\dagger\Psi^{[2]}(x,t)\\ &\quad + e^{i\theta}D_h\left(\mathfrak{f}\Psi^{[0]}(x,t)\right) + e^{i\theta}D_h\left(\mathfrak{f}\mathfrak{f}^\dagger\Psi^{[2]}(x,t)\right)\\ &= \mathfrak{f}^\dagger\mathfrak{f}\Psi^{[0]}(x,t) + \mathfrak{f}^\dagger\Psi^{[2]}(x,t) + e^{i\theta}D_h\left[\mathfrak{f}\Psi^{[0]}(x,t) + \mathfrak{f}\mathfrak{f}^\dagger\Psi^{[2]}(x,t)\right],\end{aligned}$$

whereas from the identity $\mathfrak{f}\mathfrak{f}^\dagger\mathfrak{f} = \mathfrak{f}$, there holds

$$\begin{aligned}\mathfrak{f}\Phi(x,t) &= \mathfrak{f}\left(\mathfrak{f}^\dagger\mathfrak{f}\Psi^{[0]}(x,t) + \mathfrak{f}^\dagger\Psi^{[2]}(x,t)\right) = \mathfrak{f}\Psi^{[0]}(x,t) + \mathfrak{f}\mathfrak{f}^\dagger\Psi^{[2]}(x,t),\\ -\mathfrak{f}e^{i\theta}D_h\Phi(x,t) &= e^{i\theta}D_h(\mathfrak{f}\Phi(x,t)) = e^{i\theta}D_h\left[\mathfrak{f}\Psi^{[0]}(x,t) + \mathfrak{f}\mathfrak{f}^\dagger\Psi^{[2]}(x,t)\right].\end{aligned}$$

That allows us to conclude that $\Psi(x,t)$, described as above, is equivalent to Equation (4.11), as desired. □

5 POSTSCRIPTS

5.1 Factorization of space-fractional semidiscrete operators

With the proof of Theorem 4.3, neatly summarized in Figure 2, it was established an intriguing correspondence between the null solutions of time-fractional resp. space-fractional analogs of the semidiscrete Dirac operator, treated on Sections 3.1 and 3.2, and the null solutions of the time-fractional resp. space-fractional regularizations of the semidiscrete Dirac operator (3.1).

At this stage, we would like to emphasize that, contrary to the time-fractional regularization (3.4), the corresponding space-fractional regularization (3.8) of Equation (3.1) does not factorize the space-fractional semidiscrete operator $e^{-i\theta}\partial_t + (-\Delta_h)^\alpha$, although the factorization of ${}_\theta\mathbb{D}_{h,t}^{\frac{1}{\alpha}}$ $(\beta = \frac{1}{\alpha})$ and ${}_\theta\mathcal{D}_{h,t}^{\alpha}$ was never required on the proof of main results of this paper.

To circumvent this gap, one can consider the alternative space-fractional semidiscrete variant

$${}_\theta\mathbf{D}_{h,t}^{\alpha} := (-\Delta_h)^{\frac{\alpha-1}{2}}D_h + \mathfrak{f}(-\Delta_h)^{\frac{1-\alpha}{2}}\partial_t + \mathfrak{f}^\dagger e^{-i\theta}(-\Delta_h)^{\frac{\alpha-1}{2}} \tag{5.1}$$

of ${}_\theta\mathcal{D}_{h,t} = D_h + \mathfrak{f}\partial_t + \mathfrak{f}^\dagger e^{-i\theta}$, involving the space-fractional operator $(-\Delta_h)^\sigma$ and its inverse $(-\Delta_h)^{-\sigma}$ $(\sigma = \frac{1-\alpha}{2})$.

Hereby, the discrete fractional operator $(-\Delta_h)^{-\sigma}D_h$ stands for the hypercomplex extension of the fractional Riesz-type transform considered in [15, Section 6] (see also [21, Section 21.4.3]).

