



Generalized Taylor's formula for power fractional derivatives

Hanaa Zitane¹  · Delfim F. M. Torres^{1,2} 

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Abstract

We establish a new generalized Taylor's formula for power fractional derivatives with nonsingular and nonlocal kernels, which includes many known Taylor's formulas in the literature. Moreover, as a consequence, we obtain a general version of the classical mean value theorem. We apply our main result to approximate functions in Taylor's expansions at a given point. The explicit interpolation error is also obtained. The new results are illustrated through examples and numerical simulations.

Keywords Approximation of functions · Mean value theorem · Power fractional operators · Taylor's formula

Mathematics Subject Classification 26A24 · 26A33 · 41A58

1 Introduction

Taylor's theorem is one of the central elementary tools in mathematical analysis, e.g., in numerical methods, topology optimization and optimal control [5, 11, 13]. It provides simple arithmetic formulas, in polynomial terms, to accurately compute values of various transcendental functions, such as trigonometric and exponential ones. This fundamental theorem has various significant applications in Mathematics

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✉ Delfim F. M. Torres
delfim@ua.pt, delfim@unicv.cv

Hanaa Zitane
h.zitane@ua.pt

¹ Center for Research and Development in Mathematics and Applications (CIDMA), Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal

² Research Center in Exact Sciences (CICE), Faculty of Sciences and Technology (FCT), University of Cape Verde (Uni-CV), 7943-010 Praia, Cape Verde

[19], engineering [10], and other fields of applied sciences [16, 18]. In consequence, several versions of Taylor's theorem, involving different types of fractional operators, have been established. For instance, generalizations of this formula, using fractional operators with singular kernels, such as Riemann–Liouville and Caputo derivatives, are provided in [4, 7, 15, 20]. In [8], a Taylor's theorem is proved for Atangana–Baleanu fractional derivatives in Caputo sense while in [21] it is derived for generalized weighted fractional operators.

Recently, a new generalized fractional derivative with nonsingular and nonlocal kernel was introduced [14]. The power fractional derivative (PFD) is essentially characterized by the presence of a key power parameter p , which allows one to choose the appropriate fractional operator that effectively describes the phenomena under study in a natural way, and then creating good mathematical models to represent systems and predict their future dynamical behaviors. Furthermore, this fractional operator generalizes and unifies most of fractional derivatives with nonsingular kernels, such as the Caputo–Fabrizio [6], Atangana–Baleanu [2], weighted Atangana–Baleanu [1], and weighed generalized fractional derivatives [9].

Motivated by available results, in the present paper we propose to investigate a more general and rich version of Taylor's formula involving the recently introduced power fractional derivative.

The outline of the paper is as follows. In Sect. 2, we review the necessary notions on power fractional calculus. Our main results are given in Sect. 3, where we begin by proving important lemmas and tools about power fractional operators, and their n th-order operators, that are necessary in the sequel. Furthermore, we establish a new generalized Taylor's theorem and a general version of the mean value theorem via power fractional differentiation. Then, in Sect. 4, we apply our main result to approximate functions in Taylor's series at a given point, where the explicit interpolation error of the approximation of the function by its Taylor polynomial is also characterized. We end up with Sect. 5 of conclusion and future work.

2 Preliminary definitions

Let $C([a, b])$ be the Banach space of all continuous real functions defined on $[a, b]$, where $a, b \in \mathbb{R}$, and $H^1(a, b)$ be the Sobolev space of order one defined by

$$H^1(a, b) = \{f \in L^2(a, b) : f' \in L^2(a, b)\}.$$

In what follows, we review some basic concepts and tools about power fractional calculus that are used along the text.

Definition 1 (See [14]) The power Mittag–Leffler function is given by

$${}^p E_{k,l}(\tau) = \sum_{n=0}^{+\infty} \frac{(\tau \ln p)^n}{\Gamma(kn + l)}, \quad \tau \in \mathbb{C}, \quad (1)$$

where $\min(k, l) > 0$, $p > 0$, and $\Gamma(\cdot)$ is the Gamma function [12].

Remark 1 Note that the Mittag–Leffler function of two parameters k and l is recovered when $p = e$, while the Mittag–Leffler function of one parameter k is obtained when $p = e$ and $l = 1$ [12].

Throughout the paper, we adopt the notations

$$\chi(\alpha) := \frac{1 - \alpha}{N(\alpha)}, \quad \varphi(\alpha) := \frac{\alpha}{N(\alpha)} \quad \text{and} \quad \mu_\alpha := \frac{\alpha}{1 - \alpha},$$

where $\alpha \in [0, 1)$ and $N(\alpha)$ is a normalization function such as $N(0) = N(1^-) = 1$ with $N(1^-) = \lim_{\alpha \rightarrow 1^-} N(\alpha)$.

Definition 2 (See [14]) Let $\alpha \in [0, 1)$, $\min(\beta, p) > 0$, and $f \in H^1(a, b)$. The power fractional derivative (PFD) of order α in the Caputo sense, of a function f with respect to the weight function ω , is defined by

$${}^p C D_{a,t,\omega}^{\alpha,\beta,p} f(t) = \frac{1}{\chi(\alpha)} \frac{1}{\omega(t)} \int_a^t {}^p E_{\beta,1}(-\mu_\alpha(t - \tau)^\beta) (\omega f)'(\tau) \, d\tau, \quad (2)$$

where $\omega \in C^1([a, b])$ with $\omega > 0$ on $[a, b]$.

