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Article

Variational Problems Involving a Generalized Fractional Derivative with Dependence on the Mittag–Leffler Function

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Abstract: In this paper, we investigate the necessary conditions to optimize a given functional, involving a generalization of the tempered fractional derivative. The exponential function is replaced by the Mittag–Leffler function, and the kernel depends on an arbitrary increasing function. The Lagrangian depends on time, the state function, its fractional derivative, and we add a terminal cost function to the formulation of the problem. Since this new fractional derivative is presented in a general form, some previous works are our own particular cases. In addition, for different choices of the kernel, new results can be deduced. Using variational techniques, the fractional Euler–Lagrange equation is proved, as are its associated transversality conditions. The variational problem with additional constraints is also considered. Then, the question of minimizing functionals with an infinite interval of integration is addressed. To end, we study the case of the Herglotz variational problem, which generalizes the previous one. With this work, several optimization conditions are proven that can be useful for different optimization problems dealing with various fractional derivatives.

Keywords: fractional calculus; calculus of variations; Euler–Lagrange equations; tempered fractional derivative; Mittag–Leffler function

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1. Introduction

Fractional calculus of variations [1–5] deals with calculus of variation problems, but ordinary derivatives are replaced by fractional derivatives. By doing so, we enrich the theory by allowing the dynamics to be described by a non-integer order derivative and consequently better describing the phenomena of nature, in particular when certain degrees of uncertainty are present. In fact, fractional calculus has proved to be a powerful tool in the modeling of real problems, managing to adjust theoretical models to real data in a more precise way. For example, applications can be found in Biology [6], Physics [7], Chemistry [8], Engineering [9,10], Economy [11], etc. These derivatives are non-local, conserving the memory of the process, thus being able to adjust to the current dynamics of the process. In fact, an inherent feature of this theory is the memory effect property, where the behavior at each instant is affected by the initial events. This is considered by many to be an essential feature of fractional derivatives, and any new proposal for a fractional derivative must satisfy this property [12]. This property is useful, e.g., in quantum mechanics [13,14], gravity theory [15,16], or in diffusion equation [17].

One area where fractional calculus has been applied is the calculus of variations, starting with the works of Riewe [18,19], where fractional derivatives were shown to better describe non-conservative systems. Since then, numerous works have appeared on this topic, for different fractional operators and for different formulations of the problem. For example, the works [20–23], or the books [24–26]. Optimization problems, with the presence of fractional operators, are nowadays an important issue and numerous works are being done, e.g., within the optimal control theory [27,28], stochastic processes [29,30], or machine learning [31,32].

In this work, we generalize some previous work by considering a general form of fractional derivative. This involves an arbitrary kernel function g, which extends the concepts of some known derivatives such as the Riemann–Liouville or the Hadamard fractional derivatives. In addition, we replace the exponential function that appears in the definition of tempered fractional derivative, with the Mittag–Leffler function. By doing this, some previously known results on the fractional calculus of variations are just particular cases of our new results. Our goal is to present the necessary conditions that every extremizer of a given fractional variational problem must satisfy. The Lagrange function that we will consider depends, as usual, on time, a state function, and this new generalized fractional derivative. The end time of the functional is free, and so it is also a variable of the problem. We add a terminal cost function, depending on the state function, that assigns a cost to a path taken. We first prove fractional integration by part formulae, dealing with this new operator, and with them, we obtain the fractional Euler–Lagrange equation. Furthermore, we also consider variational problems subject to additional constraints, such as isoperimetric and holonomic restrictions. Since the end point is free, we address the infinite horizon problem, that is, when the interval of integration is not finite. We end the paper by studying the Herglotz problem.

Firstly, we introduce some new definitions and notations, motivated by the concept of tempered fractional derivative [33–36]. For that, let $n \in \mathbb{N}$ and $\gamma \in \mathbb{R}^+$ with $n-1 < \gamma < n$. In addition, let $\beta \in \mathbb{R}^+$, $\lambda \in \mathbb{R}^+ \cup \{0\}$, $-\infty < a < b < +\infty$, and $g \in C^n[a,b]$ a function with $g'(t) > 0$, for all $t \in [a,b]$.

Definition 1. *The generalized left Riemann–Liouville fractional integral of a function $x : [a,b] \to \mathbb{R}$ is given by the formula*

$$\mathbb{I}_{a+}^{\gamma,\beta,\lambda,g} x(t) = \frac{\mathbb{I}_{a+}^{\gamma,g}[E_\beta(\lambda g(t))x(t)]}{E_\beta(\lambda g(t))} = \frac{1}{\Gamma(\gamma)} \int_a^t g'(\tau)(g(t)-g(\tau))^{\gamma-1} \frac{E_\beta(\lambda g(\tau))}{E_\beta(\lambda g(t))} x(\tau)\, d\tau,$$

where $\mathbb{I}_{a+}^{\gamma,g}$ denotes the left fractional integral with respect to another function (see [37]):

$$\mathbb{I}_{a+}^{\gamma,g} x(t) = \frac{1}{\Gamma(\gamma)} \int_a^t g'(\tau)(g(t)-g(\tau))^{\gamma-1} x(\tau)\, d\tau.$$

Similarly, the generalized right Riemann–Liouville fractional integral of x is given by

$$\begin{aligned}\mathbb{I}_{b-}^{\gamma,\beta,\lambda,g} x(t) &= E_\beta(\lambda g(t)) \mathbb{I}_{b-}^{\gamma,g} \left[\frac{x(t)}{E_\beta(\lambda g(t))} \right] \\ &= \frac{1}{\Gamma(\gamma)} \int_t^b g'(\tau)(g(\tau)-g(t))^{\gamma-1} \frac{E_\beta(\lambda g(t))}{E_\beta(\lambda g(\tau))} x(\tau)\, d\tau,\end{aligned}$$

where $\mathbb{I}_{b-}^{\gamma,g}$ denotes the right fractional integral with respect to another function:

$$\mathbb{I}_{b-}^{\gamma,g} x(t) = \frac{1}{\Gamma(\gamma)} \int_t^b g'(\tau)(g(\tau)-g(t))^{\gamma-1} x(\tau)\, d\tau.$$

We remark that,

- when $\beta = 1$, $E_\beta \equiv \exp$ and so we obtain the definition presented in [34];
- if $\lambda = 0$, we obtain the fractional integrals with respect to another function [37,38];
- if $\beta = 1$ and $g(t) = t$, we obtain the tempered fractional integrals [36];
- if $\lambda = 0$ and $g(t) = t$, we get the Riemann–Liouville fractional integrals [37,38];
- if $\lambda = 0$ and $g(t) = \ln t$, we get the Hadamard fractional integrals [37,38].

We now proceed with the definitions of fractional derivatives. Two types are presented: the Riemann–Liouville and Caputo type fractional derivatives.

Definition 2. *The generalized left Riemann–Liouville fractional derivative of a function $x : [a,b] \to \mathbb{R}$ is given by the formula*

$$\begin{aligned}\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}x(t) &= \frac{\mathbb{D}_{a+}^{\gamma,g}[E_\beta(\lambda g(t))x(t)]}{E_\beta(\lambda g(t))} \\ &= \frac{1}{\Gamma(n-\gamma)E_\beta(\lambda g(t))}\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^n \int_a^t g'(\tau)(g(t)-g(\tau))^{n-\gamma-1}E_\beta(\lambda g(\tau))x(\tau)\,d\tau,\end{aligned}$$

where $\mathbb{D}_{a+}^{\gamma,g}$ denotes the left Riemann–Liouville fractional derivative with respect to another function (see Ref. [37]):

$$\mathbb{D}_{a+}^{\gamma,g}x(t) = \frac{1}{\Gamma(n-\gamma)}\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^n \int_a^t g'(\tau)(g(t)-g(\tau))^{n-\gamma-1}x(\tau)\,d\tau.$$

The generalized right Riemann–Liouville fractional derivative of x is given by

$$\begin{aligned}\mathbb{D}_{b-}^{\gamma,\beta,\lambda,g}x(t) &= E_\beta(\lambda g(t))\mathbb{D}_{b-}^{\gamma,g}\left[\frac{x(t)}{E_\beta(\lambda g(t))}\right] \\ &= \frac{E_\beta(\lambda g(t))}{\Gamma(n-\gamma)}\left(-\frac{1}{g'(t)}\frac{d}{dt}\right)^n \int_t^b g'(\tau)(g(\tau)-g(t))^{n-\gamma-1}\frac{x(\tau)}{E_\beta(\lambda g(\tau))}\,d\tau,\end{aligned}$$

where $\mathbb{D}_{b-}^{\gamma,g}$ denotes the right Riemann–Liouville fractional derivative with respect to another function:

$$\mathbb{D}_{b-}^{\gamma,g}x(t) = \frac{1}{\Gamma(n-\gamma)}\left(-\frac{1}{g'(t)}\frac{d}{dt}\right)^n \int_t^b g'(\tau)(g(\tau)-g(t))^{n-\gamma-1}x(\tau)\,d\tau.$$