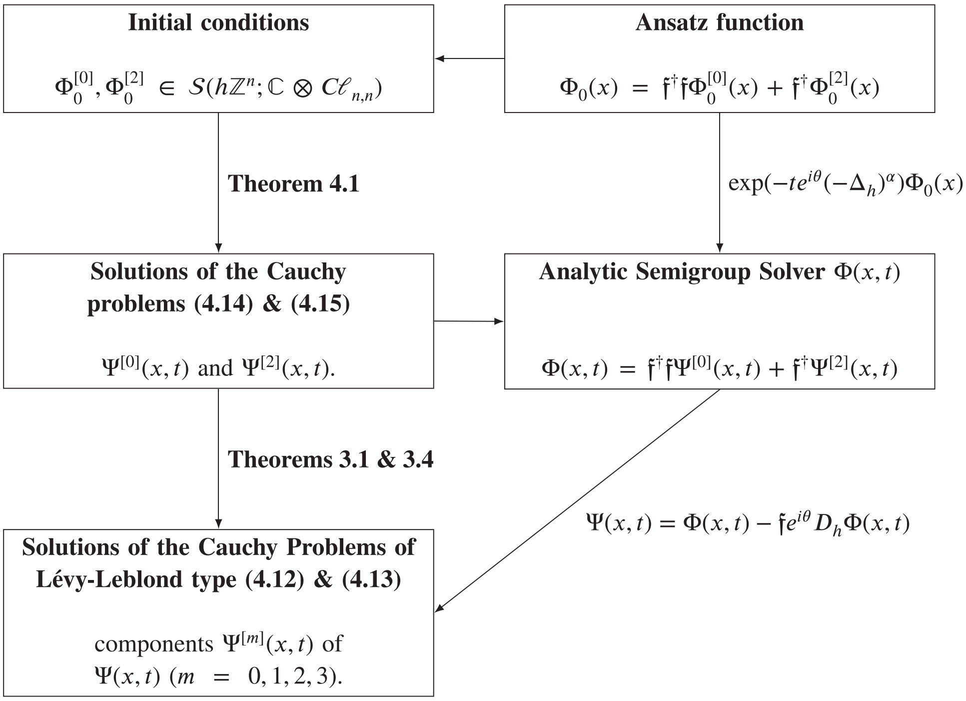

FIGURE 2 Schematic proof of Theorem 4.3.

To do so, let us now turn our attention to the fractional integral operator $(-\Delta_h)^{-\sigma}$. Owing the fact that the Fourier multiplier $(d_h(\xi)^2)^{-\sigma}$ of $\mathcal{F}_h\circ(-\Delta_h)^{-\sigma}\circ\mathcal{F}_h^{-1}$ admitting, for values of $0<\sigma<\frac{1}{2}$, the following Eulerian integral representation:

$$(d_h(\xi)^2)^{-\sigma} = \int_0^\infty \mathrm{e}^{-pd_h(\xi)^2} g_\sigma(p)dp,\ g_\sigma(p) = \frac{p^{\sigma-1}}{\Gamma(\sigma)}\ \text{(see Equation(1.2))}$$

one can guarantee that the mapping property

$$(-\Delta_h)^{-\sigma} : \mathcal{S}(h\mathbb{Z}^n;\mathbb{C}\otimes C\ell_{n,n}) \to \mathcal{S}(h\mathbb{Z}^n;\mathbb{C}\otimes C\ell_{n,n}),$$

carrying the inverse of $(-\Delta_h)^\sigma$, is fulfilled for every $0<\sigma<\frac{1}{2}$.

