LINEAR COPOSITIVE PROGRAMMING: STRONG DUAL FORMULATIONS AND THEIR PROPERTIES

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Dedicated to the memory of Professor Rafail Gabasov

Abstract. In Copositive Programming, a cost function is optimized over a cone of matrices that are positive semidefinite in the non-negative ortant. Being a fairly new field of research, Copositive Programming has already gained popularity. Duality theory is a rich and powerful area of convex optimization, which is central to understanding sensitivity analysis and infeasibility issues as well as to development of numerical methods. In this paper, we continue our recent research on dual formulations for linear Copositive Programming. The dual problems obtained in the paper satisfy the strong duality relations and do not require any additional regularity assumptions such as constraint qualifications. Different dual formulations have their own special properties, the corresponding feasible sets are described in different ways, so they can have an independent application in practice.

Keywords. Conic optimization; Copositive programming; Constraint qualification; Strong duality.

1. INTRODUCTION

Conic optimization is a subfield of convex optimization that studies the problems consisting of minimizing a convex function over an intersection of an affine subspace and a convex cone. The class of conic optimization problems includes some of the most well-known types of convex problems, such as linear and semidefinite programming problems ((LP) and (SDP), respectively).

In this paper, we deal with hte linear problems of Copositive Programming (CoP) that forms a special class of conic optimization problems. CoP is a generalization of SDP, where a linear function is optimized over a cone of matrices that are positive semidefinite in the non-negative ortant $\mathbb{R}^p_+$ (*copositive matrices*). Being a fairly new field of research, CoP has already gained popularity since its models are particularly useful in optimization, graph theory, algebra, and different applications. SDP and CoP are considered in combinatorial optimization to be valuable methods for modelling and obtaining sufficiently accurate estimates of solutions of $\mathscr{N}\mathscr{P}$-hard problems (see, e.g., [1, 2, 3]). Numerous examples of semidefinite and copositive models used in different areas of optimization and applications can be found in [1, 2, 4] and the references there.

The study of optimality and duality relations is one of the central topics of optimization ([5, 6]). Optimality conditions are a crucial issue since they allow not only to test the optimality

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Received August 16, 2022; Accepted October 19, 2022.

of a given feasible solution but also to develop efficient numerical methods. As it was mentioned in [7], the duality plays a central role in detecting infeasibility, lower-bounding of the optimal objective value, as well as in design and analysis of iterative algorithms [8]. Deriving optimality conditions is closely related to the search for *strong* (or *exact*) dual formulations, i.e., dual problems (duals), which *(i)* satisfy the weak duality property, *(ii)* have the same optimal value as the original (primal) optimization problem, and *(iii)* attain this value, when it is finite (see, for example, [9, 10] and the references therein). Often the studies on optimality conditions and strong duality use certain regularity assumptions, the so-called *constraint qualifications* (CQ). It is known that CQs fails. Therefore special attention should be paid to results that do not need additional conditions on the constraints. In the literature, there are some approaches allowing to characterize optimality and formulate strong duals without any CQ for different classes of problems (see, e.g., [11, 12, 13, 14, 15, 16, 17]). In [12, 14], for convex optimization problems, the authors developed a *polynomial ring approach* to generalize the usual concept of Lagrange multiplies, and the existence in a nonstandard polynomial form was proven. This made it possible to obtain dual characterizations of optimality that do not require any CQ. In [18], this approach was used to develop a strong duality for standard convex programming problems. In the 1980s, for convex conic optimization, Borwein and Wolkowicz developed the duality theory based on the concept of the so-called *minimal cone* (see, for example, [11]). Being quite universal in terms of its applications, the duality theory based on the minimal cone representation is quite vulnerable since it is rather abstract and, in general, there are no explicit descriptions of the minimal cone and its dual. However, the concept of minimal cone marked the beginning of a new approach to duality in the conic optimization and motivated an active research in the years that followed (see, e.g., [19, 20, 21, 22]).

To describe the minimal cone and obtain various explicit dual formulations, it is useful to take into account the specifics of the problem being solved and to use effectively its properties and structure. Thus, in [21], an *extended Lagrange-Slater dual problem* (ELSD(m_*)) was introduced for the SDP. This dual is explicitly formulated in terms of the data of the primal SDP problem and has $m_* \in \mathbb{N}$ constraints. It was shown that $m_* \leq \min\{n, p\}$, where n and p are the dimensions of the primal variables' space and the constraints' matrix space, respectively. The paper [20] generalized Ramana's dual to the context of linear conic problems over nice cones, while the authors of [22] extended Ramana's dual to linear conic problems over symmetric (i.e. self-dual and homogeneous) cones.

In [19], the authors introduced *facial reduction* cones which "encode" facial reduction algorithms. Replacing the cones of constraints by the introduced ones, the authors described strong dual problems and obtained certificates of infeasibility and weak infeasibility for linear conic problems. The proposed duals have simple entry forms and do not rely on any CQs. Some of them generalized the dual problems formulated by Ramana for the SDP. As it was mentioned above, the CoP problems are related to the problems of the SDP. Because many nice properties of semidefinite problems are not met for copositive ones, copositive programming forms a broader and more troublesome class of conic problems than the SDP. The cone of copositive matrices has a complex structure and is neither nice, nor self-dual, nor homogeneous. Some properties of the cone of copositive matrices and its dual cone can be found in [1, 2, 23, 24]. The duality issues and optimality conditions for CoP are not sufficiently well studied yet. To the best of our knowledge, no attempt has been made in CoP to obtain explicit strong dual formulations by applying approaches developed for general conic problems.

In [16], inspired by the results of [4, 21], for linear CoP, we suggested an exact explicit dual problem in the form of an extended problem (**EDP**(m_0)) which does not require any additional conditions for constraints. This dual problem was constructed using the concept of *immobile indices*, introduced in our previous works for convex semi-infinite optimization (see, e.g., [25] and the references therein). The extended dual problem (**EDP**(m_0)) can be classified as a *completely positive* problem since its variables belong to the matrix cone of the same name, it is formulated in a similar form as the dual SDP problem (ELSD(m_*)) in [21], and possesses similar properties. The size of the problem (**EDP**(m_0)) depends on the number of its constraints and is characterized by a finite parameter m_0, which is determined algorithmically. In [15], for the parameter m_0, we obtained an estimate $m_0 \leq 2n$, which is quite comparable to that obtained in [4, 21] for a much simpler case of SDP problems.

In this paper, motivated by the approach presented in [18, 19] for conic problems and using the results from [15, 16], we deduce several new strong dual formulations for copositive problems without relying on any CQs. The main aim of the paper is to study properties of the proposed strong duals and provide a detailed comparison between them.

We organize the paper as follows. In Section 2, we introduce notations and describe some preliminary results. In Section 3, for the primal copositive problem we formulate a new dual problem (**DP**) and prove that this dual satisfies the strong duality relation. The number of constraints of problem (**DP**) depends on an integer $m_0 \leq 2n$. In Section 4, we prove a more strict bound for this integer. Other dual formulations, (**EDP**) and (**FDP**), of the linear copositive problem are studied in Section 5. Reformulations of the duals (**DP**) and (**FDP**) using the polynomial ring approach developed in [12, 14, 18] are described in section 6. The paper is ended with some conclusions.

2. Notations and some Preliminary Results

Given a finite-dimensional vector space $\mathfrak{X}$ with inner product $\langle \cdot, \cdot \rangle : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}$, let us recall some common definitions.

A set $\mathscr{B} \subset \mathfrak{X}$ is convex if, for any $\mathbf{x}, \mathbf{y} \in \mathscr{B}$ and any $\alpha \in [0,1]$, it holds $\alpha \mathbf{x} + (1-\alpha)\mathbf{y} \in \mathscr{B}$. Given a set $\mathscr{B} \subset \mathfrak{X}$, denote by conv$\mathscr{B}$ its *convex hull*, i.e., the minimal (by inclusion) convex set, containing the set $\mathscr{B}$, and by span$(\mathscr{B})$ its span, i.e., the smallest linear subspace containing $\mathscr{B}$.

A set $K \subset \mathfrak{X}$ is a *cone* if, for any $\mathbf{x} \in K$ and any $\alpha > 0$, it holds $\alpha \mathbf{x} \in K$. Given a cone $K \subset \mathfrak{X}$, its dual cone K^* is given by $K^* := \{\mathbf{x} \in \mathfrak{X} : \langle \mathbf{x}, \mathbf{y} \rangle \geq 0 \ \forall \mathbf{y} \in K\}$.

In this paper, we deal with special classes of cones whose elements are symmetric matrices. In particular, we consider the cones of copositive and completely positive matrices. These cones will be defined below.

Given an integer $p > 1$, consider the vector space $\mathbb{R}^p$ with the standard orthogonal basis $\{\mathbf{e}_k,\ k = 1,2,\ldots,p\}$. Denote by $\mathbb{R}^p_+$ the set of all p - vectors with non-negative components, by S^p the space of real symmetric $p \times p$ matrices. The space S^p is considered here as a vector space with the trace product $A \bullet B := \text{trace}\,(AB)$.

