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UM ESTUDO DE OPERADORES INTEGRAIS SINGULARES COM DESLOCAMENTO

A STUDY OF SINGULAR INTEGRAL OPERATORS WITH SHIFT

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Um Estudo de Operadores Integrais Singulares com Deslocamento

A Study of Singular Integral Operators with shift

Tese apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, realizada sob a orientação científica do Doutor Luís Filipe Pinheiro de Castro, Professor Catedrático do Departamento de Matemática da Universidade de Aveiro

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Dedicated to the memory of Professor Diómedes Bárcenas.

o júri

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palavras-chave

operadores integrais singulares, operadores de deslocamento do tipo de Carleman, relações de equivalência entre operadores, propriedade de Fredholm, invertibilidade, métodos de colocação, equações integrais singulares, métodos de projecção, problemas de fronteira

resumo

Nesta tese, consideram-se operadores integrais singulares com a acção extra de um operador de deslocacamento de Carleman e com coeficientes em diferentes classes de funções essencialmente limitadas. Nomeadamente, funções contínuas por troços, funções quase-periódicas e funções possuíndo factorização generalizada.

Nos casos dos operadores integrais singulares com deslocamento dado pelo operador de reflexão ou pelo operador de salto no círculo unitário complexo, obtêm-se critérios para a propriedade de Fredholm. Para os coeficientes contínuos, uma fórmula do índice de Fredholm é apresentada. Estes resultados são consequência das relações de equivalência explícitas entre aqueles operadores e alguns operadores adicionais, tais como o operador integral singular, operadores de Toeplitz e operadores de Toeplitz mais Hankel. Além disso, as relações de equivalência permitem-nos obter um critério de invertibilidade e fórmulas para os inversos laterais dos operadores iniciais com coeficientes factorizáveis. Adicionalmente, aplicamos técnicas de análise numérica, tais como métodos de colocação de polinómios, para o estudo da dimensão do núcleo dos dois tipos de operadores integrais singulares com coeficientes contínuos por troços. Esta abordagem permite também a computação do inverso no sentido Moore-Penrose dos operadores principais.
Para operadores integrais singulares com operadores de deslocamento do tipo Carleman preservando a orientação e com funções contínuas como coeficientes, são obtidos limites superiores da dimensão do núcleo. Tal é implementado utilizando algumas estimativas e com a ajuda de relações (explícitas) de equivalência entre operadores.
Focamos ainda a nossa atenção na resolução e nas soluções de uma classe de equações integrais singulares com deslocamento que não pode ser reduzida a um problema de valor de fronteira binomial. De forma a atingir os objectivos propostos, foram utilizadas projecções complementares e identidades entre operadores. Desta forma, as equações em estudo são associadas a sistemas de equações integrais singulares. Estes sistemas são depois analisados utilizando um problema de valor de fronteira de Riemann. Este procedimento tem como consequência a construção das soluções das equações iniciais a partir das soluções de problemas de valor de fronteira de Riemann.
Motivados por uma grande diversidade de aplicações, estendemos a definição de operador integral de Cauchy para espaços de Lebesgue sobre grupos topológicos. Assim, são investigadas as condições de invertibilidade dos operadores integrais neste contexto.

keywords

singular integral operators, Carleman shift operators, operator equivalence relations, Fredholm property, invertibility, collocation methods, singular integral equations, projection methods, boundary value problems.

abstract

In this thesis we consider singular integral operators with the extra action of a Carleman shift operator and having coefficients on different classes of essentially bounded functions. Namely, continuous, piecewise continuous, semi-almost periodic and generalized factorable functions.
In the cases of the singular integral with shift action given by the reflection or the flip operator on the complex unit circle, we obtain a Fredholm criteria and, for the continuous coefficients case, an index formula is also provided. These results are consequence of explicit equivalence operator relations between those operators and some extra operators such as pure singular integral, Toeplitz and Toeplitz plus Hankel operators. Furthermore, the equivalence relations allow us to give an invertibility criterion and formulas for the left-sided and right-sided inverses of the initial operators with generalized factorable coefficients. In addition, we apply numerical analysis techniques, as polynomial collocation methods, for the study of the kernel dimension of these two kinds of singular integral operators with piecewise continuous coefficients. This approach also permits us to compute the Moore-Penrose inverse of the main operators.
For singular integral operators with generic preserving-orientation Carleman shift operators and continuous functions as coefficients, upper bounds for the kernel dimensions are obtained. This is implemented by using some estimations which are derived with the help of certain explicit operator relations.

We also focus our attention to the solvability, and the solutions, of a class of singular integral equations with shift which cannot be reduced to a binomial boundary value problem. To attain our goals, some complementary projections and operator identities are used. In this way, the equations under study are associated with systems of pure singular integral equations. These systems will be then analyzed by means of a corresponding Riemann boundary value problem. As a consequence of such a procedure, the solutions of the initial equations are constructed from the solutions of Riemann boundary value problems.

Motivated by a large diversity of applications, we extend the definition of Cauchy integral operator to the framework of Lebesgue spaces on topological groups. Thus, invertibility conditions for paired operators in this setting are investigated.

Contents

List of Symbols

$\mathcal{A}$	characteristic singular integral operator, 14
A^*	adjoint matrix (operator) of A, 74
A^+	Moore-Penrose inverse of A, 85
$\mathbb{A}/J$	quotient Banach algebra, 2
$A_{n,r}$	$:= L_n(aI_{\mathbb{T}} + bS_{\mathbb{T}})P_n(P_n - W_nP_{r-1}W_n), \quad n \in \mathbb{Z}_+$, 79
$\mathbb{A}(z, w; v)$	$= \{\lambda \in \mathbb{C} \setminus \{z, w\} \ : \ \arg \frac{\lambda - z}{\lambda - w} \in 2\pi v + 2\pi\mathbb{Z}\} \cup \{z, w\}$, 9
$\alpha(\mathfrak{T})$	$(= \dim \ker \mathfrak{T})$ dimension of the kernel of the operator $\mathfrak{T}$, 2
$\mathrm{alg}\,(a, \ldots, x)$	algebra generated by products of $a, \ldots, x$, 67
$A_p(\Gamma)$	set of Hunt–Muckenhoupt–Wheeden weights, 5
$A_p^e(\mathbb{T})$	$:= \{w \in A_p(\mathbb{T}) : \ w(-t) = w(t), \ t \in \mathbb{T}\}$, 22
$AP(\mathbb{R})$	algebra of almost-periodic functions on $\mathbb{R}$, 11
$AP^0(\mathbb{R})$	set of almost-periodic polynomials on $\mathbb{R}$, 12
$AP_\pm(\mathbb{R})$	set of function $a \in AP$ with $\Omega(a) \subset \mathbb{R}_\pm$, 12
$APW(\mathbb{R})$	Wiener subalgebra of AP, 12
$APW_\pm(\mathbb{R})$	$= APW(\mathbb{R}) \cap AP_\pm(\mathbb{R})$, 12
$a_0^{p,w}$	$: \mathcal{M}_{p,w}^0 \longrightarrow \mathbb{C}^{2\times 2}$, $a_0^{p,w}(t, \mu) := (1 - \mu)a(t - 0) + \mu a(t + 0)$, 40
$a^{p,w}$	$: \mathcal{M}_{p,w} \longrightarrow \mathbb{C}^{2\times 2}$, $a^{p,w}(t, \mu) := (1 - \mu)a(t - 0) + \mu a(t + 0)$, 40
$\arg z$	argument of z, 9
B	isometric isomorphism from $L^p(\mathbb{T}, \rho)$ onto $L^p(\mathbb{R})$, 30