As a consequence, the reformulation of $(-\Delta_h)^{\frac{1-\alpha}{2}}$ and $(-\Delta_h)^{\frac{\alpha-1}{2}}$ in terms of the discrete convolution property (2.10):

$$\begin{aligned}(-\Delta_h)^\sigma\Psi(x,t) &= (\Psi(\cdot,t)\star_h \delta_{h,\sigma})(x)\\ &:= \sum_{y\in h\mathbb{Z}^n} h^n\delta_{h,\sigma}(x-y)\Psi(y,t),\end{aligned}$$

with

$$\delta_{h,\sigma}(x-y) = \frac{1}{(2\pi)^n}\int_{\left(-\frac{\pi}{h},\frac{\pi}{h}\right]^n} \left(d_h(\xi)^2\right)^\sigma \mathrm{e}^{-i(x-y)\cdot\xi}d\xi,$$

is well defined for every $0<\sigma<\frac{1}{2}$. In case of $\sigma\to 0^+$ one has that $\delta_{h,\sigma}(x-y)$ converges to the so-called *discrete delta function* on $h\mathbb{Z}^n$. That is,

$$\lim_{\sigma\to 0^+}\delta_{h,\sigma}(x-y) = \begin{cases}\frac{1}{h^n}, & \text{if} \quad x=y\\ 0 & \text{if} \quad x\neq y.\end{cases}$$

From the spectral representation $\mathcal{F}_h \circ {}_\theta\mathbf{D}^\alpha_{h,t} \circ \mathcal{F}_h^{-1}$ in terms of its multiplier, there holds that the operators (5.1) and ${}_\theta\mathcal{D}^\alpha_{h,t}$ (see Equation (3.8)) are interrelated by the formula

$$ {}_\theta\mathbf{D}^\alpha_{h,t} = (-\Delta_h)^{\frac{\alpha-1}{2}}\, {}_\theta\mathcal{D}^\alpha_{h,t}. $$

Noteworthy, we obtain the factorization property

$$ ({}_\theta\mathbf{D}^\alpha_{h,t})^2 = \mathrm{e}^{-i\theta}\partial_t + (-\Delta_h)^\alpha. $$

5.2 | Function spaces

In this paper, it was provided a successful strategy to solve Cauchy problems involving discrete space-fractional resp. time-fractional variants of the Lévy–Leblond operator, also quoted in the literature as *parabolic Dirac -type operators*. Within this avenue of thought, already considered in the series of papers [20–22], we have built a discrete pseudo-differential calculus framework in a two-fold-way:

1. To embody the multivector structure of Clifford algebras, ubiquitous, for example, on the formulation of discrete boundary value problems of Navier–Stokes (cf. [25]) and Schrödinger type (cf.[10]);
2. To provide a reformulation of the discrete operator calculus considered on the seminal monograph [29] of Gürlebeck and Sprößig (see [29, Chap. 5]).

In descriptive terms, the central issue of this paper was to develop a whole machinery in terms of the *theory of discrete distributions* over the lattice $h\mathbb{Z}^n$, bearing in mind that the Schwartz class $\mathcal{S}(h\mathbb{Z}^n;\mathbb{C}\otimes C\ell_{n,n})$ is dense in $\ell_2(h\mathbb{Z}^n;\mathbb{C}\otimes C\ell_{n,n})$ (cf. [42, Exercise 3.1.15 of p. 302]).

The natural question that naturally arises is the following: *'How the discrete ℓ_p-spaces, Sobolev spaces and alike, already considered on the monograph* [29] *and on the papers* [10, 25], *come into play in this framework?'*

First, we would like to stress that in Stark contrast with the *continuum* setting over the Euclidean space $\mathbb{R}^n$, the *discrete Fourier transform* (2.7) maps the discrete Schwartz space $\mathcal{S}(h\mathbb{Z}^n;\mathbb{C}\otimes C\ell_{n,n})$ onto the continuous space $C^\infty\left(\left(-\frac{\pi}{h},\frac{\pi}{h}\right]^n;\mathbb{C}\otimes C\ell_{n,n}\right)$. And such isometric isomorphism can be exploited to L_p-type spaces under slightly different circumstances, how we can tacitly infer, for example, from the standard proof of Hausdorff–Young inequality (cf. [42, Corollary 3.1.24]), but also from the proof of embedding result obtained in [25, Lemma 3.1] (see also [30, p. 476]).