Definition 3. *The generalized left Caputo fractional derivative of a function $x : [a,b] \to \mathbb{R}$ is given by the formula*

$$\begin{aligned}{}^{C}\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}x(t) &= \frac{{}^{C}\mathbb{D}_{a+}^{\gamma,g}[E_\beta(\lambda g(t))x(t)]}{E_\beta(\lambda g(t))} \\ &= \frac{1}{\Gamma(n-\gamma)E_\beta(\lambda g(t))}\int_a^t g'(\tau)(g(t)-g(\tau))^{n-\gamma-1}\left(\frac{1}{g'(\tau)}\frac{d}{d\tau}\right)^n (E_\beta(\lambda g(\tau))x(\tau))\,d\tau,\end{aligned}$$

where ${}^{C}\mathbb{D}_{a+}^{\gamma,g}$ denotes the left Caputo fractional derivative with respect to another function (see Ref. [39]):

$${}^{C}\mathbb{D}_{a+}^{\gamma,g}x(t) = \frac{1}{\Gamma(n-\gamma)}\int_a^t g'(\tau)(g(t)-g(\tau))^{n-\gamma-1}\left(\frac{1}{g'(\tau)}\frac{d}{d\tau}\right)^n x(\tau)\,d\tau.$$

The generalized right Caputo fractional derivative of x is given by

$$\begin{aligned}{}^{C}\mathbb{D}_{b-}^{\gamma,\beta,\lambda,g}x(t) &= E_\beta(\lambda g(t)){}^{C}\mathbb{D}_{b-}^{\gamma,g}\left[\frac{x(t)}{E_\beta(\lambda g(t))}\right] \\ &= \frac{E_\beta(\lambda g(t))}{\Gamma(n-\gamma)}\int_t^b g'(\tau)(g(\tau)-g(t))^{n-\gamma-1}\left(-\frac{1}{g'(\tau)}\frac{d}{d\tau}\right)^n\left(\frac{x(\tau)}{E_\beta(\lambda g(\tau))}\right)d\tau,\end{aligned}$$

where ${}^{C}\mathbb{D}_{b-}^{\gamma,g}$ denotes the right Caputo fractional derivative with respect to another function:

$${}^{C}\mathbb{D}_{b-}^{\gamma,g}x(t) = \frac{1}{\Gamma(n-\gamma)}\int_t^b g'(\tau)(g(\tau)-g(t))^{n-\gamma-1}\left(-\frac{1}{g'(\tau)}\frac{d}{d\tau}\right)^n x(\tau)\,d\tau.$$

The work is organized in the following way. We start with Section 2, in which we deduce integration by parts formulae for the two Caputo type fractional derivatives, which will be a crucial formulae for the remainder of this work. The main results are proven in Section 3. We obtain the fractional Euler–Lagrange, which is a necessary condition of optimization that every curve that minimizes or maximizes a given functional must fulfill. The terminal time of the functional is free, and so the respective natural boundary conditions are derived. In Section 4 we impose some constraints on the variational problem, such as isoperimetric and holonomic restrictions, and functionals with an unbounded interval of integration are studied. We end by considering the Herglotz variational problem, which is a generalization of the usual calculus of variations.

2. Fractional Integration by Parts

In the following result, we present a similar formula to integration by parts, but dealing with the previous fractional operators. As it can be seen, although in the left-hand side of the formula we have the Caputo fractional derivatives, on the right-hand side it depends the Riemann–Liouville fractional integrals and derivatives.

Theorem 1. *Let $x, y : [a,b] \to \mathbb{R}$ be two functions of class C^n. Then,*

$$\int_a^b x(t)\,{}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} y(t)\,dt = \int_a^b y(t)g'(t)\mathbb{D}_{b-}^{\gamma,\beta,\lambda,g}(x(t)/g'(t))\,dt$$
$$+\left[\sum_{k=0}^{n-1}\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{n-k-1}(E_\beta(\lambda g(t))y(t))\cdot\left(-\frac{1}{g'(t)}\frac{d}{dt}\right)^k\frac{\mathbb{I}_{b-}^{n-\gamma,\beta,\lambda,g}(x(t)/g'(t))}{E_\beta(\lambda g(t))}\right]_a^b$$

and

$$\int_a^b x(t)\,{}^C\mathbb{D}_{b-}^{\gamma,\beta,\lambda,g} y(t)\,dt = \int_a^b y(t)g'(t)\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}(x(t)/g'(t))\,dt$$
$$-\left[\sum_{k=0}^{n-1}\left(-\frac{1}{g'(t)}\frac{d}{dt}\right)^{n-k-1}\left(\frac{y(t)}{E_\beta(\lambda g(t))}\right)\cdot\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^k\left(\mathbb{I}_{a+}^{n-\gamma,\beta,\lambda,g}(x(t)/g'(t))E_\beta(\lambda g(t))\right)\right]_a^b.$$

Proof. Interchange the order of integration, and we get the following

$$\begin{aligned}\int_a^b x(t)\,{}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} y(t)\,dt &= \int_a^b\int_a^t \frac{x(t)}{\Gamma(n-\gamma)E_\beta(\lambda g(t))}\\ &\quad\times g'(\tau)(g(t)-g(\tau))^{n-\gamma-1}\left(\frac{1}{g'(\tau)}\frac{d}{d\tau}\right)^n(E_\beta(\lambda g(\tau))y(\tau))\,d\tau\,dt\\ &= \int_a^b\int_t^b \frac{x(\tau)}{\Gamma(n-\gamma)E_\beta(\lambda g(\tau))}\\ &\quad\times g'(t)(g(\tau)-g(t))^{n-\gamma-1}\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^n(E_\beta(\lambda g(t))y(t))\,d\tau\,dt\\ &= \int_a^b \frac{d}{dt}\left[\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{n-1}(E_\beta(\lambda g(t))y(t))\right]\\ &\quad\times\left[\frac{1}{\Gamma(n-\gamma)}\int_t^b (g(\tau)-g(t))^{n-\gamma-1}\frac{x(\tau)}{E_\beta(\lambda g(\tau))}\,d\tau\right]dt.\end{aligned}$$

Performing one integration by parts, we obtain

$$\begin{aligned}
&\left[\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{n-1}\big(E_\beta(\lambda g(t))y(t)\big) \times \frac{1}{\Gamma(n-\gamma)}\int_t^b (g(\tau)-g(t))^{n-\gamma-1}\frac{x(\tau)}{E_\beta(\lambda g(\tau))}\,d\tau\right]_a^b \\
&\quad - \int_a^b \left[\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{n-1}\big(E_\beta(\lambda g(t))y(t)\big)\right] g'(t) \\
&\quad \times \left(\frac{1}{g'(t)}\frac{d}{dt}\right)\left[\frac{1}{\Gamma(n-\gamma)}\int_t^b (g(\tau)-g(t))^{n-\gamma-1}\frac{x(\tau)}{E_\beta(\lambda g(\tau))}\,d\tau\right] dt \\
&= \left[\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{n-1}\big(E_\beta(\lambda g(t))y(t)\big) \times \frac{\mathbb{I}_{b-}^{n-\gamma,\beta,\lambda,g}(x(t)/g'(t))}{E_\beta(\lambda g(t))}\right]_a^b \\
&\quad + \int_a^b \frac{d}{dt}\left[\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{n-2}\big(E_\beta(\lambda g(t))y(t)\big)\right] \\
&\quad \times \left(-\frac{1}{g'(t)}\frac{d}{dt}\right)\left[\frac{1}{\Gamma(n-\gamma)}\int_t^b (g(\tau)-g(t))^{n-\gamma-1}\frac{x(\tau)}{E_\beta(\lambda g(\tau))}\,d\tau\right] dt.
\end{aligned}$$

Performing a second integration by parts, we arrive at

$$\begin{aligned}
&\left[\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{n-1}\big(E_\beta(\lambda g(t))y(t)\big) \times \frac{\mathbb{I}_{b-}^{n-\gamma,\beta,\lambda,g}(x(t)/g'(t))}{E_\beta(\lambda g(t))}\right]_a^b \\
&+ \left[\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{n-2}\big(E_\beta(\lambda g(t))y(t)\big) \times \left(-\frac{1}{g'(t)}\frac{d}{dt}\right)\frac{\mathbb{I}_{b-}^{n-\gamma,\beta,\lambda,g}(x(t)/g'(t))}{E_\beta(\lambda g(t))}\right]_a^b \\
&\quad + \int_a^b \left[\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{n-2}\big(E_\beta(\lambda g(t))y(t)\big)\right] g'(t) \\
&\quad \times \left(-\frac{1}{g'(t)}\frac{d}{dt}\right)^2\left[\frac{1}{\Gamma(n-\gamma)}\int_t^b (g(\tau)-g(t))^{n-\gamma-1}\frac{x(\tau)}{E_\beta(\lambda g(\tau))}\,d\tau\right] dt \\
&= \left[\sum_{k=0}^{1}\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{n-k-1}\big(E_\beta(\lambda g(t))y(t)\big) \times \left(-\frac{1}{g'(t)}\frac{d}{dt}\right)^k \frac{\mathbb{I}_{b-}^{n-\gamma,\beta,\lambda,g}(x(t)/g'(t))}{E_\beta(\lambda g(t))}\right]_a^b \\
&\quad + \int_a^b \frac{d}{dt}\left[\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{n-3}\big(E_\beta(\lambda g(t))y(t)\big)\right] g'(t) \\
&\quad \times \left(-\frac{1}{g'(t)}\frac{d}{dt}\right)^2\left[\frac{1}{\Gamma(n-\gamma)}\int_t^b (g(\tau)-g(t))^{n-\gamma-1}\frac{x(\tau)}{E_\beta(\lambda g(\tau))}\,d\tau\right] dt.
\end{aligned}$$

Repeating this procedure, we prove the first formula; the second one can be obtained in a similar way. □

We remark that, when $0<\gamma<1$, from Theorem 1 we conclude that

$$\int_a^b x(t)\,{}^{C}\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}y(t)\,dt = \int_a^b y(t)g'(t)\mathbb{D}_{b-}^{\gamma,\beta,\lambda,g}\frac{x(t)}{g'(t)}\,dt + \left[y(t)\cdot\mathbb{I}_{b-}^{1-\gamma,\beta,\lambda,g}\frac{x(t)}{g'(t)}\right]_a^b.$$