Remark 2 The PFD (2) generalizes and includes various cases of fractional derivative operators available in the literature, such as:

- when $p = e$, $\beta = 1$, and $\omega(t) \equiv 1$, we obtain the Caputo-Fabrizio fractional derivative [6] defined by

$${}^p C D_{a,t,1}^{\alpha,1,e} f(t) = \frac{1}{\chi(\alpha)} \int_a^t \exp(-\mu_\alpha(t - \tau)) f'(\tau) \, d\tau;$$

- when $p = e$, $\beta = \alpha$, and $\omega(t) \equiv 1$, we retrieve the Atangana–Baleanu fractional derivative [2] given by

$${}^p C D_{a,t,1}^{\alpha,\alpha,e} f(t) = \frac{1}{\chi(\alpha)} \int_a^t E_\alpha(-\mu_\alpha(t - \tau)) f'(\tau) \, d\tau;$$

- when $p = e$ and $\beta = \alpha$, we obtain the weighted Atangana–Baleanu fractional derivative [1] given by

$${}^p C D_{a,t,\omega}^{\alpha,\alpha,e} f(t) = \frac{1}{\chi(\alpha)} \frac{1}{\omega(t)} \int_a^t E_\alpha(-\mu_\alpha(t - \tau)^\alpha) (\omega f)'(\tau) \, d\tau;$$

- when $p = e$, we get the weighted generalized fractional derivative [9] defined as follows:

$${}^p C D_{a,t,\omega}^{\alpha,\beta,e} f(t) = \frac{1}{\chi(\alpha)} \frac{1}{\omega(t)} \int_a^t E_\beta(-\mu_\alpha(t - \tau)^\beta) (\omega f)'(\tau) \, d\tau.$$

Now we provide the power fractional integral (PFI) associated with the power fractional derivative (2).

Definition 3 (See [14]) The power fractional integral of order α , of a function f with respect to the weight function ω , is given by

$${}^p I_{a,t}^{\alpha,\beta,p} f(t) = \chi(\alpha) f(t) + \ln p \cdot \varphi(\alpha) {}^{RL} I_{a,\omega}^{\beta} f(t), \quad (3)$$

where ${}^{RL} I_{a,\omega}^{\beta}$ is the standard weighted Riemann–Liouville fractional integral of order β defined by

$${}^{RL} I_{a,\omega}^{\beta} f(t) = \frac{1}{\Gamma(\beta)} \frac{1}{\omega(t)} \int_a^t (t - \tau)^{\beta-1} (\omega f)(\tau) d\tau.$$

Remark 3 If we let $p = e$ in (3), then we retrieve the generalized fractional integral operator given in [9]. Moreover, if we let $p = e$, $\beta = \alpha$, and $\omega(t) \equiv 1$ in (3), then we obtain the Atangana–Baleanu fractional integral operator introduced in [2].

For the sake of simplicity, we shall denote ${}^p D_{a,t,\omega}^{\alpha,\beta,p}$ and ${}^p I_{a,t,\omega}^{\alpha,\beta}$ by ${}^p D_{a,\omega}^{\alpha,\beta}$ and ${}^p I_{a,\omega}^{\alpha,\beta}$, respectively.

3 Taylor's formula via power fractional derivatives

In this section, we establish a new generalized Taylor's formula in the framework of the power fractional derivative. We first prove some fundamental lemmas and results about power fractional operators and their n th-order that are needed in the proof of the main theorem.

Lemma 1 The power fractional derivative ${}^p D_{a,\omega}^{\alpha,\beta}$ can be expressed as follows:

$${}^p D_{a,\omega}^{\alpha,\beta} f(t) = \frac{1}{\chi(\alpha)} \sum_{n=0}^{+\infty} (-\mu_{\alpha} \ln p)^n {}^{RL} I_{a,\omega}^{\beta n+1} \left(\frac{(\omega f)'}{\omega} \right) (t). \quad (4)$$

This series converges locally and uniformly in t for any a , α , β , p , ω and f , verifying the conditions laid out in Definition 2.

Proof The power Mittag–Leffler function ${}^p E_{k,l}(s)$ is an entire function of s . Since it is locally uniformly convergent in the whole complex plane, it implies that the PFD

may be rewritten as follows:

$$\begin{aligned}
 {}^pD_{a,\omega}^{\alpha,\beta} f(t) &= \frac{1}{\chi(\alpha)} \frac{1}{\omega(t)} \sum_{n=0}^{+\infty} \frac{(-\mu_\alpha \ln p)^n}{\Gamma(\beta n + 1)} \int_a^t (t - \tau)^{\beta n} (\omega f)'(\tau) \, d\tau \\
 &= \frac{1}{\chi(\alpha)} \sum_{n=0}^{+\infty} (-\mu_\alpha \ln p)^n \frac{1}{\Gamma(\beta n + 1)} \frac{1}{\omega(t)} \int_a^t (t - \tau)^{\beta n} (\omega f)'(\tau) \, d\tau \\
 &= \frac{1}{\chi(\alpha)} \sum_{n=0}^{+\infty} (-\mu_\alpha \ln p)^n {}^{RL}I_{a,\omega}^{\beta n + 1} \left(\frac{(\omega f)'}{\omega} \right) (t),
 \end{aligned}$$

as required. □

Remark 4 The new formula (4) of the power fractional derivative (2) is easier to handle for certain computational purposes.

