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Positive bidiagonal factorization of tetradiagonal Hessenberg matrices

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Recently, a spectral Favard theorem was presented for bounded banded lower Hessenberg matrices that possess a positive bidiagonal factorization. The paper establishes conditions, expressed in terms of continued fractions, under which an oscillatory tetradiagonal Hessenberg matrix can have such a positive bidiagonal factorization. Oscillatory tetradiagonal Toeplitz matrices are examined as a case study of matrices that admit a positive bidiagonal factorization. Furthermore, the paper proves that oscillatory banded Hessenberg matrices are organized in rays, where the origin of the ray does not have a positive bidiagonal factorization, but all the interior points of the ray do have such a positive bidiagonal factorization.

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Continued fractions
Gauss–Borel factorization
Bidiagonal factorization
Oscillatory retracted matrices

Contents

1. Introduction

In this paper, we will study for the tetradiagonal Hessenberg matrix of the form

$$T=\begin{bmatrix} c_0 & 1 & 0 & \cdots & & \\ b_1 & c_1 & 1 & \ddots & & \\ a_2 & b_2 & c_2 & 1 & & \\ 0 & a_3 & b_3 & c_3 & 1 & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{1}$$

where we assume that $a_n > 0$, whether it is possible or not to find a *positive bidiagonal factorization* (PBF) given by the expression:

$$T = L_1 L_2 U, \tag{2}$$

where L_1, L_2, and U are bidiagonal matrices.

$$L_1=\begin{bmatrix} 1 & 0 & 0 & \cdots \\ \alpha_2 & 1 & 0 & \cdots \\ 0 & \alpha_5 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}, \quad L_2=\begin{bmatrix} 1 & 0 & 0 & \cdots \\ \alpha_3 & 1 & 0 & \cdots \\ 0 & \alpha_6 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}, \quad U=\begin{bmatrix} \alpha_1 & 1 & 0 & \cdots \\ 0 & \alpha_4 & 1 & \ddots \\ 0 & 0 & \alpha_7 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{3}$$

with the fulfillment of the following positivity requirement

$$\alpha_j > 0, \qquad j \in \mathbb{N}.$$

In [8], it was demonstrated that this factorization is sufficient for a Favard theorem for bounded banded Hessenberg semi-infinite matrices (with $p+2$ diagonals), i.e., for the existence of certain positive measures. These measures ensure that the recursion polynomials are multiple orthogonal polynomials, and the Hessenberg matrix serves as the recursion matrix for this sequence of multiple orthogonal polynomials. We extended this Favard theorem to bounded banded matrices with a positive bidiagonal factorization. In this context, we encountered a mixed multiple orthogonal polynomials [10].

Additionally, we would like to mention [16], where the authors present an important application of this factorization in obtaining stochastic bidiagonal factorizations of stochastic Hessenberg matrices.

Finite truncations of matrices with this PBF result in oscillatory matrices. In this paper, we will be dealing with totally nonnegative matrices that are oscillatory matrices. Therefore, we need to introduce some definitions and properties briefly.

Totally nonnegative (TN) matrices are those with all their minors nonnegative [13,15], and the set of nonsingular TN matrices is denoted by InTN. Oscillatory matrices [15] are totally nonnegative, irreducible [17], and nonsingular matrices. It is worth noting that the set of oscillatory matrices is denoted by IITN (irreducible invertible totally nonnegative) in [13]. An oscillatory matrix A can also be defined as a totally nonnegative matrix such that there exists an integer n for which A^n is totally positive (i.e., all minors are positive). The Cauchy–Binet Theorem implies that these sets of matrices are closed under the usual matrix product. Thus, according to [13, Theorem 1.1.2], the product of matrices in InTN is again InTN (similar statements hold for TN or oscillatory matrices). We have an important result known as the Gantmacher–Krein Criterion:

Theorem 1.1 *(Gantmacher–Krein Criterion). [15, Chapter 2, Theorem 10] A totally nonnegative matrix A is oscillatory if and only if it is nonsingular and the elements on the first subdiagonal and first superdiagonal are positive.*

Let us discuss the connection of oscillatory matrices with the standard case, i.e., tridiagonal semi-infinite matrices, which are commonly referred to in the literature as Jacobi matrices.

Definition 1.2. Jacobi matrices are tridiagonal real matrices of the form

$$J := \begin{bmatrix} m_0 & 1 & 0 & \cdots & \\ \ell_1 & m_1 & 1 & \ddots & \\ 0 & \ell_2 & m_2 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ & & & & \end{bmatrix}, \tag{4}$$

with $\ell_j > 0$, $j = 1, 2, \dots$. Given such matrix J we denote by $J^{[N]}$ the finite truncation, i.e., the leading principal submatrix defined by

$$J^{[N]} := \begin{bmatrix} m_0 & 1 & 0 & \cdots & \cdots & 0 \\ \ell_1 & m_1 & 1 & \ddots & & \vdots \\ 0 & \ell_2 & m_2 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \ell_N & m_N \end{bmatrix} \in \mathbb{R}^{(N+1)\times(N+1)}, \qquad \Delta_N := \det J^{[N]}, \tag{5}$$

and we let $J^{[N,k]}$ stand for the principal submatrices defined by

$$J^{[N,k]} := \begin{bmatrix} m_k & 1 & 0 & \cdots & \cdots & 0 \\ \ell_{k+1} & m_{k+1} & 1 & \ddots & & \vdots \\ 0 & \ell_{k+2} & m_{k+2} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \ell_N & m_N \end{bmatrix}$$

$$\in \mathbb{R}^{(N+1-k)\times(N+1-k)}, \ \Delta_{N,k} := \det J^{[N,k]}. \tag{6}$$

Regarding oscillatory Jacobi matrices, we have:

Theorem 1.3 *(Oscillatory Jacobi matrices). [14, Chapter XIII, §9] and [15, Chapter 2, Theorem 11]. A tridiagonal matrix is oscillatory if and only if,*

i) *The matrix entries of the first subdiagonal and first superdiagonal are positive.*
ii) *All leading principal minors are positive.*

If the semi-infinite matrix J is bounded, i.e., $\|J\|_\infty < \infty$, all the possible eigenvalues of the submatrices $J^{[N]}$ belong to the disk $D(0, \|J\|_\infty)$. As all the eigenvalues are real, let us consider those that are negative, and let b be the supremum of the absolute values of all negative eigenvalues. Notice that $b \leqslant \|J\|_\infty$.

Theorem 1.4. *For $s > b$, the matrix $J_s = J + sI$ is oscillatory.*

Proof. Take $s > b$, then J_s has the eigenvalues of its leading principal submatrices $J_s^{[N]} = J^{[N]} + sI_{N+1}$ all positive. The corresponding characteristic polynomials are $P_{N+1}(x - s) = \det(xI_{N+1} - J_s^{[N]})$, so that $\det J_s^{[N]} = (-1)^{N+1} P_{N+1}(-s)$. As $-s$ is a lower bound for any possible zero of this monic polynomial, we have that $(-1)^{N+1} P_{N+1}(-s) > 0$. Hence, the leading principal minors of J_s are all positive, and the entries on the subdiagonal and superdiagonal are positive. Thus, we conclude, according to Theorem 1.3, that J_s is an oscillatory matrix. □

A very important consequence of the fact that there exists a positive s such that $J + sI$ is oscillatory is that all eigenvalues are simple, and the polynomial $P_{N+1}(x - s)$ interlaces P_N and $P^{(1)}_{N+1}$. In other words, the characteristic polynomial of the oscillatory matrix $J_s^{[N]}$, denoted as $P_{N+1}(x - s)$, interlaces the characteristic polynomials of its submatrices $J_s^{[N]}(1) = J_s^{[N,1]}$ (given by $P^{(1)}_{N+1}(x - s)$) and $J_s^{[N]}(N + 1) = J_s^{[N-1]}$ (given by $P_N(x - s)$).

As a consequence, we can deduce the positivity of the Christoffel coefficients from the oscillatory character of $J_s^{[N]}$. This allows us to apply the spectral Favard theorem, as presented in [25, §4.1], to bounded Jacobi matrices that are oscillatory up to an appropriate shift. Consequently, all the interlacing properties follow immediately.

Now, let's demonstrate that for Jacobi matrices, the positive bidiagonal factorization (PBF) and the oscillatory properties are equivalent.

Proposition 1.5. *A Jacobi matrix is oscillatory if and only if it admits a PBF.*

Proof. Let us assume that the Jacobi matrix $J^{[N,1]}$ is oscillatory. Then, the Gauss–Borel factorization of $J^{[N,1]}$, given by

$$\begin{bmatrix} m_1 & 1 & 0 & \cdots & & 0 \\ \ell_2 & m_2 & 1 & \ddots & & \vdots \\ 0 & \ell_3 & m_3 & 1 & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & & 0 & \ell_N & m_N \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ \beta_2 & \ddots & \ddots & & \vdots \\ 0 & \beta_4 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \beta_{2N-2} & 1 \end{bmatrix} \begin{bmatrix} \beta_1 & 1 & 0 & \cdots & 0 \\ 0 & \beta_3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \beta_5 & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & & 0 & \beta_{2N-1} \end{bmatrix},$$

leads to $m_1 = \beta_1$ as well as $m_n = \beta_{2n-2} + \beta_{2n-1}$ and $\ell_n = \beta_{2n-2}\beta_{2n-3}$, with $\beta_0 \coloneqq 0$. Hence, as $\ell_n > 0$, $n \in \{2, 3, \dots\}$, we get that $\beta_n > 0$ for $n \in \mathbb{N}$.