B_0	isometric isomorphism from $L^\infty(\mathbb{R})$ onto $L^\infty(\mathbb{T})$, 12
B_2	diagonal matrix operator $\mathrm{diag}(B, B)$, 33
B^2	Besicovitch space, 64
$\beta(\mathfrak{T})$	$(= \dim\operatorname{coker}\mathfrak{T})$ dimension of the cokernel of the operator $\mathfrak{T}$, 2
$\mathbb{C}$	complex plane, 4
$C(\mathbb{R})$	space of continuous functions on $\mathbb{R}$, 11
$C(\overline{\mathbb{R}})$	set of continuous functions with finite limits at $\pm\infty$, 11
$C_0(\mathbb{R})$	set continuous functions vanishing at $-\infty$ and $+\infty$, 11
$\chi_{\mathbb{T}_+}$	characteristic function of $\mathbb{T}_+$, 22
$\operatorname{coker}\mathfrak{T}$	quotient $Y/\operatorname{Im}\mathfrak{T}$ with $\mathfrak{T} \in \mathcal{L}(X, Y)$, 2
$\mathbb{D}$	complex unit disk, 116
$d(c)$	geometric mean value of c, 64
$E^p(\mathbb{D}, \rho)$	weighted Smirnov class, 116
$\mathcal{G}\mathbb{A}$	group of all invertible elements on a Banach algebra $\mathbb{A}$, 2
$G^{\pm 1}(t)$	diagonal matrix function $\mathrm{diag}(1, t^{\pm 1})$, 27
Γ	Carleson curve, 4
$\Gamma(t, \epsilon)$	portion of Γ contained on the disk of radius ϵ centered at t, 4
$\mathcal{H}_b$	Hankel operator with Fourier symbol b, 35
$\mathcal{H}^s(\mathbb{T})$	Hölder-Zygmund space, 82
$\mathcal{H}(z, w; u, v)$	horn between z and w whose boundary arcs are $\mathbb{A}(z, w; u)$ and $\mathbb{A}(z, w; v)$, 10
$H^p_\pm(\Gamma, w)$	weighted Hardy spaces on Γ, 7
$\widetilde{H}^p_-(\mathbb{T}, \sigma)$	$:= H^p_-(\mathbb{T}, \sigma) \oplus \mathbb{C}$, 7
$\Im m z$	imaginary part of a complex number z, 10
$\operatorname{Im}\mathfrak{T}$	image of the operator $\mathfrak{T}$, 2
$\operatorname{Ind}\mathfrak{T}$	$(= \dim\ker\mathfrak{T} - \dim\operatorname{coker}\mathfrak{T})$ Fredholm index of $\mathfrak{T}$, 2
$\widetilde{J}$	flip operator, 19

ℓ_0	the zero extension operator from $\mathbb{T}_+$ to $\mathbb{T}$, 22
$l^2(\mathbb{R})$	collection of all functions $f : \mathbb{R} \longrightarrow \mathbb{C}$ for which the set $\{\lambda \in \mathbb{R} \,:\, f(\lambda) \neq 0\}$ is at most countable and $\|f\|^2_{l^2(\mathbb{R})} := \sum |f(\lambda)|^2 < \infty$, 65
$L^p(\Gamma)$	classic Lebesgue space, 4
$L^p(\Gamma, w)$	weighted Lebesgue space, 5
$L^\infty(\Gamma)$	Banach algebra of essentially bounded functions on Γ, 5
$[L^\infty(\Gamma)]^{N\times N}$	the algebra of matrix functions with entries in $L^\infty(\Gamma)$, 5
$l^\infty(\mathbb{R})$	set of all functions $f : \mathbb{R} \longrightarrow \mathbb{C}$ such that $\|f\|_{l^\infty(\mathbb{R})} := \sup_{\lambda\in\mathbb{R}} |f(\lambda)| < \infty$, 65
$\mathcal{L}(X)$	$\mathcal{L}(X) \equiv \mathcal{L}(X, X)$, 1
$\mathcal{L}(X, Y)$	space bounded and linear operators from X into Y, 1
$M(a)$	Bohr mean value of a function $a \in AP(\mathbb{R})$, 12
$\mathcal{M}_{p,w}$	$:= \bigcup_{t\in\mathbb{T}}(\{t\} \times \mathcal{H}(0, 1; \nu_t^-(p, w), \nu_t^+(p, w)))$, 40
$\mathcal{M}^0_{p,w}$	$:= \bigcup_{t\in\mathbb{T}}(\{t\} \times \mathbb{A}(0, 1; \nu_t^0(p, w)))$, $\nu_t^0(p, w) = \frac{1}{2}(\nu_t^-(p, w) + \nu_t^+(p, w))$, 40
$M_{\mathbb{R}_+}$	operator in $\mathcal{L}([L^p(\mathbb{R}_+)]^2, L^p(\mathbb{R}))$, 31
$M_{\mathbb{T}_+}$	operator in $\mathcal{L}([L^p(\mathbb{T}_+, w)]^2, L^p(\mathbb{T}, w))$, 22
$N_{\mathbb{R}_+}$	operator in $\mathcal{L}([L^p(\mathbb{R}_+, |x|^{-1/2p})]^2, [L^p(\mathbb{R}_+)]^2)$, 31
$N_{\mathbb{T}_+}$	operator in $\mathcal{L}([L^p(\mathbb{T}, w)]^2, [L^p(\mathbb{T}_+, w)]^2)$, 27
$\mathcal{O}$	singular integral operator with shift, 20
$\Omega(a)$	Bohr-Fourier spectrum of $a \in AP(\mathbb{R})$, 12
$P_\pm$	Riesz projections, 7
P_k	complementary projections, $k = 1, 2$, 118
$PC(\mathbb{R})$	space of piecewise continuous functions on $\mathbb{R}$, 11
$\varphi^\pm(x_0)$	$= \lim_{x\to x_0^\pm} \varphi(x)$, 11
$\psi_\mathbb{T}$	complex-valued function defined by $(\upsilon_\mathbb{T} - \vartheta_\mathbb{T})^{-1}(\upsilon_\mathbb{T} + \vartheta_\mathbb{T})$, 28
$R_{\mathbb{R}_+}$	linear and bounded matrix operator on $[L^p(\mathbb{R}_+)]^2$, 31

$\mathbb{R}$	real line, 4
$\mathbb{R}_+$	positive semi-line $[0, \infty)$, 4
$\mathbb{R}_-$	negative semi-line $(-\infty, 0]$, 4
$r_{_{\mathbb{T}_+}}$	restriction operator to $\mathbb{T}_+$, 23
$\mathcal{R}(a)$	essential range of a, 9
$\mathcal{R}(\mathbb{T})$	algebra of all rational functions, 82
$SAP(\mathbb{R})$	algebra of semi-almost periodic functions on $\mathbb{R}$, 11
S_Γ	Cauchy singular integral operator along Γ, 5
$s_k(A)$	the k-th approximation number of an operator $A \in \mathcal{L}(F)$, 74
$\operatorname{sp} T$	spectrum of $T \in \mathcal{L}(X)$, 102
$\operatorname{sp}_{\mathrm{ess}} T$	essential spectrum of $T \in \mathcal{L}(X)$, 8
$\mathbb{T}$	complex unit circle, 4
$\mathbb{T}_+$	$:= \{\tau \in \mathbb{T} : 0 < \arg \tau < \pi\}$, 22
$\mathfrak{T}^-$	generalized inverse of an operator $\mathfrak{T}$, 1
$\mathfrak{T}^{(-1)}$	left (respectively right) inverse of operator $\mathfrak{T}$, 1
$\mathcal{T}_a$	Toeplitz operator with Fourier symbol a, 8
$\mathcal{T}_a + \mathcal{H}_b$	Toeplitz plus Hankel operator with Fourier symbols a and b, 35
$u_{\mathbb{R}}$	complex-valued matrix function on $\mathbb{R}$, 33
$u_{\mathbb{T}}$	$= \operatorname{diag}(B_0, B_0) u_{\mathbb{R}}$, 33
$\upsilon_{\mathbb{T}}$	complex-valued matrix function on $\mathbb{T}$, 28
$v_{\mathbb{R}}$	complex-valued matrix function on $\mathbb{R}$, 33
$v_{\mathbb{T}}$	$= \operatorname{diag}(B_0, B_0) v_{\mathbb{R}}$, 33
$\vartheta_{\mathbb{T}}$	complex-valued matrix function on $\mathbb{T}$, 28
$\operatorname{wind} \varphi$	winding number of a continuous function φ, 19
$W_{\mathbb{R}}$	reflection shift operator on $L^p(\mathbb{R})$, 18
$[X]^N$	direct sum of N copies of X, 1
$\otimes$	Kronecker's product, 66

Introduction

The first formulations of linear boundary values problems (BVP's) for analytic functions were due to Riemann in 1853 (see, [78]), while the theory of singular integral equations with their integral in the sense of its principal value was originated almost directly after the development of the classical theory of integral equations by E. Fredholm in 1903. Singular integral equations were investigated by D. Hilbert [45, 46] and H. Poincaré [74], while studying two different problems, Hilbert when investigating some boundary value problems of analytic functions and Poincaré while studying the theory of tides. J. Plemelj [73] applied further the Cauchy singular integral as a mathematical device for solving boundary value problems. However, the theory of singular integral equations (SIE's) did not receive the attention of the mathematicians for some time. The general properties of singular integral equations were further established by F. Nöther [71], while the solutions of singular integral equations of the convolution type were further improved by N. Wiener and E. Hopf [99].