3. Fractional Calculus of Variations

Fractional calculus of variations deals with variational problems, but the integer-order derivatives are replaced by some type of fractional integral or fractional derivative [24–26]. In the present study, we will consider a functional with free end-time, where the dynamics of the state function is described by the fractional derivative given in Definition 3. The basic problem consists of finding the curves x and the terminal time T for which the following functional

$$\mathcal{F}(x,T) = \int_a^T \mathcal{L}(t, x(t), {}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+} x(t))\, dt + \Phi(T, x(T)), \tag{1}$$

defined on $\Delta := C^1([a,b],\mathbb{R}) \times [a,b]$, attains a minimum value, where the Lagrange function $\mathcal{L} : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ and the terminal cost function $\Phi : [a,b] \times \mathbb{R} \to \mathbb{R}$ are continuously differentiable functions, and the fractional order γ belongs to the interval $(0,1)$. To simplify the writing, we introduce the following notation:

$$[x](t) := (t, x(t), {}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+} x(t)).$$

The Fundamental Problem

The main result of this section is known as the fractional Euler–Lagrange equation, and provides a necessary condition that all solutions of the variational problem must satisfy. The classical one, studied independently by Euler and Lagrange, reads as

$$\frac{\partial \mathcal{L}}{\partial x}(t, x(t), x'(t) = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial x'}(t, x(t), x'(t),$$

in the case where the functional involves a first-order derivative:

$$\mathcal{F}(x) = \int_a^b \mathcal{L}(t, x(t), x'(t))\, dt.$$

If x' is replaced by ${}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+} x$, then instead of an ODE, we obtain a fractional differential equation. An initial condition $x(a) = X_a$, with $X_a \in \mathbb{R}$, may be imposed in the formulation of the problem. If not, a transversality condition is obtained.

Theorem 2. *Let $(\overline{x}, \overline{T}) \in \Delta$ be a minimizer of functional (1). Then, the following equation is verified for all $t \in [a, \overline{T}]$:*

$$\frac{\partial \mathcal{L}[\overline{x}](t)}{\partial x} + g'(t)\mathbb{D}^{\gamma,\beta,\lambda,g}_{\overline{T}-}\left(\frac{\partial \mathcal{L}[\overline{x}](t)}{\partial {}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+} x} \cdot \frac{1}{g'(t)}\right) = 0. \tag{2}$$

In addition, at $t = \overline{T}$,

$$\begin{cases} \mathbb{I}^{1-\gamma,\beta,\lambda,g}_{\overline{T}-}\left(\dfrac{\partial \mathcal{L}[\overline{x}](t)}{\partial {}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+} x} \cdot \dfrac{1}{g'(t)}\right) + \dfrac{\partial \Phi(t, \overline{x}(t))}{\partial x} = 0, \\[2ex] \mathcal{L}[\overline{x}](t) + \dfrac{\partial \Phi(t, \overline{x}(t))}{\partial t} + \dfrac{\partial \Phi(t, \overline{x}(t))}{\partial x}\overline{x}'(t) = 0. \end{cases}$$

In addition, if $x(a)$ is free, then the boundary condition

$$\mathbb{I}^{1-\gamma,\beta,\lambda,g}_{\overline{T}-}\left(\frac{\partial \mathcal{L}[\overline{x}](t)}{\partial {}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+} x} \cdot \frac{1}{g'(t)}\right) = 0.$$

is satisfied when evaluated at $t = a$.

Proof. The result is proven using variational arguments. Consider a variation of the optimal solution of type

$$(\overline{x}(t) + \delta\theta(t), \overline{T} + \delta s), \quad (\theta, s) \in \Delta, \, -r < \delta < r,$$

where $r \in \mathbb{R}$ is a small positive number. Observe that, in case $x(a)$ is fixed, then we need to assume that $\theta(a) = 0$. Considering the function $f(\delta) := \mathcal{F}(\overline{x}(t) + \delta\theta(t), \overline{T} + \delta s)$, its first variation must be zero when we evaluate it at $(\overline{x}, \overline{T})$. Thus, we conclude that

$$\int_a^{\overline{T}} \left[\frac{\partial \mathcal{L}[\overline{x}](t)}{\partial x} \theta(t) + \frac{\partial \mathcal{L}[\overline{x}](t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x} {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} \theta(t) \right] dt + s\mathcal{L}[\overline{x}](\overline{T})$$
$$+ s \frac{\partial \Phi(\overline{T}, \overline{x}(\overline{T}))}{\partial t} + \frac{\partial \Phi(\overline{T}, \overline{x}(\overline{T}))}{\partial x} (s\overline{x}'(\overline{T}) + \theta(\overline{T})) = 0,$$

and integrating by parts the second term of the integral (Theorem 1):

$$\int_a^{\overline{T}} \left[\frac{\partial \mathcal{L}[\overline{x}](t)}{\partial x} + g'(t)\mathbb{D}_{\overline{T}-}^{\gamma,\beta,\lambda,g} \left(\frac{\partial \mathcal{L}[\overline{x}](t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x} \cdot \frac{1}{g'(t)} \right) \right] \theta(t)\, dt$$
$$+ s \left[\mathcal{L}[\overline{x}](\overline{T}) + \frac{\partial \Phi(\overline{T}, \overline{x}(\overline{T}))}{\partial t} + \frac{\partial \Phi(\overline{T}, \overline{x}(\overline{T}))}{\partial x} \overline{x}'(\overline{T}) \right]$$
$$+ \frac{\partial \Phi(\overline{T}, \overline{x}(\overline{T}))}{\partial x} \theta(\overline{T}) + \left[\theta(t) \mathbb{I}_{\overline{T}-}^{1-\gamma,\beta,\lambda,g} \left(\frac{\partial \mathcal{L}[\overline{x}](t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x} \cdot \frac{1}{g'(t)} \right) \right]_a^{\overline{T}} = 0. \quad (3)$$

Since s is arbitrary, and θ is also arbitrary in $[a, \overline{T}]$ (or in $(a, \overline{T}]$ if case $x(a)$ is fixed a priori), we prove the desired formulas. □

Equation (2) is known as the fractional Euler–Lagrange equation associated with the functional (1).

The transversality conditions obtained at $t = \overline{T}$ can be written in another form, with the help of the total variation

$$\delta_x = (\overline{x} + \theta)(\overline{T} + s) - \overline{x}(\overline{T}),$$

with s and θ as given in the proof of Theorem 2. Assuming that $\theta'(\overline{T}) = 0$ and that $|s| \ll 1$, by Taylor's formula, we obtain the relation

$$\delta_x = (\overline{x} + \theta)(\overline{T}) + \overline{x}'(\overline{T})s + O(s)^2 - \overline{x}(\overline{T}),$$

and so

$$\theta(\overline{T}) = \delta_x - \overline{x}'(\overline{T})s + O(s)^2. \quad (4)$$

Replacing (4) into (3), and assuming that the Euler–Lagrange Equation (2) holds and that $\theta(a) = 0$, we prove that

$$s \left[\mathcal{L}[\overline{x}](\overline{T}) + \frac{\partial \Phi(\overline{T}, \overline{x}(\overline{T}))}{\partial t} + \frac{\partial \Phi(\overline{T}, \overline{x}(\overline{T}))}{\partial x} \overline{x}'(\overline{T}) \right]$$
$$+ (\delta_x - \overline{x}'(\overline{T})s) \left[\frac{\partial \Phi(\overline{T}, \overline{x}(\overline{T}))}{\partial x} + \left[\mathbb{I}_{\overline{T}-}^{1-\gamma,\beta,\lambda,g} \left(\frac{\partial \mathcal{L}[\overline{x}](t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x} \cdot \frac{1}{g'(t)} \right) \right]_{t=\overline{T}} \right] + O(s)^2 = 0,$$

and so

$$s\left[\mathcal{L}[\overline{x}](\overline{T}) + \frac{\partial\Phi(\overline{T},\overline{x}(\overline{T}))}{\partial t} - \overline{x}'(\overline{T})\left[\mathbb{I}^{1-\gamma,\beta,\lambda,g}_{\overline{T}-}\left(\frac{\partial\mathcal{L}[\overline{x}](t)}{\partial^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+}x}\cdot\frac{1}{g'(t)}\right)\right]_{t=\overline{T}}\right] + \delta_x\left[\frac{\partial\Phi(\overline{T},\overline{x}(\overline{T}))}{\partial x} + \left[\mathbb{I}^{1-\gamma,\beta,\lambda,g}_{\overline{T}-}\left(\frac{\partial\mathcal{L}[\overline{x}](t)}{\partial^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+}x}\cdot\frac{1}{g'(t)}\right)\right]_{t=\overline{T}}\right] + O(s)^2 = 0.$$

In conclusion, the transversality conditions can be written as

$$\begin{cases} \mathcal{L}[\overline{x}](\overline{T}) + \dfrac{\partial\Phi(\overline{T},\overline{x}(\overline{T}))}{\partial t} - \overline{x}'(\overline{T})\left[\mathbb{I}^{1-\gamma,\beta,\lambda,g}_{\overline{T}-}\left(\dfrac{\partial\mathcal{L}[\overline{x}](t)}{\partial^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+}x}\cdot\dfrac{1}{g'(t)}\right)\right]_{t=\overline{T}} = 0, \\ \dfrac{\partial\Phi(\overline{T},\overline{x}(\overline{T}))}{\partial x} + \left[\mathbb{I}^{1-\gamma,\beta,\lambda,g}_{\overline{T}-}\left(\dfrac{\partial\mathcal{L}[\overline{x}](t)}{\partial^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+}x}\cdot\dfrac{1}{g'(t)}\right)\right]_{t=\overline{T}} = 0. \end{cases}$$