If the Jacobi matrix $J^{[N,1]}$ admits a PBF, we deduce that it is oscillatory from the Gantmacher–Krein Criterion Theorem 1.1. □

The structure of the paper is as follows. Section 2 is devoted to introducing truncations and continued fractions needed for the discussion. Next, in §3, we prove the existence of the PBF for tetradiagonal matrices in the finite case, using positive finite continued fractions. We then extend this PBF to the semi-infinite case, where it is shown to happen when certain nonnegative infinite continued fraction is indeed positive. In §4, we discuss oscillatory tetradiagonal Toeplitz–Hessenberg matrices as a case study of matrices that admit a PBF. Finally, in §5, we demonstrate that from any given oscillatory tetradiagonal matrix, we can construct tetradiagonal matrices with a PBF in several ways. Additionally, we find that oscillatory matrices are organized in rays, where the origin of the ray is an oscillatory matrix that doesn't have a PBF, and all the interior points of the ray are PBF matrices. We also show that for any PBF tetradiagonal matrix, there is a retraction that is oscillatory but without a PBF, and vice versa.

2. Factorization properties

We now discuss some aspects of the Gauss–Borel factorization and PBF of oscillatory tetradiagonal matrices. For more on Gauss–Borel factorization and orthogonal polynomials, see for example [22] and references therein.

First, let us introduce some convenient notation for the tetradiagonal case. We denote by $T^{[N]} = T[\{0,1,\dots,N\}] \in \mathbb{R}^{(N+1)\times(N+1)}$ the $(N+1)$-th leading principal submatrix of the banded Hessenberg matrix T:

$$T^{[N]} := \begin{bmatrix} c_0 & 1 & 0 & \cdots & & & 0 \\ b_1 & c_1 & 1 & \ddots & & & \vdots \\ a_2 & b_2 & c_2 & 1 & \ddots & & \vdots \\ 0 & a_3 & b_3 & c_3 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & a_{N-1} & b_{N-1} & c_{N-1} & 1 \\ 0 & \cdots & \cdots & 0 & a_N & b_N & c_N \end{bmatrix}, \qquad \delta^{[N]} := \det T^{[N]}. \tag{7}$$

Further truncations are

$$T^{[N,k]} := \begin{bmatrix} c_k & 1 & 0 & \cdots & & & 0 \\ b_{k+1} & c_{k+1} & 1 & \ddots & & & \vdots \\ a_{k+2} & b_{k+2} & c_{k+2} & 1 & \ddots & & \vdots \\ 0 & a_{k+3} & b_{k+3} & c_{k+3} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & a_{N-1} & b_{N-1} & c_{N-1} & 1 \\ 0 & \cdots & \cdots & 0 & a_N & b_N & c_N \end{bmatrix}$$

$$\in \mathbb{R}^{(N+1-k)\times(N+1-k)}, \quad k \in \{0,1,\dots,N\},$$

where it is worth noting that $T^{[N,N+1]} := 1$ and $T^{[N]} = T^{[N,0]}$.

The Gauss–Borel factorization of the leading principal submatrices $T^{[N]}$ in (7) is

$$T^{[N]} = L^{[N]}U^{[N]}, \quad L^{[N]} := \begin{bmatrix} 1 & 0 & \cdots & & & 0 \\ m_1 & 1 & \ddots & & & \vdots \\ \ell_2 & m_2 & 1 & \ddots & & \\ 0 & \ell_3 & m_3 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \ell_N & m_N & 1 \end{bmatrix},$$

$$U^{[N]} := \begin{bmatrix} \alpha_1 & 1 & 0 & \cdots & 0 \\ 0 & \alpha_4 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \alpha_7 & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \alpha_{3N+1} \end{bmatrix}. \tag{8}$$

Proposition 2.1. *The Gauss–Borel factorization* (8) *exists if and only if all leading principal minors* $\delta^{[n]}$, $n \in \{0,1,\dots,N\}$, *of* $T^{[N]}$ *are nonzero, i.e.,* $\delta^{[n]} \neq 0$ *for* $n \in \{0,1,\dots,N\}$. *For* $n \in \mathbb{N}$, *the following expressions for the coefficients hold:*

$$\ell_{n+1} = \frac{a_{n+1}\delta^{[n-2]}}{\delta^{[n-1]}}, \qquad m_n = c_n - \frac{\delta^{[n]}}{\delta^{[n-1]}}, \qquad \alpha_{3n-2} = \frac{\delta^{[n-1]}}{\delta^{[n-2]}}, \tag{9}$$

where $\delta^{[-1]} = 1$ *and* $a_1 = 0$. *The recurrence relation for the determinants is given by:*

$$\delta^{[n]} = a_n\delta^{[n-3]} - b_n\delta^{[n-2]} + c_n\delta^{[n-1]}, \tag{10}$$

and it is satisfied for $n \in \{0,1,\dots,N\}$.

Proof. Notice that $\delta^{[N]} = \det T^{[N]} = \det U^{[N]} = \alpha_1\alpha_4\cdots\alpha_{3N+1}$. Hence, we get $\alpha_{3N+1} = \frac{\delta^{[N]}}{\delta^{[N-1]}}$. From the last row of the LU factorization, we get:

$$a_N = \ell_N\alpha_{3N-5}, \qquad b_N = \ell_N + m_N\alpha_{3N-2}, \qquad c_N = m_N + \alpha_{3N+1},$$

which leads to:

$$\ell_N = \frac{\delta^{[N-3]}}{\delta^{[N-2]}}a_N, \qquad m_N = c_N - \frac{\delta^{[N]}}{\delta^{[N-1]}},$$

and:

$$b_N - a_N\frac{\delta^{[N-3]}}{\delta^{[N-2]}} - \frac{\delta^{[N-1]}}{\delta^{[N-2]}}\left(c_N - \frac{\delta^{[N]}}{\delta^{[N-1]}}\right) = 0,$$

which confirms the recurrence relation (10). □

Note that, in this proof, we obtained (10) by expanding the determinant $\delta^{[N]}$ along the last row.

Proposition 2.2. *Assuming that $T^{[N]}$ given in (7) is an oscillatory matrix, we can deduce that:*

$$a_n, b_n, c_n > 0 \quad \textit{for } n \in \{1, 2, \dots, N\}.$$

Proof. According to the Gantmacher–Krein Criterion, Theorem 1.1, [15, II.7 Theorem 10], we have $b_n > 0$ for $n \in \{1, 2, \dots, N\}$. Moreover, since $T^{[N]}$ is an invertible totally nonnegative matrix (InTN) according to [13, page 50, Chapter 2], we also deduce that c_n is positive for the same range of n. □

We introduce some auxiliary submatrices that will be instrumental in the following developments.

Definition 2.3 *(Auxiliary submatrices).* Given the lower triangular factor $L^{[N]}$, determined by the Gauss–Borel factorization (8), we consider its complementary submatrix by deleting the first row and last column. This submatrix is called the auxiliary Jacobi matrix and is denoted as $J^{[N,1]} = L^{[N]}(\{1\}, \{N+1\}) \in \mathbb{R}^{N\times N}$, following the definition in (6) with $k = 1$.

Additionally, we define $T_1^{[N]} = T(\{1\}, \{N+1\}) \in \mathbb{R}^{N\times N}$ as the complementary submatrix obtained by removing the first row and last column of $T^{[N]}$. Thus, $T_1^{[N]}$ is given by:

$$T_1^{[N]} := \begin{bmatrix} b_1 & c_1 & 1 & 0 & \cdots & & 0 \\ a_2 & b_2 & c_2 & 1 & \ddots & & \vdots \\ 0 & a_3 & b_3 & c_3 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 1 \\ \vdots & & & \ddots & a_{N-1} & b_{N-1} & c_{N-1} \\ 0 & \cdots & \cdots & \cdots & 0 & a_N & b_N \end{bmatrix}, \quad \delta_1^{[N]} := \det T_1^{[N]}. \tag{11}$$

Furthermore, we define other auxiliary complementary submatrices as follows:

$$T_1^{[N,k]} := \begin{bmatrix} b_{k+1} & c_{k+1} & 1 & 0 & \cdots & & 0 \\ a_{k+2} & b_{k+2} & c_{k+2} & 1 & \ddots & & \vdots \\ 0 & a_{k+3} & b_{k+3} & c_{k+3} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 1 \\ \vdots & & & \ddots & a_{N-1} & b_{N-1} & c_{N-1} \\ 0 & \cdots & \cdots & \cdots & 0 & a_N & b_N \end{bmatrix}, \quad \delta_1^{[N,k]} := \det T_1^{[N,k]}, \tag{12}$$

where it should be noted that $T_1^{[N,0]} = T_1^{[N]}$ and $\delta_1^{[N,0]} = \delta_1^{[N]}$.

Finally, we define the upper bidiagonal matrix $U^{[N-1,k]}$ as:

$$U^{[N-1,k]}=\begin{bmatrix}\alpha_{3k+1} & 1 & 0 & \cdots & 0\\ 0 & \alpha_{3k+4} & \ddots & \ddots & \vdots\\ \vdots & \ddots & \alpha_{3k+7} & \ddots & 0\\ \vdots & & \ddots & \ddots & 1\\ 0 & \cdots & \cdots & 0 & \alpha_{3N-2}\end{bmatrix}.$$

A set of finite continued fractions will be needed in the subsequent analysis. For detailed information on continued fractions, interested readers can refer to [19,21,26].