The Cauchy singular integrals as a mathematical tool to various fields of science as *Mathematical Physics and Mechanical, Economics, Medicine*, among others, and numerous applications in: *Theory of elasticity, plasticity, aerodynamics, hydromechanics engineering, heat and mass transfer, oscillation theory, fluid dynamics, filtration theory, electrostatics, electrodynamics, biomechanics, game theory, control, queuing theory, electrical engineering* has been extensively studied by several authors: M. Riesz, N.I. Muskhelishvili, A.P. Calderón & A. Zygmund, F.D. Gakhov, I.N. Vekua, A. Dzhuraev, E.M. Stein, S.G. Mikhlin & S. Prössdorf, E. Meister, H. Widom are probably some of the most known among them. More information can be found, for one-dimensional equations, on the classics books [39], [69], [93], [96] and references therein.

Thus, due to these wide range of applications, it was primordial to develop the mathematical fundamentals of the theory of singular integral operators (SIO's). The monographs of Clancey & Gohberg [29], Gohberg & Krupnik [41] and Mikhlin & Prössdorf [67] are some resources that collect much of this groundwork. The same reasons mentioned above led to generalizations of the Cauchy integral operator. For example, defining these operators on different type of curves [67], in octonionic spaces [62], as well as defining the so-called Calderón-Zygmund type operators on surfaces [72, 82], singular convolution operators on the Heisenberg group [65] and right convolution operators on homogeneous groups [77], are some of the most popular generalizations related to the Cauchy integral operator. Since then, a lot of works in this subject was already published and, owing to this development, new and more sophisticated methods are now available to solve singular integral equations. See, e.g. [43], [76], [81].

Singular integral equations with a shift (SIES's) are connected with such boundary values problems in a natural way. So, subsequent to the works of Riemann and Hilbert, Haseman (1907) and Carleman (1932) carried out an analogous investigation of problems of this type. Historically, the paper by Haseman [44] was the first work in which the boundary value problem with a shift (BVPS) was considered for analytic functions. The successful development of the theory of SIE and BVP naturally stimulated the study of SIES and BVPS. The papers by D. Kveselava [59, 60] were fundamental in this direction. The study of singular integral operators with conjugation began in the years 40-50 of the last century precisely with the investigation of boundary values problems for analytic functions with conjugation, namely by N. Markushevich (1946), N. Vekua (1952), Boiarskii (1952). The paper by N. Vekua [95] is identified as the first paper in which SIES were considered.

Up to the present many publications were devoted to these problems, and an important part of these investigations was devoted to the following two main directions: (i) The study of the Fredholm theory of SIES and (ii) The solvability theory of SIE and BVPS. The following monographs synthesize a lot of these developments: F.D. Gakhov [39], N. Karapetiants and S. Samko [48], V.G. Kravchenko and G.S. Litvinchuk [55], G.S. Litvinchuk [64], S.G. Mikhlin and S. Prössdorf [67]; cf. also the references therein.

As a result of all knowledge on this topic, several applications have been developed to the theories as: *Theory of the cavity currents in an ideal liquid, theory of infinitesimal bounds of surfaces with positive curvature, contact theory of elasticity and physics of plasma.* However, despite this development, the mathematical fundamental understanding of singular integral operators with shift (SIOS's) is far to be complete.

Hence, in this thesis we are going to investigate mathematical fundamental theory of that kind of operators. More precisely, we will study regularity properties of singular integral operators with shift, i.e., those properties that arise from a direct influence of the kernel and image of the operators. In particular, we will give criteria for the Fredholm property, explicit representation of (generalized, left, right) inverses and (bounds) dimensions of their kernels. This will be achieved by means of certain relations between operators, that allow us to derive the regularity properties of these singular integral operators with shift from the regularity properties of other operators (as pure singular integral, Toeplitz and Toeplitz plus Hankel) for which results concerning to regularity properties are already known. Besides, in order to apply numerical tools to the study of the solutions of singular integral equations with shift, we will provided conditions under which singular integral operators with shift can be rewritten in a convenient way to use numerical analysis machinery as approximation methods.

In addition, we will deal with the problem of finding solvability conditions for the existence and uniqueness, as well as the representations of solutions of a class singular integral equations with (weighted) Carleman shift, which cannot be reduced to a binomial boundary value problem. The approach implemented here consists in the use of some complementary projections which relate the solutions of the original equation with the solutions of a (2×2)-system of (pure) equations. In this fashion, the desired solutions can be constructed from the solutions of the equations in the related system, instead of the use of the gluing technique or by a Haseman problem (cf., e.g., [64]). Even more, this method only requires solutions of each equation separately and not the full solution of the (2×2)-system.

Finally, we will give a generalization of the Cauchy integral operator acting on Lebesgue spaces over compact abelian groups. In this framework, we will investigate the invertibility conditions of paired singular integral operators for different kinds of coefficients.

Let us now give a survey of the chapters.

The nature of the first chapter is completely expository, the basic concepts and results from classical Operator Theory that will be needed further on are introduced. Also this chapter serves to formalize the notation used in the sequel. In particular, are introduced the definitions of: weighted Lebesgue and Hardy spaces, Cauchy integral, Toeplitz and shift operators, so as well as, the different classes of essentially bounded functions which will play a central role in our study of singular integral operators with shift. Only selected results closely related with the main goal of our interests are proved. So, that chapter can be skipped and be used as a reference while reading the others at the reader's discretion.

In Chapter 2 we will show several explicit equivalence operator relations between singular integral operators with shift and some extra operators. Namely, we are going to consider the class of singular integral operators with the extra action of the reflection shift operator on the complex unit circle, as well as the class of singular integral operators with so-called flip operator. Proceeding as in [49, 50], for the first class we will exhibit an equivalence relation with a pure matrix singular integral operator, while for the second one, will be constructed an equivalence after extension relation with a pure matrix singular integral operator and, in addition as in [36], a similarity relation with a class of matricial Toeplitz plus Hankel operators. All of these equivalence relations given in this chapter will allow us to describe, in subsequent chapters, the Fredholm characteristics of these operators and also permit us to build different types of inverses. Even more, due to these equivalence relations, we can compute the dimensions of the kernel of these operators (under the Fredholm property), as well as to study the applicability of numerical methods like: polynomial collocation, approximation and projection methods, or inclusively, study stability and the order of convergence of that mentioned methods in those operators.