Let $\mathscr{COP}^p$ be the cone of symmetric copositive $p \times p$ matrices defined as

$$\mathscr{COP}^p := \{D \in S^p : \mathbf{t}^\top D \mathbf{t} \geq 0 \ \forall \mathbf{t} \in \mathbb{R}^p_+\}.$$

Consider a compact subset of $\mathbb{R}^p_+$ in the form of a simplex

$$T := \{\mathbf{t} \in \mathbb{R}^p_+ : \mathbf{e}^\top \mathbf{t} = 1\} \tag{2.1}$$

with $\mathbf{e} = (1,1,...,1)^\top \in \mathbb{R}^p$. It is evident that the cone $\mathscr{COP}^p$ can be equivalently described in the form

$$\mathscr{COP}^p = \{D \in S^p : \ \mathbf{t}^\top D\mathbf{t} \geq 0 \ \forall \mathbf{t} \in T\}. \tag{2.2}$$

The dual cone to $\mathscr{COP}^p$ is the cone of *completely positive matrices*, defined as

$$(\mathscr{COP}^p)^* = \mathscr{CP}^p := \mathrm{conv}\{\mathbf{x}\mathbf{x}^\top : \mathbf{x} \in \mathbb{R}^p_+\}.$$

The cones of copositive and completely positive matrices are known to be *proper* cones, which means that they are closed, convex, pointed, and full-dimensional. Let us formulate some definitions from convex analysis for the cone $\mathscr{COP}^p$ of copositive matrices.

A nonempty convex subset $\mathscr{F}$ of $\mathscr{COP}^p$ is called a *face* of $\mathscr{COP}^p$ if it follows from the condition $\alpha A + (1-\alpha)B \in \mathscr{F}$ with $A, B \in \mathscr{COP}^p$ and $\alpha \in (0,1)$ that $A, B \in \mathscr{F}$. A face $\mathscr{F}$ of $\mathscr{COP}^p$ is *exposed* if there exists a matrix $A \in \mathscr{CP}^p$ such that $\mathscr{F} = \{D \in \mathscr{COP}^p : D \bullet A = 0\}$. Such a face is hereinafter referred to as an *exposed face generated by* $A \in \mathscr{CP}^p$ and is denoted by $\mathscr{F}(A)$.

Consider a linear copositive programming problem in the form

$$\textbf{COP:} \qquad \min_{\mathbf{x} \in \mathbb{R}^n} \mathbf{c}^\top \mathbf{x} \ \text{ s.t. } \mathscr{A}(\mathbf{x}) \in \mathscr{COP}^p,$$

where $\mathbf{x} = (x_1, ..., x_n)^\top$ is the vector of decision variables, the constraint matrix function $\mathscr{A}(\mathbf{x})$ is defined as $\mathscr{A}(\mathbf{x}) := \sum_{s=1}^{n} A_s x_s + A_0$; the vector $\mathbf{c} \in \mathbb{R}^n$ and the matrices $A_s \in S^p$, $s = 0, 1, \ldots, n$ are given.

It follows from the equivalent description (2.2) of the cone $\mathscr{COP}^p$ that problem (**COP**) is equivalent to the following Semi-infinite Programming (SIP) problem:

$$\min_{\mathbf{x} \in \mathbb{R}^n} \ \mathbf{c}^\top \mathbf{x} \ \text{ s.t. } \mathbf{t}^\top \mathscr{A}(\mathbf{x}) \mathbf{t} \geq 0 \ \forall \mathbf{t} \in T, \tag{2.3}$$

where the (index) set T is defined in (2.1).

Suppose that problem (**COP**) is feasible. Then, without loss of generality, we can assume that $A_0 \in \mathscr{COP}^p$. Denote by X the set of feasible solutions in (**COP**), $X = \{\mathbf{x} \in \mathbb{R}^n : \mathscr{A}(\mathbf{x}) \in \mathscr{COP}^p\}$. We say that problem (**COP**) satisfies the Slater condition if, for some $\bar{\mathbf{x}} \in X$, it holds $\mathbf{t}^\top \mathscr{A}(\bar{\mathbf{x}}) \mathbf{t} > 0 \ \forall \mathbf{t} \in T$. For problem (**COP**), the Lagrange dual problem defined in [4] takes the form

$$\textbf{DLP}: \ \ \max -U \bullet A_0, \text{ s.t. } U \bullet A_s = c_s \ \forall s = 1,2,...,n; \ U \in \mathscr{CP}^p, \tag{2.4}$$

where matrix U is the decision variable.

In what follows, for an optimization problem (P), we will denote by $val(P)$ the optimal value of its objective function. Taking into account the SIP-reformulation (2.3) of problem (**COP**) and the duality results from the SIP theory, it is not difficult to prove the following theorem.

Theorem 2.1. *If the constraints of problem* (**COP**) *satisfy the Slater condition and* $val(\textbf{COP}) > -\infty$, *then, for the pair of problems* (**COP**) *and* (**DLP**), *the strong duality relations hold true, which means that the optimal values* $val(\textbf{COP})$ *and* $val(\textbf{DLP})$ *of these problems are equal and the dual problem attains its maximal value.*

If the constraints of problem (**COP**) do not satisfy the Slater condition, then, in general, for the pair of problems (**COP**) and (**DLP**), the strong duality relations may be not valid. Let us illustrate the latter statement by a small example. Consider problem (**COP**) with the following data:

$$n=2,\ p=3,\ \mathbf{c}^\top=(0,\ -1);$$
$$A_0=\begin{pmatrix} a & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\ A_1=\begin{pmatrix} 0 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix},\ A_2=\begin{pmatrix} -1 & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & 0\end{pmatrix}, \tag{2.5}$$

where $a>0$.

It is easy to see that, for $\mathbf{t}^*=(0,0,1)^\top\in T$, $(\mathbf{t}^*)^\top\mathscr{A}(\mathbf{x})\mathbf{t}^*=0$ for all $\mathbf{x}\in\mathbb{R}^2$. Hence the constraints of this problem do not satisfy the Slater condition. It easy to check that, for this example, $\mathscr{A}(\mathbf{x})\in\mathscr{COP}^p$ iff $x_1\le 0, x_2\le 0$. Hence vector $\mathbf{x}^0=(-1,0)^\top$ is an optimal solution of this problem and the optimal value of the cost function is equal to $val(\mathbf{COP})=\mathbf{c}^\top\mathbf{x}^0=0$.

Let us consider the corresponding Lagrange dual problem (**DLP**), which can be rewritten in the form

$$\begin{aligned} &\max_U(-U\bullet A_0),\quad \text{s.t. } U\bullet A_1=0,\ U\bullet A_2=-1,\\ &\text{with } U:=\sum_{i=1}^{p_*}\mathbf{t}(i)\mathbf{t}^\top(i),\ \mathbf{t}(i)\in\mathbb{R}^3_+,\ i=1,\dots,p_*,\end{aligned} \tag{2.6}$$

where $p_*=p(p+1)/2=6$. In this example, problem (2.6) takes the form

$$\max_{\mathbf{t}(i),i=1,\dots,p_*}(-a\sum_{i=1}^{p_*}t_1^2(i)),$$
$$\text{s.t. } \sum_{i=1}^{p_*}t_2^2(i)=0;\sum_{i=1}^{p_*}(-t_1^2(i)-2t_2(i)t_3(i))=-1,\ t_k(i)\ge 0,\ i=1,\dots,p_*;\ k=1,2,3.$$

It follows from the constraints of the dual problem above that, for any dual feasible solution it holds $t_2(i)=0$ for all $i=1,\dots,p_*$ and $\sum\limits_{i=1}^{p_*}t_1^2(i)=1$. Hence the optimal value of this problem is $val(\mathbf{DLP})=-a<0$. Consequently, the duality gap is positive: $val(\mathbf{COP})-val(\mathbf{DLP})=a>0$.

3. A New Dual Formulation

In this section, for problem (**COP**), we formulate a new dual problem and prove that this dual satisfies the strong duality relation.

Given a finite integer $m_0\ge 0$, let us consider the following problem:

$$\max -(U+W_{m_0})\bullet A_0,$$
$$\text{s.t. } (U+W_{m_0})\bullet A_s=c_s\ \forall s=1,2,\dots,n;\ U\in\mathscr{CP}^p,\ W_0=\mathbb{O}_p; \tag{3.1}$$
$$\mathbf{DP}:\quad (U_m+W_{m-1})\bullet A_s=0\ \forall s=0,1,\dots,n,\ \forall m=1,\dots,m_0, \tag{3.2}$$
$$U_m\in\mathscr{CP}^p,\ W_m\in(\mathscr{F}(U_m))^*\ \forall m=1,\dots,m_0, \tag{3.3}$$

where $\mathscr{F}(U)$ is the exposed face of $\mathscr{COP}^p$ generated by $U\in\mathscr{CP}^p$ and $\mathbb{O}_p$ stays for the $p\times p$ null matrix. Here matrices $W_0, U_m, W_m, m=1,\dots,m_0, U$ are the decision variables.

Actually, problem (**DP**) should be denoted by (**DP(**m_0**)**) since the number of its constraints depends on some integer value m_0. For the sake of simplicity, we use a more short notation, but remember that this problem contains parameter m_0. When $m_0 = 0$, we suppose that the set $\{1,...,m_0\}$ in empty and consequently the corresponding problem (**DP**) coincides with Lagrange dual problem (2.4).