Definition 2.4 *(Finite continued fractions).* We introduce the following expressions for finite continued fractions:

$$\mathcal{K}[n,k]:=m_k-\cfrac{\ell_{k+1}}{m_{k+1}-\cfrac{\ell_{k+2}}{m_{k+2}-\cfrac{\ell_{k+3}}{\ddots\, m_{n-1}-\cfrac{\ell_n}{m_n}}}},\qquad n\in\{k+1,k+2,\dots\},$$

$$\mathcal{K}[k+1,k]:=m_k.$$

We now present a collection of results regarding factorizations, determinants, and relations of the introduced matrices. These results will be relevant in the subsequent developments.

Theorem 2.5 *(Determinants and continued fractions). Assuming that $T^{[N]}$ as defined in (7) is an oscillatory matrix, the following results hold:*

- i) *The triangular factors in the Gauss–Borel factorization (8) of $T^{[N]}$, namely $L^{[N]}$ and $U^{[N]}$, belong to the set of invertible totally nonnegative matrices.*
- ii) *The matrix entries of the triangular factors in the Gauss–Borel factorization of $T^{[N]}$ are positive: $\ell_2,\dots,\ell_N,m_1,\dots,m_N,\alpha_1,\alpha_4,\dots,\alpha_{3N-2}>0$.*
- iii) *For $k\in\mathbb{N}$, the recurrence relation*

$$D(n+1)=m_{k+n}D(n)-\ell_{k+n}D(n-1),\qquad n\in\mathbb{N},\tag{13}$$

for the initial conditions $D(0)=1, D(1)=m_k$ has a solution $D(n)=\Delta_{k+n-1,k}$, and for the initial conditions $D(0)=0$ and $D(1)=1$ has a solution $D(n)=\Delta_{k+n-1,k+1}$. The determinants $\Delta_{N,k}$ were defined in (6).

iv) *The ratio of consecutive determinants is bounded as follows:*

$$\frac{a_{n+2}}{b_{n+2}} < \frac{\delta^{[n]}}{\delta^{[n-1]}} < c_n, \qquad \frac{\ell_{n+1}}{m_{n+1}} < \frac{\Delta_{n,1}}{\Delta_{n-1,1}} < m_n.$$

v) *For $k \in \mathbb{N}$, the continued fraction given in Definition 2.4 is the ratio of consecutive determinants defined in Equation (6):*

$$\mathcal{K}[n,k] = \frac{\Delta_{n,k}}{\Delta_{n,k+1}}.$$

vi) *For $k = 1$, the determinants in (6) are positive, i.e., $\Delta_{n,1} > 0$.*

Proof. i) As $T^{[N]}$ is totally nonnegative, we know it has a Gauss–Borel factorization with totally nonnegative factors, as shown in [13, Theorem 2.4.1].

ii) For $n \in \{2,3,\dots,N\}$, we have $a_n = \ell_n \alpha_{3n-5}$ and, consequently, since $a_n \neq 0$, we deduce that $\ell_2,\dots,\ell_N,\alpha_1,\alpha_4,\dots,\alpha_{3N-2} > 0$. Moreover,

$$\begin{bmatrix} m_n & 1 \\ \ell_{n+1} & m_{n+1} \end{bmatrix} \geqslant 0,$$

and as $\ell_{n+1} > 0$, we deduce that $m_n m_{n+1} \neq 0$, and thus all $m_1,\dots,m_N > 0$.

iii) Expand the determinant $\Delta_{n,k}$ along the last row. One can verify that the initial conditions lead to the sequence of determinants.

iv) From (9) and $m_n > 0$, we get $b_n\delta^{[n-2]} > a_n\delta^{[n-3]}$, $c_n\delta^{[n-1]} > \delta^{[n]}$, and the first inequality follows. For the second inequality, we use the proof of Proposition 1.5. From the factorization, we get $\Delta_{n,1} = \beta_1 \cdots \beta_{2n-1}$ and, consequently, $\beta_{2n-1} = \dfrac{\Delta_{n,1}}{\Delta_{n-1,1}}$ and $\beta_{2n-2} = m_n - \dfrac{\Delta_{n,1}}{\Delta_{n-1,1}}$. As $\beta_n > 0$, $n \in \mathbb{N}$, we deduce that $m_n > \dfrac{\Delta_{n,1}}{\Delta_{n-1,1}}$. As for the oscillatory case, where we require $\Delta_n > 0$, the recursion relation (13), i.e., $\Delta_{n,1} = m_n\Delta_{n-1,1} - \ell_n\Delta_{n-2,1}$, implies that $m_n\Delta_{n-1,1} > \ell_n\Delta_{n-2,1}$, and the lower bound follows immediately.

v) Use the Euler–Wallis theorem for continued fractions, as shown in, for example, [12, Theorem 9.2].

vi) The first two determinants are positive, and then we apply induction. Let us assume that $\Delta_{n-1,1} > 0$ and that $\Delta_{n,1} = 0$. Then, for $k = 0$, Equation (13) implies that $\Delta_{n+1,1} = -\ell_{n+1}\Delta_{n-1,1} < 0$, which is in contradiction with the fact that $\Delta_{n+1,1} \geqslant 0$. □

Theorem 2.6 *(Factorizations and oscillatory matrices). For the submatrices of $T^{[N]}$ and their determinants introduced in Definition 2.3, the following results hold:*

i) *The auxiliary Jacobi matrix $J^{[N,1]}$ is oscillatory.*

ii) *The following factorizations are valid:*

$$T_1^{[N]} = J^{[N,1]}U^{[N-1]}, \tag{14}$$

$$T_1^{[N,k]} - m_{k+1}E_{1,1} = J^{[N,k+1]}U^{[N-1,k]}. \tag{15}$$

Moreover, $\delta_1^{[N]} > 0$*, and the following relation between determinants holds:*

$$\Delta_{N,1} = \frac{\delta_1^{[N]}}{\delta^{[N-1]}}, \tag{16}$$

$$\Delta_{N,k+1} = \alpha_1 \cdots \alpha_{3k-2}\frac{\delta_1^{[N,k]} - m_{k+1}\delta_1^{[N,k+1]}}{\delta^{[N-1]}}. \tag{17}$$

(Recall that $\Delta^{[N,k]} := \det J^{[N,k]}$*,* $\delta_1^{[N]} := \det T_1^{[N]}$*, and* $\delta_1^{[N,k]} := \det T_1^{[N,k]}$*.)*

iii) *The submatrix* $T_1^{[N]}$ *is oscillatory.*

iv) *The submatrices* $J^{[N,k+1]}$ *and* $T_1^{[N,k]}$ *are also oscillatory. In particular,* $\Delta_{N,k+1}, \delta_1^{[N,k]} > 0$*.*

v) *The following relations are satisfied:*

$$\Delta_{N,2}\delta^{[N-1]} = c_0\delta_1^{[N,1]} - a_2\delta_1^{[N,2]},$$

$$\frac{\Delta_{N,1}}{\Delta_{N,2}} = \frac{\delta_1^{[N]}}{c_0\delta_1^{[N,1]} - a_2\delta_1^{[N,2]}}. \tag{18}$$

vi) *The recursion relation in* k *is satisfied:*

$$\Delta_{N,k+1} = m_{k+1}\Delta_{N,k+2} - \ell_{k+2}\Delta_{N,k+3}. \tag{19}$$

Proof. i) According to Theorem 1.3, see [14, Chapter XIII,§9] and [15, Chapter 2, Theorem 11], the Jacobi matrix $J^{[N,1]}$ is oscillatory if and only if,

(a) The matrix entries $\ell_2, \dots, \ell_N$ are positive.

(b) All leading principal minors $\Delta_{n,1}$ are positive.

As we have seen in previous points, both requirements are satisfied.

ii) Equations (14) and (15) follow directly from the Gauss–Borel factorization of $T^{[N]}$. Taking determinants and expanding the determinant along the first row we get (17). From Equation (14) we conclude that $\alpha_1\alpha_4\cdots\alpha_{3n-2}\Delta_n = \det T_1^{[n]}$. As previously said all $\alpha_1, \alpha_4, \dots, \alpha_{3n-2} > 0$ and $\Delta_{n,1} > 0$. Therefore, $\delta_1^{[N]} \neq 0$.

iii) Given that the matrix $T_1^{[N]}$ belongs to InTN with $a_n, b_n, c_n > 0$, see Proposition 2.2, the Gantmacher–Krein Criterion, Theorem 1.1, leads to the oscillatory character of this submatrix.

iv) If a matrix A is oscillatory then so is any submatrix $A[\boldsymbol{\alpha}]$ for any contiguous subset of indexes $\boldsymbol{\alpha}$, see [15, Chapter 2, §7] and [13, Corollary 2.6.7]. Then, $J^{[N,k+1]} =$

$J^{[N]}(\{k+1,\dots,N\})$ and $T_1^{[N,k]} = T_1^{[N]}(\{k+1,\dots,N\})$ are oscillatory and, consequently, $\Delta_{N,k+1} = \det J^{[N]}(\{k+1,\dots,N\}) > 0$ and $\delta_1^{[N,k]} = \det T_1^{[N]}(\{k+1,\dots,N\}) > 0$.

v) Put $k = 1$ in (17) and recall that $\alpha_1 = c_0$ and $\alpha_1 \ell_2 = a_2$. For Equation (18) use (16).

vi) Expand the determinants along the first row. □

A set of convergent infinite continued fractions is essential in what follows.