Some particular cases can now be easily deduced. If we assume that the terminal point must belong to a vertical line, that is, $\overline{T}$ is fixed, then $s = 0$ and δ_x is arbitrary, and so the transversality condition reduces to

$$\frac{\partial\Phi(\overline{T},\overline{x}(\overline{T}))}{\partial x} + \left[\mathbb{I}^{1-\gamma,\beta,\lambda,g}_{\overline{T}-}\left(\frac{\partial\mathcal{L}[\overline{x}](t)}{\partial^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+}x}\cdot\frac{1}{g'(t)}\right)\right]_{t=\overline{T}} = 0.$$

On the other hand, if the terminal point belongs to an horizontal line, then $\delta_x = 0$ and s is arbitrary and we produce the condition

$$L[\overline{x}](\overline{T}) + \frac{\partial\Phi(\overline{T},\overline{x}(\overline{T}))}{\partial t} - \overline{x}'(\overline{T})\left[\mathbb{I}^{1-\gamma,\beta,\lambda,g}_{\overline{T}-}\left(\frac{\partial\mathcal{L}[\overline{x}](t)}{\partial^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+}x}\cdot\frac{1}{g'(t)}\right)\right]_{t=\overline{T}} = 0.$$

Finally, suppose that the terminal point is given by a prescribed curve. More precisely, let $\eta \in C^1([a,b],\mathbb{R})$ be a curve and consider the restriction $x(\overline{T}) = \eta(\overline{T})$ as being imposed in the formulation of the variational problem. In such a case, by Taylor's Theorem,

$$\delta_x = \eta(\overline{T}+s) - \eta(\overline{T}) = \eta'(\overline{T})s + O(s)^2,$$

and consequently only one transversality condition is obtained:

$$L[\overline{x}](\overline{T}) + \frac{\partial\Phi(\overline{T},\overline{x}(\overline{T}))}{\partial t} - \overline{x}'(\overline{T})\left[\mathbb{I}^{1-\gamma,\beta,\lambda,g}_{\overline{T}-}\left(\frac{\partial\mathcal{L}[\overline{x}](t)}{\partial^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+}x}\cdot\frac{1}{g'(t)}\right)\right]_{t=\overline{T}} + \eta'(\overline{T})\left[\frac{\partial\Phi(\overline{T},\overline{x}(\overline{T}))}{\partial x} + \left[\mathbb{I}^{1-\gamma,\beta,\lambda,g}_{\overline{T}-}\left(\frac{\partial\mathcal{L}[\overline{x}](t)}{\partial^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+}x}\cdot\frac{1}{g'(t)}\right)\right]_{t=\overline{T}}\right] = 0.$$

We can describe the Euler–Lagrange Equation (2) within the framework of the fractional Hamiltonian formulation [40]. Define the fractional canonical momenta p as

$$p := \frac{\partial\mathcal{L}[\overline{x}]}{\partial^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+}x} \tag{5}$$

and the fractional canonical Hamilton H as

$$H = p\,{}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+}x - L.$$

The total differential of H is

$$dH = dp\, {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x + p\, d\,{}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x - \frac{\partial \mathcal{L}[\overline{x}]}{\partial t}\, dt - \frac{\partial \mathcal{L}[\overline{x}]}{\partial x}\, dx - \frac{\partial \mathcal{L}[\overline{x}]}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x}\, d\,{}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x.$$

Using (5), we get

$$dH = dp\, {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x - \frac{\partial \mathcal{L}[\overline{x}]}{\partial t}\, dt - \frac{\partial \mathcal{L}[\overline{x}]}{\partial x}\, dx,$$

and then taking into account the Euler–Lagrange Equation (2), we get the following

$$dH = dp\, {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x - \frac{\partial \mathcal{L}[\overline{x}]}{\partial t}\, dt + g'(t)\mathbb{D}_{\overline{T}-}^{\gamma,\beta,\lambda,g}\left(\frac{\partial \mathcal{L}[\overline{x}](t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x} \cdot \frac{1}{g'(t)}\right) dx.$$

Thus, looking at the Hamiltonian as a function of three variables $H = H(t,x,p)$, then

$$dH = \frac{\partial H}{\partial t}\, dt + \frac{\partial H}{\partial x}\, dx + \frac{\partial H}{\partial p}\, dp,$$

and so

$$\begin{cases} \frac{\partial H}{\partial t} = -\frac{\partial \mathcal{L}[\overline{x}]}{\partial t} \\ \frac{\partial H}{\partial x} = g'(t)\mathbb{D}_{\overline{T}-}^{\gamma,\beta,\lambda,g}\left(\frac{\partial \mathcal{L}[\overline{x}](t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x} \cdot \frac{1}{g'(t)}\right) \\ \frac{\partial H}{\partial p} = {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x. \end{cases}$$

Observe that, in contrast to the classical case, in which only first order derivatives appear in the Lagrangian, the fractional Hamiltonian is not constant even when the Lagrangian does not explicitly depend on time.

For the case where the state function is m-dimensional, that is, $x = (x_1, \ldots, x_m)$, and in this case ${}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x(t) = ({}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x_1, \ldots, {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x_m)$ and

$$\mathcal{F}(x,T) = \int_a^T \mathcal{L}(t, x_1(t), \ldots, x_m(t), {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x_1(t), \ldots, {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x_m(t))\, dt + \Phi(T, x(T)), \quad (6)$$

where for each $k = 1, \ldots, m$, $x_k \in C^1([a,b], \mathbb{R})$, a similar result can be proven:

Theorem 3. *Let $(\overline{x}, \overline{T})$ be a minimizer of functional (6), where $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_m)$. Then, the following equation is verified for all $t \in [a, \overline{T}]$ and for all $k = 1, \ldots, m$:*

$$\frac{\partial \mathcal{L}[\overline{x}](t)}{\partial x_k} + g'(t)\mathbb{D}_{\overline{T}-}^{\gamma,\beta,\lambda,g}\left(\frac{\partial \mathcal{L}[\overline{x}](t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x_k} \cdot \frac{1}{g'(t)}\right) = 0.$$

In addition, at $t = \overline{T}$,

$$\begin{cases} \mathbb{I}_{\overline{T}-}^{1-\gamma,\beta,\lambda,g}\left(\frac{\partial \mathcal{L}[\overline{x}](t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x_k} \cdot \frac{1}{g'(t)}\right) + \frac{\partial \Phi(t, \overline{x}(t))}{\partial x_k} = 0, \quad \forall k \in \{1, \ldots, m\} \\ \mathcal{L}[\overline{x}](t) + \frac{\partial \Phi(t, \overline{x}(t))}{\partial t} + \sum_{i=1}^{m} \frac{\partial \Phi(t, \overline{x}(t))}{\partial x_i} \overline{x}_i'(t) = 0. \end{cases}$$

In addition, if $x_k(a)$ is free, then

$$\mathbb{I}_{\overline{T}-}^{1-\gamma,\beta,\lambda,g}\left(\frac{\partial \mathcal{L}[\overline{x}](t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x_k} \cdot \frac{1}{g'(t)}\right) = 0.$$

is satisfied when evaluated at $t = a$, for all $k = 1, \ldots, m$.

4. Some Generalizations

In this section, we present some other variational problems. Namely, we will consider the cases when integral or holonomic constraints are imposed when formulating the problem.

The isoperimetric problem is one of the oldest mathematical problems we can find, and it is formulated as follows: among all the closed curves, without self-intersections and a fixed perimeter, which one has the largest area? Nowadays, we can formulate this problem by using the language of the calculus of variations, as we intend to maximize a given functional, with an integral restriction that the set of admissible curves must satisfy. In the variational problem with holonomic constraints, an equation that depends on time and state variables is imposed on the problem. Such restrictions do not reduce the dimension of space, but they reduce the dimension of space for the possible movement of objects. For example, the space of admissible curves for the problem can be on a curve or on a surface.

4.1. Isoperimetric Problems

Two types of isoperimetric problems will be considered: in the first one, the integral constraint depends on the terminal time, and in the second one, it is independent of $\overline{T}$.

Case I: Suppose that we intend to minimize functional (1), subject to the restrictions

- $x(a) = X_a \in \mathbb{R}$ is fixed;
- the set of admissible functions must verify the relation

$$I(x,T) := \int_a^T i(t, x(t), {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x(t))\, dt = \chi(T),$$

where $i : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ is a function of class C^1 and $\chi : [a,b] \to \mathbb{R}$ is a given function. The next result presents the necessary conditions that every solution must verify.