The main purpose of Chapter 3 is to use the equivalence operator relations presented in Chapter 2, in order to establish Fredholm and invertibility criteria for singular integral operators with the extra action of the reflection shift and flip operators. These results will depend on the class of essentially bounded functions to which the coefficients of the operators belong. For Fredholm criteria, we consider the coefficients in the class of continuous, piecewise continuous and semi-almost-periodic functions, whereas for the invertibility criterion we consider generalized factorizable essentially bounded functions as coefficients. The mentioned explicit equivalence operator relations allow us to extract the Fredholm characteristics, as well as the invertibility conditions, from the related equivalent operators, i.e., we will take advantage of the Fredholm results given by Gohberg-Krein and Douglas concerning to Toeplitz operators with continuous and piecewise continuous symbols (see, [11]) and the results by T. Ehrhardt about Toeplitz plus Hankel operators with the same kind of symbols [36]. Also, we will adapt a result by A. Böttcher, Y.I. Karlovich and I.M. Spitkovsky [12], which deals with the Fredholmness of singular integral operators with semi-almost-periodic coefficients, to the case of Toeplitz plus Hankel operators with the same class of coefficients. The invertibility conclusions will be obtained from a generalization of the well–known Simonenko's invertibility Theorem for pure matrix singular integral operators with coefficients admitting a generalized factorization in L^p (see e.g., [67]). Several examples are presented in order to show the applicability of the results.

Chapter 4 is devoted to study the dimension of the kernels of Fredholm singular integral operators with shift. The chapter will be divided in two main parts. More precisely: In the first part we are going to compute the dimension of the kernel of the singular integral operators with reflection and flip having piecewise continuous functions as coefficients (since in Chapter

3 we characterize the Fredholm property of them). To attain such a goal, we are going to use stable finite dimension projection methods (in the form of elements in a standard C^*-algebra [43]), collocation polynomial methods and the so-called k-splitting property. That kernel dimension will depend on the number of singular values, tending to zero, of the projection method associated to the singular integral operators; therefore, we will study the convergence speed of that singular values. In addition, the used strategy will allows us to compute the Moore-Penrose inverses of those singular integral operators (subjected to appropriate conditions). In the second part of this chapter, upper bounds for the kernel dimension of singular integral operators with generic orientation-preserving weighted Carleman shift and continuous coefficients are obtained. This is implemented by using a Fredholm characterization of an equivalent singular integral operator (up to an invertibility criterion of a related functional operator [55]) and some estimations which are derived with the help of certain explicit operator relations. In particular, the interplay between classes of operators with and without Carleman shifts have a preponderant importance to achieve the mentioned bounds. Examples showing the techniques developed here are properly provided.

In Chapter 5, we will present conditions that guarantee a convenient representation of singular integral operators with anti-commutative Carleman shift and piecewise continuous symbols. The convenience of this representation arises from the fact that it allows, in a more easy way, the use of numerical analysis tools such as Garlekin or finite section methods to compute the solutions of equations modeled by singular integral operators with shift. In order to establish such a representation, we will use a symbol calculus from C^*-algebra generated by two idempotents and a flip given in [79].

The solvability of a class of singular integral equations with reflection and factorable essentially bounded coefficients, in weighted Lebesgue spaces is analyzed, and the corresponding solutions are obtained in Chapter 6. The main techniques are based on the consideration of certain complementary projections and operator identities used in [27, 28, 94]. Therefore, the equations under study are associated with systems of pure singular integral equations. These systems will be then analyzed by means of a corresponding Riemann boundary value problem. As a consequence of such a procedure, the solutions of the initial equations are constructed from the solutions of Riemann boundary value problems. The method is also applied to singular integral equations with the so-called *commutative* and *anti-commutative* weighted Carleman shifts. In the final part of the chapter we will consider a simpler case of singular integral equations on Carleson curves with commutative and anti-commutative weighted Carleman shifts having continuous coefficients.

Using the Fourier's coefficient representation of functions defined on Lebesgue spaces over a compact, connected, multiplicative and abelian group, see e.g. [1, 66, 86], in Appendix A, we define bounded operators which can be considered to be a version of singular integral operators over this kind of groups. The main purpose of this part is to investigate conditions that guarantee the existence of the inverse, and in such case the form of the (lateral) inverse(s), of the mentioned operators. We consider as coefficients for these operators, trigonometrical polynomial functions and essentially bounded functions satisfying a factorization concept introduced by L. Rodman and I.M. Spitkovsky in [83], which is analogous to the notion of factorization in Banach algebras (cf., page 12). In order to obtain that mentioned generalization, the topological group is provided of an order which plays a principal role in the whole appendix, mainly because both the operator definition and the factorization notion depend on it.

We would like to point out that the new results presented in this thesis are mainly based in works of the author, which are published in journals and conference proceedings [14, 15, 17, 18, 19, 20]. The non–published material appears in the following author's accepted works [21, 23, 24] as well as in the submitted works [2, 16, 22]. The auxiliary results of other authors included in the manuscript are properly referred.

Chapter 1

Notation, Definitions and Auxiliary Statements

Let us start by recalling some basic definitions which will be used in what follows. Most of the results in this chapter are well-known and can be found in classical Functional Analysis and Operator Theory textbooks. Therefore, we do not here refer to a particular source. However, some of them are directly connected with the operators under study, in which case we give a proof and the proper reference.

1.1 General operator theoretic preliminaries

Considering Banach spaces X and Y, the space of all *bounded linear operators* from X into Y is denoted as $\mathcal{L}(X,Y)$ and $\mathcal{L}(X,X) \equiv \mathcal{L}(X)$. $[X]^N := \bigoplus_{i=1}^{N} X_i$ is the *direct sum of N copies of X*, and the elements in this space can be written as column vectors of length N with entries on X. By $\mathcal{K}(X,Y)$ we mean the *algebra of all compact operators* from X into Y which is an ideal of the Banach algebra $\mathcal{L}(X,Y)$.

An operator $\mathfrak{T} \in \mathcal{L}(X,Y)$ is said to be *left* (resp. *right*) *invertible* if there is an operator $\mathfrak{T}^{(-1)} \in \mathcal{L}(Y,X)$ such that

$$\mathfrak{T}^{(-1)}\mathfrak{T}x = x, \quad x \in X \qquad (\mathfrak{T}\mathfrak{T}^{(-1)}y = y, \quad y \in Y).$$

The operator $\mathfrak{T}^{(-1)}$ is then called a *left* (resp. *right*) *inverse* of $\mathfrak{T}$. If an operator $\mathfrak{T}$ is both left and right invertible, then all left and right inverses are equal to each other and coincide with the inverse $\mathfrak{T}^{-1}$ of $\mathfrak{T}$. Recall also that an operator $\mathfrak{T}^{-} : Y \longrightarrow X$ is called a *generalized inverse* of a bounded linear operator $\mathfrak{T} : X \longrightarrow Y$ if it satisfies the relation

$$\mathfrak{T}\mathfrak{T}^{-}\mathfrak{T} = \mathfrak{T}.$$

The *group of all invertible elements in a Banach algebra* $\mathbb{A}$ will be denoted by $\mathcal{G}\mathbb{A}$. If J is an ideal in $\mathbb{A}$, then the quotient $\mathbb{A}/J$ becomes a Banach algebra with the norm $\|\widehat{x}\| = \|x + J\| = \inf\{\|x + j\| \ : \ j \in J\}$.

1.1.1 The Fredholm property

In this part we introduce the notion of Fredholmness of an operator $\mathfrak{T} \in \mathcal{L}(X, Y)$ and we mention some corresponding properties that will be useful in the sequel. More information about the Fredholm property can be found, for instance, in [11, 13, 41, 48, 55, 67].

The *kernel* $\ker \mathfrak{T}$ and the *image* $\operatorname{Im} \mathfrak{T}$ of the operator $\mathfrak{T}$ are linear subspaces of X and Y, respectively, which are defined as follows:

$$\ker \mathfrak{T} := \{x \in X \ : \ \mathfrak{T}x = 0\}, \quad \operatorname{Im} \mathfrak{T} := \{\mathfrak{T}x \ : \ x \in X\}.$$

In case that $\operatorname{Im} \mathfrak{T}$ is closed, we call operator $\mathfrak{T}$ to be *normally solvable*. Notice that $\ker \mathfrak{T}$ is always closed. Assuming that $\mathfrak{T}$ is normally solvable, the *cokernel* of $\mathfrak{T}$ is defined by the quotient:

$$\operatorname{coker} \mathfrak{T} := Y / \operatorname{Im} \mathfrak{T}.$$

The *defect numbers* of $\mathfrak{T}$ are the integers

$$\alpha(\mathfrak{T}) := \dim \ker \mathfrak{T}, \qquad (\textit{nullity})$$

and

$$\beta(\mathfrak{T}) := \dim \operatorname{coker} \mathfrak{T}, \qquad (\textit{deficiency}).$$

A normally solvable operator $\mathfrak{T}$ is called a *Fredholm* if both $\alpha(\mathfrak{T})$ and $\beta(\mathfrak{T})$ are finite. In this case, the *Fredholm index* of $\mathfrak{T}$ is defined by the finite number

$$\operatorname{Ind} \mathfrak{T} := \alpha(\mathfrak{T}) - \beta(\mathfrak{T}).$$

Several characterizations of the Fredholm property are known.