The power fractional integral and derivative satisfy the following composition property.

Proposition 1 Let $\alpha \in [0, 1)$, $p, \beta > 0$ and $f \in H^1(a, b)$. Then, it holds that

$${}^pI_{a,\omega}^{\alpha,\beta} ({}^pD_{a,\omega}^{\alpha,\beta} f) (t) = f(t) - \frac{(\omega f)(a)}{\omega(t)}. \tag{5}$$

Proof According with Definition 3, we have

$${}^pI_{a,\omega}^{\alpha,\beta} ({}^pD_{a,\omega}^{\alpha,\beta} f) (t) = \chi(\alpha) {}^pD_{a,\omega}^{\alpha,\beta} f(t) + \ln p \cdot \varphi(\alpha) {}^{RL}I_{a,\omega}^\beta ({}^pD_{a,\omega}^{\alpha,\beta} f) (t).$$

Moreover, by virtue of Lemma 1, one obtains that

$$\begin{aligned}
 {}^pI_{a,\omega}^{\alpha,\beta} ({}^pD_{a,\omega}^{\alpha,\beta} f) (t) &= \sum_{n=0}^{+\infty} (-\mu_\alpha \ln p)^n {}^{RL}I_{a,\omega}^{\beta n + 1} \left(\frac{(\omega f)'}{\omega} \right) (t) \\
 &\quad + \mu_\alpha \ln p {}^{RL}I_{a,\omega}^\beta \left[\sum_{n=0}^{+\infty} (-\mu_\alpha \ln p)^n {}^{RL}I_{a,\omega}^{\beta n + 1} \left(\frac{(\omega f)'}{\omega} \right) (t) \right] \\
 &= \sum_{n=0}^{+\infty} (-\mu_\alpha \ln p)^n {}^{RL}I_{a,\omega}^{\beta n + 1} \left(\frac{(\omega f)'}{\omega} \right) (t) - \sum_{n=0}^{+\infty} (-\mu_\alpha \ln p)^{n+1} {}^{RL}I_{a,\omega}^{\beta(n+1)+1} \left(\frac{(\omega f)'}{\omega} \right) (t) \\
 &= \sum_{n=0}^{+\infty} (-\mu_\alpha \ln p)^n {}^{RL}I_{a,\omega}^{\beta n + 1} \left(\frac{(\omega f)'}{\omega} \right) (t) - \sum_{n=1}^{+\infty} (-\mu_\alpha \ln p)^n {}^{RL}I_{a,\omega}^{\beta n + 1} \left(\frac{(\omega f)'}{\omega} \right) (t) \\
 &= {}^{RL}I_{a,\omega}^1 \left(\frac{(\omega f)'}{\omega} \right) (t) \\
 &= \frac{1}{\omega(t)} \int_a^t (\omega f)'(\tau) \, d\tau \\
 &= f(t) - \frac{(\omega f)(a)}{\omega(t)},
 \end{aligned}$$

which completes the proof. □

Next result provides the n th-order power fractional integral formula.

Lemma 2 Let $n \in \mathbb{N}$ and $f \in C([a, b])$. Then,

$${}^p I_{a,\omega}^{n[\alpha,\beta]} f(t) = \sum_{m=0}^n C_n^m \chi(\alpha)^{n-m} (\ln p \cdot \varphi(\alpha))^m \left({}^{RL} I_{a,\omega}^{m\beta} f(t) \right), \quad (6)$$

where $\alpha \in [0, 1)$, $p, \beta > 0$, $t \in [a, b]$ and ${}^p I_{a,\omega}^{n[\alpha,\beta]} = {}^p I_{a,\omega}^{[\alpha,\beta]} \dots {}^p I_{a,\omega}^{[\alpha,\beta]}$, n -times.

Proof For $n = 0$, formula (6) holds. Indeed, one has ${}^p I_{a,\omega}^{0 \times [\alpha,\beta]} f(t) = f(t)$ and

$$\sum_{m=0}^0 C_0^m \chi(\alpha)^{0-m} (\ln p \cdot \varphi(\alpha))^m \left({}^{RL} I_{a,\omega}^{m\beta} f(t) \right) = {}^{RL} I_{a,\omega}^{0 \times \beta} f(t) = f(t).$$