Theorem 4. *Let $(\overline{x}, \overline{T}) \in \Delta$ be a minimizer of functional (1), subject to the restrictions as explained above. If, for some $t \in [a, \overline{T}]$,*

$$\frac{\partial i[\overline{x}](t)}{\partial x} + g'(t)\mathbb{D}_{\overline{T}-}^{\gamma,\beta,\lambda,g}\left(\frac{\partial i[\overline{x}](t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x} \cdot \frac{1}{g'(t)}\right) \neq 0, \tag{7}$$

then there exists $\lambda \in \mathbb{R}$ such that, for all $t \in [a, \overline{T}]$,

$$\frac{\partial H[\overline{x}](t)}{\partial x} + g'(t)\mathbb{D}_{\overline{T}-}^{\gamma,\beta,\lambda,g}\left(\frac{\partial H[\overline{x}](t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x} \cdot \frac{1}{g'(t)}\right) = 0, \tag{8}$$

where $H = \mathcal{L} + \lambda i$. At $t = \overline{T}$,

$$\begin{cases} \mathbb{I}_{\overline{T}-}^{1-\gamma,\beta,\lambda,g}\left(\dfrac{\partial H[\overline{x}](t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x} \cdot \dfrac{1}{g'(t)}\right) + \dfrac{\partial \Phi(t, \overline{x}(t))}{\partial x} = 0, \\ \\ H[\overline{x}](t) + \dfrac{\partial \Phi(t, \overline{x}(t))}{\partial t} + \dfrac{\partial \Phi(t, \overline{x}(t))}{\partial x}\overline{x}'(t) - \lambda \chi'(t) = 0. \end{cases}$$

Proof. The variations that we will consider depend on two parameters:

$$(\overline{x}(t) + \delta_1\theta_1(t) + \delta_2\theta_2(t), \overline{T} + \delta_1 s),\ -r < \delta_1, \delta_2 < r,$$

where $\theta_1, \theta_2 \in C^1([a,b], \mathbb{R})$, with $\theta_1(a) = \theta_2(a) = 0$, and $s \in [a,b]$. Define the two following functions:

$$f(\delta_1, \delta_2) := \mathcal{F}(\overline{x}(t) + \delta_1\theta_1(t) + \delta_2\theta_2(t), \overline{T} + \delta_1 s)$$

and

$$h(\delta_1,\delta_2) := I(\overline{x}(t)+\delta_1\theta_1(t)+\delta_2\theta_2(t),\overline{T}+\delta_1 s)-\chi(\overline{T}+\delta_1 s).$$

Then,

$$\begin{aligned}\frac{\partial h}{\partial \delta_2}(0,0) &= \int_a^{\overline{T}}\left[\frac{\partial i[\overline{x}](t)}{\partial x}\theta_2(t)+\frac{\partial i[\overline{x}](t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}x}{}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}\theta_2(t)\right]dt\\ &= \int_a^{\overline{T}}\left[\frac{\partial i[\overline{x}](t)}{\partial x}+g'(t)\mathbb{D}_{\overline{T}-}^{\gamma,\beta,\lambda,g}\left(\frac{\partial i[\overline{x}](t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}x}\cdot\frac{1}{g'(t)}\right)\right]\theta_2(t)\,dt\\ &\quad+\left[\theta_2(t)\mathbb{I}_{\overline{T}-}^{1-\gamma,\beta,\lambda,g}\left(\frac{\partial i[\overline{x}](t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}x}\cdot\frac{1}{g'(t)}\right)\right]_a^{\overline{T}}.\end{aligned}$$

Since (7) holds, there must exist some function θ_2 for which

$$\frac{\partial h}{\partial \delta_2}(0,0)\neq 0.$$

Besides this, $h(0,0)=0$ and so, by the Implicit Function Theorem, there exists some function $\varpi:(-k,k)\to\mathbb{R}$ for which $h(\delta_1,\varpi(\delta_1))=0$, that is, there exists a family of variations fulfilling the integral constraint.

We now proceed by proving the desired necessary conditions. For that, observe that the variational problem is equivalent to the following one:

$$\text{minimize } f, \text{ under the restriction } h(\delta_1,\delta_2)=0,$$

and since $(0,0)$ is a solution and we just proved that $\nabla h(0,0)\neq(0,0)$, using the technique of the Lagrange multiplier rule, we conclude that there exists a real λ such that $\nabla(f+\lambda h)(0,0)=(0,0)$, and so $\partial(f+\lambda h)/\partial\delta_1(0,0)=0$. Computing $\partial(f+\lambda h)/\partial\delta_1(0,0)$, we arrive at the formula

$$\begin{aligned}&\int_a^{\overline{T}}\left[\frac{\partial \mathcal{L}[\overline{x}](t)}{\partial x}\theta_1(t)+\frac{\partial \mathcal{L}[\overline{x}](t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}x}{}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}\theta_1(t)\right]dt+s\mathcal{L}[\overline{x}](\overline{T})\\ &\quad+s\frac{\partial\Phi(\overline{T},\overline{x}(\overline{T}))}{\partial t}+\frac{\partial\Phi(\overline{T},\overline{x}(\overline{T}))}{\partial x}(s\overline{x}'(\overline{T})+\theta_1(\overline{T}))\\ &+\lambda\left(\int_a^{\overline{T}}\left[\frac{\partial i[\overline{x}](t)}{\partial x}\theta_1(t)+\frac{\partial i[\overline{x}](t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}x}{}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}\theta_1(t)\right]dt+si[\overline{x}](\overline{T})-s\chi'(\overline{T})\right)\\ &=\int_a^{\overline{T}}\left[\frac{\partial H[\overline{x}](t)}{\partial x}+g'(t)\mathbb{D}_{\overline{T}-}^{\gamma,\beta,\lambda,g}\left(\frac{\partial H[\overline{x}](t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}x}\cdot\frac{1}{g'(t)}\right)\right]\theta_1(t)\,dt\\ &\quad+s\left[H[\overline{x}](\overline{T})+\frac{\partial\Phi(\overline{T},\overline{x}(\overline{T}))}{\partial t}+\frac{\partial\Phi(\overline{T},\overline{x}(\overline{T}))}{\partial x}\overline{x}'(\overline{T})-\lambda\chi'(\overline{T})\right]\\ &\quad+\frac{\partial\Phi(\overline{T},\overline{x}(\overline{T}))}{\partial x}\theta_1(\overline{T})+\left[\theta_1(t)\mathbb{I}_{\overline{T}-}^{1-\gamma,\beta,\lambda,g}\left(\frac{\partial H[\overline{x}](t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}x}\cdot\frac{1}{g'(t)}\right)\right]_a^{\overline{T}}=0.\end{aligned}$$

The rest of the proof follows from the arbitrariness of θ_1 and s. □

Case II: Suppose that we intend to minimize functional (1), subject to the restrictions

- $x(a)=X_a\in\mathbb{R}$ is fixed;
- The set of admissible functions must verify the relation

$$I(x) := \int_a^b i(t,x(t),{}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}x(t))\,dt=\chi,$$

where $i : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ is a function of class C^1 and χ is a constant.

Theorem 5. *Let $(\overline{x}, \overline{T}) \in \Delta$ be a minimizer of functional (1), subject to the restrictions above. If, for some $t \in [a,b]$,*

$$\frac{\partial i[\overline{x}](t)}{\partial x} + g'(t)\mathbb{D}_{b-}^{\gamma,\beta,\lambda,g}\left(\frac{\partial i[\overline{x}](t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}x} \cdot \frac{1}{g'(t)}\right) \neq 0,$$

then there exists $\lambda \in \mathbb{R}$ such that, for all $t \in [a, \overline{T}]$,

$$\frac{\partial L[\overline{x}](t)}{\partial x} + \lambda\frac{\partial i[\overline{x}](t)}{\partial x} + g'(t)\mathbb{D}_{\overline{T}-}^{\gamma,\beta,\lambda,g}\left(\frac{\partial L[\overline{x}](t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}x} \cdot \frac{1}{g'(t)}\right) + \lambda g'(t)\mathbb{D}_{b-}^{\gamma,\beta,\lambda,g}\left(\frac{\partial i[\overline{x}](t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}x} \cdot \frac{1}{g'(t)}\right) = 0,$$

and for all $t \in [\overline{T}, b]$,

$$\lambda\frac{\partial i[\overline{x}](t)}{\partial x} + \lambda g'(t)\mathbb{D}_{b-}^{\gamma,\beta,\lambda,g}\left(\frac{\partial i[\overline{x}](t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}x} \cdot \frac{1}{g'(t)}\right) = 0.$$

In addition, at $t = \overline{T}$,

$$\begin{cases} \mathbb{I}_{\overline{T}-}^{1-\gamma,\beta,\lambda,g}\left(\dfrac{\partial L[\overline{x}](t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}x} \cdot \dfrac{1}{g'(t)}\right) + \dfrac{\partial \Phi(t,\overline{x}(t))}{\partial x} = 0, \\ L[\overline{x}](t) + \dfrac{\partial \Phi(t,\overline{x}(t))}{\partial t} + \dfrac{\partial \Phi(t,\overline{x}(t))}{\partial x}\overline{x}'(t) = 0, \end{cases}$$

and at $t = b$,

$$\mathbb{I}_{b-}^{1-\gamma,\beta,\lambda,g}\left(\frac{\partial i[\overline{x}](t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}x} \cdot \frac{1}{g'(t)}\right) = 0.$$

Proof. Similarly as in the previous theorem, define

$$f(\delta_1,\delta_2) := \mathcal{F}(\overline{x}(t) + \delta_1\theta_1(t) + \delta_2\theta_2(t), \overline{T} + \delta_1 s)$$

and

$$h(\delta_1,\delta_2) := I(\overline{x}(t) + \delta_1\theta_1(t) + \delta_2\theta_2(t)) - \chi,$$

and so

$$\begin{aligned} \frac{\partial h}{\partial \delta_2}(0,0) &= \int_a^b \left[\frac{\partial i[\overline{x}](t)}{\partial x} + g'(t)\mathbb{D}_{b-}^{\gamma,\beta,\lambda,g}\left(\frac{\partial i[\overline{x}](t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}x} \cdot \frac{1}{g'(t)}\right)\right]\theta_2(t)\,dt \\ &\quad + \left[\theta_2(t)\mathbb{I}_{b-}^{1-\gamma,\beta,\lambda,g}\left(\frac{\partial i[\overline{x}](t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}x} \cdot \frac{1}{g'(t)}\right)\right]_a^b. \end{aligned}$$