Definition 2.7 *(Infinite continued fraction and tails).* We introduce the following infinite continued fraction

$$\mathcal{K}[1] := m_1 - \cfrac{\ell_2}{m_2 - \cfrac{\ell_3}{m_3 - \ddots}} \tag{20}$$

and its tails

$$\mathcal{K}[k+1] := m_{k+1} - \cfrac{\ell_{k+2}}{m_{k+2} - \cfrac{\ell_{k+3}}{m_{k+3} - \ddots}}, \qquad k \in \mathbb{N}.$$

Corollary 2.8. *The infinite continued fraction in (20) can be computed as the following large N limit ratio*

$$\mathcal{K}[1] = \lim_{N\to\infty} \frac{\delta_1^{[N]}}{c_0\delta_1^{[N,1]} - a_2\delta_1^{[N,2]}}, \tag{21}$$

of determinants given in (11) and (12).

Proof. This is a direct consequence of (18). □

Now, an important result follows regarding the behavior of these infinite continued fractions.

Theorem 2.9 *(Infinite continued fractions). For the infinite continued fractions $\mathcal{K}[1]$ given in (20), we observe the following properties:*

i) *For $k \in \mathbb{N}_0$, the sequences $\{\mathcal{K}[n,k]\}_{n=k+1}^{\infty}$ of the finite continued fractions given in Definition 2.4 are positive and strictly decreasing.*

ii) *The infinite continued fraction $\mathcal{K}[1]$ converges and is nonnegative.*

iii) *The tails converge and are positive, i.e. $\mathcal{K}[k+1] > 0$ for $k \in \mathbb{N}$.*

Proof. i) The positivity follows directly from the positivity of $\Delta_{N,k}$. From (19), we have

$$\frac{\Delta_{N+1,k+1}}{\Delta_{N+1,k+2}} = m_{k+1} - \frac{\ell_{k+2}}{\frac{\Delta_{N+1,k+2}}{\Delta_{N+1,k+3}}}, \qquad \frac{\Delta_{N,k+1}}{\Delta_{N,k+2}} = m_{k+1} - \frac{\ell_{k+2}}{\frac{\Delta_{N,k+2}}{\Delta_{N,k+3}}}. \tag{22}$$

Since m_k and $\Delta_{N,k+1}$ are both positive, the inequality

$$\frac{\Delta_{N+1,k+1}}{\Delta_{N+1,k+2}} < \frac{\Delta_{N,k+1}}{\Delta_{N,k+2}}, \tag{23}$$

can be written as

$$m_{k+1} - \frac{\ell_{k+2}}{\frac{\Delta_{N+1,k+2}}{\Delta_{N+1,k+3}}} < m_{k+1} - \frac{\ell_{k+2}}{\frac{\Delta_{N,k+2}}{\Delta_{N,k+2}}},$$

where we have used (22). Therefore, (23) is equivalent to the inequality

$$\frac{\Delta_{N+1,k+2}}{\Delta_{N+1,k+3}} < \frac{\Delta_{N,k+2}}{\Delta_{N,k+3}}.$$

Hence, if the inequality

$$\frac{\Delta_{N+1,N-1}}{\Delta_{N+1,N}} < \frac{\Delta_{N,N-1}}{\Delta_{N,N}} \tag{24}$$

holds for $k = N - 2$, then the inequality (23) will also hold. But,

$$\frac{\Delta_{N+1,N-1}}{\Delta_{N+1,N}} = m_N - \frac{\ell_{N+1}}{m_{N+1}}, \qquad \frac{\Delta_{N,N-1}}{\Delta_{N,N}} = m_N,$$

and (24) is indeed satisfied.

ii) This result follows directly from the previous observation and the fact that any positive, strictly decreasing sequence is convergent to a nonnegative number.

iii) For $k \in \mathbb{N}$, we have $\mathcal{K}[k] = m_k - \frac{\ell_{k+1}}{\mathcal{K}[k+1]}$, and since $\ell_{k+1} > 0$, the convergence of $\mathcal{K}[k]$ requires $\mathcal{K}[k + 1] > 0$. □

3. Positive bidiagonal factorization of tetradiagonal Hessenberg matrices

We will now explore how the Gauss–Borel factorization can be employed to obtain a bidiagonal factorization of the banded Hessenberg matrix. This, in turn, will introduce continued fractions into our theory.

Lemma 3.1. *The factorization of any lower triangular matrix of the form*

$$L^{[N]}=\begin{bmatrix} 1 & 0 & \cdots & & & 0\\ m_1 & 1 & \ddots & & & \vdots\\ \ell_2 & m_2 & 1 & & & \\ 0 & \ell_3 & m_3 & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & \ell_N & m_N & 1\end{bmatrix},$$

can be decomposed into bidiagonal factors, denoted as $L_1^{[N]}$ and $L_2^{[N]}$, given by

$$L^{[N]}=L_1^{[N]}L_2^{[N]},\quad L_1^{[N]}=\begin{bmatrix} 1 & 0 & \cdots & 0\\ \alpha_2 & \ddots & \ddots & \vdots\\ 0 & \alpha_5 & \ddots & 0\\ \vdots & \ddots & \ddots & \\ 0 & \cdots\ 0 & \alpha_{3N-1} & 1\end{bmatrix},$$

$$L_2^{[N]}=\begin{bmatrix} 1 & 0 & \cdots & 0\\ \alpha_3 & \ddots & \ddots & \vdots\\ 0 & \alpha_6 & \ddots & 0\\ \vdots & \ddots & \ddots & \\ 0 & \cdots\ 0 & \alpha_{3N} & 1\end{bmatrix}, \tag{25}$$

where the elements α_2, α_3, α_5, α_6, etc., are uniquely determined in terms of each other and α_2 using the following infinite continued fractions:

$$\alpha_{3n}=m_n-\cfrac{\ell_n}{m_{n-1}-\cfrac{\ell_{n-1}}{m_{n-2}-\cfrac{\ell_{n-2}}{\ddots\, m_2-\cfrac{\ell_2}{m_1-\alpha_2}}}},\quad \alpha_{3n-1}=\cfrac{\ell_n}{m_{n-1}-\cfrac{\ell_{n-1}}{m_{n-2}-\cfrac{\ell_{n-2}}{\ddots\, m_2-\cfrac{\ell_2}{m_1-\alpha_2}}}}. \tag{26}$$

The factorization exists if and only if $\alpha_{3n}\neq 0$ for $n\in\{1,\dots,N-1\}$.

Proof. The factorization (25) implies the following relationships:

$$m_n=\alpha_{3n-1}+\alpha_{3n},\qquad n\in 1,\dots,N,\qquad \ell_n=\alpha_{3n-1}\alpha_{3n-3},\qquad n\in 2,\dots,N.$$

These equations can be solved recursively as follows:

$$\alpha_3 = m_1 - \alpha_2, \qquad \alpha_5 = \frac{\ell_2}{m_1 - \alpha_2},$$

$$\alpha_6 = m_2 - \frac{\ell_2}{m_1 - \alpha_2}, \qquad \alpha_8 = \frac{\ell_3}{m_2 - \frac{\ell_2}{m_1-\alpha_2}},$$

$$\alpha_9 = m_3 - \frac{\ell_3}{m_2 - \frac{\ell_2}{m_1-\alpha_2}}, \qquad \alpha_{10} = \frac{\ell_4}{m_3 - \frac{\ell_3}{m_2 - \frac{\ell_2}{m_1-\alpha_2}}},$$

and so on. The result follows by induction.

Thus, for a given value of α_2, the factorization exists if and only if $\alpha_{3n} \neq 0$, for $n \in 1, \dots, N-1$. □

Proposition 3.2. *For each $\alpha_2 < \mathcal{K}[N,1]$, where $\mathcal{K}[N,1]$ is the finite continued fraction introduced in Definition 2.4, the factorization of $L^{[N]}$ into bidiagonal factors given by*

$$L^{[N]} = L_1^{[N]} L_2^{[N]}, \qquad L_1^{[N]} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \alpha_2 & \ddots & \ddots & & \vdots \\ 0 & \alpha_5 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_{3N-1} & 1 \end{bmatrix},$$

$$L_2^{[N]} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \alpha_3 & \ddots & \ddots & & \vdots \\ 0 & \alpha_6 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_{3N} & 1 \end{bmatrix},$$

where $\alpha_3, \alpha_5, \alpha_6, \alpha_8, \dots, \alpha_{3n-1}, \alpha_{3n} > 0$, exists and is unique. If $\alpha_2 \in [0, \mathcal{K}[N,1])$, then both $L_1^{[N]}$ and $L_2^{[N]}$ are invertible totally nonnegative matrices.

Proof. In the solution provided by Equation (26), it is required that $\alpha_3, \alpha_5, \alpha_6, \alpha_8, \dots, \alpha_{3N-1}, \alpha_{3N} > 0$. Let's proceed step by step. Firstly, if $\alpha_2 < m_1$, we can observe that α_3 and α_5 are both positive.

In the next step, if we assume $\alpha_2 < m_1$ and $\alpha_2 < m_1 - \frac{\ell_2}{m_2}$, we then have $\alpha_3, \alpha_5, \alpha_6$, and α_8 as positive. We notice that since the sequence $\mathcal{K}[N,1] > 0$ is decreasing, it holds that $m_1 - \frac{\ell_2}{m_2} < m_1$, so only one condition is needed.

Continuing, in the next step, we deduce that in order to ensure $\alpha_3, \alpha_5, \alpha_6, \alpha_8, \alpha_9$, and α_{10} are positive, we require $\alpha_2 < m_1 - \dfrac{\ell_2}{m_2 - \frac{\ell_3}{m_3}}$.