Theorem 1.1. *$\mathfrak{T} \in \mathcal{L}(X, Y)$ is Fredholm if and only if $\widehat{\mathfrak{T}}$ is invertible in the Calkin algebra $\mathcal{L}(X, Y)/\mathcal{K}(X, Y)$.*

The operator $\mathfrak{T}$ is said to admit a *left regularization* if there exists an operator $\mathfrak{R}_l \in \mathcal{L}(Y, X)$ such that

$$\mathfrak{R}_l \mathfrak{T} = I_X + K_X,$$

where I_X is the identity operator and K_X is a compact operator on X. The operator $\mathfrak{R}_l$ is called a *left regularizer* of $\mathfrak{T}$. We say that the operator $\mathfrak{T}$ admits a *right regularization* if there exists an operator $\mathfrak{R}_r \in \mathcal{L}(Y, X)$ such that

$$\mathfrak{T}\mathfrak{R}_r = I_Y + K_Y,$$

with I_Y being the identity operator and K_Y a compact operator on Y. In that case $\mathfrak{R}_r$ is referred to as a *right regularizer* of $\mathfrak{T}$. If an operator $\mathfrak{T}$ admits both a left and right regularization, then $\mathfrak{T}$ is said to admit a *two-sided regularization.*

In order to establish our Fredholm criteria the following well-known results will be necessary:

Theorem 1.2. *For an operator $A \in \mathcal{L}(X, Y)$ the following properties are equivalent:*

(i) *A is a Fredholm operator.*

(ii) *A admits a two-sided regularization.*

Theorem 1.3. *Let A be a Fredholm operator and suppose that $T : X \longrightarrow Y$ is compact. Then $A + T$ is Fredholm and* $\operatorname{Ind}(A + T) = \operatorname{Ind} A$.

1.1.2 Equivalence relations between operators

To study certain linear bounded operators, very frequently we need to transfer properties between one operator to another somehow equivalent operator. In view of this, we introduce some operator relations for bounded linear operators $\mathfrak{T} : X_1 \to X_2$ and $\mathfrak{S} : Y_1 \to Y_2$, acting between Banach spaces. More precisely, the operators $\mathfrak{T}$ and $\mathfrak{S}$ are said to be *equivalent* [3, 26] if there are two boundedly invertible linear operators, $\mathfrak{E} : Y_2 \to X_2$ and $\mathfrak{F} : X_1 \to Y_1$, such that

$$\mathfrak{T} = \mathfrak{E}\, \mathfrak{S}\, \mathfrak{F}. \tag{1.1}$$

Additionally, in the particular case of $\mathfrak{E} = \mathfrak{F}^{-1}$ in (1.1), we will say that we have a *similarity relation* between the operators $\mathfrak{T}$ and $\mathfrak{S}$. In the sequel of the work we will also use the notion of *equivalence after extension relation* (cf., e.g., [3]): the operators $\mathfrak{T}$ and $\mathfrak{S}$ are called *equivalent after extension* if

Banach spaces Z and W exist such that $\mathfrak{T} \oplus I_Z$ and $\mathfrak{S} \oplus I_W$ are equivalent operators. I.e.,

$$\begin{pmatrix} \mathfrak{T} & 0 \\ 0 & I_Z \end{pmatrix} = \mathfrak{E} \begin{pmatrix} \mathfrak{S} & 0 \\ 0 & I_W \end{pmatrix} \mathfrak{F}. \tag{1.2}$$

It follows from (1.1) (and (1.2)) that if two operators are equivalent (after extension), then they belong to the same regularity class [25, 92]. Here, by *regularity class* we mean all that properties arising from a direct influence of the kernel and image of an operator. Thus, one of these operators is invertible, one-sided invertible, Fredholm, one-sided regularizable, generalized invertible or normally solvable, if and only if the other operator enjoys the same property. This important consequence of the equivalence operator relations will have a global preponderance also in this thesis.

1.2 Cauchy integral and Toeplitz operators

1.2.1 Weighted Lebesgue spaces

We refer to a subset Γ of the complex plane $\mathbb{C}$ as an *arc* if it is homeomorphic to a connected subset of the real line $\mathbb{R}$ which contains at least two distinct points. Equivalently, $\Gamma \subset \mathbb{C}$ is an arc if and only if Γ is homeomorphic to one of the sets $[0,1]$, $\mathbb{R}_+ := [0,\infty)$ ($\mathbb{R}_- := (-\infty, 0]$) or $(-\infty,\infty)$. A subset Γ of $\mathbb{C}$ is referred to as a *Jordan curve* if it homeomorphic to the complex unit circle $\mathbb{T} := \{t \in \mathbb{C} \ : \ |t| = 1\}$.

Now suppose that Γ is a curve, and for a point $t \in \Gamma$ and a number $\epsilon \in (0,\infty)$, let $\Gamma(t,\epsilon) := \{\tau \in \Gamma : |\tau - t| < \epsilon\}$ stand for the portion of Γ contained on the open disk of radius ϵ centered at t. If all these arcs are rectifiable and the sum of their lengths is finite, we say that $\Gamma(t,\epsilon)$ is *rectifiable*. The curve Γ is *locally rectifiable* if $\Gamma(t,\epsilon)$ is rectifiable for every $t \in \Gamma$ and $\epsilon \in (0,\infty)$.

Let Γ be a rectifiable curve and equip Γ with the Lebesgue length measure. The measure of a measurable subset $\gamma \subset \Gamma$ will be denoted by $|\gamma|$. The curve Γ is said to be a *Carleson curve* (a *Jordan-Carleson curve*) if

$$C_\Gamma := \sup_{t\in\Gamma} \sup_{\epsilon>0} \frac{|\Gamma(t,\epsilon)|}{\epsilon} < \infty.$$

In other words, Γ is a Carleson curve if and only if there is a constant C_Γ such that $|\Gamma(t,\epsilon)| \leq C_\Gamma \epsilon$ and all $\epsilon > 0$.

For a Carleson curve, let $L^p(\Gamma)$ ($1 \leq p < \infty$) be the classic *Banach space of all measurable functions φ on Γ which are absolutely integrable in the p-th power* and given a weight $w : \Gamma \longrightarrow [0,+\infty]$, whose preimage $w^{-1}(\{0,\infty\})$

has measure zero, we let $L^p(\Gamma, w)$ $(1 \leq p < \infty)$ stand for the weighted Lebesgue space with the norm

$$\|\varphi\|_{p,w} := \left(\int_\Gamma |\varphi(\tau)|^p w(\tau)^p |d\tau| \right)^{1/p}.$$

By $L^\infty(\Gamma)$ we denote the Banach algebra of all *essentially bounded and Lebesgue measurable* complex valued functions defined on Γ endowed with the norm

$$\|\phi\|_\infty := \operatorname{ess\,sup}\{|\phi(t)| \;:\; t \in \Gamma\}.$$

In addition, $[L^\infty(\Gamma)]^{N\times N}$ stands for the algebra of all *matrix functions* $a : \Gamma \longrightarrow \mathbb{C}^{N\times N}$ *with entries in* $L^\infty(\Gamma)$.

We would like to point out that, usually, in this thesis we will use as the curve Γ the unitary circle $\mathbb{T}$, the real line $\mathbb{R}$ or the semi-lines $\mathbb{R}_\pm$.