Lemma 3.1 (Weak duality)**.** *For any* $\mathbf{x} \in X$ *and any feasible solution*

$$(W_0, U_m, W_m, m = 1,...,m_0, U) \tag{3.4}$$

*of problem (**DP**), the following inequality holds true:*

$$\mathbf{c}^\top \mathbf{x} \geq -(U + W_{m_0}) \bullet A_0. \tag{3.5}$$

Proof. Notice that it follows from (3.3) that

$$\forall D \in \mathcal{COP}^p \;\; \exists \theta = \theta(D) > 0 \text{ such that } (\theta U_m + W_m) \bullet D \geq 0 \; \forall m = 1,...,m_0. \tag{3.6}$$

For any $\mathbf{x} \in X$ and any feasible solution (3.4) of problem (**DP**), we have

$$\begin{aligned} \mathbf{c}^\top \mathbf{x} &= \sum_{s=1}^{n} (U + W_{m_0}) \bullet A_s x_s + (U + W_{m_0}) \bullet A_0 - (U + W_{m_0}) \bullet A_0 \\ &= (U + W_{m_0}) \bullet \mathscr{A}(\mathbf{x}) - (U + W_{m_0}) \bullet A_0. \end{aligned} \tag{3.7}$$

It follows from (3.2) that $(U_m + W_{m-1}) \bullet \mathscr{A}(\mathbf{x}) = 0$, $\forall m = 1,...,m_0$. These equalities imply that, for any $\theta \in \mathbb{R}$, $\sum\limits_{m=1}^{m_0} \theta^{m_0-m}(U_m + W_{m-1}) \bullet \mathscr{A}(\mathbf{x}) = 0$. The latter equality can be rewritten as follows:

$$\begin{aligned} \theta^{m_0-2}(\theta U_1 + W_1) \bullet \mathscr{A}(\mathbf{x}) + \theta^{m_0-3}(\theta U_2 + W_2) \bullet \mathscr{A}(\mathbf{x}) + ... \\ + (\theta U_{m_0-1} + W_{m_0-1}) \bullet \mathscr{A}(\mathbf{x}) + U_{m_0} \bullet \mathscr{A}(\mathbf{x}) = 0. \end{aligned} \tag{3.8}$$

For any $\mathbf{x} \in X$, taking into account the conditions $\mathscr{A}(\mathbf{x}) \in \mathcal{COP}^p$ and (3.6), one can conclude that there exists $\theta(\mathbf{x}) > 0$ such that $(\theta(\mathbf{x})U_m + W_m) \bullet \mathscr{A}(\mathbf{x}) \geq 0$ for all $m = 1,...,m_0$ and $U_{m_0} \bullet \mathscr{A}(\mathbf{x}) \geq 0$. Taking into account these inequalities and equality (3.8), we have $U_{m_0} \bullet \mathscr{A}(\mathbf{x}) = 0$ for all $\mathbf{x} \in X$. This equality and the inequality $(\theta(\mathbf{x})U_{m_0} + W_{m_0}) \bullet \mathscr{A}(\mathbf{x}) \geq 0$ above imply that $W_{m_0} \bullet \mathscr{A}(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in X$ and all feasible solutions of the problem (**DP**) . Notice that $U \in \mathcal{CP}^p$. Hence

$$(U + W_{m_0}) \bullet \mathscr{A}(\mathbf{x}) \geq 0 \tag{3.9}$$

for all feasible solutions of problems (**COP**) and (**DP**). It is evident that (3.7) and (3.9) imply inequality (3.5). The lemma is proved. □

Theorem 3.2 (Strong duality)**.** *Let problem* (**COP**) *be consistent and* $val(\textbf{COP}) > -\infty$. *Then there exists* m_0, $0 \leq m_0 \leq 2n$, *such that, for the pair of problems* (**COP**) *and* (**DP**), *the strong duality relations hold true, i.e., problem* (**DP**) *has an optimal solution*

$$(W_0^0,\; U_m^0,\; W_m^0, m = 1,...,m_0,\; U^0) \tag{3.10}$$

and the equality

$$val(\textbf{COP}) = -(U^0 + W_{m_0}^0) \bullet A_0 \tag{3.11}$$

holds true.

Note that it follows from (3.11) that $val(\textbf{COP}) = val(\textbf{DP})$ and the duality gap for the corresponding pair of problems is zero.

Proof. Suppose that problem (**COP**) is consistent and $val(\textbf{COP}) > -\infty$. Based on the results from [15], it is easy to demonstreate that there exist an integer m_0, $0 \le m_0 \le 2n$, and a set of matrices

$$(W_0^{(*)},\, U_m^{(*)},\, W_m^{(*)},\, D_m^{(*)},\, m = 1, ..., m_0,\, U^{(*)}), \tag{3.12}$$

where $U_m^{(*)} \in S^p$, $D_m^{(*)} \in S^p$, $W_m^{(*)} \in \mathbb{R}^{p\times p}, m = 1, ..., m_0$, satisfying conditions (3.1), (3.2), and the conditions

$$\begin{pmatrix} U_m^{(*)} & W_m^{(*)} \\ (W_m^{(*)})^\top & D_m^{(*)} \end{pmatrix} \in \mathscr{CP}^{2p} \;\; \forall m = 1, ..., m_0, \tag{3.13}$$

such that

$$val(\textbf{COP}) = -(U^{(*)} + W_{m_0}^{(*)}) \bullet A_0. \tag{3.14}$$

For $m = 1, ..., m_0$, it follows from (3.13) that there exists a matrix B_m with non-negative elements in the form

$$B_m = \begin{pmatrix} V_m \\ \Lambda_m \end{pmatrix} \in \mathbb{R}^{2p\times k(m)} \text{ with } V_m \in \mathbb{R}^{p\times k(m)},\ \Lambda_m \in \mathbb{R}^{p\times k(m)},$$

such that

$$\begin{pmatrix} U_m^{(*)} & W_m^{(*)} \\ (W_m^{(*)})^\top & D_m^{(*)} \end{pmatrix} = B_m B_m^\top = \begin{pmatrix} V_m \\ \Lambda_m \end{pmatrix} (V_m^\top \;\; \Lambda_m^\top).$$

Then, for $m = 1, ..., m_0$, the matrices $U_m^{(*)}$, $W_m^{(*)}$, $D_m^{(*)}$ admit representations

$$U_m^{(*)} = V_m V_m^\top,\;\; W_m^{(*)} = V_m \Lambda_m^\top,\; D_m^{(*)} = \Lambda_m \Lambda_m^\top \tag{3.15}$$

with some matrices

$$V_m = (\tau^m(i), i \in I_m),\ \Lambda_m = (\lambda^m(i), i \in I_m), \tag{3.16}$$
$$\text{where }\; \tau^m(i) \in \mathbb{R}^p_+,\ \lambda^m(i) \in \mathbb{R}^p_+,\ i \in I_m,\ |I_m| = k(m).$$

Notice that, for any $m = 1, ..., m_0$ and any $A \in S^p$,

$$(U_m^{(*)} + W_m^{(*)}) \bullet A = (U_m^{(*)} + \overline{W}_m^{(*)}) \bullet A \text{ with } \overline{W}_m^{(*)} = 0.5(W_m^{(*)} + (W_m^{(*)})^\top) \in S^p.$$

By construction,

$$U_m^{(*)} \in \mathscr{CP}^p,\;\; \forall m = 1, ..., m_0. \tag{3.17}$$

Now we show that

$$\overline{W}_m^{(*)} \in (\mathscr{F}(U_m^{(*)}))^*,\;\; \forall m = 1, ..., m_0. \tag{3.18}$$

Let $m \in \{1, ..., m_0\}$. Taking into account (3.15) that conditions (3.18) can be rewritten in the form $(V_m L_m^\top + L_m V_m^\top) \in (\mathscr{F}(V_m V_m^\top))^*$, which, in turn, can be formulated as follows:

$$L_m^\top D V_m = \sum_{i\in I_m} (\lambda^m(i))^\top D \tau^m(i) \ge 0,\ \forall D \in \mathscr{F}(V_m V_m^\top). \tag{3.19}$$

Note that the cone $\mathscr{F}(V_m V_m^\top) := \{D \in \mathscr{COP}^p : D \bullet V_m V_m^\top = 0\}$ can be presented as

$$\mathscr{F}(V_m V_m^\top) := \{D \in \mathscr{COP}^p : V_m^\top D V_m = 0\} = \{D \in \mathscr{COP}^p : (\tau^m(i))^\top D \tau^m(i) = 0,\ \forall i \in I_m\}.$$

Then the following inequalities hold true (see Proposition 2.4 in [26]) $D\tau^m(i)\geq 0$ for all $i\in I_m$, $D\in\mathscr{F}(V_mV_m^\top)$. These inequalities together with the inequalities $\lambda^m(i)\geq 0\ \forall i\in I_m$, imply (3.19) and, consequently, (3.18). From (3.17) and (3.18), one can conclude that the set of matrices

$$(W_0^0=0, U_m^0=U_m^{(*)},\ W_m^0=\overline{W}_m^{(*)}, m=1,...,m_0,\ U^0=U^{(*)}) \tag{3.20}$$

is a feasible solution of problem (**DP**) and equality (3.11) is satisfied. Taking into account (see Lemma 3.1) that for all feasible solutions to problem (**DP**), the inequality $\mathbf{c}^\top\mathbf{x}^0\geq-(U+W_{m_0})\bullet A_0$ holds true. We conclude that (3.20) is an optimal solution to the problem. The theorem is proved. □

4. A NEW BOUND FOR THE INTEGER m_0 IN DUAL PROBLEM (**DP**)

In the previous section, we showed that in problem (**DP**), parameter m_0 is less or equal to $2n$. The aim of this section is to show that this bound can be corrected and integer parameter m_0 can be estimated as follows: $m_0\leq\min\{2n,p^*\}$. Here and in what follows, we set $p^*:=p(p+1)/2$, where p is the order of matrices in $\mathscr{COP}^p$.

Let us first, recall some definitions and properties described in [27]. Given a face $\mathscr{F}$ of $\mathscr{COP}^p$, the set

$$T_0(\mathscr{F}):=\{\mathbf{t}\in T:\ \mathbf{t}^\top D\mathbf{t}=0\ \ \forall D\in\mathscr{F}\}$$

is called the *set of zeros* of $\mathscr{F}$. The set $T_0(\mathscr{F})$ is empty if $\mathscr{F}=\mathscr{COP}^p$ and is the union of a finite number of convex bounded polyhedra otherwise.