Finally, using induction, we can show that the same pattern continues for higher values of n, leading to the result. □

Theorem 3.3 *(Positive bidiagonal factorization in the finite case). Let us assume that the matrix $T^{[N]}$ given in (7) is oscillatory. Then, for each $\alpha_2 < \mathcal{K}[N,1]$, we can determine a positive sequence $\alpha_1, \alpha_3, \alpha_4, \alpha_5, \dots, \alpha_{3N+1}$ that satisfies the factorization:*

$$T^{[N]} = L_1^{[N]} L_2^{[N]} U^{[N]},$$

where the bidiagonal matrices are given by:

$$L_1^{[N]} = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ \alpha_2 & \ddots & \ddots & & \vdots \\ 0 & \alpha_5 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_{3N-1} & 1 \end{bmatrix}, \qquad L_2^{[N]} = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ \alpha_3 & \ddots & \ddots & & \vdots \\ 0 & \alpha_6 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_{3N} & 1 \end{bmatrix},$$

$$U^{[N]} = \begin{bmatrix} \alpha_1 & 1 & 0 & \cdots & 0 \\ 0 & \alpha_4 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \alpha_7 & \ddots & 0 \\ & & \ddots & \ddots & 1 \\ 0 & \cdots & & 0 & \alpha_{3N+1} \end{bmatrix},$$

Proof. It follows as a consequence of ii) in Theorem 2.5 and Proposition 3.2. □

Theorem 3.4 *(PBF in the semi-infinite case). Let us assume that the banded Hessenberg matrix T in (1) is oscillatory. For each $\alpha_2 < \mathcal{K}[1]$, where $\mathcal{K}[1]$ is the infinite continued fraction in (20), there exists a unique positive sequence $\{\alpha_1, \alpha_3, \alpha_4, \dots\}$ such that the PBF (2), (3) holds. Additionally, if $\alpha_2 \in [0, \mathcal{K}[1])$, then L_1, L_2, and U belong to* InTN.

Furthermore, the matrix entries satisfy the following relations:

$$\begin{cases} c_n = \alpha_{3n+1} + \alpha_{3n} + \alpha_{3n-1}, \\ b_n = \alpha_{3n}\alpha_{3n-2} + \alpha_{3n-1}\alpha_{3n-2} + \alpha_{3n-1}\alpha_{3n-3}, \\ a_n = \alpha_{3n-1}\alpha_{3n-3}\alpha_{3n-5}. \end{cases}$$

It is known that the infinite continued fraction $\mathcal{K}[1]$ in (20) could be zero, and this poses an important issue in the constructions of the spectral measure representation for the banded Hessenberg matrix T, as α_2 cannot be assumed to be a positive number (cf. [8]).

Notice that in [2], this assumption was taken for granted. However, for hypergeometric multiple orthogonal polynomials [5,20], and Jacobi–Piñeiro multiple orthogonal polynomials [2,6], there are parameter regions where it is indeed true that $\alpha_2 > 0$.

4. Oscillatory Toeplitz tetradiagonal matrices

We now consider the uniform case that arises when the matrix T satisfies the conditions:

$$a_n = a > 0, \qquad b_n = b \geqslant 0, \qquad c_n = c \geqslant 0. \tag{27}$$

This means that the Hessenberg matrix T is a banded Toeplitz matrix given by:

$$T = \begin{bmatrix} c & 1 & 0 & \cdots & & \\ b & c & 1 & \ddots & & \\ a & b & c & 1 & & \\ 0 & a & b & c & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{28}$$

Proposition 4.1 *(Edrei–Schoenberg). The Toeplitz matrix (28) is oscillatory if and only if there exist positive numbers $\beta_1 \geqslant \beta_2 \geqslant \beta_3 > 0$ such that:*

$$a = \beta_1\beta_2\beta_3, \qquad b = \beta_1\beta_2 + \beta_1\beta_3 + \beta_2\beta_3, \qquad c = \beta_1 + \beta_2 + \beta_3. \tag{29}$$

Proof. According to the Edrei–Schoenberg Theorem [11,24], the matrix T is TN if and only if the generating function can be written in the form:

$$f(t) = 1 + ct + bt^2 + at^3 = (1 + \beta_1 t)(1 + \beta_2 t)(1 + \beta_3 t),$$

where $\beta_1 \geqslant \beta_2 \geqslant \beta_3 \geqslant 0$. By using these β's, we can express the matrix entries in terms of them, as shown in (29). Since $a > 0$, we must have $\beta_1 \geqslant \beta_2 \geqslant \beta_3 > 0$. Consequently, $b > 0$, and by applying the Gantmacher–Krein criterion (see Theorem 1.1), we can conclude that the Toeplitz matrix is oscillatory. □

We will now demonstrate that all tetradiagonal Toeplitz matrices (28) are oscillatory if and only if they admit a PBF:

Proposition 4.2. *If T is an oscillatory banded Toeplitz matrix as in (28) with $\beta_1 > \beta_2 > \beta_3 > 0$, then the determinants $\delta^{[N]} = \det T^{[N]}$ are explicitly given in terms of $\{\beta_1, \beta_2, \beta_3\}$ as follows:*

$$\delta^{[n]} = \frac{\beta_1^{n+2}}{(\beta_1-\beta_2)(\beta_1-\beta_3)} + \frac{\beta_2^{n+2}}{(\beta_2-\beta_1)(\beta_2-\beta_3)} + \frac{\beta_3^{n+2}}{(\beta_3-\beta_1)(\beta_3-\beta_2)}. \tag{30}$$

Proof. The determinants $\delta^{[n]} = \det T^{[n]}$ are subject to the recursion relation

$$\delta^{[n]} - c\delta^{[n-1]} + b\delta^{[n-2]} - a\delta^{[n-3]} = 0, \tag{31}$$

with initial conditions $\delta^{[-2]} = \delta^{[-1]} = 0$ and $\delta^{[0]} = 1$.

Following the theory of recursion relations, as presented in [12], we consider the characteristic polynomial

$$p(\lambda) = \lambda^3 - c\lambda^2 + b\lambda - a,$$

and notice that $p(\lambda) = \lambda^3 f\left(-\frac{1}{\lambda}\right)$. Hence, the characteristic roots are $\beta_1, \beta_2, \beta_3 > 0$. If the roots are distinct, i.e., simple, then the general solution to the recursion (31) will be

$$C_1\beta_1^n + C_2\beta_2^n + C_3\beta_3^n,$$

where C_1, C_2, and C_3 are constants determined by the initial conditions:

$$\begin{bmatrix} 1 & 1 & 1 \\ \frac{1}{\beta_1} & \frac{1}{\beta_2} & \frac{1}{\beta_3} \\ \frac{1}{\beta_1^2} & \frac{1}{\beta_2^2} & \frac{1}{\beta_3^2} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

which ensures that (30) holds. □

Corollary 4.3. *For the matrix T as described in Proposition 4.1, and considering the large N ratio asymptotics of the determinants $\delta^{[N]}$, we find the following:*

$$\lim_{N\to\infty} \frac{\delta^{[N]}}{\delta^{[N-1]}} = \beta_1. \tag{32}$$

Proof. In the case of distinct characteristic roots $\beta_1 > \beta_2 > \beta_3 > 0$, this follows directly from Proposition 4.2.

When the two smaller characteristic roots coincide, i.e., $\beta_2 = \beta_3$, the general solution to the recursion relation will be

$$C_1\beta_1^n + (C_2 + C_3 n)\beta_2^n,$$

but the large N ratio asymptotics of the determinant remain unchanged.

However, when the largest characteristic root is degenerate, with multiplicity two or three, the determinant will have a dominant term $q(n)\beta_1^n$ asymptotically, where q is a polynomial with $\deg q = 1$ or 2, respectively. In this case, we recover the result in (32) for the large N ratio asymptotics. □

Theorem 4.4 *(Infinite continued fractions and harmonic mean). The continued fraction considered in (20) for an oscillatory tetradiagonal Toeplitz matrix T as in (28) represents half of the harmonic mean of the two largest characteristic roots, i.e.,*

$$\mathcal{K}[1] = \frac{\beta_1\beta_2}{\beta_1+\beta_2}.$$

Proof. To compute the continued fraction $\mathcal{K}[1]$ according to (21), we only require the use of the determinants $\delta_1^{[N]}$ of the matrix $T_1^{[N]}$ given by

$$T_1^{[N]} = \begin{bmatrix} b & c & 1 & 0 & \cdots & \cdots & 0 \\ a & b & c & 1 & \ddots & & \vdots \\ 0 & a & b & c & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 1 \\ \vdots & & & \ddots & a & b & c \\ 0 & \cdots & \cdots & \cdots & 0 & a & b \end{bmatrix} \in \mathbb{R}^{N\times N},$$

as in this Toeplitz case, we have $\delta_1^{[N,k]} = \delta_1^{[N-k]}$. These determinants are subject to the following uniform recursion relation:

$$\delta_1^{[n]} - b\delta_1^{[n-1]} + ac\delta_1^{[n-2]} - a^2\delta_1^{[n-3]} = 0, \tag{33}$$

as can be deduced from an expansion along the last column. The initial conditions are $\delta_1^{[-2]} = \delta_1^{[-1]} = 0$ and $\delta_1^{[0]} = 1$. The characteristic roots $\gamma_1, \gamma_2, \gamma_3$ are the zeros of the polynomial

$$q(t) = t^3 - bt^2 + act - a^2,$$

and we find that $p(t) = -\frac{t^3}{a^2}q\left(\frac{a}{t}\right)$ or $q(t) = -\frac{t^3}{a}p\left(\frac{a}{t}\right)$. Hence, the characteristic roots are $\gamma = \frac{a}{\beta}$, which, arranged in decreasing order, can be written as follows:

$$\gamma_1 = \beta_1\beta_2, \qquad \gamma_2 = \beta_1\beta_3, \qquad \gamma_3 = \beta_3\beta_2.$$