To be more precise, based on the L_p-extension problem posed in [30, Section 2.1.] from a probabilistic perspective, one can conjecture that the mapping property

$$ \mathcal{F}_h : \ell_q(h\mathbb{Z}^n;\mathbb{C}\otimes C\ell_{n,n}) \to L_p\left(\left(-\frac{\pi}{h},\frac{\pi}{h}\right]^n;\mathbb{C}\otimes C\ell_{n,n}\right), $$

yields an isometric isomorphism between the $\mathbb{C}\otimes C\ell_{n,n}$-*Banach modules* $\ell_q(h\mathbb{Z}^n;\mathbb{C}\otimes C\ell_{n,n}) := \ell_q(h\mathbb{Z}^n)\otimes(\mathbb{C}\otimes C\ell_{n,n})$ and

$$ L_p\left(\left(-\frac{\pi}{h},\frac{\pi}{h}\right]^n;\mathbb{C}\otimes C\ell_{n,n}\right) := L_p\left(\left(-\frac{\pi}{h},\frac{\pi}{h}\right]^n\right)\otimes(\mathbb{C}\otimes C\ell_{n,n})(1\leq p<\infty) $$

in the following situations (cf. [30, Section 2.1.b] and [30, Example 2.2.8 of p. 94]):

- $q = 2$ and $1 \leq p < \infty$;
- $p \leq q \leq 2$.

The systematic treatment of such function spaces properties on the crossroads of pseudo-differential calculus and probability theory will be postponed to forthcoming research papers. At the moment, we can mention the recent paper of

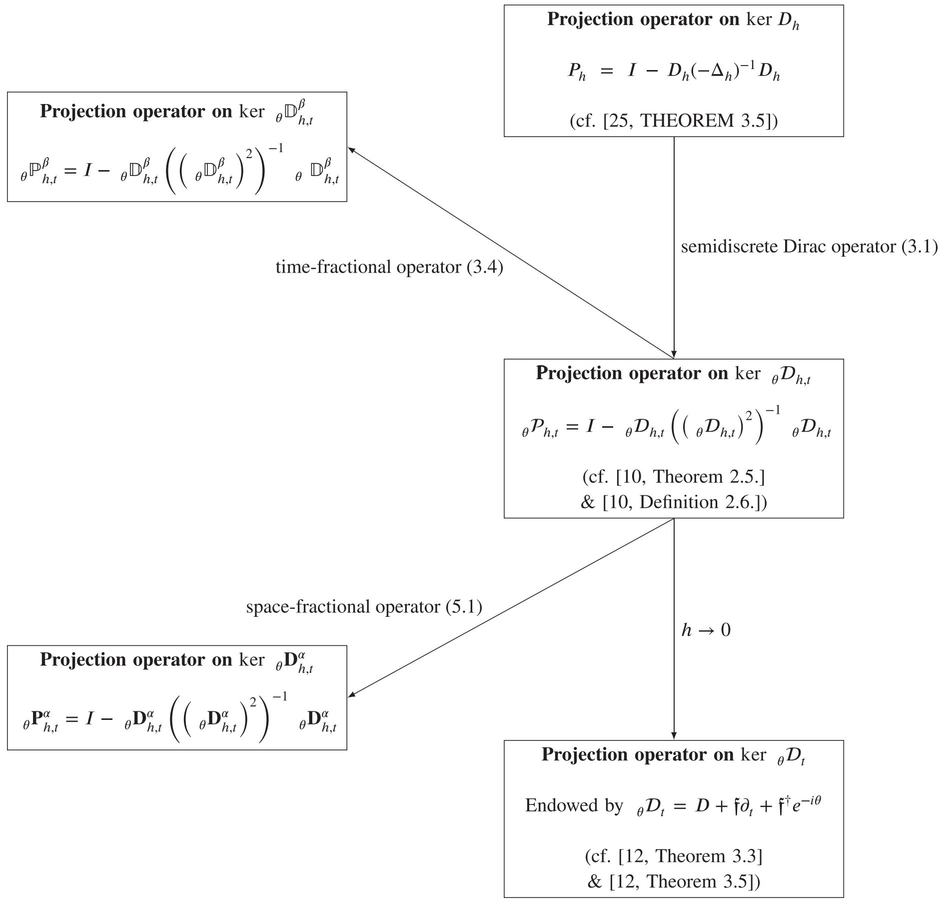

FIGURE 3 The Helmholtz–Leray picture on the fractional semidiscrete case.