Now, one assumes that formula (6) is satisfied and we prove that

$${}^p I_{a,\omega}^{(n+1)[\alpha,\beta]} f(t) = \sum_{m=0}^{n+1} C_{n+1}^m \chi(\alpha)^{n+1-m} (\ln p \cdot \varphi(\alpha))^m \left({}^{RL} I_{a,\omega}^{m\beta} f(t) \right)$$

holds true. Indeed, from Definition 3, we have

$$\begin{aligned} {}^p I_{a,\omega}^{(n+1)[\alpha,\beta]} f(t) &= \chi(\alpha) \left({}^p I_{a,\omega}^{n[\alpha,\beta]} f(t) \right) + \ln p \cdot \varphi(\alpha) {}^{RL} I_{a,\omega}^{\beta} \left({}^p I_{a,\omega}^{n[\alpha,\beta]} f(t) \right) \\ &= \chi(\alpha) \left(\sum_{m=0}^n C_n^m \chi(\alpha)^{n-m} (\ln p \cdot \varphi(\alpha))^m \left({}^{RL} I_{a,\omega}^{m\beta} f(t) \right) \right) \\ &\quad + \ln p \cdot \varphi(\alpha) {}^{RL} I_{a,\omega}^{\beta} \left(\sum_{m=0}^n C_n^m \chi(\alpha)^{n-m} (\ln p \cdot \varphi(\alpha))^m \left({}^{RL} I_{a,\omega}^{m\beta} f(t) \right) \right) \\ &= \sum_{m=0}^n C_n^m \chi(\alpha)^{n+1-m} (\ln p \cdot \varphi(\alpha))^m \left({}^{RL} I_{a,\omega}^{m\beta} f(t) \right) \\ &\quad + \sum_{m=0}^n C_n^m \chi(\alpha)^{n-m} (\ln p \cdot \varphi(\alpha))^{m+1} \left({}^{RL} I_{a,\omega}^{m\beta} f(t) \right) \\ &= \chi(\alpha)^{n+1} f(t) + \sum_{m=1}^n C_n^m \chi(\alpha)^{n+1-m} (\ln p \cdot \varphi(\alpha))^m \left({}^{RL} I_{a,\omega}^{m\beta} f(t) \right) \\ &\quad + \sum_{m=1}^n C_n^{m-1} \chi(\alpha)^{n+1-m} (\ln p \cdot \varphi(\alpha))^m \left({}^{RL} I_{a,\omega}^{m\beta} f(t) \right) \\ &\quad + (\ln p \cdot \varphi(\alpha))^{n+1} \left({}^{RL} I_{a,\omega}^{(n+1)\beta} f(t) \right). \end{aligned}$$

Using the fact that $C_{n+1}^m = C_n^m + C_n^{m-1}$, it follows that

$${}_p I_{a,\omega}^{(n+1)[\alpha,\beta]} f(t) = \sum_{m=0}^{n+1} C_{n+1}^m \chi(\alpha)^{n+1-m} (\ln p \cdot \varphi(\alpha))^m \left({}^{RL} I_{a,\omega}^{m\beta} f(t) \right),$$

which completes the proof. □

The following result allows us to easily construct our generalized Taylor’s formula via power fractional derivatives.

Theorem 1 *Assume that ${}_p D_{a,\omega}^{n[\alpha,\beta]} f \in C([a, b])$ and ${}_p D_{a,\omega}^{(n+1)[\alpha,\beta]} \in C([a, b])$ for $\alpha \in [0, 1)$, and $p, \beta > 0$. Then it holds that*

$$\begin{aligned} & {}_p I_{a,\omega}^{n[\alpha,\beta]} {}_p D_{a,\omega}^{n[\alpha,\beta]} f(t) - I_{a,\omega}^{(n+1)[\alpha,\beta]} {}_p D_{a,\omega}^{(n+1)[\alpha,\beta]} f(t) \\ &= \frac{\omega(a)}{\omega(t)} \left({}_p D_{a,\omega}^{n[\alpha,\beta]} f(a) \right) \sum_{m=0}^n C_n^m \chi(\alpha)^{n-m} (\ln p \cdot \varphi(\alpha))^m \left(\frac{(t-a)^{m\beta}}{\Gamma(m\beta+1)} \right), \end{aligned} \tag{7}$$

where $t \in [a, b]$ and ${}_p D_{a,\omega}^{n[\alpha,\beta]} = {}_p D_{a,\omega}^{[\alpha,\beta]} \dots {}_p D_{a,\omega}^{[\alpha,\beta]}$, n -times.

Proof We have

$$\begin{aligned} {}_p I_{a,\omega}^{n[\alpha,\beta]} {}_p D_{a,\omega}^{n[\alpha,\beta]} f(t) - {}_p I_{a,\omega}^{(n+1)[\alpha,\beta]} {}_p D_{a,\omega}^{(n+1)[\alpha,\beta]} f(t) &= {}_p I_{a,\omega}^{n[\alpha,\beta]} \left({}_p D_{a,\omega}^{n[\alpha,\beta]} f(t) - {}_p I_{a,\omega}^{[\alpha,\beta]} {}_p D_{a,\omega}^{(n+1)[\alpha,\beta]} f(t) \right) \\ &= {}_p I_{a,\omega}^{n[\alpha,\beta]} \left({}_p D_{a,\omega}^{n[\alpha,\beta]} f(t) - {}_p I_{a,\omega}^{[\alpha,\beta]} {}_p D_{a,\omega}^{[\alpha,\beta]} \left({}_p D_{a,\omega}^{n[\alpha,\beta]} f(t) \right) \right). \end{aligned}$$