Consider the convex hull $\operatorname{conv}T_0(\mathscr{F})$ of $T_0(\mathscr{F})$. The set

$$V_0(\mathscr{F}):=\{\tau^0(j), j\in J_0\},$$

composed by all vertices of the polyhedron $\operatorname{conv}T_0(\mathscr{F})$, is called *the set of minimal zeros* of $\mathscr{F}$. Evidently, the set $J_0:=J_0(\mathscr{F})$ of the vertex indices of the polyhedron $\operatorname{conv}T_0(\mathscr{F})$ is finite: $0\leq|J_0|<\infty$. Given a vector $\mathbf{t}=(t_k,k\in P)^\top\in\mathbb{R}^p_+$ with $P:=\{1,2,...,p\}$, denote $P_+(\mathbf{t}):=\{k\in P:\ t_k>0\}$. For a finite set $V\subset T$, one introduces the corresponding number and sets

$$\begin{aligned}\sigma(V)&:=\min\{t_k,k\in P_+(\mathbf{t}),\ \mathbf{t}\in V\}>0,\\ \Omega(V)&:=\{\mathbf{t}\in T:\rho(\mathbf{t},\operatorname{conv}V)\geq\sigma(V)\},\end{aligned}\tag{4.1}$$

where $\rho(\mathbf{a},\mathscr{B}):=\min\limits_{\tau\in\mathscr{B}}\sum\limits_{k\in P}|a_k-\tau_k|$ is the distance between a set $\mathscr{B}\subset\mathbb{R}^p$ and a point $\mathbf{a}\in\mathbb{R}^p$. If the set V is empty, we consider that $\Omega(V)=T$.

Proposition 4.1. *Let $\overline{\mathscr{F}}$ and $\mathscr{F}$ be faces of $\mathscr{COP}^p$ such that $\overline{\mathscr{F}}\subset\mathscr{F}$. Assume that there exists $\tau\in T_0(\overline{\mathscr{F}})\setminus\operatorname{conv}(V_0(\mathscr{F}))$, where $V_0(\mathscr{F})$ is the set of minimal zeros of $\mathscr{F}$. Then $\dim\overline{\mathscr{F}}<\dim\mathscr{F}$.*

Proof. Since $\tau\notin T_0(\mathscr{F})$ and $\tau\in T_0(\overline{\mathscr{F}})$, then there exists a matrix $\widehat{D}\in\mathscr{F}$ such that

$$\tau^\top\widehat{D}\tau>0 \text{ and } \tau^\top D\tau=0\ \forall D\in\overline{\mathscr{F}}. \tag{4.2}$$

By definition, $\operatorname{span}(\overline{\mathscr{F}})=\{D=\sum\limits_{j=1}^{p_*}\alpha_jD_j,\ \alpha_j\in\mathbb{R},\ D_j\in\overline{\mathscr{F}},\ j=1,...,p_*\}$. As $\overline{\mathscr{F}}\subset\mathscr{F}$, one has $\operatorname{span}(\overline{\mathscr{F}})\subset\operatorname{span}(\mathscr{F})$. It is evident that $\widehat{D}\in\operatorname{span}(\mathscr{F})$. Let us show that $\widehat{D}\notin\operatorname{span}(\overline{\mathscr{F}})$. Suppose the contrary: $\widehat{D}\in\operatorname{span}(\overline{\mathscr{F}})$. Then $\widehat{D}$ admits a representation $\widehat{D}=\sum\limits_{j=1}^{p_*}\alpha_jD_j$, where $\alpha_j\in\mathbb{R}$, $D_j\in\overline{\mathscr{F}},\ j=1,...,p_*$. From this representation and the equalities in (4.2), it follows that the equality $\tau^\top \widehat{D}\tau = \sum_{j=1}^{p_*} \alpha_j \tau^\top D_j \tau = 0$ contradicts the inequality in (4.2). Therefore, we have shown that $\widehat{D} \in \text{span}(\mathscr{F})$ and $\widehat{D} \notin \text{span}(\overline{\mathscr{F}})$. Hence $\text{span}(\overline{\mathscr{F}}) \neq \text{span}(\mathscr{F})$. From the last inequality and the inclusion $\text{span}(\overline{\mathscr{F}}) \subset \text{span}(\mathscr{F})$, it follows that $\dim\overline{\mathscr{F}} < \dim\mathscr{F}$. The proposition is proved. □

Lemma 4.2. *The statement of Theorem 3.2 is valid for* $m_0 \leq \min\{2n,\, p^*\}$.

Proof. If $2n \leq p^*$, then the statement of this lemma follows from Theorem 3.2. Suppose $2n > p^*$. In Theorem 3.2, we can choose $m_0 \leq p^* < 2n$. To prove this estimate of the number m_0, we construct iteratively a set of matrices forming a dual feasible solution (3.10) satisfying (3.11). This can be done using the following algorithm.

Iteration #0. Consider a Semi-infinite Programming problem

$$\mathbf{SIP}(0): \qquad \max_{\mathbf{x}\in\mathbb{R}^n,\ \mu\in\mathbb{R}} \mu, \ \text{ s.t. } \mathbf{t}^\top \mathscr{A}(\mathbf{x})\mathbf{t} \geq \mu \ \forall \mathbf{t} \in T.$$

Notice that the constraints of this problem satisfy the Slater condition.

If $(\mathbf{SIP}(0))$ admits a feasible solution $(\bar{\mathbf{x}}, \bar{\mu})$ with $\bar{\mu} > 0$, then set $m_* = 0$ and go to the *Final step*.

Suppose that $val(\mathbf{SIP}(0)) = 0$. Hence the vector $(\mathbf{x} = \mathbf{0},\ \mu = 0)$ is an optimal solution to problem $(\mathbf{SIP}(0))$ and it follows from the optimality conditions for this solution that there exist an index set $I_1,\ |I_1| \leq n+1$, numbers and vectors $\gamma_i > 0,\ \tau(i) \in T,\ i \in I_1$, such that

$$\sum_{i\in I_1} \gamma_i (\tau(i))^\top A_s \tau(i) = 0 \ \forall s = 0,1,...,n, \ \sum_{i\in I_1} \gamma_i = 1. \tag{4.3}$$

Denote

$$U_1^0 := \sum_{i\in I_1} \gamma_i \tau(i)(\tau(i))^\top \in \mathscr{CP}^p, \tag{4.4}$$

and consider the exposed face $\mathscr{F}(U_1^0)$ of $\mathscr{COP}^p$ generated by $U_1^0 \in \mathscr{CP}^p$:

$$\mathscr{F}(U_1^0) := \{D \in \mathscr{COP}^p : D \bullet U_1^0 = 0\} = \{D \in \mathscr{COP}^p : (\tau(i))^\top D\tau(i) = 0 \ \forall i \in I_1\}.$$

It is evident that equalities (4.3) can be rewritten as $A_s \bullet U_1^0 = 0$ for all $s = 0,1,...,n$. It follows from (4.3) that $\mathscr{A}(\mathbf{x}) \in \mathscr{F}(U_1^0)\ \forall \mathbf{x} \in \mathbb{R}^n$. Let $V_1 = \{\xi^1(j), j \in J_1\}$ be the set of minimal zeros of $\mathscr{F}(U_1^0)$. Then $(\xi^1(j))^\top \mathscr{A}(\mathbf{x})\xi^1(j) = 0 \ \forall j \in J_1, \forall \mathbf{x} \in \mathbb{R}^n$. Go to the next iteration.

Iteration #1. Consider a semi-infinite problem

$$\begin{aligned} & \max_{\mathbf{x}\in\mathbb{R}^n,\mu\in\mathbb{R}} \mu, \\ \mathbf{SIP}(1): \qquad & \text{s.t. } \ \mathbf{t}^\top \mathscr{A}(\mathbf{x})\mathbf{t} \geq \mu \ \forall \mathbf{t} \in \Omega(V_1), \\ & \mathscr{A}(\mathbf{x})\xi^1(j) \geq 0 \ \forall j \in J_1, \end{aligned}$$

where the set $\Omega(V_1)$ is defined in (4.1) with V replaced by V_1. Notice that the constraints of this problem satisfy regularity conditions since there is *a finite* number of *linear* inequality constraints $\mathscr{A}(\mathbf{x})\xi^1(j) \geq 0 \ \forall j \in J_1$, and there exists a feasible solutions $(\tilde{\mathbf{x}} = \mathbf{0},\ \tilde{\mu} = -1)$ such that $\mathbf{t}^\top \mathscr{A}(\tilde{\mathbf{x}})\mathbf{t} > \tilde{\mu} \ \forall \mathbf{t} \in \Omega(V_1)$. If $(\mathbf{SIP}(1))$ admits a feasible solution $(\bar{\mathbf{x}}, \bar{\mu})$ with $\bar{\mu} > 0$, then set $m_* = 1$ and go to the *Final step*. Suppose that $val(\mathbf{SIP}(1)) = 0$. Hence the vector $(\mathbf{x} = \mathbf{0},\ \mu = 0)$ is an optimal solution to problem $(\mathbf{SIP}(1))$. Consequently, taking into account the regularity conditions mentioned above, we conclude that there exist a set $\Delta I_1: \ |\Delta I_1| \leq n+1$, numbers and vectors $\gamma_i > 0, \ \tau(i) \in \Omega(V_1), \ i \in \Delta I_1, \ \lambda^1(j) \in \mathbb{R}^p_+, \ j \in J_1$, such that

$$\sum_{i\in\Delta I_1} \gamma_i(\tau(i))^\top A_s\tau(i) + \sum_{j\in J_1} (\lambda^1(j))^\top A_s\xi^1(j) = 0 \ \forall s = 0,1,...,n, \ \sum_{i\in\Delta I_1} \gamma_i = 1. \tag{4.5}$$