Again, we conclude that there exists a family of variations fulfilling the integral constraint and that there exists a real λ such that $\nabla(f + \lambda h)(0,0) = (0,0)$, and, in particular, $\partial(f + \lambda h)/\partial\delta_1(0,0) = 0$. Computing $\partial(f + \lambda h)/\partial\delta_1(0,0)$, we arrive at the desired formulas. □

4.2. Holonomic Constraint

An holonomic constraint is a restriction imposed on the set of admissible functions, and relates time to space coordinates. More precisely, the problem is formulated as follows. Consider the functional with two state functions x_1, x_2 given as

$$\mathcal{F}_H(x_1, x_2, T) = \int_a^T \mathcal{L}_H(t, x_1(t), x_2(t), {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x_1(t), {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x_2(t))\, dt + \Phi_H(T, x_1(T), x_2(T)), \quad (9)$$

defined on $\Delta_H := C^1([a,b],\mathbb{R}) \times C^1([a,b],\mathbb{R}) \times [a,b]$, where $\mathcal{L}_H : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ and $\Phi_H : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ are two continuously differentiable functions. Define

$$[x]_H(t) := (t, x_1(t), x_2(t), {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x_1(t), {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x_2(t)).$$

The goal is to minimize $\mathcal{F}_H$, subject to the restrictions

- $(x_1(a), x_2(a)) = (X_{a,1}, X_{a,2}) \in \mathbb{R}^2$ is fixed;
- The set of admissible functions must verify the relation

$$\Lambda(t, x_1(t), x_2(t)) = 0, \quad (10)$$

where $\Lambda : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ is a function of class C^1.

The next result provides an answer to this variational problem.

Theorem 6. *Let $(\overline{x}_1, \overline{x}_2, \overline{T}) \in \Delta_H$ be a minimizer of functional (9), subject to the restrictions above. If, for all $t \in [a, \overline{T}]$,*

$$\frac{\partial \Lambda}{\partial x_2}(t, \overline{x}_1(t), \overline{x}_2(t)) \neq 0,$$

then there exists a continuous function $\lambda : [a, \overline{T}] \to \mathbb{R}$ such that, for all $t \in [a, \overline{T}]$,

$$\frac{\partial \mathcal{L}_H[\overline{x}]_H(t)}{\partial x_1} + g'(t)\mathbb{D}_{\overline{T}-}^{\gamma,\beta,\lambda,g}\left(\frac{\partial \mathcal{L}_H[\overline{x}]_H(t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x_1} \cdot \frac{1}{g'(t)}\right) + \lambda(t)\frac{\partial \Lambda}{\partial x_1}(t, \overline{x}_1(t), \overline{x}_2(t)) = 0, \quad (11)$$

and

$$\frac{\partial \mathcal{L}_H[\overline{x}]_H(t)}{\partial x_2} + g'(t)\mathbb{D}_{\overline{T}-}^{\gamma,\beta,\lambda,g}\left(\frac{\partial \mathcal{L}_H[\overline{x}]_H(t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x_2} \cdot \frac{1}{g'(t)}\right) + \lambda(t)\frac{\partial \Lambda}{\partial x_2}(t, \overline{x}_1(t), \overline{x}_2(t)) = 0. \quad (12)$$

Moreover, at $t = \overline{T}$,

$$\begin{cases} \mathbb{I}_{\overline{T}-}^{1-\gamma,\beta,\lambda,g}\left(\dfrac{\partial \mathcal{L}_H[\overline{x}]_H(t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x_1} \cdot \dfrac{1}{g'(t)}\right) + \dfrac{\partial \Phi_H(t, \overline{x}_1(t), \overline{x}_2(t))}{\partial x_1} = 0, \\[2ex] \mathbb{I}_{\overline{T}-}^{1-\gamma,\beta,\lambda,g}\left(\dfrac{\partial \mathcal{L}_H[\overline{x}]_H(t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x_2} \cdot \dfrac{1}{g'(t)}\right) + \dfrac{\partial \Phi_H(t, \overline{x}_1(t), \overline{x}_2(t))}{\partial x_2} = 0, \\ \mathcal{L}_H[\overline{x}]_H(t) + \dfrac{\partial \Phi_H(t, \overline{x}_1(t), \overline{x}_2(t))}{\partial t} + \dfrac{\partial \Phi_H(t, \overline{x}_1(t), \overline{x}_2(t))}{\partial x_1}\overline{x}_1'(t) \\ \quad + \dfrac{\partial \Phi_H(t, \overline{x}_1(t), \overline{x}_2(t))}{\partial x_2}\overline{x}_2'(t) = 0. \end{cases}$$

.

Proof. Define the function $\lambda : [a, \overline{T}] \to \mathbb{R}$ as

$$\lambda(t) = -\frac{\frac{\partial \mathcal{L}_H[\overline{x}]_H(t)}{\partial x_2} + g'(t)\mathbb{D}^{\gamma,\beta,\lambda,g}_{\overline{T}-}\left(\frac{\partial \mathcal{L}_H[\overline{x}]_H(t)}{\partial {}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+}x_2}\cdot\frac{1}{g'(t)}\right)}{\frac{\partial \Lambda}{\partial x_2}(t, \overline{x}_1(t), \overline{x}_2(t))}. \quad (13)$$

Then Equation (12) is proved. To prove the remaining conditions, consider variations depending on two functions $\theta_1, \theta_2 \in C^1([a,b], \mathbb{R})$ with $\theta_1(a) = 0$ and $\theta_2(a) = 0$:

$$(\overline{x}_1(t) + \delta\theta_1(t), \overline{x}_2(t) + \delta\theta_2(t), \overline{T} + \delta s), \quad s, \delta \in \mathbb{R},$$

and define the function f in a neighborhood of zero as

$$f_H(\delta) := \mathcal{F}_H(\overline{x}_1(t) + \delta\theta_1(t), \overline{x}_2(t) + \delta\theta_2(t), \overline{T} + \delta s).$$

These variations must satisfy relation (10), that is,

$$\Lambda(t, \overline{x}_1(t) + \delta\theta_1(t), \overline{x}_2(t) + \delta\theta_2(t)) = 0.$$

Differentiating this last equation with respect to δ and setting $\delta = 0$, we get

$$\frac{\partial \Lambda}{\partial x_1}(t, \overline{x}_1(t), \overline{x}_2(t))\theta_1(t) + \frac{\partial \Lambda}{\partial x_2}(t, \overline{x}_1(t), \overline{x}_2(t))\theta_2(t) = 0. \quad (14)$$

Since $\frac{\partial f_H}{\partial \delta}(0,0) = 0$, we obtain the following

$$\begin{aligned}\int_a^{\overline{T}} \Bigg[&\frac{\partial \mathcal{L}_H[\overline{x}]_H(t)}{\partial x_1}\theta_1(t) + \frac{\partial \mathcal{L}_H[\overline{x}]_H(t)}{\partial x_2}\theta_2(t) + \frac{\partial \mathcal{L}_H[\overline{x}]_H(t)}{\partial {}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+}x_1}{}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+}\theta_1(t) \\ &+ \frac{\partial \mathcal{L}_H[\overline{x}]_H(t)}{\partial {}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+}x_2}{}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+}\theta_2(t)\Bigg] dt + s\mathcal{L}_H[\overline{x}]_H(\overline{T}) + s\frac{\partial \Phi_H(\overline{T}, \overline{x}_1(\overline{T}), \overline{x}_2(\overline{T}))}{\partial t} \\ &+ \frac{\partial \Phi_H(\overline{T}, \overline{x}_1(\overline{T}), \overline{x}_2(\overline{T}))}{\partial x_1}(s\overline{x}_1'(\overline{T}) + \theta_1(\overline{T})) + \frac{\partial \Phi_H(\overline{T}, \overline{x}_1(\overline{T}), \overline{x}_2(\overline{T}))}{\partial x_2}(s\overline{x}_2'(\overline{T}) + \theta_2(\overline{T})) = 0.\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}\int_a^{\overline{T}} &\left[\frac{\partial \mathcal{L}_H[\overline{x}]_H(t)}{\partial x_1} + g'(t)\mathbb{D}^{\gamma,\beta,\lambda,g}_{\overline{T}-}\left(\frac{\partial \mathcal{L}_H[\overline{x}]_H(t)}{\partial {}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+}x_1}\cdot\frac{1}{g'(t)}\right)\right]\theta_1(t) \\ &+ \left[\frac{\partial \mathcal{L}_H[\overline{x}]_H(t)}{\partial x_2} + g'(t)\mathbb{D}^{\gamma,\beta,\lambda,g}_{\overline{T}-}\left(\frac{\partial \mathcal{L}_H[\overline{x}]_H(t)}{\partial {}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+}x_2}\cdot\frac{1}{g'(t)}\right)\right]\theta_2(t)\,dt \\ &+ s\mathcal{L}_H[\overline{x}]_H(\overline{T}) + s\frac{\partial \Phi_H(\overline{T}, \overline{x}_1(\overline{T}), \overline{x}_2(\overline{T}))}{\partial t} \\ &+ \frac{\partial \Phi_H(\overline{T}, \overline{x}_1(\overline{T}), \overline{x}_2(\overline{T}))}{\partial x_1}(s\overline{x}_1'(\overline{T}) + \theta_1(\overline{T})) + \frac{\partial \Phi_H(\overline{T}, \overline{x}_1(\overline{T}), \overline{x}_2(\overline{T}))}{\partial x_2}(s\overline{x}_2'(\overline{T}) + \theta_2(\overline{T})) \\ &+ \left[\theta_1(t)\mathbb{I}^{1-\gamma,\beta,\lambda,g}_{\overline{T}-}\left(\frac{\partial \mathcal{L}_H[\overline{x}]_H(t)}{\partial {}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+}x_1}\cdot\frac{1}{g'(t)}\right)\right]_a^{\overline{T}} + \left[\theta_2(t)\mathbb{I}^{1-\gamma,\beta,\lambda,g}_{\overline{T}-}\left(\frac{\partial \mathcal{L}_H[\overline{x}]_H(t)}{\partial {}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+}x_2}\cdot\frac{1}{g'(t)}\right)\right]_a^{\overline{T}} = 0. \quad (15)\end{aligned}$$