Using Proposition 1, one obtains that

$$\begin{aligned} {}_p I_{a,\omega}^{n[\alpha,\beta]} {}_p D_{a,\omega}^{n[\alpha,\beta]} f(t) - {}_p I_{a,\omega}^{(n+1)[\alpha,\beta]} {}_p D_{a,\omega}^{(n+1)[\alpha,\beta]} f(t) &= {}_p I_{a,\omega}^{n[\alpha,\beta]} \left(\frac{\omega(a) {}_p D_{a,\omega}^{n[\alpha,\beta]} f(a)}{\omega(t)} \right) \\ &= \omega(a) \left({}_p D_{a,\omega}^{n[\alpha,\beta]} f(a) \right) {}_p I_{a,\omega}^{n[\alpha,\beta]} \left(\frac{1}{\omega(t)} \right). \end{aligned}$$

Then, by virtue of Lemma 2, it follows

$$\begin{aligned} & {}_p I_{a,\omega}^{n[\alpha,\beta]} {}_p D_{a,\omega}^{n[\alpha,\beta]} f(t) - {}_p I_{a,\omega}^{(n+1)[\alpha,\beta]} {}_p D_{a,\omega}^{(n+1)[\alpha,\beta]} f(t) \\ &= \omega(a) \left({}_p D_{a,\omega}^{n[\alpha,\beta]} f(a) \right) \sum_{m=0}^n C_n^m \chi(\alpha)^{n-m} (\ln p \cdot \varphi(\alpha))^m \left({}^{RL} I_{a,\omega}^{m\beta} \left(\frac{1}{\omega(t)} \right) \right) \\ &= \frac{\omega(a)}{\omega(t)} \left({}_p D_{a,\omega}^{n[\alpha,\beta]} f(a) \right) \sum_{m=0}^n C_n^m \chi(\alpha)^{n-m} (\ln p \cdot \varphi(\alpha))^m \left(\frac{(t-a)^{m\beta}}{\Gamma(m\beta+1)} \right). \end{aligned}$$

The proof is complete. □

Now, we are able to provide our main result.

Theorem 2 (Taylor’s formula for power fractional derivatives) *Assume that ${}^pD_{a,\omega}^{m[\alpha,\beta]} \in C([a, b])$ for $\alpha \in [0, 1)$, $p, \beta > 0$ and $m = 0, 1, \dots, n + 1$. Then,*

$$f(t) = \frac{1}{\omega(t)} \left[\omega(a) \sum_{l=0}^n {}^pD_{a,\omega}^{l[\alpha,\beta]} f(a) \sum_{m=0}^l C_l^m \chi(\alpha)^{l-m} (\ln p \cdot \varphi(\alpha))^m \frac{(t-a)^{m\beta}}{\Gamma(m\beta+1)} \right. \\ \left. + \omega(\lambda) {}^pD_{a,\omega}^{(n+1)[\alpha,\beta]} f(\lambda) \sum_{m=0}^{n+1} C_{n+1}^m \chi(\alpha)^{n+1-m} (\ln p \cdot \varphi(\alpha))^m \frac{(t-a)^{m\beta}}{\Gamma(m\beta+1)} \right], \tag{8}$$

where $t \in [a, b]$, $\lambda \in [a, t]$, and ${}^pD_{a,\omega}^{l[\alpha,\beta]} = {}^pD_{a,\omega}^{[\alpha,\beta]} \dots {}^pD_{a,\omega}^{[\alpha,\beta]}$, l -times.

Proof By virtue of formula (7), one has

$$\sum_{l=0}^n \left({}^pI_{a,\omega}^{l[\alpha,\beta]} {}^pD_{a,\omega}^{l[\alpha,\beta]} f(t) - {}^pI_{a,\omega}^{(l+1)[\alpha,\beta]} {}^pD_{a,\omega}^{(l+1)[\alpha,\beta]} f(t) \right) \\ = \frac{\omega(a)}{\omega(t)} \sum_{l=0}^n \left({}^pD_{a,\omega}^{l[\alpha,\beta]} f(a) \right) \sum_{m=0}^l C_l^m \chi(\alpha)^{l-m} (\ln p \cdot \varphi(\alpha))^m \left(\frac{(t-a)^{m\beta}}{\Gamma(m\beta+1)} \right),$$

which implies

$$f(t) - {}^pI_{a,\omega}^{(n+1)[\alpha,\beta]} {}^pD_{a,\omega}^{(n+1)[\alpha,\beta]} f(t) = \frac{\omega(a)}{\omega(t)} \sum_{l=0}^n \left({}^pD_{a,\omega}^{n[\alpha,\beta]} f(a) \right) \\ \times \sum_{m=0}^l C_l^m \chi(\alpha)^{l-m} (\ln p \cdot \varphi(\alpha))^m \left(\frac{(t-a)^{m\beta}}{\Gamma(m\beta+1)} \right).$$

Moreover, from Lemma 2, one obtains

$$f(t) = \frac{\omega(a)}{\omega(t)} \sum_{l=0}^n \left({}^pD_{a,\omega}^{l[\alpha,\beta]} f(a) \right) \sum_{m=0}^l C_l^m \chi(\alpha)^{l-m} (\ln p \cdot \varphi(\alpha))^m \left(\frac{(t-a)^{m\beta}}{\Gamma(m\beta+1)} \right) \\ + \sum_{m=0}^{n+1} C_{n+1}^m \chi(\alpha)^{n+1-m} (\ln p \cdot \varphi(\alpha))^m \left({}^{RL}I_{a,\omega}^{m\beta} {}^pD_{a,\omega}^{(n+1)[\alpha,\beta]} f(t) \right).$$