From (4.3) and (4.5), it follows that $\Delta I_1 \neq \emptyset$ and

$$\sum_{i\in I_2} \gamma_i(\tau(i))^\top A_s\tau(i) + \sum_{j\in J_1} (\lambda^1(j))^\top A_s\xi^1(j) = 0 \ \forall s = 0,1,...,n, \tag{4.6}$$

where $I_2 := I_1 \cup \Delta I_1$. Consider matrices

$$U^0_2 := \sum_{i\in I_2} \gamma_i\tau(i)(\tau(i))^\top \in \mathscr{CP}^p, \ W^0_1 := 0.5 \sum_{j\in J_1} \Big(\lambda^1(j)(\xi^1(j))^\top + \xi^1(j)(\lambda^1(j))^\top\Big),$$

and the exposed face of $\mathscr{COP}^p$ generated by $U^0_2 \in \mathscr{CP}^p$:

$$\mathscr{F}(U^0_2) := \{D \in \mathscr{COP}^p : D \bullet U^0_2 = 0\} = \{D \in \mathscr{COP}^p : (\tau(i))^\top D\tau(i) = 0 \ \forall i \in I_2\}.$$

It is evident that equalities (4.6) can be rewritten as $A_s \bullet (U^0_2 + W^0_1) = 0$ for all $s = 0,1,...,n$. By construction, $\mathscr{F}(U^0_2) \subset \mathscr{F}(U^0_1)$, $\tau(i) \in T_0(\mathscr{F}(U^0_2))$, and $\tau(i) \in \Omega(V_1) \subset T \setminus \mathrm{conv} V_1$ for all $i \in \Delta I_1 \neq \emptyset$. Hence, it follows from Proposition 4.1 that

$$\mathscr{F}(U^0_2) \subset \mathscr{F}(U^0_1), \ \dim\mathscr{F}(U^0_2) < \dim F(U^0_1).$$

Notice that $U^0_1 \in \mathscr{CP}^p$. Let us show that

$$W^0_1 \in (\mathscr{F}(U^0_1))^*. \tag{4.7}$$

This condition is equivalent to

$$D \bullet W^0_1 = \sum_{j\in J_1} (\lambda^1(j))^\top D\xi^1(j) \geq 0, \ \forall D \in \mathscr{F}(U^0_1). \tag{4.8}$$

For $j \in J_1$, by construction, it holds $\xi^1(j) \in T_0(\mathscr{F}(U^0_1))$, where we conclude that $D\xi^1(j) \geq 0$ for all $D \in \mathscr{F}(U^0_1)$. From the latter inequalities and the conditions $\lambda^1(j) \in \mathbb{R}^p_+$ for all $j \in J_1$, we conclude that relations (4.8). Consequently, inclusion (4.7) hold true. Let $V_2 = \{\xi^2(j), j \in J_2\}$ be the set of minimal zeros of $\mathscr{F}(U^0_2)$. Go to the next iteration.

Iteration #m, $m \geq 2$. At the beginning of this iteration, the following data is available:

• the indices, numbers and sets $\tau(i) \in T, \ \gamma_i > 0, i \in I_m = I_{m-1} \cup \Delta I_{m-1}$, and some vectors $\lambda^{m-1}(j) \in \mathbb{R}^p_+, \ j \in J_{m-1}$;

• the matrices

$$U^0_m = \sum_{i\in I_m} \gamma_i\tau(i)(\tau(i))^\top \in \mathscr{CP}^p, \ U^0_{m-1} = \sum_{i\in I_{m-1}} \gamma_i\tau(i)(\tau(i))^\top \in \mathscr{CP}^p,$$

$$W^0_{m-1} = 0.5 \sum_{j\in J_{m-1}} [\lambda^{m-1}(j)(\xi^{m-1}(j))^\top + \xi^{m-1}(j)(\lambda^{m-1}(j))^\top],$$

satisfying the equalities $A_s \bullet (U^0_m + W^0_{m-1}) = 0 \ \ \forall s = 0,1,...,n$, that can be rewritten in the form

$$\sum_{i\in I_m} \gamma_i(\tau(i))^\top A_s\tau(i) + \sum_{j\in J_{m-1}} (\lambda^{m-1}(j))^\top A_s\xi^{m-1}(j) = 0 \ \forall s = 0,1,...,n, \tag{4.9}$$

• the exposed faces of $\mathcal{COP}^p$ generated by U^0_{m-1} and U^0_m:

$$\mathcal{F}(U^0_{m-1}) = \{D \in \mathcal{COP}^p : (\tau(i))^\top D\tau(i) = 0\ \forall i \in I_{m-1}\},$$
$$\mathcal{F}(U^0_m) = \{D \in \mathcal{COP}^p : (\tau(i))^\top D\tau(i) = 0\ \forall i \in I_m,\} \tag{4.10}$$

such that $\mathcal{F}(U^0_m) \subset \mathcal{F}(U^0_{m-1})$, $\dim\mathcal{F}(U^0_m) < \dim\mathcal{F}(U^0_{m-1})$;

• the sets $V_{m-1} = \{\xi^{m-1}(j), j \in J_{m-1}\}$ and $V_m = \{\xi^m(j), j \in J_m\}$ of minimal zeros of the faces $\mathcal{F}(U^0_{m-1})$ and $\mathcal{F}(U^0_m)$, respectively.

Consider a SIP problem

$$\mathbf{SIP}(m): \qquad \max_{\mathbf{x}\in\mathbb{R}^n,\ \mu\in\mathbb{R}} \mu, \quad \text{s.t. } \mathbf{t}^\top \mathcal{A}(\mathbf{x})\mathbf{t} \geq \mu \ \ \forall \mathbf{t} \in \Omega(V_m), \quad \mathcal{A}(\mathbf{x})\xi^m(j) \geq 0 \ \ \forall j \in J_m,$$

where the set $\Omega(V_m)$ is defined in (4.1) with V replaced by V_m.

If problem ($\mathbf{SIP}(m)$) admits a feasible solution $(\bar{\mathbf{x}}, \bar{\mu})$ with $\bar{\mu} > 0$, then set $m_* = m$ and go to the *Final step*. Suppose that *val*($\mathbf{SIP}(m)$)$= 0$. Hence the vector ($\mathbf{x} = \mathbf{0}$, $\mu = 0$) is an optimal solution to problem ($\mathbf{SIP}(m)$). Consequently, there exist numbers and vectors $\gamma_i > 0,\ \tau(i) \in \Omega(V_m),\ i \in \Delta I_m,\ |\Delta I_m| \leq n+1,\ \mathbf{w}^m(j) \in \mathbb{R}^p_+,\ j \in J_m$, such that

$$\sum_{i\in\Delta I_m} \gamma_i(\tau(i))^\top A_s\tau(i) + \sum_{j\in J_m} (\mathbf{w}^m(j))^\top A_s\xi^m(j) = 0\ \forall s = 0,1,\dots,n, \quad \sum_{i\in\Delta I_m} \gamma_i = 1.$$

Hence $\Delta I_m \neq \emptyset$ and it follows from these equalities and (4.9) that

$$\sum_{i\in I_{m+1}} \gamma_i(\tau(i))^\top A_s\tau(i) + \sum_{j\in J_{m-1}} (\lambda^{m-1}(j))^\top A_s\xi^{m-1}(j) + \sum_{j\in J_m} (\mathbf{w}^m(j))^\top A_s\xi^m(j) = 0 \quad \forall s = 0,1,\dots,n, \tag{4.11}$$

where $I_{m+1} = I_m \cup \Delta I_m$. As $\mathcal{F}(U^0_m) \subset \mathcal{F}(U^0_{m-1})$, for all $j \in J_{m-1}$, the following holds true:

$$\xi^{m-1}(j) \in T_0(\mathcal{F}(U^0_m)) \implies \xi^{m-1}(j) \in \text{conv}V_m \implies$$
$$\xi^{m-1}(j) = \sum_{i\in J_m} \alpha_{ij}\xi^m(i),\ \sum_{i\in J_m} \alpha_{ij} = 1,\ \alpha_{ij} \geq 0, i \in J_m.$$

Taking into account the equalities above, we can present (4.11) as follows:

$$\sum_{i\in I_{m+1}} \gamma_i(\tau(i))^\top A_s\tau(i) + \sum_{j\in J_m} (\lambda^m(j))^\top A_s\xi^m(j) = 0\ \forall s = 0,1,\dots,n, \tag{4.12}$$

where $\lambda^m(j) = \mathbf{w}^m(j) + \sum_{i\in J_{m-1}} \alpha_{ji}\lambda^{m-1}(i) \geq 0$, $j \in J_m$. Denote

$$U^0_{m+1} := \sum_{i\in I_{m+1}} \gamma_i\tau(i)(\tau(i))^\top \in \mathcal{CP}^p,$$
$$W^0_m := 0.5\sum_{j\in J_m} \left(\lambda^m(j)(\xi^m(j))^\top + \xi^m(j)(\lambda^m(j))^\top\right) \tag{4.13}$$

and consider the exposed face of $\mathcal{COP}^p$ generated by U^0_{m+1}:

$$\mathcal{F}(U^0_{m+1}) := \{D \in \mathcal{COP}^p : D \bullet U^0_{m+1} = 0\} = \{D \in \mathcal{COP}^p : (\tau(i))^\top D\tau(i) = 0\ \forall i \in I_{m+1}\}.$$

It is evident that equalities (4.12) can be rewritten in the form

$$A_s \bullet (U^0_{m+1} + W^0_m) = 0 \;\forall s = 0, 1, ..., n. \tag{4.14}$$

By construction, $\mathscr{F}(U^0_{m+1}) \subset \mathscr{F}(U^0_m)$, $\tau(i) \in T_0(\mathscr{F}(U^0_{m+1}))$ and $\tau(i) \in \Omega(V_m) \subset T \setminus \text{conv}V_m$ for all $i \in \Delta I_m \neq \emptyset$. Hence, it follows from Proposition 4.1 that

$$\mathscr{F}(U^0_{m+1}) \subset \mathscr{F}(U^0_m),\ \dim\mathscr{F}(U^0_{m+1}) < \dim\mathscr{F}(U^0_m). \tag{4.15}$$

Notice that $U^0_m \in \mathscr{C}\mathscr{P}^p$. Let us show that

$$W^0_m \in (\mathscr{F}(U^0_m))^*. \tag{4.16}$$

Recall that this condition is equivalent to the following:

$$D \bullet W^0_m = \sum_{j \in J_m} (\lambda^m(j))^\top D\xi^m(j) \geq 0 \;\forall D \in \mathscr{F}(U^0_m). \tag{4.17}$$

As for $j \in J_m$, by construction, it holds $\xi^m(j) \in T_0(\mathscr{F}(U^0_m))$. We obtain $D\xi^m(j) \geq 0$ for all $D \in \mathscr{F}(U^0_m)$. From the latter inequalities and conditions $\lambda^m(j) \in \mathbb{R}^p_+$ for all $j \in J_m$, we conclude that inequalities (4.17). Consequently, inclusion (4.16) hold true.