By Equations (13) and (14) we get that

$$\begin{aligned}&\left[\frac{\partial \mathcal{L}_H[\overline{x}]_H(t)}{\partial x_2} + g'(t)\mathbb{D}^{\gamma,\beta,\lambda,g}_{\overline{T}-}\left(\frac{\partial \mathcal{L}_H[\overline{x}]_H(t)}{\partial {}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+}x_2}\cdot\frac{1}{g'(t)}\right)\right]\theta_2(t) \\ &\qquad = -\lambda(t)\frac{\partial \Lambda}{\partial x_2}(t, \overline{x}_1(t), \overline{x}_2(t))\theta_2(t) = \lambda(t)\frac{\partial \Lambda}{\partial x_1}(t, \overline{x}_1(t), \overline{x}_2(t))\theta_1(t),\end{aligned}$$

and replacing this condition in (15), we prove that

$$\int_a^{\overline{T}} \left[\frac{\partial \mathcal{L}_H[\overline{x}]_H(t)}{\partial x_1} + g'(t)\mathbb{D}_{\overline{T}-}^{\gamma,\beta,\lambda,g}\left(\frac{\partial \mathcal{L}_H[\overline{x}]_H(t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x_1} \cdot \frac{1}{g'(t)} \right) + \lambda(t)\frac{\partial \Lambda}{\partial x_1}(t,\overline{x}_1(t),\overline{x}_2(t)) \right]$$
$$\times \theta_1(t)\,dt + s\left[\mathcal{L}_H[\overline{x}]_H(t) + \frac{\partial \Phi_H(t,\overline{x}_1(t),\overline{x}_2(t))}{\partial t} \right.$$
$$\left. + \frac{\partial \Phi_H(t,\overline{x}_1(t),\overline{x}_2(t))}{\partial x_1}\overline{x}_1'(t) + \frac{\partial \Phi_H(t,\overline{x}_1(t),\overline{x}_2(t))}{\partial x_2}\overline{x}_2'(t) \right]$$
$$+ \theta_1(\overline{T})\left[\mathbb{I}_{\overline{T}-}^{1-\gamma,\beta,\lambda,g}\left(\frac{\partial \mathcal{L}_H[\overline{x}]_H(t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x_1} \cdot \frac{1}{g'(t)} \right) + \frac{\partial \Phi_H(t,\overline{x}_1(t),\overline{x}_2(t))}{\partial x_1} \right]_{t=\overline{T}}$$
$$+ \theta_2(\overline{T})\left[\mathbb{I}_{\overline{T}-}^{1-\gamma,\beta,\lambda,g}\left(\frac{\partial \mathcal{L}_H[\overline{x}]_H(t)}{\partial {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x_2} \cdot \frac{1}{g'(t)} \right) + \frac{\partial \Phi_H(t,\overline{x}_1(t),\overline{x}_2(t))}{\partial x_2} \right]_{t=\overline{T}} = 0.$$

Condition (11) and the three transversality conditions follow from the arbitrariness of θ and s. □

4.3. Infinite Horizon Problem

In this section, we consider the case when the phenomenon spreads over time, and so we are interested in optimizing functionals where the interval of integration is the infinite interval $[a,+\infty)$. Consider the set $\Delta_\infty := C^1([a,+\infty),\mathbb{R})$ and the functional

$$\mathcal{F}_\infty(x) = \int_a^{+\infty} \mathcal{L}_\infty(t, x(t), {}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g} x(t))\,dt, \tag{16}$$

where $\mathcal{L}_\infty : [a,+\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is a continuously differentiable function. In addition, we assume a fixed initial condition: $x(a) = X_a$, with $X_a \in \mathbb{R}$.

We remark that the improper integral may diverge to $-\infty$, and thus there may be more than one minimal candidate curve. We therefore have to decide on a new criterion as to which of these curves would be the optimal solution to the problem. To overcome this issue, different concepts of minimal curves for functionals with an unbounded range of integration are studied. Here we will follow the one presented in Ref. [41]: we say that $\overline{x}$ is a weakly minimal curve for $\mathcal{F}_\infty$ if

$$\lim_{T\to+\infty} \inf_{\overline{T}\geq T} \int_a^{\overline{T}} \mathcal{L}_\infty[x](t) - \mathcal{L}_\infty[\overline{x}](t)\,dt \geq 0,$$

for all $x \in \Delta_\infty$. For our next result, we will use the following notations: given $\delta \in \mathbb{R}\setminus\{0\}$, $\overline{T} \geq a$, $\overline{x}, \theta \in \Delta_\infty$,

$$\begin{cases} \Psi_1(\delta,\overline{T}) & := \displaystyle\int_a^{\overline{T}} \frac{\mathcal{L}_\infty[\overline{x}+\delta\theta](t) - \mathcal{L}_\infty[\overline{x}](t)}{\delta}\,dt, \\ \Psi_2(\delta,T) & := \displaystyle\inf_{\overline{T}\geq T}\int_a^{\overline{T}} \mathcal{L}_\infty[\overline{x}+\delta\theta](t) - \mathcal{L}_\infty[\overline{x}](t)\,dt, \\ \Psi_3(\delta) & := \displaystyle\lim_{T\to+\infty} \Psi_2(\delta,T). \end{cases}$$

Theorem 7. *Let $\overline{x} \in \Delta_\infty$ be a weakly minimal curve for functional* (16)*. Assume that*

- $\displaystyle\lim_{\delta\to 0}\frac{\Psi_2(\delta,T)}{\delta}$ *exists for all $T \geq a$;*
- $\displaystyle\lim_{T\to+\infty}\frac{\Psi_2(\delta,T)}{\delta}$ *exists uniformly for all $\delta \neq 0$;*

- *for all* $\overline{T} \geq a$ *and* $\delta \neq 0$, *there exists a sequence* $(\Psi_1(\delta, \overline{T}_n))_{n\in\mathbb{N}}$ *such that* $\lim_{n\to+\infty} \Psi_1(\delta, \overline{T}_n) = \inf_{\overline{T}\geq T} \Psi_1(\delta, \overline{T})$ *uniformly for all* $\delta \neq 0$.

Then, for all $\overline{T} \geq t \geq a$,

$$\frac{\partial \mathcal{L}_\infty[\overline{x}](t)}{\partial x} + g'(t)\mathbb{D}^{\gamma,\beta,\lambda,g}_{\overline{T}-}\left(\frac{\partial \mathcal{L}_\infty[\overline{x}](t)}{\partial {}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+}x} \cdot \frac{1}{g'(t)}\right) = 0. \tag{17}$$

Moreover,

$$\lim_{T\to+\infty} \inf_{\overline{T}\geq T} \mathbb{I}^{1-\gamma,\beta,\lambda,g}_{\overline{T}-}\left(\frac{\partial \mathcal{L}_\infty[\overline{x}](t)}{\partial {}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+}x} \cdot \frac{1}{g'(t)}\right)\Big|_{t=\overline{T}} = 0.$$

Proof. Let $\overline{x}(t) + \delta\theta(t)$ be a variation of $\overline{x}$, with $\theta(a) = 0$. Observe that $\Psi_3(\delta) \geq 0$, for all $\delta \neq 0$, and $\Psi_3(0) = 0$. Therefore, $\Psi_3'(0) = 0$ and so

$$\begin{aligned}
0 &= \lim_{\delta\to 0} \frac{\Psi_3(\delta)}{\delta} = \lim_{\delta\to 0} \lim_{T\to+\infty} \frac{\Psi_2(\delta, T)}{\delta} = \lim_{T\to+\infty} \lim_{\delta\to 0} \frac{\Psi_2(\delta, T)}{\delta} \\
&= \lim_{T\to+\infty} \lim_{\delta\to 0} \inf_{\overline{T}\geq T} \Psi_1(\delta, \overline{T}) = \lim_{T\to+\infty} \lim_{\delta\to 0} \lim_{n\to+\infty} \Psi_1(\delta, \overline{T}_n) \\
&= \lim_{T\to+\infty} \lim_{n\to+\infty} \lim_{\delta\to 0} \Psi_1(\delta, \overline{T}_n) = \lim_{T\to+\infty} \inf_{\overline{T}\geq T} \lim_{\delta\to 0} \Psi_1(\delta, \overline{T}) \\
&= \lim_{T\to+\infty} \inf_{\overline{T}\geq T} \lim_{\delta\to 0} \int_a^{\overline{T}} \frac{\mathcal{L}_\infty[\overline{x} + \delta\theta](t) - \mathcal{L}_\infty[\overline{x}](t)}{\delta}\, dt \\
&= \lim_{T\to+\infty} \inf_{\overline{T}\geq T} \int_a^{\overline{T}} \left[\frac{\partial \mathcal{L}_\infty[\overline{x}](t)}{\partial x}\theta(t) + \frac{\partial \mathcal{L}_\infty[\overline{x}](t)}{\partial {}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+}x} {}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+}\theta(t)\right] dt \\
&= \lim_{T\to+\infty} \inf_{\overline{T}\geq T} \left(\int_a^{\overline{T}} \left[\frac{\partial \mathcal{L}_\infty[\overline{x}](t)}{\partial x} + g'(t)\mathbb{D}^{\gamma,\beta,\lambda,g}_{\overline{T}-}\left(\frac{\partial \mathcal{L}_\infty[\overline{x}](t)}{\partial {}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+}x} \cdot \frac{1}{g'(t)}\right)\right]\theta(t)\, dt \right. \\
&\quad \left. + \left[\theta(t)\mathbb{I}^{1-\gamma,\beta,\lambda,g}_{\overline{T}-}\left(\frac{\partial \mathcal{L}_\infty[\overline{x}](t)}{\partial {}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+}x} \cdot \frac{1}{g'(t)}\right)\right]_{t=\overline{T}} \right) = 0.
\end{aligned}$$