1.2.2 The Cauchy integral operator

As usual, we denote by $A_p(\Gamma)$ the set of all *weights* $w : \Gamma \longrightarrow [0, +\infty]$ *such that* $w \in L^p(\Gamma)$, $w^{-1} \in L^q(\Gamma)$, $\frac{1}{p} + \frac{1}{q} = 1$, *and*

$$c_w := \sup_{t\in\Gamma} \sup_{\varepsilon>0} \left(\frac{1}{\varepsilon} \int_{\Gamma(t,\varepsilon)} w(\tau)^p |d\tau| \right)^{1/p} \left(\frac{1}{\varepsilon} \int_{\Gamma(t,\varepsilon)} w(\tau)^{-q} |d\tau| \right)^{1/q} < \infty. \quad (1.3)$$

The condition (1.3) is called the *Hunt–Muckenhoupt–Wheeden condition*, and $A_p(\Gamma)$ is referred to as the set of *Hunt–Muckenhoupt–Wheeden weights.*

Suppose henceforth that Γ is oriented with the natural orientation in the counterclockwise sense.

Definition 1.1. The Cauchy singular integral operator S_Γ of a function $f : \Gamma \longrightarrow \mathbb{C}$ at a point $t \in \Gamma$ is given by

$$(S_\Gamma f)(t) := \lim_{\varepsilon\to 0} \frac{1}{\pi i} \int_{\Gamma\setminus\Gamma(t,\varepsilon)} \frac{f(\tau)}{\tau - t} |d\tau|.$$

Theorem 1.4 (cf., e.g., [8])**.** *Let Γ be a Carleson curve, $1 < p < \infty$, and $w \in A_p(\Gamma)$. Then, $(S_\Gamma f)(t)$ is well-defined and finite for every $t \in \Gamma$ and every $f \in L^p(\Gamma, w)$.*

Proof. The proof runs by steps. First we are going to prove the desired fact for the truncate Cauchy integral operator

$$(S_\varepsilon f)(t) := \frac{1}{\pi i} \int_{\Gamma\setminus\Gamma(t,\varepsilon)} \frac{f(\tau)}{\tau - t} |d\tau|.$$

If Γ is bounded, then the proof is direct, since $w \in L^p(\Gamma)$, $w^{-1} \in L^q(\Gamma)$ and thus $L^p(\Gamma, w) \subset L^1(\Gamma)$. So assume that Γ is unbounded. For $f \in L^p(\Gamma, w)$, we have

$$|(S_\varepsilon f)(t)| \leq \left(\int_{|\tau - t| \geq \varepsilon} |f(\tau)|^p w(\tau)^p |d\tau| \right)^{1/p} \left(\int_{|\tau - t| \geq \varepsilon} \frac{w(\tau)^{-q}}{|\tau - t|^q} |d\tau| \right)^{1/q},$$

where the integration over $|\tau - t| \geq \varepsilon$ means integration over $\Gamma \setminus \Gamma(t, \varepsilon)$. The first factor on the right is at most $\|f\|_{p,w}$ and hence the assertion will follow as soon as we have shown that the second factor is finite.

Since $w \in A_p(\Gamma)$, it follows that $w^{-1} \in A_q(\Gamma)$. From Corollary 2.32 in [8], it can be deduced that there is a $r \in (1, q)$ such that $w^{-q/r} \in A_r(\Gamma)$, which implies the existence of a constant $C < \infty$ such that

$$\left(\int_{|\tau - t| < y} w(\tau)^{-q} |d\tau| \right)^{1/r} \left(\int_{|\tau - t| < y} w(\tau)^{qs/r} |d\tau| \right)^{1/s} \leq Cy$$

with $\frac{1}{s} + \frac{1}{r} = 1$ for every $y > \varepsilon$. Thus, furthermore,

$$\left(\int_{\varepsilon \leq |\tau - t| < y} w(\tau)^{-q} |d\tau| \right)^{1/r} \left(\int_{|\tau - t| < \varepsilon} w(\tau)^{qs/r} |d\tau| \right)^{1/s} \leq Cy. \tag{1.4}$$

By the Hölder inequality

$$|\Gamma(t, \varepsilon)| = \int_{|\tau - t| < \varepsilon} |d\tau| \leq \left(\int_{|\tau - t| < \varepsilon} w(\tau)^{-q} |d\tau| \right)^{1/r} \left(\int_{|\tau - t| < \varepsilon} w(\tau)^{qs/r} |d\tau| \right)^{1/s}. \tag{1.5}$$

From (1.4) and (1.5), we conclude that

$$\left(\int_{\varepsilon \leq |\tau - t| < y} w(\tau)^{-q} |d\tau| \right)^{1/r} |\Gamma(t, \varepsilon)| \left(\int_{|\tau - t| < \varepsilon} w(\tau)^{-q} |d\tau| \right)^{-1/r} \leq Cy$$

whence

$$\int_{\varepsilon \leq |\tau - t| < y} w^{-q}(\tau) |d\tau| \leq \frac{C^r}{|\Gamma(t, \varepsilon)|^r} \left(\int_{|\tau - t| < \varepsilon} w(\tau)^{-q} |d\tau| \right) y^r =: N y^r. \tag{1.6}$$

Using (1.6) with $y = 2^{k+1}\varepsilon$, we obtain

$$\begin{aligned} \int_{|\tau - t| \geq \varepsilon} \frac{w(\tau)^{-q}}{|\tau - t|^q} |d\tau| &= \sum_{k=0}^{\infty} \int_{2^k \varepsilon \leq |\tau - t| < 2^{k+1}\varepsilon} \frac{w(\tau)^{-q}}{|\tau - t|^q} |d\tau| \\ &\leq \sum_{k=0}^{\infty} \frac{1}{2^{kq} \varepsilon^q} \int_{\varepsilon \leq |\tau - t| < 2^{k+1}\varepsilon} w(\tau)^{-q} |d\tau| \\ &\leq \sum_{k=0}^{\infty} \frac{N}{2^{kq} \varepsilon^q} (2^{k+1}\varepsilon)^r = \frac{2^r N}{\varepsilon^{q-r}} \sum_{k=0}^{\infty} \left(\frac{1}{2^{q-r}} \right)^k < \infty. \end{aligned}$$

Now, in order to conclude the proof we need to check that $\lim_{\varepsilon\to 0}(S_\varepsilon f)(t)$, is well defined and finite. Notice that

$$(S_\varepsilon f)(t) = \frac{1}{\pi i}\int_{\Gamma\setminus\Gamma(t,\varepsilon)} \frac{f(\tau)-f(t)}{\tau - t}|d\tau| + \frac{f(t)}{\pi i}\int_{\Gamma\setminus\Gamma(t,\varepsilon)} \frac{1}{\tau - t}|d\tau| \tag{1.7}$$

where, from [8, Theorem 4.2] the limit $\varepsilon \to 0$ of the second term on the right of (1.7) exists and is finite. Moreover, if Γ is an arc with a starting point A and endpoint B, then

$$\lim_{\varepsilon\to 0}\frac{1}{\pi i}\int_{\Gamma\setminus\Gamma(t,\varepsilon)} \frac{1}{\tau - t}|d\tau| = \frac{1}{\pi i}\log\left(\frac{B-t}{A-t}\right) - 1.$$

Here, $\log\left(\frac{B-t}{A-t}\right)$ is the boundary value of the branch of the function $\log\left(\frac{B-z}{A-z}\right)$ which is analytic in $\mathbb{C}\setminus\Gamma$ and vanishes at infinity as z approaches t from the left.

Also, note that the first term on the right of (1.7) does not possess a singularity in case that f is smooth enough. Having this in mind, following the procedures of §4.2 of [8] it follows that this first term is also finite. Thus the proof is done. $\square$

The problem of characterizing the Γ, p, ω for which S_Γ is bounded on $L^p(\Gamma,\omega)$ has been studied by many mathematicians for a long time. Here is the final result.