Let $V_{m+1} = \{\xi^{m+1}(j), j \in J_{m+1}\}$ be the set of minimal zeros of $\mathscr{F}(U^0_{m+1})$. Go to the next Iteration #$(m+1)$ with the following data:

- the indices and numbers $\tau(i) \in T$, $\gamma_i > 0, i \in I_{m+1} = I_m \cup \Delta I_m$,
- the matrices U^0_{m+1} and W^0_m defined in (4.13) and satisfying equalities (4.14) that can be rewritten in the form (4.12),
- the exposed faces $\mathscr{F}(U^0_m)$ and $\mathscr{F}(U^0_{m+1})$,
- the sets of minimal zeros $V_m = \{\xi^m(j), j \in J_m\}$ and $V_{m+1} = \{\xi^{m+1}(j), j \in J_{m+1}\}$ of the faces $\mathscr{F}(U^0_m)$ and $\mathscr{F}(U^0_{m+1})$ and some vectors $\lambda^m(j) \in \mathbb{R}^p_+$, $j \in J_m$.

It follows from (4.15) that the algorithm performs a finite number m_*, $m_* \leq p^*$, of iterations, after which it proceeds to the *Final step*.

Final step. At this step, we have that, for some m_*, $0 \leq m_* \leq p^*$, problem (**SIP**(m_*)) has a feasible solution $(\bar{\mathbf{x}}, \bar{\mu})$ with $\bar{\mu} > 0$. Moreover, if $m_* > 0$, we have a set of matrices U^0_m, W^0_{m-1}, $m = 1, ..., m_*$; $W^0_0 = \mathbb{O}_p$, satisfying the conditions

$$(U^0_m + W^0_{m-1}) \bullet A_s = 0 \;\forall s = 0, 1, ..., n,\ \forall m = 1, ..., m_*, \tag{4.18}$$

$$U^0_m \in \mathscr{C}\mathscr{P}^p,\ W^0_m \in (\mathscr{F}(U^0_m))^* \;\forall m = 1, ..., m_* - 1,\ U^0_{m_*} \in \mathscr{C}\mathscr{P}^p. \tag{4.19}$$

Note that, just as was done in [26, 27], it can be shown that the set of feasible solutions to the problem (**SIP**(m_*)) with $\mu = 0$ coincides with X. Consequently, $\bar{\mathbf{x}} \in X$. If $m_* = 0$, then, by construction, $\mathbf{t}^\top \mathscr{A}(\bar{\mathbf{x}})\mathbf{t} \geq \bar{\mu} > 0$ for all $\mathbf{t} \in T$. Hence the constraints of the problem (**COP**) satisfy the Slater condition and it follows from Theorem 2.1 that, for the pair of problems (**COP**) and (**DP**) with $m_0 = 0$, the strong duality relations hold true.

Suppose now that $m_* > 0$. By construction, the following inequalities hold true for the previously found vector $\bar{\mathbf{x}} \in X$:

$$\mathscr{A}(\bar{\mathbf{x}})\xi^{m_*}(j) \geq 0 \;\forall j \in J_{m_*};\ \mathbf{t}^\top \mathscr{A}(\bar{\mathbf{x}})\mathbf{t} \geq \bar{\mu} > 0 \;\forall \mathbf{t} \in \Omega(V_{m_*}). \tag{4.20}$$

Consider a SIP problem

$$\mathbf{RP}: \qquad \min_{\mathbf{x}\in\mathbb{R}^n} \mathbf{c}^\top \mathbf{x} \ \text{ s.t. } \mathbf{t}^\top \mathscr{A}(\mathbf{x})\mathbf{t} \geq 0 \ \forall \mathbf{t} \in \Omega(V_{m_*}),\ \mathscr{A}(\mathbf{x})\xi^{m_*}(j) \geq 0 \ \forall j \in J_{m_*},$$

where the sets $V_{m_*} = \{\xi^{m_*}(j), j \in J_{m_*}\}$ and $\Omega(V_{m_*})$ are the same as in the problem (**SIP**(m_*)).

The problem (**RP**) has the following properties.

a) It is not difficult to show (see for example [26]) that $(\xi^{m_*}(j))^\top \mathscr{A}(\mathbf{x})\xi^{m_*}(j) = 0$ for all $j \in J_{m_*}$, $\mathbf{x} \in X$. Then (see Theorem 1 in [27]), the sets of feasible solutions in the problems (**COP**) and (**RP**) coincide, which implies the equivalence of these problems.

b) Relations (4.20) hold true and hence the first group of constraints in (**RP**) satisfies the Slater condition.

c) The inequalities in the second group of constraints in (**RP**) are formulated in terms of *linear* w.r.t. $\mathbf{x}$ functions and the number of these constraints is *finite*.

It follows from property a) that $val(\mathbf{COP}) = val(\mathbf{RP})$. Taking into account properties b) and c) and applying Theorem 1 from [28], we conclude that there exist vectors $\mathbf{t}(i) \in \Omega(V_{m_*})$, $i \in I$, $|I| \leq n$, such that the Linear Programming (LP) problem

$$\mathbf{LP}: \qquad \min_{\mathbf{x}\in\mathbb{R}^n} \mathbf{c}^\top \mathbf{x} \ \text{ s.t. } (\mathbf{t}(i))^\top \mathscr{A}(\mathbf{x})\mathbf{t}(i) \geq 0 \ \forall i \in I,\ \mathscr{A}(\mathbf{x})\xi^{m_*}(j) \geq 0 \ \forall j \in J_{m_*},$$

has the same optimal value as problem (**RP**):

$$val(\mathbf{LP}) = val(\mathbf{RP}) = val(\mathbf{COP}) > -\infty. \tag{4.21}$$

Poblem (**LP**) is consistent since any feasible solution in the problem (**COP**) is feasible for (**LP**) too. Hence problem (**LP**) has an optimal solution $\mathbf{x}^*$ and there exist numbers and vectors $\gamma(i) \geq 0$, $i \in I$, $\lambda(j) \in \mathbb{R}^p_+$, $j \in J_{m_*}$, such that

$$\sum_{i\in I} \gamma(i)(\mathbf{t}(i))^\top A_s \mathbf{t}(i) + \sum_{j\in J_{m_*}} (\lambda(j))^\top A_s \xi^{m_*}(j) = c_s \ \forall s = 1, ..., n, \tag{4.22}$$

$$\gamma(i)(\mathbf{t}(i))^\top \mathscr{A}(\mathbf{x}^*)\mathbf{t}(i) = 0 \ \forall i \in I,\ (\lambda(j))^\top \mathscr{A}(\mathbf{x}^*)\xi^{m_*}(j) = 0 \ \forall j \in J_{m_*}. \tag{4.23}$$

Denote

$$U^0 := \sum_{i\in I} \gamma(i)\mathbf{t}(i)(\mathbf{t}(i))^\top,\ W^0_{m_*} := 0.5 \sum_{j\in J_{m_*}} \Big((\lambda(j))^\top \xi^{m_*}(j) + (\xi^{m_*}(j))^\top \lambda(j)\Big). \tag{4.24}$$

Then relations (4.22) can be rewritten in the form

$$(U^0 + W^0_{m_*}) \bullet A_s = c_s \ \forall s = 1, ..., n, \tag{4.25}$$

and it follows from (4.23) that

$$U^0 \bullet \mathscr{A}(\mathbf{x}^*) = 0,\ \ W^0_{m_*} \bullet \mathscr{A}(\mathbf{x}^*) = 0. \tag{4.26}$$

It is evident that $U^0 \in \mathscr{CP}^p$ and, as before, one can show that

$$W^0_{m_*} \in (\mathscr{F}(U^0_{m_*}))^*. \tag{4.27}$$

It follows from the inclusion $U^0 \in \mathscr{CP}^p$ and relations (4.18), (4.19),(4.25), and (4.27) that the set of matrices

$$(W^0_0, U^0_m, W^0_m, m = 1, ..., m_0,\ U^0), \tag{4.28}$$

constructed in (4.4), (4.13), and (4.24), is a feasible solution of the problem (**DP**) with $m_0 = m_*$. It follows from (4.25) that

$$val(\mathbf{LP}) = \mathbf{c}^\top \mathbf{x}^* = (U^0 + W^0_{m_0}) \bullet \mathscr{A}(\mathbf{x}^*) - (U^0 + W^0_{m_0}) \bullet A_0,$$

wherefrom, taking into account (4.21), (4.26), we have

$$val(\mathbf{COP}) = val(\mathbf{LP}) = -(U^0 + W^0_{m_0}) \bullet A_0.$$

Thus we have proved that the statements of Theorem 3.2 hold true with $m_0 = m_* \le p^*$. □

5. Some Other Dual Formulations for the Linear Copositive Problem

In [15], for problem (**COP**), we considered an (extended) dual problem in the form

$$\begin{aligned} &\max -(U + W_{m_0}) \bullet A_0, \\ \mathbf{EDP:} \quad &\text{s.t.} \quad (3.1),\ (3.2), \text{ and} \\ &\begin{pmatrix} U_m & W_m \\ (W_m)^\top & D_m \end{pmatrix} \in \mathscr{CP}^{2p},\ m = 1, \ldots, m_0. \end{aligned} \tag{5.1}$$

Here matrices W_0, U_m, W_m, D_m, $m = 1, \ldots, m_0$, U are the decision variables.