If $\theta(\overline{T}) = 0$, then

$$\lim_{T\to+\infty} \inf_{\overline{T}\geq T} \left(\int_a^{\overline{T}} \left[\frac{\partial \mathcal{L}_\infty[\overline{x}](t)}{\partial x} + g'(t)\mathbb{D}^{\gamma,\beta,\lambda,g}_{\overline{T}-}\left(\frac{\partial \mathcal{L}_\infty[\overline{x}](t)}{\partial {}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+}x} \cdot \frac{1}{g'(t)}\right)\right]\theta(t)\, dt = 0\right.$$

and so, by ([42] Lemma 1), for all $t \geq a$ and for all $\overline{T} \geq t$, Equation (17) holds. Then, we can also conclude that

$$\lim_{T\to+\infty} \inf_{\overline{T}\geq T} \left[\mathbb{I}^{1-\gamma,\beta,\lambda,g}_{\overline{T}-}\left(\frac{\partial \mathcal{L}_\infty[\overline{x}](t)}{\partial {}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+}x} \cdot \frac{1}{g'(t)}\right)\right]_{t=\overline{T}} = 0.$$

□

4.4. The Herglotz Problem

The Herglotz variational problem [43] is a generalization of the usual variational problem, but instead of being given by an integral, the action functional is described by a differential equation. In this way, it can be more realistic to describe certain physical processes and can be applied to solve non-conservative problems for which the classical variational principle is not applicable [44,45]. It was formulated in the following way:

Determine the curves $x, z \in C^1([a,b],\mathbb{R})$ for which $z(b)$ attains an extreme value, where x and z are related by the differential equation

$$\begin{cases} z'(t) = \mathcal{L}(t, x(t), x'(t), z(t)), \quad t \in [a,b], \\ x(a) = X_a,\ z(a) = Z_a, \quad X_a, Z_a \in \mathbb{R}. \end{cases}$$

In case the Lagrange function L does not depend on z', this variational problem reduces to the classical one: Find an extreme value for the functional

$$\mathcal{F}(x) = \int_a^b \mathcal{L}(t, x(t), x'(t))\, dt, \quad x(a) = X_a.$$

By replacing the first order derivative by the generalized left Caputo fractional derivative, we obtain an extension of the classical Herglotz problem. The fractional Herglotz problem is formulated in the following way. Determine the trajectories $x, z \in C^1([a,b],\mathbb{R})$ and $T \in [a,b]$ for which $z(T)$ attains a minimum value, where (x,z) must obey the system

$$\begin{cases} z'(t) = \mathcal{L}_{He}(t, x(t), {}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+} x(t), z(t)), \quad t \in [a,b], \\ x(a) = X_a,\ z(a) = Z_a, \quad X_a, Z_a \in \mathbb{R}. \end{cases} \tag{18}$$

Observe that the curve z does not actually depend on time t but also on the trajectory x, $z = z[t,x]$. The Lagrange function $\mathcal{L}_{He} : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$ is of class C^1 and we introduce the abbreviation $[x,z](t) := (t, x(t), {}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+} x(t), z(t))$.

Theorem 8. *Let $(\overline{x}, \overline{z}, \overline{T}) \in C^1([a,b],\mathbb{R}) \times C^1([a,b],\mathbb{R}) \times [a,b]$ be a solution for the fractional Herglotz variational problem. Define the function $\chi : [a,b] \to \mathbb{R}$ by*

$$\chi(t) = \exp\left(-\int_a^t \frac{\partial \mathcal{L}_{He}[\overline{x},\overline{z}](\tau)}{\partial z}\, d\tau \right).$$

Then, for all $t \in [a, \overline{T}]$:

$$\chi(t) \frac{\partial \mathcal{L}[\overline{x},\overline{z}](t)}{\partial x} + g'(t) \mathbb{D}^{\gamma,\beta,\lambda,g}_{\overline{T}-} \left(\chi(t) \frac{\partial \mathcal{L}[\overline{x},\overline{z}](t)}{\partial {}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+} x} \cdot \frac{1}{g'(t)} \right) = 0.$$

In addition, at $t = \overline{T}$,

$$\mathbb{I}^{1-\gamma,\beta,\lambda,g}_{\overline{T}-} \left(\chi(t) \frac{\partial \mathcal{L}[\overline{x},\overline{z}](t)}{\partial {}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+} x} \cdot \frac{1}{g'(t)} \right) = 0.$$

If $\overline{T} < b$ then $\mathcal{L}_{He}[\overline{x},\overline{z}](\overline{T}) = 0$.

Proof. The variation along $\overline{x}$ is denoted by $\overline{x}(t) + \delta\theta(t)$, where $\theta(a) = 0$. The variation of $\overline{z}$, along the direction of θ, is given by

$$\eta(t) = \frac{d\overline{z}[\overline{x} + \delta\theta, t]}{d\delta}\Big|_{\delta=0}.$$

Since these variations must satisfy the differential Equation (18), we get that for all $t \in [a,b]$,

$$\frac{d\overline{z}[\overline{x} + \delta\theta, t]}{dt} = \mathcal{L}_{He}(t, \overline{x} + \delta\theta, {}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+} \overline{x} + \delta {}^C\mathbb{D}^{\gamma,\beta,\lambda,g}_{a+} \theta, \overline{z}[\overline{x} + \delta\theta, t]).$$

Observe that

$$\begin{aligned}
\eta'(t) &= \frac{d}{dt}\frac{d\bar{z}[\bar{x}+\delta\theta,t]}{d\delta}\Big|_{\delta=0} = \frac{d}{d\delta}\frac{d\bar{z}[\bar{x}+\delta\theta,t]}{dt}\Big|_{\delta=0} \\
&= \frac{d}{d\delta}\mathcal{L}_{He}(t,\bar{x}+\delta\theta,{}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}\bar{x}+\delta^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}\theta,\bar{z}[\bar{x}+\delta\theta,t])\Big|_{\delta=0} \\
&= \frac{\partial\mathcal{L}_{He}[\bar{x},\bar{z}](t)}{\partial x}\theta(t) + \frac{\partial\mathcal{L}_{He}[\bar{x},\bar{z}](t)}{\partial^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}x}{}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}\theta(t) + \frac{\partial\mathcal{L}_{He}[\bar{x},\bar{z}](t)}{\partial z}\eta(t),
\end{aligned}$$

and so we obtain the differential equation

$$\eta'(t) - \frac{\partial\mathcal{L}_{He}[\bar{x},\bar{z}](t)}{\partial z}\eta(t) = \frac{\partial\mathcal{L}_{He}[\bar{x},\bar{z}](t)}{\partial x}\theta(t) + \frac{\partial\mathcal{L}_{He}[\bar{x},\bar{z}](t)}{\partial^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}x}{}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}\theta(t),$$

whose solution satisfies

$$\eta(\overline{T})\chi(\overline{T}) - \eta(a)\chi(a) = \int_a^{\overline{T}}\left[\frac{\partial\mathcal{L}_{He}[\bar{x},\bar{z}](t)}{\partial x}\theta(t) + \frac{\partial\mathcal{L}_{He}[\bar{x},\bar{z}](t)}{\partial^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}x}{}^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}\theta(t)\right]\chi(t)\,dt.$$

Since $\eta(a) = 0$ and $\eta(\overline{T}) = 0$, and integrating by parts, we get

$$\begin{aligned}
\int_a^{\overline{T}}\left[\chi(t)\frac{\partial\mathcal{L}[\bar{x},\bar{z}](t)}{\partial x} + g'(t)\mathbb{D}_{\overline{T}-}^{\gamma,\beta,\lambda,g}\left(\chi(t)\frac{\partial\mathcal{L}[\bar{x},\bar{z}](t)}{\partial^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}x}\cdot\frac{1}{g'(t)}\right)\right]\theta(t)\,dt \\
+\left[\theta(t)\mathbb{I}_{\overline{T}-}^{1-\gamma,\beta,\lambda,g}\left(\chi(t)\frac{\partial\mathcal{L}_{He}[\bar{x}](t)}{\partial^C\mathbb{D}_{a+}^{\gamma,\beta,\lambda,g}x}\cdot\frac{1}{g'(t)}\right)\right]_a^{\overline{T}} = 0.
\end{aligned}$$

The rest of the proof follows from the arbitrariness of θ. □

5. Conclusions and Future Work

In this work, we considered some calculus of variation problems, where the dynamics of the state function is ruled by a generalized fractional derivative. This fractional operator is a generalization of some well-known ones, and therefore some particular cases of our new results may be useful in different fields. The necessary conditions for optimizing a certain class of functionals are described as fractional differential equations, and, in general, no analytic method exists to solve them. So, an important open problem that arises is how to develop numerical tools to solve such fractional equations.

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