Theorem 1.5. ([9, Theorem 1.1]) *Let $1 < p < \infty$, let Γ be a rectifiable Carleson curve, and let ω be a weight on Γ. The operator S_Γ is bounded on $L^p(\Gamma,\omega)$ if and only if $\omega \in A_p(\Gamma)$.*

This theorem allows us introduce the complementary *Riesz projections* operators:

$$P_+ := \frac{1}{2}(I_\Gamma + S_\Gamma) \quad \text{and} \quad P_- := \frac{1}{2}(I_\Gamma - S_\Gamma) \tag{1.8}$$

where I_Γ is the identity operator on $L^p(\Gamma,w)$. The sets $H^p_+(\Gamma,w)\equiv P_+L^p(\Gamma,w)$, $H^p_-(\Gamma,w) \equiv P_-L^p(\Gamma,w)$ and $\widetilde{H}^p_-(\mathbb{T},w) := H^p_-(\mathbb{T},w) + \mathbb{C}$ are subspaces of $L^p(\Gamma,w)$ which are called the *weighted Hardy spaces.* Notice that $L^p(\Gamma,w)$ decompose into the direct sum $H^p_-(\Gamma,w)\oplus H^p_+(\Gamma,w)$.

For a function φ on $L^1(\mathbb{T},w)$ the *Fourier coefficients* are defined by

$$\varphi_n = \frac{1}{2\pi}\int_0^\infty \varphi\left(e^{i\theta}\right)e^{-in\theta}d\theta, \qquad n\in\mathbb{Z}$$

and the (formal) *Fourier series*

$$\varphi\left(e^{i\theta}\right) \sim \sum_{n\in\mathbb{Z}} \varphi_n e^{in\theta},$$

can be associated to φ. Thus, in terms of the Fourier coefficients of a function $f \in L^p(\mathbb{T}, w)$, $1 < p < \infty$, the Cauchy integral operator acts according to the rule

$$S_{\mathbb{T}} f : \sum_{n \in \mathbb{Z}} f_n e^{in\theta} \longmapsto \sum_{n \geq 0} f_n e^{in\theta} - \sum_{n<0} f_n e^{in\theta}, \tag{1.9}$$

and the Riesz projections act as

$$P_+ f : \sum_{n \in \mathbb{Z}} f_n e^{in\theta} \longmapsto \sum_{n \geq 0} f_n e^{in\theta}, \quad P_- f : \sum_{n \in \mathbb{Z}} f_n e^{in\theta} \longmapsto \sum_{n<0} f_n e^{in\theta}.$$

Notice that for $1 \leq p < \infty$ The *Hardy space* $H^p_+(\mathbb{T}, w)$ is then defined as

$$H^p_+(\mathbb{T}, w) := \{\varphi \in L^p(\mathbb{T}, w) \ : \ \varphi_n = 0 \quad \text{for all} \quad n < 0\}$$

and the Hardy space $H^p_-(\mathbb{T}, w)$, consisting of the image of the projection operator P_-, by

$$H^p_-(\mathbb{T}, w) := \{\varphi \in L^p(\mathbb{T}, w) \ : \ \varphi_n = 0 \quad \text{for all} \quad n \geq 0\}.$$

1.2.3 The Toeplitz operator

The *Toeplitz operator* acting on the space $H^p_+(\Gamma, w)$, $1 < p < \infty$, with a generating function $a \in L^\infty(\Gamma)$, is defined by

$$\mathcal{T}_a : H^p_+(\mathbb{T}, w) \ni f \longmapsto P_+(af) \in H^p_+(\mathbb{T}, w).$$

It is also convenient to rewrite this in operator theoretic notation as

$$\mathcal{T}_a := P_+ a P_+, \tag{1.10}$$

where the left factor P_+ is understood as the Riesz projection acting from $L^p(\Gamma, w)$ onto $H^p_+(\Gamma, w)$, and the right factor P_+ is understood as the embedding operator form $H^p_+(\Gamma, w)$ to $L^p(\Gamma, w)$. We are using aI_Γ as the *operator of multiplication* by a on $L^p(\Gamma, w)$. In case aI_Γ follows another operator, T say, we omit the I_Γ and abbreviate $aI_\Gamma T$ to aT.

Toeplitz operators enjoy a property which will be useful for our purposes in Chapter 3. That is: they *are invertible if and only if they are Fredholm of index zero* ("Coburn's Lemma"); see e.g. [13, 30, 41]. Since $\mathcal{T}_a - \lambda I = \mathcal{T}_{a-\lambda}$, we may deduce from the Coburn's Lemma that the spectrum of $\mathcal{T}_a$, $\operatorname{sp} \mathcal{T}_a$, is the union of the essential spectrum of $\mathcal{T}_a$,

$$\operatorname{sp}_{\mathrm{ess}} \mathcal{T}_a := \{\lambda \in \mathbb{C} \ : \ \mathcal{T}_a - \lambda I \quad \text{is not Fredholm on } H^p_+(\Gamma, w)\},$$

and of the set of all $\lambda \in \mathbb{C}$ for which $\mathcal{T}_a - \lambda I$ is Fredholm of nonzero index. As a rule, the latter set may be found without serious difficulty once only $\mathrm{sp}_{\mathrm{ess}}\, \mathcal{T}_a$ is available.

However, even in the case where Γ is the unit circle $\mathbb{T}$, the weight w is identically 1 and $p = 2$, there are no satisfactory descriptions of spectrum, or the essential spectrum, for general $a \in L^\infty(\Gamma)$. This motivates the consideration of symbols a in certain subclasses of $L^\infty(\Gamma)$.

In 1952, I. Gohberg provided a description of the spectrum of a Toeplitz operator with continuous coefficient (which was generalized for general Carleson curves and general Muckenhoupt weights by A. Böttcher and Yu.I. Karlovich in 1995, see [9, Theorem 1.2]). The story of describing the essential spectrum for $a \in PC(\Gamma)$ has its beginning in the sixties, when several mathematicians, including I.B. Simonenko, A.P. Calderón, F. Spitzer, H. Widom, A. Devinatz, I. Gohberg, and N. Krupnik, realized that if Γ is a piecewise smooth curve, then the essential spectrum of $\mathcal{T}_a$ on $H^2_+(\Gamma)$ is the closed continuous curve resulting from the essential range of a by filling in a line segment between the endpoints of each jump:

$$\mathrm{sp}_{\mathrm{ess}}\, \mathcal{T}_a = \mathcal{R}(a) \bigcup_{t \in \Lambda_a} [a^-(t), a^+(t)],$$

where $\mathcal{R}(a)$ is the essential range of a and $\Lambda_a = \{t \in \Gamma \ : \ a^-(t) \neq a^+(t)\}$. However, when considering the case $p \neq 2$, H. Widom [98] as well as I. Gohberg and N. Krupnik (see, [41]) observed that then the line segments mentioned above go over into circular arcs. Given two points $z, w \in \mathbb{C}$ and a number $v \in (0, 1)$, define the circular arc between z and w whose shape is determined by v

$$\mathbb{A}(z, w; v) := \{\lambda \in \mathbb{C} \setminus \{z, w\} \ : \ \arg \frac{\lambda - z}{\lambda - w} \in 2\pi v + 2\pi\mathbb{Z}\} \cup \{z, w\}$$

which can be seen in the following way: Denote by Y_v the horizontal line $Y_v := \{\gamma \in \mathbb{C} \ : \ \Im m \gamma = v\}$. Then $\{e^{2\pi\gamma} \ : \ \gamma \in Y_v\}$ is a ray starting at the origin and making the angle $2\pi v$ with the real axis. Let $M_{z,w}(\zeta) := (w\zeta - z)/(\zeta - 1)$ be the Möbius transformation mapping 0 and ∞ to z and w, respectively. So, we may write

$$\mathbb{A}(z, w; v) = \{M_{z,w}(e^{2\pi\gamma}) \ : \ \gamma \in Y_v\} \cup \{z, w\}.$$