Note that it would be more correct to denote problem (**EDP**) by (**EDP**(m_0)) since the number of its constraints depends on some integer m_0. For the sake of simplicity, we use a more short notation, but remember that this problem contains the parameter m_0. Based on the results from [15], it is easy to show that the pair of problems (**COP**) and (**EDP**) satisfies the strong duality relations for any $m_0 \ge 2n$. It follows from the proof of Theorem 3.2 that, in general, the set of feasible solutions of the dual problem (**DP**) is bigger then the set of feasible solutions of the problem (**EDP**).

Some other strong dual formulations for conic optimization problems were considered in [19]. The theorem 2 from [19], applied to the problem (**COP**), is as follows.

Theorem 5.1. *For all sufficiently large integer values of the parameter m_0, the following problem:*

$$\begin{aligned} &\max -Y_{m_0+1} \bullet A_0, \\ \mathbf{FDP}: \quad &\text{s.t.} \quad Y_m \bullet A_s = 0,\ s = 0, 1, \ldots, n,\ m = 1, \ldots, m_0; \\ &Y_{m_0+1} \bullet A_s = c_s,\ s = 1, 2, \ldots, n; \\ &(Y_1, Y_2, \ldots, Y_{m_0+1}) \in \mathrm{FR}_{m_0+1}(\mathscr{COP}^p) \end{aligned} \tag{5.2}$$

*is a strong dual for problem (**COP**).*

Here matrices Y_1, $Y_2, \ldots, Y_{m_0+1}$ are the decision variables and for an integer $k \ge 1$, $\mathrm{FR}_k(\mathscr{K})$ denotes the *facial reduction cone of order k* of a cone $\mathscr{K}$:

$$\mathrm{FR}_k(\mathscr{K}) := \{(Y_1, Y_2, \ldots, Y_k) : Y_1 \in \mathscr{K}^*,\ Y_m \in (\mathscr{K} \cap Y_1^\perp \cap \ldots \cap Y_{m-1}^\perp)^*, m = 2, \ldots, k\}.$$

Thus, the variables of problem (**FDP**) (the dual variables) belong to the facial reduction cone of order $m_0 + 1$ of the cone $\mathscr{COP}^p$. Notice here that it was shown in [19] that, for any $k \ge 1$, the cone $\mathrm{FR}_k(\mathscr{COP}^p)$ is convex and for any $k > 1$, it is not closed.

As above (in case of problems (**DP**) and (**EDP**)), we use shorter notation (**FDP**) instead of a more accurate (**FDP**(m_0)). Notice that it follows from the Theorem 1 in [19] that we can here set $m_0 = p_* := p(p+1)/2$.

Lemma 5.2. *Let* $(W_0, U_m, W_m, m = 1, \dots, m_0, U)$ *be a feasible solution to problem* (**DP**)*. Then*

$$(Y_1 = U_1,\, Y_m = U_m + W_{m-1},\, m = 2, \dots, m_0,\ Y_{m_0+1} = U + W_{m_0}) \tag{5.3}$$

is a feasible solution to problem (**FDP**)*.*

Proof. It follows from the formulations of problems (**DP**) and (**FDP**) that, to prove the lemma, we have to show that the set of matrices (5.3) satisfies condition (5.2). Notice that $Y_1 = U_1 \in \mathscr{CP}^p = (\mathscr{COP}^p)^*$. Let us show that

$$D \bullet Y_2 \geq 0\ \forall D \in \{D \in \mathscr{COP}^p : D \bullet Y_1 = 0\} = \mathscr{COP}^p \cap Y_1^{\perp}, \tag{5.4}$$

$$D \bullet U_2 = 0,\ D \bullet W_1 = 0\ \forall D \in \{D \in \mathscr{COP}^p : D \bullet Y_1 = 0, D \bullet Y_2 = 0\} = \mathscr{COP}^p \cap Y_1^{\perp} \cap Y_2^{\perp}. \tag{5.5}$$

By construction, it holds

$$D \bullet Y_2 = D \bullet U_2 + D \bullet W_1,\ U_2 \in \mathscr{CP}^p,\ W_1 \in (\mathscr{F}(U_1))^*. \tag{5.6}$$

Since $U_2 \in \mathscr{CP}^p$, we have

$$D \bullet U_2 \geq 0\ \forall D \in \mathscr{COP}^p. \tag{5.7}$$

Taking into account that $Y_1 = U_1$ and $D \bullet W_1 \geq 0$ for all $D \in \mathscr{F}(U_1)$, we conclude that

$$D \bullet W_1 \geq 0\ \forall D \in \mathscr{COP}^p \cap Y_1^{\perp}. \tag{5.8}$$

It is evident that relations (5.6)-(5.8) imply relations (5.4) and (5.5). Suppose that, for some m, $2 \leq m \leq m_0$,

$$D \bullet Y_m \geq 0\ \forall D \in \mathscr{COP}^p \cap Y_1^{\perp} \cap \dots \cap Y_{m-1}^{\perp}, \tag{5.9}$$

$$D \bullet U_m = 0,\ D \bullet W_{m-1} = 0\ \forall D \in \mathscr{COP}^p \cap Y_1^{\perp} \cap \dots \cap Y_{m-1}^{\perp} \cap Y_m^{\perp}. \tag{5.10}$$

Let us show that the following relations are satisfied:

$$D \bullet Y_{m+1} \geq 0\ \forall D \in \mathscr{COP}^p \cap Y_1^{\perp} \cap \dots \cap Y_{m-1}^{\perp} \cap Y_m^{\perp}, \tag{5.11}$$

$$D \bullet U_{m+1} = 0,\ D \bullet W_m = 0\ \forall D \in \mathscr{COP}^p \cap Y_1^{\perp} \cap \dots \cap Y_m^{\perp} \cap Y_{m+1}^{\perp}. \tag{5.12}$$

By construction, we have

$$D \bullet Y_{m+1} = D \bullet U_{m+1} + D \bullet W_m,\ U_{m+1} \in \mathscr{CP}^p,\ W_m \in (\mathscr{F}(U_m))^*. \tag{5.13}$$

Since $U_{m+1} \in \mathscr{CP}^p$ and $W_m \in (\mathscr{F}(U_m))^*$, then

$$D \bullet U_{m+1} \geq 0\ \forall D \in \mathscr{COP}^p, \tag{5.14}$$

$$D \bullet W_m \geq 0\ \forall D \in \mathscr{F}(U_m). \tag{5.15}$$

Due to conditions (5.10), we have

$$D \bullet U_m = 0\ \forall D \in \mathscr{COP}^p \cap Y_1^{\perp} \cap \dots \cap Y_{m-1}^{\perp} \cap Y_m^{\perp},$$

wherefrom, taking into account (5.15), we conclude that

$$D \bullet W_m \geq 0\ \forall D \in \mathscr{COP}^p \cap Y_1^{\perp} \cap \dots \cap Y_{m-1}^{\perp} \cap Y_m^{\perp}. \tag{5.16}$$

Relations (5.11) follow from (5.13), (5.14), and (5.16). From (5.11), (5.13), (5.14), and (5.16), we obtain (5.12).

Thus, we have proved that relations (5.9) and (5.10) hold true for all $m = 1, 2, \dots, m_0$. It follows from (5.9) that

$$Y_1 \in \mathscr{CP}^p,\ Y_m \in (\mathscr{COP}^p \cap Y_1^{\perp} \cap \dots \cap Y_{m-1}^{\perp})^*\ \forall m = 2, \dots, m_0. \tag{5.17}$$

Now, let us show that

$$Y_{m_0+1} := U + W_{m_0} \in (\mathscr{COP}^p \cap Y_1^\perp \cap \ldots \cap Y_{m_0-1}^\perp \cap Y_{m_0}^\perp)^*. \tag{5.18}$$

In fact, by construction, $U \in \mathscr{CP}^p$ and hence

$$D \bullet U \geq 0 \; \forall D \in \mathscr{COP}^p. \tag{5.19}$$

Moreover, due to (5.16) with $m = m_0$, we have

$$D \bullet W_{m_0} \geq 0 \; \forall D \in \mathscr{COP}^p \cap Y_1^\perp \cap \ldots \cap Y_{m_0-1}^\perp \cap Y_{m_0}^\perp.$$

It follows from these inequalities and (5.19) that $D \bullet (U + W_{m_0}) \geq 0$ for all $D \in \mathscr{COP}^p \cap Y_1^\perp \cap \ldots \cap Y_{m_0-1}^\perp \cap Y_{m_0}^\perp$. Hence, inclusion (5.18) holds true. It follows from (5.17) and (5.18) that the set of matrices (5.3) satisfies condition (5.2) and hence this set is a feasible solution of the problem (**FDP**). The lemma is proved. □

It follows from this lemma that the set of feasible solutions of problem (**FDP**) is wider than the set of feasible solutions of problem (**DP**). The following corollary is a consequence of Lemma 5.2.