Let us assume that the roots are distinct, i.e., simple; the other degenerate cases can be treated similarly. Then, the general solution to the recursion (33) will be

$$C_1\gamma_1^n + C_2\gamma_2^n + C_3\gamma_3^n,$$

for some constants C_1, C_2, C_3 determined by the initial conditions:

$$\begin{bmatrix} 1 & 1 & 1 \\ \frac{1}{\gamma_1} & \frac{1}{\gamma_2} & \frac{1}{\gamma_3} \\ \frac{1}{\gamma_1^2} & \frac{1}{\gamma_2^2} & \frac{1}{\gamma_3^2} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Proceeding as above, we find

$$\delta_1^{[n]} = \frac{\beta_1^{n+1}\beta_2^{n+1}}{(\beta_2-\beta_3)(\beta_1-\beta_3)} + \frac{\beta_1^{n+1}\beta_3^{n+1}}{(\beta_3-\beta_2)(\beta_1-\beta_2)} + \frac{\beta_2^{n+1}\beta_3^{n+1}}{(\beta_3-\beta_1)(\beta_2-\beta_1)}.$$

Hence, according to (21) with $c_0 = c$ and $a_2 = a$, we have

$$\begin{aligned}\mathcal{K}[1] &= \lim_{n\to\infty} \frac{\dfrac{\beta_1^{n+1}\beta_2^{n+1}}{(\beta_2-\beta_3)(\beta_1-\beta_3)}}{(\beta_1+\beta_2+\beta_3)\dfrac{\beta_1^{n}\beta_2^{n}}{(\beta_2-\beta_3)(\beta_1-\beta_3)} - \beta_1\beta_2\beta_3\dfrac{\beta_1^{n-1}\beta_2^{n-1}}{(\beta_2-\beta_3)(\beta_1-\beta_3)}}\\ &= \frac{\beta_1^2\beta_2^2}{(\beta_1+\beta_2+\beta_3)\beta_1\beta_2 - \beta_1\beta_2\beta_3},\end{aligned}$$

and after some manipulations, the representation for $\mathcal{K}[1]$ follows. □

Theorem 4.5. *A tetradiagonal Toeplitz matrix is oscillatory if and only if it admits a PBF.*

Remark 4.6. Note that the determinants $\delta^{[n]}$ and $\delta_1^{[n]}$ of Toeplitz matrices can be alternatively computed with the aid of the Widom theorem on Toeplitz matrices [4, Theorem 2.8].

Remark 4.7. Tetradiagonal Toeplitz matrices and properties of their eigenvalues have been studied in [3]. The authors describe all types of the limiting sets, classify their exceptional points, and establish asymptotic formulas for the analytic arcs near their endpoints.

5. Retractions and tails

We will demonstrate that from any given oscillatory tetradiagonal matrix, we can construct tetradiagonal matrices with PBF in several ways. Additionally, we will find that oscillatory matrices are organized in rays. The origin of the ray is an oscillatory matrix that does not have a PBF, while all the interior points of the ray have a PBF.

Let A be a TN matrix. It follows from [13, Section 9.5] on retractions of TN matrices, and in particular, the proof of [13, Lemma 9.5.2], that $A + sE_{1,1}$ is also TN for $s \geqslant -\frac{\det A}{\det A(\{1\})}$. This is known as a retraction, and interestingly, when s is a negative number, a retraction can transform an oscillatory matrix with $\alpha_2 = 0$ into an oscillatory matrix with a PBF and vice versa. In this context, we use the notation $E_{1,2}(s) := I + sE_{1,2}$ to represent the bidiagonal matrix with a nonzero contribution only possibly at the entry in the second row and first column.

Theorem 5.1 *(Retractions and PBF).*

i) *For an oscillatory tetradiagonal matrix* T *as in* (1), *the tetradiagonal matrix*

$$T_s = E_{1,2}(s)T = \begin{bmatrix} c_0 & 1 & 0 & \cdots & & \\ b_1 + sc_0 & c_1 + s & 1 & \ddots & & \\ a_2 & b_2 & c_2 & 1 & & \\ 0 & a_3 & b_3 & c_3 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{34}$$

has a PBF for $s > -\mathcal{K}[1]$.

ii) *For a tetradiagonal matrix* T *as in* (1) *that admits a PBF, i.e., with* $\mathcal{K}[1] > 0$, *the tetradiagonal matrix*

$$\begin{aligned}\tilde{T} &= E_{1,2}\Big(-\mathcal{K}[1]\Big)T \\ &= \begin{bmatrix} c_0 & 1 & 0 & \cdots & & \\ b_1 - \mathcal{K}[1]c_0 & c_1 - \mathcal{K}[1] & 1 & \ddots & & \\ a_2 & b_2 & c_2 & 1 & & \\ 0 & a_3 & b_3 & c_3 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},\end{aligned}$$

is an oscillatory matrix that does not admit a PBF, i.e. $\tilde{\mathcal{K}}[1] = 0$.

Proof. i) The Jacobi matrix $J_s^{[N,1]} = J^{[N,1]} + sE_{1,1}$ is TN for $s \geqslant -\frac{\Delta_{N,1}}{\Delta_{N,2}}$ and InTN for $s > -\frac{\Delta_{N,1}}{\Delta_{N,2}} = -\mathcal{K}[N,1]$. Thus, according to Theorem 1.1, it is an oscillatory matrix for $s > -\mathcal{K}[N,1]$. Consequently, the corresponding lower unitriangular matrix $L_s^{[N]}$ that has $J_s^{[N,1]}$ as a complementary submatrix, obtained by deleting the first row and last column, is InTN for $s > -\mathcal{K}[N,1]$. This is a consequence of [13, Lemma 3.3.4]. The continued fraction $\mathcal{K}_s[N,1]$, corresponding to the oscillatory Jacobi matrix $J_s^{[N,1]}$, is $\mathcal{K}[N,1] + s$.

Now, let us consider the banded Hessenberg matrix defined as $T_s^{[N]} = L_s^{[N]}U^{[N]}$, which is clearly InTN for $s > -\mathcal{K}[N,1]$ since its factors are. A direct computation shows that

$$T_s^{[N]} = \begin{bmatrix} c_0 & 1 & 0 & \cdots & & & & 0 \\ b_1 + sc_0 & c_1 + s & 1 & \ddots & & & & \vdots \\ a_2 & b_2 & c_2 & 1 & & & & \\ 0 & a_3 & b_3 & c_3 & 1 & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & 0 \\ & & & a_{N-1} & b_{N-1} & c_{N-1} & & 1 \\ 0 & \cdots & & 0 & a_N & b_N & & c_N \end{bmatrix}.$$

Observe that $m_1 = \frac{b_1}{c_0} = \mathcal{K}[2,1]$, and recall that $\{\mathcal{K}[n,1]\}_{n=2}^{\infty}$ is a positive and decreasing sequence. Consequently, $b_1 + sc_0 > 0$ for $s > -\mathcal{K}[N,1]$ and $N \in \{2,3,\dots\}$. Therefore, using the Gantmacher–Krein Criterion, we conclude that $T_s^{[N]}$ is oscillatory. Finally, for the large N limit with $s > -\mathcal{K}[1]$, the matrix T_s is oscillatory and has $\mathcal{K}_s[1] > 0$.

ii) The retraction $\tilde{J}^{[N,1]} = J^{[N,1]} - \mathcal{K}[1]E_{1,1}$ of the Jacobi matrix of $T^{[N]}$ is oscillatory. This is a direct consequence of the fact that $0 < \mathcal{K}[1] < \mathcal{K}[n,1]$ for all $n \in \mathbb{N}$. The associated finite continued fraction is $\tilde{\mathcal{K}}[N,1] = \mathcal{K}[N,1] - \mathcal{K}[1]$.

To continue, let us consider as in the previous discussion the unitriangular matrix $\tilde{L}^{[N]}$ such that $\tilde{J}^{[N,1]} = \tilde{L}^{[N]}(\{1\},\{N+1\})$ is a complementary submatrix, obtained by deleting the first row and last column. This triangular matrix is InTN, and the matrix $\tilde{T}^{[N]} = \tilde{L}^{[N]}U^{[N]}$ is oscillatory as well, with associated continued fraction $\tilde{\mathcal{K}}[N,1] = \mathcal{K}[N,1]-\mathcal{K}[1]$. Thus, in the large N limit, the semi-infinite banded Hessenberg matrix $\tilde{T}$ is oscillatory with $\tilde{\mathcal{K}}[1] = 0$, i.e. no PBF exists. □

Corollary 5.2. *If the tetradiagonal matrix T given in (1) has $\mathcal{K}[1] = 0$, then T_s as in (34) has a PBF for $s > 0$.*

The next result is based on the fact that the terms $\mathcal{K}[2], \mathcal{K}[3], \dots$ of the continued fraction $\mathcal{K}[1]$ are positive.