Cerejeiras, Kähler and Lucas [11], which treats several aspects of [weighted] ℓ_q- spaces in overlap with symbol classes (see, e.g., [11, Theorems 4 and 8]).

5.3 | Toward Helmholtz–Leray-type decompositions

In the monograph [29], Gürlebeck and Sprößig have popularized the strategy of solving boundary value problems in the context of hypercomplex variables, including the stationary Navier–Stokes equations. Such strategy, successfully exploited to the discrete setting (cf. [10, 25]), entails the following steps:

(i) Compute the fundamental solution of a Dirac-type operator;
(ii) Determine the right inverse of the Dirac-type operator, from the knowledge of the fundamental solution and an analog of the Borel–Pompeiu formula to incorporate the boundary conditions;
(iii) Determine projection operators of the Helmholtz–Leray type;
(iv) Convert the boundary value problem into an integral equation and solve it, if necessary, by a fixed point scheme.

In the context of our framework, one can establish, for example, a parallel with the approach developed in [25], by considering the pseudo-differential operator $T_h = D_h(-\Delta_h)^{-1}$—the so-called *discrete Teodorescu operator* (cf. [29, p. 239] and [11, Section 5])—as the right inverse of the discrete Dirac operator D_h ($D_hT_h = I$), as well as $P_h = I - D_h(-\Delta_h)^{-1}D_h$ for the hypercomplex analog of the so-called *Helmholtz–Leray projection*, due to the following set of properties:

P1 *Projection property:* $P_h^2 = P_h$ in $\mathcal{S}(h\mathbb{Z}^n; \mathbb{C}\otimes C\ell_{n,n})$;
P2 *Null property:* $D_hP_h = 0$ in $\mathcal{S}(h\mathbb{Z}^n; \mathbb{C}\otimes C\ell_{n,n})$;

P3 *Direct sum property:* $P_h\Psi = \Psi$, for all $\Psi \in \ker D_h$;
P4 *Multivector functions coming from a potential term :* $P_h(D_h\Psi) = 0$, for all $\Psi \in \ker D_h$.

Noteworthy, it should be emphasized that such properties seamlessly resemble to the set of properties used in [29] to construct a discrete analog of the Cauchy integral operator (see [29, Theorem 5.3.]).

Also, by taking into account the factorization of the semidiscrete Dirac-type operators—considered throughout this paper—as well as the theory of [fractional semidiscrete] analytic semigroups besides the proof of Theorems 4.1 and 4.3, the construction of the projection operators depicted in Figure 3 can be naturally adopted, bearing in mind a possible exploitation of the techniques used, for example, in [10, 12], to obtain hypercomplex formulations of boundary value problems, as well as possible applications on the crossroads of PDEs and stochastics (see, e.g., the models considered, for example, in [17, 32, 46]). For the technical details, involving the definition of the inverse of the time-fractional/space-fractional operators depicted on the left side of Figure 3, we refer to [43, Section 26].

Last but not least, it would be stressed that an important aspect besides the study of Lévy–Leblond or parabolic-type operators—already quoted on the pioneering work of Cerejeiras, Kähler and Sommen (cf. [12])—is the ability to treat non-stationary boundary value problems as stationary ones. Thus, the major reason for including this extra discussion in the end of this paper is mainly for a preliminary motivation for further research studies in the streamlines of this work.

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CONFLICT OF INTEREST STATEMENT

The author declares no conflicts of interest.

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ENDNOTE

[1] The published version of the book chapter [21] contains a constant typo in formula (4.2), which has been corrected here.

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