Corollary 5.3. *In problem* (**FDP**)*, one can choose the value of the parameter m_0 satisfying the condition $m_0 \leq \min\{2n, p^*\}$.*

Notice that the condition (3.3) in problem (**DP**) can reformulated as

$$(U_m, W_m) \in \mathrm{FR}_2(\mathscr{COP}^p) \; \forall m = 1, \ldots, m_0. \tag{5.20}$$

Comparing dual problems (**EDP**), (**DP**), and (**FDP**), we can state the following.

1) Problems (**EDP**), (**DP**), and (**FDP**) differ from each other in constraints (3.3), (5.1), and (5.2).

2) Problem (**EDP**) can be considered as a completely positive problem since its constraints are formulated in terms of completely positive matrices. Problems (**DP**) and (**FDP**) are conic problems whose variables belong to the cones $\mathrm{FR}_2(\mathscr{COP}^p)$ and $\mathrm{FR}_{m_0+1}(\mathscr{COP}^p)$ respectively.

3) Dual problems (**DP**) and (**EDP**) contain m_0 separate simple conditions (3.3) and (5.1), respectively, for each $m = 1, \ldots, m_0$. In problem (**FDP**), instead of these m_0 constraints, there is a single but more complex constraint (5.2) in a recursive form (this constraint can be considered as a kind of "aggregation" of the mentioned above "simple" constraints in the problem (**EDP**)).

4) The facial reduction cone $\mathrm{FR}_{m_0+1}(\mathscr{COP}^p)$ used in the constraints of problem (**FDP**) (see (5.2)) is not explicitly described. The dimension of this cone is large, which greatly complicates the solution of this problem.

5) Each feasible solution of problem (**EDP**) generates a feasible solution to problem (**DP**), and each feasible solution of the last problem generates a feasible solution to problem (**FDP**).

6. Reformulations of Problems (DP) and (FDP) Using a Polynomial Ring Approach

In [18], the authors used a *polynomial ring approach* developed in [12, 14] to formulate a strong dual for a standard convex optimization program. The aim of this section is to show how the strong dual problems (**DP**) and (**FDP**) considered in this paper for the copositive problem (**COP**) can be reformulated in terms of this approach.

Let us recall some of notations used in [12, 14] to obtain polynomial Lagrange multipliers for convex programs, and adapt them to the case of the finite-dimensional space S^p. Let $\mathscr{P}$ denote the vector space of real polynomials in one indeterminate θ and let $\mathscr{P}_m$ denote the subspace of polynomials of degree not more than m. A polynomial $\pi(\theta) = \sum\limits_{i=0}^{m} a_i\theta^i \in \mathscr{P}_m$ is termed positive if the coefficient of the highest non-vanishing power is positive, which is denoted by $\pi(\theta) > 0$. The inequality $\pi(\theta) \geq 0$ refers to either $\pi(\theta) > 0$ or $\pi(\theta) \equiv 0$. Alternatively,

$$\pi(\theta) \geq 0 \iff \pi(\bar{\theta}) \geq 0 \text{ for all sufficiently large } \bar{\theta}. \tag{6.1}$$

Let $\mathscr{P}_m^p$ denote the set of $p \times p$ symmetric matrix polynomials of degree not more than m:

$$\mathscr{P}_m^p := \{D(\theta) = \sum_{i=0}^{m} Y_i\theta^i,\ Y_i \in S^p,\ i = 0, 1, \ldots, m\}.$$

For the cone $\mathscr{COP}^p$ and any non-negative integer m, denote

$$(\mathscr{COP}^p)_m^* := \{D(\theta) \in \mathscr{P}_m^p : D(\theta) \bullet A \geq 0\ \forall A \in \mathscr{COP}^p\}.$$

Here $\pi(\theta) := D(\theta) \bullet A \in \mathscr{P}_m$ and the inequality $\pi(\theta) \geq 0$ is understood as it was mentioned above. Notice that $(\mathscr{COP}^p)_0^* = (\mathscr{COP}^p)^* = \mathscr{CP}^p$, $\mathscr{CP}^p \subset (\mathscr{COP}^p)_m^*$ for any $m \geq 0$. The set $(\mathscr{COP}^p)_m^*$ can be considered as a generalization of the cone $\mathscr{CP}^p$ that is dual to $\mathscr{COP}^p$. Using the notations introduced above, we can reformulate condition (5.2) in the dual problem (**FDP**) as follows:

$$\sum_{i=1}^{m_0+1} \theta^{m_0+1-i} Y_i \in (\mathscr{COP}^p)_{m_0}^* \tag{6.2}$$

and condition (3.3) in the dual problem (**DP**) as

$$U_m\theta + W_m \in (\mathscr{COP}^p)_1^*\ \ \forall m = 1, \ldots, m_0. \tag{6.3}$$

With the equivalent presentation (6.2) of constraints (5.2), we can reformulate problem (**FDP**) by using the polynomial ring approach:

$$\textbf{RDP}: \quad \max_{X_0, X_1, \ldots, X_{m_0}} \lim_{\theta\to\infty} (-A_0 \bullet X(\theta)),$$

$$\text{s.t.} \ \lim_{\theta\to\infty} A_s \bullet X(\theta) = c_s\ \forall s = 1, \ldots, n;\ \ X(\theta) = \sum_{i=0}^{m_0} X_i\theta^i \in (\mathscr{COP}^p)_{m_0}^*.$$

Here, as above, the notation (**RDP**) stays for (**RDP(m_0)**), where m_0 is some integer parameter. In fact, for any matrix polynomial $X(\theta) = \sum\limits_{i=0}^{m_0} X_i\theta^i$ and any $s = 1, \ldots, n$, we have

$$\lim_{\theta\to\infty} A_s \bullet X(\theta) = \begin{cases} +\infty & \text{if } A_s \bullet X_{i_0} > 0,\ A_s \bullet X_i = 0\ \forall i = i_0+1, \ldots, m_0, \text{ for some } 1 \leq i_0 \leq m_0; \\ -\infty & \text{if } A_s \bullet X_{i_0} < 0;,\ A_s \bullet X_i = 0\ \forall i = i_0+1, \ldots, m_0, \text{ for some } 1 \leq i_0 \leq m_0; \\ A_s \bullet X_0 & \text{if } A_s \bullet X_i = 0\ \forall i = 1, \ldots, m_0. \end{cases}$$

Hence, to satisfy constraints of the problem (**RDP**), the following equalities should hold true:

$$A_s \bullet X_0 = c_s,\ A_s \bullet X_i = 0\ \forall i = 1, \ldots, m_0;\ \forall s = 1, \ldots, n. \tag{6.4}$$

Above, without loss of generality, we have supposed that $A_0 \in \mathscr{COP}^p$. Hence, for any $X(\theta) \in (\mathscr{COP}^p)^*_{m_0}$, it holds $-A_0 \bullet X(\theta) \leq 0$. Consequently,

$$\lim_{\theta \to \infty} -A_0 \bullet X(\theta) = \begin{cases} -A_0 \bullet X_0 & \text{if } A_0 \bullet X_i = 0 \ \forall i = 1, \dots, m_0, \\ -\infty & \text{otherwise.} \end{cases}$$

Because our goal is to maximize the cost function of the problem (**RDP**), $\lim\limits_{\theta \to \infty} -A_0 \bullet X(\theta)$, we should consider such sets of matrices $X_i, i = 0, 1, \dots, m_0$, which satisfy the equalities

$$A_0 \bullet X_i = 0 \ \forall i = 1, \dots, m_0. \tag{6.5}$$

Moreover, the condition $X(\theta) \in (\mathscr{COP}^p)^*_{m_0}$ implies that the following inclusions should be satisfied:

$$X_{m_0} \in \mathscr{COP}^p, \ X_i \in \mathscr{COP}^p \cap X^{\perp}_{i+1} \cap X^{\perp}_{i+2} \dots \cap X^{\perp}_{m_0} \ \ \forall i = m_0 - 1, \dots, 0. \tag{6.6}$$

From relations (6.4)-(6.6), one can conclude that problems (**FDP**) and (**RDP**) are equivalent. It is evident that matrices $X_i, i = 0, 1, \dots, m_0$, satisfying conditions (6.4)-(6.6) and matrices $Y_i, i = 1, \dots m_0 + 1$, forming a feasible solution of problem (**FDP**) are related as follows: $Y_{m_0+1} = X_0$, $Y_{m_0} = X_1, \ \dots, \ X_{m_0} = Y_1$. Notice that the dual formulation (**RDP**) is closely related to the dual problem proposed for the standard convex program by Kortanek et al. [18], but it is not a direct consequence of this result.

Problems (**FDP**) and (**RDP**) are equivalent but written in different forms, with the problem (**RDP**) having a compact form that resembles the form of the Lagrange dual problem (2.4).

7. Conclusions

The main contribution of the paper is to deduce new dual problems for the copositive problem and to study their properties. All of these problems satisfy the strong duality relations. The results of the paper provide templates for creating other strong dual formulations for linear/convex copositive problems. These formulations can be used for various purposes, both theoretical and practical. They can be used to obtain new optimality conditions and to analyze some numerical methods for solving Convex Optimization problems that do not satisfy the regularity conditions.

Acknowledgements
This work was partially supported state research program ”Convergence”(Republic of Belarus), Task 1.3.01, by Portuguese funds through CIDMA - Center for Research and Development in Mathematics and Applications, and FCT - Portuguese Foundation for Science and Technology, within the project UIDB/04106/2020. The authors thank the anonymous reviewer for the valuable comments and suggestions which helped to improve the presentation of this article.

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