Then, by applying the integral mean value theorem, we deduce that

$$f(t) = \frac{1}{\omega(t)} \left[\omega(a) \sum_{l=0}^n {}^pD_{a,\omega}^{l[\alpha,\beta]} f(a) \sum_{m=0}^l C_l^m \chi(\alpha)^{l-m} (\ln p \cdot \varphi(\alpha))^m \frac{(t-a)^{m\beta}}{\Gamma(m\beta+1)} \right. \\ \left. + \omega(\lambda) {}^pD_{a,\omega}^{(n+1)[\alpha,\beta]} f(\lambda) \sum_{m=0}^{n+1} C_{n+1}^m \chi(\alpha)^{n+1-m} (\ln p \cdot \varphi(\alpha))^m \frac{(t-a)^{m\beta}}{\Gamma(m\beta+1)} \right],$$

which completes the proof. □

Remark 5 Our Taylor’s formula for the power fractional derivative, as stated by Theorem 2, includes most of Taylor’s formulas without singular kernels that exist in the literature, such as

- Taylor’s formula involving the generalized weighted fractional derivative [21], obtained when $p = e$;
- Taylor’s formula involving the weighted Atangana–Baleanu derivative in Caputo sense [1, 21], obtained when $p = e$ and $\beta = \alpha$;
- Taylor’s formula involving the Atangana–Baleanu derivative in Caputo sense [8], obtained when $p = e$, $\beta = \alpha$, and $\omega(t) \equiv 1$.

As a consequence of Theorem 2, we obtain a mean value theorem in the framework of power fractional derivative operators.

Corollary 1 (Mean value theorem for power fractional derivatives) *Suppose that $f \in C([a, b])$ and ${}^pD_{a,\omega}^{[\alpha,\beta]} f \in C([a, b])$ for $\alpha \in [0, 1)$ and $p, \beta > 0$. Then,*

$$f(t) = \frac{1}{\omega(t)} \left[\omega(a)f(a) + \omega(\lambda) {}^pD_{a,\omega}^{[\alpha,\beta]} f(\lambda) \left(\chi(\alpha) + \ln p \cdot \varphi(\alpha) \frac{(t-a)^\beta}{\Gamma(\beta+1)} \right) \right], \tag{9}$$

where $t \in [a, b]$ and $\lambda \in [a, t]$.

Proof The proof follows directly from Theorem 2 by taking $n = 0$. □

Remark 6 If we let $p = e$, $\omega(t) \equiv 1$ and $\alpha = \beta = 1$ in Corollary 1, we obtain the classical mean value theorem.

4 Application: Approximation of functions

In this section, we apply the developed Taylor’s formula for power fractional derivatives (8) to approximate functions at a given point. The approximation method is described in the following result.

Theorem 3 *Suppose that ${}^pD_{a,\omega}^{m[\alpha,\beta]} \in C([a, b])$ for $\alpha \in [0, 1)$, $p, \beta > 0$ and $m = 0, 1, \dots, n + 1$. If $t \in [a, b]$, then*

$$f(t) \simeq {}^pA_n^{\alpha,\beta}(t) = \frac{\omega(a)}{\omega(t)} \sum_{l=0}^n {}^pD_{a,\omega}^{l[\alpha,\beta]} f(a) \sum_{m=0}^l C_l^m \chi(\alpha)^{l-m} (\ln p \cdot \varphi(\alpha))^m \frac{(t-a)^{m\beta}}{\Gamma(m\beta+1)}. \tag{10}$$

In addition, the interpolation error ${}^pR_N^{\alpha,\beta}$ can be expressed as

$$\begin{aligned} {}^pR_N^{\alpha,\beta}(t) &:= f(t) - {}^pA_N^{\alpha,\beta}(t) \\ &= \omega(\lambda) {}^pD_{a,\omega}^{(N+1)[\alpha,\beta]} f(\lambda) \sum_{m=0}^{N+1} C_{N+1}^m \chi(\alpha)^{N+1-m} (\ln p \cdot \varphi(\alpha))^m \frac{(t-a)^{m\beta}}{\Gamma(m\beta+1)}, \end{aligned} \tag{11}$$

where $\lambda \in [a, t]$.

Proof The proof follows directly from Theorem 2. \square

In order to handle suitably our examples, we first establish the following technical lemma.

Lemma 3 *The l -th-order of the power fractional derivative ${}^p C D_{a,t,1}^{l[\alpha,\beta]}$ can be expressed as*

$${}^p D_{a,1}^{l[\alpha,\beta]} f(t) = \frac{1}{\chi(\alpha)^l} \sum_{q=0}^{+\infty} C_{q+l-1}^{l-1} (-\mu_\alpha \ln p)^q \left({}^{RL} I_{a,1}^{\beta q+1} f'(t) \right),$$

where ${}^p D_{a,1}^{l[\alpha,\beta]} = {}^p D_{a,1}^{[\alpha,\beta]} \dots {}^p D_{a,1}^{[\alpha,\beta]}$, l -times.