Theorem 5.3 *(Tails and positive bidiagonal factorization). If T is an oscillatory banded Hessenberg matrix as in (1), then the matrices*

$$T^{(2)} := \begin{bmatrix} c_1 - \frac{b_1}{c_0} & 1 & 0 & \cdots & \\ b_2 - \frac{a_2}{c_0} & c_2 & 1 & \ddots & \\ a_3 & b_3 & c_3 & 1 & \\ 0 & a_4 & b_4 & c_4 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

$$T^{(k+1)} := \begin{bmatrix} \alpha_{3k+1} & 1 & 0 & \cdots & & \\ (c_{k+1}+\alpha_{3k+4})\alpha_{3k+1} & c_{k+1} & 1 & \ddots & & \\ a_{k+2} & b_{k+2} & c_{k+2} & 1 & \ddots & \\ 0 & a_{k+3} & b_{k+3} & c_{k+3} & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

$k \in \{2, 3, \dots\}$,

have a PBF with associated continued fractions $\mathcal{K}[2]$ *and* $\mathcal{K}[k+1]$ *for* $k \in \{2, 3, \dots\}$, *respectively.*

Proof. For $k \in \mathbb{N}$, the tail $\mathcal{K}[k+1]$ corresponds to the continued fraction of the Jacobi matrix

$$J^{(k+1)} := \begin{bmatrix} m_{k+1} & 1 & 0 & \cdots & \\ \ell_{k+2} & m_{k+2} & 1 & \ddots & \\ 0 & \ell_{k+3} & m_{k+3} & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{35}$$

which is an oscillatory submatrix of

$$L^{(k+1)} = \begin{bmatrix} 1 & 0 & \cdots & & \\ m_{k+1} & 1 & \ddots & & \\ \ell_{k+2} & m_{k+2} & 1 & \ddots & \\ 0 & \ell_{k+3} & m_{k+3} & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

where all its leading principal submatrices are InTN. We then define the upper triangular

$$U^{(k+1)} = \begin{bmatrix} \alpha_{3k+1} & 1 & 0 & \cdots \\ 0 & \alpha_{3k+4} & 1 & \ddots \\ 0 & 0 & \alpha_{3k+7} & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix},$$

where all its leading principal submatrices are InTN. The corresponding tetradiagonal matrix $T^{(k+1)} = L^{(k+1)}U^{(k+1)}$ is given by

$$
T^{(k+1)} = \begin{bmatrix} c_k - m_k & 1 & 0 & \cdots & & \\ b_{k+1} - \ell_{k+1} & c_{k+1} & 1 & \ddots & & \\ a_{k+2} & b_{k+2} & c_{k+2} & 1 & \ddots & \\ 0 & a_{k+3} & b_{k+3} & c_{k+3} & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},
$$

where $c_k - m_k = \alpha_{3k+1} > 0$ and $b_{k+1} - \ell_{k+1} = m_{k+1}\alpha_{3k+1} = (c_{k+1} + \alpha_{3k+4})\alpha_{3k+1} > 0$. According to the Gantmacher–Krein Criterion, this matrix is oscillatory, and its continued fraction $\mathcal{K}^{(k+1)}[1]$ is equal to the tail $\mathcal{K}[k+1]$, which is positive. To obtain $T^{(2)}$, recall that $\alpha_1 = c_0$, $m_1\alpha_1 = b_1$, and $\ell_2\alpha_1 = a_2$. □

We conclude the paper with a result that guarantees the existence of associated tetradiagonal matrices with a PBF for any given oscillatory tetradiagonal matrix:

Theorem 5.4. *Let's assume that the Hessenberg matrix T in (1) is oscillatory. Then, matrices*

$$
\check{T} := \begin{bmatrix} b_1 & 1 & 0 & \cdots & & \\ a_2c_1 & b_2 & 1 & \ddots & & \\ a_2a_3 & a_3c_2 & b_3 & 1 & \ddots & \\ 0 & a_3a_4 & a_4c_3 & b_4 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},
$$

$$
\check{T}^{[k]} := \begin{bmatrix} b_{k+1} - m_{k+1} & 1 & 0 & \cdots & & \\ a_{k+2}c_{k+1} & b_{k+2} & 1 & \ddots & & \\ a_{k+2}a_{k+3} & a_{k+3}c_{k+2} & b_{k+3} & 1 & \ddots & \\ 0 & a_{k+3}a_{k+4} & a_{k+4}c_{k+3} & b_{k+4} & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \quad k \in \mathbb{N},
$$

admit a PBF.

Proof. Theorem 2.6 guarantees that the submatrix $T^{[N]}1$ and its submatrices $T^{[N,k]}1$ are oscillatory. Additionally, using Equation (15)and the fact that $J^{[N,k]}$ is oscillatory and $U^{[N-1,k]}$ is InTN (notice that the product is InTN, and the elements in the first superdiagonal and subdiagonal are positive, then use the Gantmacher–Krein Criterion, Theorem 1.1), we can deduce that the retraction $T^{[N,k]}1 - \ell k + 1E_{1,1}$ is also oscillatory. As these matrices are upper Hessenberg, their transposition will transform them into lower Hessenberg matrices:

$$
\left(T_1^{[N]}\right)^\top := \begin{bmatrix}
b_1 & a_2 & 0 & \cdots & & & 0\\
c_1 & b_2 & a_3 & & & & \vdots\\
1 & c_2 & b_3 & a_4 & & & \\
0 & 1 & c_3 & b_4 & a_5 & & \\
\vdots & & \ddots & \ddots & \ddots & \ddots & 0\\
 & & & & & b_{N-1} & a_N\\
0 & \cdots & 0 & 1 & & c_{N-1} & b_N
\end{bmatrix},
$$

$$
\left(T_1^{[N,k]} - m_{k+1}E_{1,1}\right)^\top :=
$$

$$
\begin{bmatrix}
b_{k+1}-m_{k+1} & a_{k+2} & 0 & \cdots & & & 0\\
c_{k+1} & b_{k+2} & a_{k+3} & & & & \vdots\\
1 & c_{k+2} & b_{k+3} & a_{k+4} & & & \\
0 & 1 & c_{k+3} & b_{k+4} & a_{k+5} & & \\
\vdots & & \ddots & \ddots & \ddots & \ddots & 0\\
 & & & & & b_{N-1} & a_N\\
0 & \cdots & 0 & 1 & & c_{N-1} & b_N
\end{bmatrix}.
$$

Although they are not normalized to be monic on the first superdiagonal, they are monic on the second subdiagonal. We can perform a similarity transformation $T \mapsto ATA^{-1}$ using the diagonal matrix

$$
A = \operatorname{diag}(1, a_2, a_2a_3, \ldots, a_2a_3\cdots a_N)
$$

to obtain a monic banded Hessenberg matrix:

$$
\check{T}^{[N-1]} := A\left(T_1^{[N]}\right)^\top A^{-1}
$$

$$
= \begin{bmatrix}
b_1 & 1 & 0 & \cdots & & & 0\\
a_2c_1 & b_2 & 1 & & & & \vdots\\
a_2a_3 & a_3c_2 & b_3 & 1 & & & \\
0 & a_3a_4 & a_4c_3 & b_4 & 1 & & \\
\vdots & & \ddots & \ddots & \ddots & \ddots & 0\\
 & & & & & b_{N-1} & 1\\
0 & \cdots & 0 & a_{N-1}a_N & & a_Nc_{N-1} & b_N
\end{bmatrix}
$$

which happens to be oscillatory. Similarly, the transformation $T \mapsto A_kTA_k^{-1}$ using the diagonal matrix $A_k = \operatorname{diag}(1, a_{k+2}, a_{k+2}a_{k+3}, \ldots, a_{k+2}a_{k+3}\cdots a_N)$ yields another monic banded Hessenberg matrix:

$$
\check{T}^{[N-1,k]} := A_k\left(T_1^{[N,k]} - m_{k+1}E_{1,1}\right)^\top A_k^{-1}
$$

$$
= \begin{bmatrix} b_{k+1} - m_{k+1} & 1 & 0 & \cdots & & & 0 \\ a_{k+2}c_{k+1} & b_{k+2} & 1 & \ddots & & & \vdots \\ a_{k+2}a_{k+3} & a_{k+3}c_{k+2} & b_{k+3} & 1 & \ddots & & \\ 0 & a_{k+3}a_{k+4} & a_{k+4}c_{k+3} & b_{k+4} & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & a_{N-1}a_N & a_N c_{N-1} & & b_N \end{bmatrix},
$$

which is also oscillatory. According to (14) and (15), these matrices admit the factorizations:

$$
\begin{aligned}
&\check{T}^{[N-1]} = \check{L}^{[N-1]}\check{J}^{[N]}, && \check{L}^{[N-1]} := A\big(U^{[N-1]}\big)^{\top}A^{-1}, && \check{J}^{[N,1]} = A\big(J^{[N,1]}\big)^{\top}A^{-1}, \\
&\check{T}^{[N-1,k]} = \check{L}^{[N-1,k]}\check{J}^{[N,1]}, && \check{L}^{[N-1,k]} := A_k\big(U^{[N-1,k]}\big)^{\top}A_k^{-1}, && \check{J}^{[N,k+1]} = A_k\big(J^{[N,k+1]}\big)^{\top}A_k^{-1},
\end{aligned} \tag{36}
$$

with (recalling that $\alpha_1 = c_0$)

$$
\check{L}^{[N-1]} = \begin{bmatrix} c_0 & 0 & \cdots & & 0 \\ a_2 & \alpha_4 & \ddots & & \vdots \\ 0 & a_3 & \alpha_7 & \ddots & \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_N & \alpha_{3N-2} \end{bmatrix}, \quad \check{J}^{[N,1]} = \begin{bmatrix} m_1 & \frac{\ell_2}{a_2} & 0 & \cdots & & 0 \\ a_2 & m_2 & \frac{\ell_3}{a_3} & \ddots & & \vdots \\ 0 & a_3 & m_3 & \frac{\ell_4}{a_4} & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ & & & \ddots & \ddots & \frac{\ell_N}{a_N} \\ 0 & \cdots & & 0 & a_N & m_N \end{bmatrix},
$$

$$
\check{L}^{[N-1,k]} = \begin{bmatrix} \alpha_{3k+1} & 0 & \cdots & & 0 \\ a_{k+2} & \alpha_{3k+4} & \ddots & & \vdots \\ 0 & a_{k+3} & \alpha_{3k+7} & \ddots & \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_N & \alpha_{3N-2} \end{bmatrix}, \quad \check{J}^{[N,k+1]} = \begin{bmatrix} m_{k+1} & \frac{\ell_{k+2}}{a_{k+2}} & 0 & \cdots & & 0 \\ a_{k+2} & m_{k+2} & \frac{\ell_{k+3}}{a_{k+3}} & \ddots & & \vdots \\ 0 & a_{k+3} & m_{k+3} & \frac{\ell_{k+4}}{a_{k+4}} & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ & & & \ddots & \ddots & \frac{\ell_N}{a_N} \\ 0 & \cdots & & 0 & a_N & m_N \end{bmatrix}.
$$