Proof From Lemma 1, one has

$$\begin{aligned} {}^p D_{a,1}^{l[\alpha,\beta]} f(t) &= \left[\frac{1}{\chi(\alpha)} \sum_{n=0}^{+\infty} (-\mu_\alpha \ln p)^n {}^{RL} I_{a,\omega}^{\beta n+1} \frac{d}{dt} \right]^l f(t) \\ &= \frac{1}{\chi(\alpha)^l} \sum_{n_1, \dots, n_l} (-\mu_\alpha \ln p)^{\sum n_k} {}^{RL} I_{a,\omega}^{\beta \sum n_k+1} \frac{d}{dt} f(t) \\ &= \frac{1}{\chi(\alpha)^l} \sum_{q=0}^{+\infty} C_{q+l-1}^{l-1} (-\mu_\alpha \ln p)^q \left({}^{RL} I_{a,1}^{\beta q+1} f'(t) \right), \end{aligned}$$

which completes the proof. \square

Now, we approximate some basic functions about the point $a = 0$ using the generalized Taylor series given by Theorem 3.

Example 1 Consider the exponential function

$$f(t) = \exp(\delta t), \quad \delta > 0.$$

For $\omega(t) \equiv 1$, the weighted fractional integral ${}^{RL} I_{a,1}^\beta$ coincides with the Riemann–Liouville one ${}^{RL} I_a^\beta$ [17] with $\beta > 0$. Therefore, by virtue of Lemma 3, one has

$${}^p D_{a,1}^{l[\alpha,\beta]} f(t) = \frac{1}{\chi(\alpha)^l} \sum_{q=0}^{+\infty} C_{q+l-1}^{l-1} (-\mu_\alpha \ln p)^q (\delta^{-\beta q} \exp(\delta t)), \quad (12)$$

for $\alpha \in [0, 1)$ and $\beta, p > 0$. Then, by applying Theorem 3, and taking into account (12), it follows that

$$\exp(\delta t) \simeq \sum_{l=0}^n \frac{1}{\chi(\alpha)^l} \sum_{q=0}^{+\infty} C_{q+l-1}^{l-1} (-\mu_\alpha \ln p)^q (\delta^{-\beta q})$$

$$\times \sum_{m=0}^l C_l^m \chi(\alpha)^{l-m} (\ln p \cdot \varphi(\alpha))^m \frac{t^{m\beta}}{\Gamma(m\beta + 1)},$$

where the remainder has the form

$$\begin{aligned} {}^pR_N^{\alpha,\beta}(t) = & \frac{1}{\chi(\alpha)^{N+1}} \sum_{q=0}^{+\infty} C_{q+N}^N (-\mu_\alpha \ln p)^q (\delta^{-\beta q} \exp(\delta\lambda)) \\ & \times \sum_{m=0}^{N+1} C_{N+1}^m \chi(\alpha)^{N+1-m} (\ln p \cdot \varphi(\alpha))^m \frac{t^{m\beta}}{\Gamma(m\beta + 1)} \end{aligned}$$

with $\lambda \in [0, t]$.

In contrast with Example 1, we now consider a cosine function instead of an exponential.

Example 2 Consider the function

$$f(t) = \cos(\delta t), \quad \delta > 0.$$

From Lemma 3, one has

$${}^pD_{a,1}^{l[\alpha,\beta]} f(t) = \frac{1}{\chi(\alpha)^l} \sum_{q=0}^{+\infty} C_{q+l-1}^{l-1} (-\mu_\alpha \ln p)^q \left(\delta^{-\beta q} \cos\left(\delta t - \beta q \frac{\pi}{2}\right) \right), \quad (13)$$

for $\alpha \in [0, 1)$ and $\beta, p > 0$. Then, using (10), (11), and (13), the function $\cos(\delta t)$ can be approximated by

$$\begin{aligned} \cos(\delta t) \simeq & \sum_{l=0}^n \frac{1}{\chi(\alpha)^l} \sum_{q=0}^{+\infty} C_{q+l-1}^{l-1} (-\mu_\alpha \ln p)^q \left(\delta^{-\beta q} \cos\left(\beta q \frac{\pi}{2}\right) \right) \\ & \times \sum_{m=0}^l C_l^m \chi(\alpha)^{l-m} (\ln p \cdot \varphi(\alpha))^m \frac{t^{m\beta}}{\Gamma(m\beta + 1)}, \end{aligned}$$

where the interpolation error is given by

$$\begin{aligned} {}^pR_N^{\alpha,\beta}(t) = & \frac{1}{\chi(\alpha)^{N+1}} \sum_{q=0}^{+\infty} C_{q+N}^N (-\mu_\alpha \ln p)^q \left(\delta^{-\beta q} \cos\left(\delta\lambda - \beta q \frac{\pi}{2}\right) \right) \\ & \times \sum_{m=0}^{N+1} C_{N+1}^m \chi(\alpha)^{N+1-m} (\ln p \cdot \varphi(\alpha))^m \frac{t^{m\beta}}{\Gamma(m\beta + 1)} \end{aligned}$$

with $\lambda \in [0, t]$.

Finally, we expand the sine function as a power Taylor series.