The Jacobi matrices involved in these factorizations are oscillatory and consequently admit a PBF. Hence, the existence of a PBF follows. □

6. Conclusions and outlook

In [8], it was demonstrated that for bounded banded Hessenberg matrices (having $p+2$ diagonals) with a PBF, a spectral Favard theorem [25] for multiple orthogonal polynomials can be established. This marked the first extension of the spectral Favard theory to multiple orthogonality.[1] Therefore, it is crucial to explore the relationship between the oscillatory character of a banded Hessenberg matrix and the existence of this PBF. We have already shown that for the well-known tridiagonal case, the corresponding Jacobi matrix, when conveniently shifted, becomes oscillatory, and oscillatory and PBF properties coincide in this case. The next step is to consider the tetradiagonal case, which is the focus of this study. We aim to investigate the existence of a PBF for a generic tetradiagonal oscillatory Hessenberg matrix.

Regarding continued fractions, in the finite case, we establish the existence of a bidiagonal positive factorization in the tetradiagonal scenario, while in the infinite case, we provide a bound for the existence of such factorization. We specifically examine oscillatory Toeplitz matrices and prove their admittance of PBF. Moreover, we demonstrate that if an oscillatory tetradiagonal matrix lacks a PBF, there are several ways to find associated oscillatory matrices that possess a PBF.

In [9], we focus on the tetradiagonal case and explore corresponding multiple orthogonal polynomials in the step-line with two weights. We present the PBF factorization in terms of the values of the orthogonal polynomials of type I and II at 0, which consequently offers a spectral interpretation of the Darboux transformation.

Looking ahead, we aim to understand the behavior of Hessenberg matrices with more diagonals. Preliminary attempts suggest that the role of continued fractions may need to be replaced by more general objects, possibly branched continued fractions. Further research in this direction will provide valuable insights into the properties and factorization of higher-dimensional Hessenberg matrices.

In the context of constant Toeplitz matrices, it is essential to investigate whether the oscillatory property guarantees the existence of a PBF. This question remains open and requires further research to determine the conditions under which such matrices admit a PBF.

Furthermore, extending the analysis to banded recursion matrices with multiple superdiagonals and subdiagonals introduces new challenges. The behavior of such matrices may differ significantly from those with a smaller number of diagonals. Understanding the relationship between the oscillatory nature of these matrices and the existence of a PBF will require careful investigation and potentially lead to new insights in the field of matrix factorizations. These questions provide intriguing avenues for future studies in the area of oscillatory matrices and their factorizations.

Ethical approval

Not applicable.

[1] For an in-depth study of multiple orthogonal polynomials, see [23] and [18] (chapter written by Van Assche). Additionally, [1] provides a description of multiple orthogonal polynomials in terms of a Gauss–Borel factorization and its connection with integrable systems, while [7] discusses Pearson equations and Christoffel formulas for general Christoffel/Geronimus transformations.

CRediT authorship contribution statement

All the authors have contributed equally.

Declaration of competing interest

The authors declare no conflict of interest.

Data availability

No data was used for the research described in the article.

Declaration of generative AI and AI-assisted technologies in the writing process

During the preparation of this work the authors used ChatGPT in order to improve English grammar, syntax, spelling and wording. After using this tool/service, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

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References

[1] Carlos Álvarez-Fernández, Ulises Fidalgo, Manuel Mañas, Multiple orthogonal polynomials of mixed type: Gauss–Borel factorization and the multi-component 2D Toda hierarchy, Adv. Math. 227 (2011) 1451–1525.

[2] Alexander Aptekarev, Valery Kaliaguine, Jeannette Van Iseghem, The genetic sums' representation for the moments of a system of Stieltjes functions and its application, Constr. Approx. 16 (2000) 487–524.

[3] Albrecht Böttcher, Juanita Gasca, Sergei M. Grudsky, Anatoli V. Kozak, Eigenvalue clusters of large tetradiagonal Toeplitz matrices, Integral Equ. Oper. Theory 93 (2021) 8.

[4] Albrecht Böttcher, Sergei M. Grudsky, Spectral Properties of Banded Toeplitz Matrices, SIAM, Philadelphia, 2005.

[5] Amílcar Branquinho, Juan E.F. Díaz, Ana Foulquié-Moreno, Manuel Mañas, Hypergeometric multiple orthogonal polynomials and random walks, arXiv:2107.00770.

[6] Amílcar Branquinho, Juan E.F. Díaz, Ana Foulquié-Moreno, Manuel Mañas, Carlos Álvarez-Fernández, Jacobi-Piñeiro random walks, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. (2023), in press.

[7] Amílcar Branquinho, Ana Foulquié-Moreno, Manuel Mañas, Multiple orthogonal polynomials: Pearson equations and Christoffel formulas, Anal. Math. Phys. 12 (2022) 129.

[8] Amílcar Branquinho, Ana Foulquié-Moreno, Manuel Mañas, Oscillatory banded Hessenberg matrices, multiple orthogonal polynomials and Markov chains, Phys. Scr. (2023), https://doi.org/10.1088/1402-4896/ace93d.

[9] Amílcar Branquinho, Ana Foulquié-Moreno, Manuel Mañas, Bidiagonal factorization of tetradiagonal matrices and Darboux transformations, Anal. Math. Phys. 13 (2023) 42.

[10] Amílcar Branquinho, Ana Foulquié-Moreno, Manuel Mañas, Spectral theory for bounded banded matrices with positive bidiagonal factorization and mixed multiple orthogonal polynomials, Adv. Math. (2023), in press.

[11] Albert Edrei, Proof of a conjecture of Schoenberg on the generating function of a totally positive sequence, Can. J. Math. 5 (1953) 86–94.

[12] Saber Elaydi, An Introduction to Difference Equations, third edition, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 2005.

[13] Shaun M. Fallat, Charles R. Johnson, Totally Nonnegative Matrices, Princeton Series in Applied Mathematics, Princeton University Press, Princeton, 2011.

[14] Felix P. Gantmacher, Matrix Theory, Volume Two, Chelsea Publishing Company, New York, 1974.

[15] Felix P. Gantmacher, Mark G. Krein, Oscillation and Kernels and Small Vibrations of Mechanical Systems, revised second edition, AMS Chelsea Publishing, American Mathematical Society, Providence, Rhode Island, 2002.

[16] F. Alberto Grünbaum, Manuel D. de la Iglesia, An urn model for the Jacobi–Piñeiro polynomials, Proc. Am. Math. Soc. 150 (2022) 3613–3625.

[17] Roger A. Horn, Charles R. Johnson, Matrix Analysis, second edition, corrected reprint, Cambridge University Press, Cambridge, 2018.

[18] Mourad E.H. Ismail, Classical and Quantum Orthogonal Polynomials in One Variable, Encyclopedia of Mathematics and Its Applications, vol. 98, Cambridge University Press, 2009.

[19] William B. Jones, Wolfgang J. Thron, Continued Fractions: Analytic Theory and Applications, Encyclopedia of Mathematics and Its Applications, vol. 11, Cambridge University Press, Cambridge, 1984.

[20] Hélder Lima, Ana Loureiro, Multiple orthogonal polynomials with respect to Gauss' hypergeometric function, Stud. Appl. Math. 148 (2022) 154–185.

[21] Lisa Lorentzen, Haakon Waadeland, Continued Fractions: 1. Convergence Theory, 2nd edition, Atlantis Press/World Scientific, Paris, 2008.

[22] Manuel Mañas, Revisiting biorthogonal polynomials. An LU factorization discussion, in: E. Huertas, F. Marcellán (Eds.), Orthogonal Polynomials: Current Trends and Applications, in: SEMA SIMAI Springer Series, vol. 22, Springer, 2021, pp. 273–308.

[23] Evgenii M. Nikishin, Vladimir N. Sorokin, Rational Approximations and Orthogonality, Translations of Mathematical Monographs, vol. 92, American Mathematical Society, Providence, 1991.

[24] Isaac J. Schoenberg, Some analytical aspects of the problem of smoothing, in: Studies and Essays Presented to R. Courant on His 60th Anniversary, Interscience Publishers, New York, 1948, pp. 351–370.

[25] Barry Simon, Operator Theory. A Comprehensive Course in Analysis, Part 4, American Mathematical Society, Providence, Rhode Island, 2015.

[26] Hubert S. Wall, Analytic Theory of Continued Fractions, American Mathematical Society Chelsea, 1948.