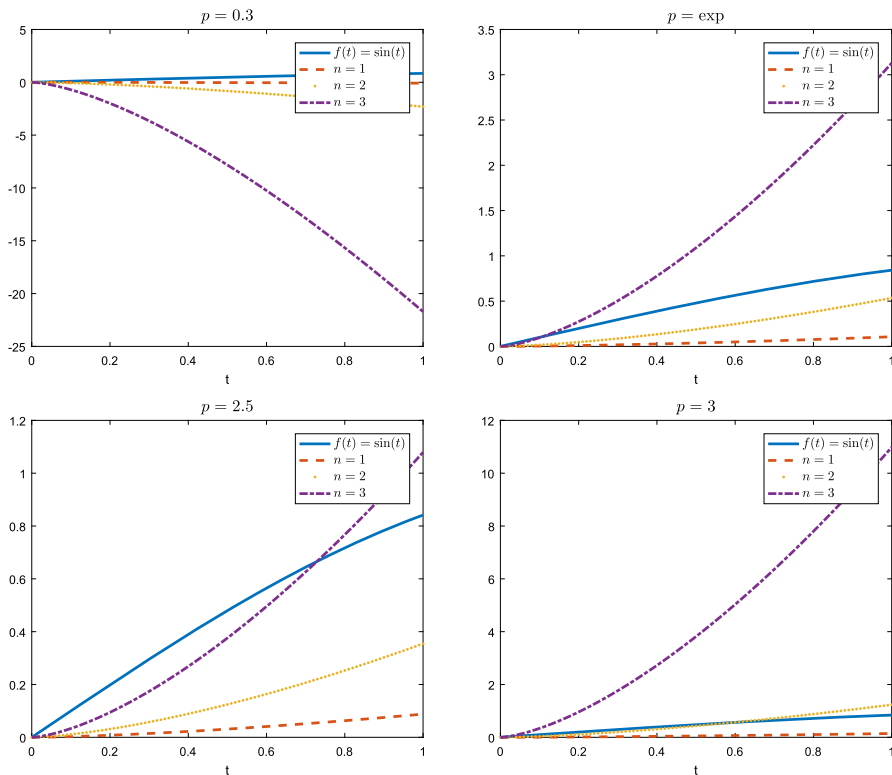


Fig. 1 The function $f(t) = \sin(t)$ and the corresponding Taylor polynomials ${}^pA_n^{\alpha,\beta}(t)$ of order $n = 1, 2, 3$, centered at $t = 0$, for $\alpha = 0.1, \beta = 1.5$ and different values of p

Example 3 Consider the function

$$f(t) = \sin(\delta t), \quad \delta > 0.$$

From Lemma 3, we have

$${}^pD_{a,1}^{l[\alpha,\beta]} f(t) = \frac{1}{\chi(\alpha)^l} \sum_{q=0}^{+\infty} C_{q+l-1}^{l-1} (-\mu_\alpha \ln p)^q \left(\delta^{-\beta q} \sin \left(\delta t - \beta q \frac{\pi}{2} \right) \right), \quad (14)$$

for $\alpha \in [0, 1)$ and $\beta, p > 0$. Then, using Theorem 3, the approximation of the function $\sin(\delta t)$, in the neighborhood of $a = 0$, is given by

$$\sin(\delta t) \simeq \sum_{l=0}^n \frac{1}{\chi(\alpha)^l} \sum_{q=0}^{+\infty} C_{q+l-1}^{l-1} (-\mu_\alpha \ln p)^q \left(-\delta^{-\beta q} \sin \left(\beta q \frac{\pi}{2} \right) \right)$$

$$\times \sum_{m=0}^l C_l^m \chi(\alpha)^{l-m} (\ln p \cdot \varphi(\alpha))^m \frac{t^{m\beta}}{\Gamma(m\beta + 1)} \tag{15}$$

with the remainder

$$\begin{aligned} {}^pR_N^{\alpha,\beta}(t) &= \frac{1}{\chi(\alpha)^{N+1}} \sum_{q=0}^{+\infty} C_{q+N}^N (-\mu_\alpha \ln p)^q \left(\delta^{-\beta q} \sin\left(\delta\lambda - \beta q \frac{\pi}{2}\right) \right) \\ &\times \sum_{m=0}^{N+1} C_{N+1}^m \chi(\alpha)^{N+1-m} (\ln p \cdot \varphi(\alpha))^m \frac{t^{m\beta}}{\Gamma(m\beta + 1)}, \end{aligned}$$

where $\lambda \in [0, t]$.

We plot the function $f(t) = \sin(\delta t)$ and its Taylor polynomials ${}^pA_n^{\alpha,\beta}(t)$ given by (15) around the origin $t = 0$, for $\delta = 1, \alpha = 0.1, \beta = 1.5$ and different values of the order n , over the interval $[0, 1]$. We also consider different values of the power parameter p to see its effect on the function approximation. The results are summarized in Fig. 1.

5 Conclusion

We proved a new generalized Taylor’s theorem for power fractional derivatives, which extends those available in the literature with nonsingular kernels. The proof is based on the establishment of new formulas for n th-order of the power fractional operators. We also obtained a general version of the mean value theorem. As an application, we used our main result to approximate functions at a given point, where the interpolation error of the approximation is also given explicitly. Examples and simulations are presented.

Our developed results can be used to solve fractional differential equations with non constant coefficients in series form. This will be addressed elsewhere.

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Declarations

Conflict of interest The authors have no conflicts or competing of interests to declare.

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