Notice that $\mathbb{A}(z, w; \frac{1}{2})$ is nothing but the line segment $[z, w]$. The Widom-Gohberg-Krupnik result says that if Γ is smooth, w is identically 1, $p \in (1, \infty)$, $a \in PC(\Gamma)$, then

$$\mathrm{sp}_{\mathrm{ess}}\, \mathcal{T}_a = \mathcal{R}(a) \bigcup_{t \in \Lambda_a} \mathbb{A}\left(a^-(t), a^+(t); \frac{1}{p}\right).$$

I. Gohberg and N. Krupnik (see [41]) also studied spaces with so-called *power weights* (*Khvedelidze weights*), that is, with weights of the form

$$w(\tau) = \prod_{j=1}^{n} |\tau - t_j|^{\lambda_j} \qquad (\tau \in \Gamma)$$

where $t_1, \ldots, t_n$ are distinct points on Γ and $\lambda_1, \ldots, \lambda_n$ are nonzero real numbers. The weight w belongs to $A_p(\Gamma)$ if and only if $-1/p < \lambda_j < 1/q$ $(1/p + 1/q = 1)$ for all j. This result has been well known under several additional hypotheses and was obtained in the work of G.H. Hardy, J.E. Littlewood, M. Riesz, S.G. Mikhlin, K.I. Babenko, B.V. Khvedelidze, H. Helson, G. Szegö, H. Widom, F. Forelli, I.I. Danilyuk, V.Yu. Shelepov, A.P. Calderón, and others. For general Carleson curves a proof was first given by E.A. Danilov (for the proof, see [8]). Gohberg and Krupnik showed that for piecewise smooth curves with the weight w one has

$$\mathrm{sp}_{\mathrm{ess}}\, \mathcal{T}_a = \mathcal{R}(a) \bigcup_{t \in \Lambda_a} \mathbb{A}\left(a^-(t), a^+(t); \frac{1}{p} + \lambda_t\right).$$

where $\lambda_t = 0$ for $t \notin \{t_1, \ldots, t_n\}$ and $\lambda_{t_j} = \lambda_j$. Thus, although now the circular arcs participating in the spectrum may have different shapes, they nevertheless remain circular arcs. In 1990, I.M. Spitkovsky considered again the case of a piecewise smooth curve Γ, but he admitted arbitrary Muckenhoupt weights $w \in A_p(\Gamma)$ $(1 < p < \infty)$. His result says that the presence of Muckenhoupt weights may metamorphose the circular arc into so-called horns. A *horn* is a closed subset of the plane which is bounded by two circular arcs. Given two numbers $u, v \in (0,1)$ satisfying $u \le v$, denote by $Y_{u,v}$ the closed stripe between the horizontal lines through iu and iv, i.e., $Y_{u,v} = \{\gamma \in \mathbb{C} \;:\; u \le \Im m\gamma \le v\}$. Then $\{e^{2\pi\gamma} \;:\; \gamma \in Y_{u,v}\}$ is an angular sector with vertex at the origin. With $M_{z,w}(\zeta) = (w\zeta - z)/(\zeta - 1)$ as above put

$$\mathcal{H}(z, w; u, v) := \{M_{z,w}(e^{2\pi\gamma}) \;:\; \gamma \in Y_{u,v}\} \cup \{z, w\}.$$

Thus, $\mathcal{H}(z, w; u, v)$ is the horn between z and w whose boundary arcs are $\mathbb{A}(z, w; u)$ and $\mathbb{A}(z, w; v)$. I.M. Spitkovsky associated two numbers u_t and v_t with each point $t \in \Gamma$ which, in a sense, measure the "powerlikeness" of the weight w at t and proved that

$$\mathrm{sp}_{\mathrm{ess}}\, \mathcal{T}_a = \mathcal{R}(a) \bigcup_{t \in \Lambda_a} \mathcal{H}\left(a^-(t), a^+(t); \frac{1}{p} + u_t, \frac{1}{p} + v_t\right).$$

The metamorphose of this circular arcs, in the case of the more complicated Carleson curve, as well as corresponding examples and properties can be found in [8, 9, 11].

1.3 Classes of essentially bounded functions

Some regularity properties of convolution type operators like Wiener-Hopf, Toeplitz, Hankel and singular integral operators, depend on the characteristics of their coefficients symbols, see, [33, 35, 87, 88, 100].

In this section we present the coefficient symbols which we will deal. More specifically, they belong to the algebras of: continuous, piecewise continuous, almost periodic and semi-almost periodic functions. Also we are dealing with generalized factorable essentially bounded functions. More information and historical background about these classes of functions can be found e.g. in [6, 7, 13, 31, 61].

As usual, $C(\mathbb{R})$ denotes the space of all *continuous functions* on $\mathbb{R}$, and let $PC(\mathbb{R})$ stand for the space of all *bounded piecewise continuous functions* on $\mathbb{R}$, i.e., functions $\varphi \in L^\infty(\mathbb{R})$ for which the one-sided limits:

$$\varphi^\pm(x_0) := \lim_{x \to x_0^\pm} \varphi(x)$$

exist for each $x_0 \in \mathbb{R}$.

We denote by $C(\overline{\mathbb{R}})$ (here, $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$) the set of all *complex-valued continuous functions* ζ on $\mathbb{R}$ which *have finite limits* $\zeta(\pm\infty)$ *at* $-\infty$ *and* $+\infty$. More precisely,

$$C(\overline{\mathbb{R}}) := C(\mathbb{R}) \cap PC(\mathbb{R}).$$

The algebra $AP(\mathbb{R})$ of (Bohr) continuous *almost-periodic functions* is defined as

$$AP(\mathbb{R}) := \operatorname{alg}_{L^\infty(\mathbb{R})}\{e_\lambda : \lambda \in \mathbb{R}\}, \quad e_\lambda(x) := e^{i\lambda x} \quad (x \in \mathbb{R}).$$

Finally, the algebra $SAP(\mathbb{R})$ of the *semi-almost periodic functions* is the smallest closed subalgebra of $L^\infty(\mathbb{R})$ containing $C(\overline{\mathbb{R}}) \cup AP(\mathbb{R})$. That is,

$$SAP(\mathbb{R}) := \operatorname{alg}_{L^\infty(\mathbb{R})}(AP(\mathbb{R}), C(\overline{\mathbb{R}})).$$

It is well–known that every $a \in SAP(\mathbb{R})$ can be written in the form

$$a = (1-u)a_l + ua_r + a_0, \tag{1.11}$$

where $u \in C(\overline{\mathbb{R}})$ is any fixed function such that

$$0 \le u \le 1, \quad u(-\infty) = 0, \quad u(+\infty) = 1,$$

a_l and a_r belong to $AP(\mathbb{R})$, and a_0 is in $C_0(\mathbb{R})$, i.e., a_0 belongs to the set of all *continuous functions vanishing at* $-\infty$ *and* $+\infty$. Moreover, a_l and a_r are uniquely determined by a and the maps

$$a \mapsto a_l \quad \text{and} \quad a \mapsto a_r$$

are C^*-algebras homomorphisms of $SAP(\mathbb{R})$ onto $AP(\mathbb{R})$ (cf. [88] or see e.g. [11, Theorem 1.21]). The functions a_l and a_r are referred to as the *almost-periodic representatives of* a *at* $-\infty$ *and* $+\infty$, respectively.

The set of all *almost-periodic polynomials* will be denoted by $AP^0(\mathbb{R})$ (recall that the algebra $AP(\mathbb{R})$ can be defined as the closure of $AP^0(\mathbb{R})$ in $L^\infty(\mathbb{R})$). The *Bohr mean value* of a function $a \in AP(\mathbb{R})$ is the element

$$M(a) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} a(x)dx,$$

and the *Bohr-Fourier spectrum* $\Omega(a)$ of $a \in AP(\mathbb{R})$ is the at most countable set

$$\Omega(a) := \{\lambda \in \mathbb{R} \ : \ M(ae_\lambda) \neq 0\}.$$

As usual, we set

$$AP_\pm(\mathbb{R}) = \{a \in AP \ : \ \Omega(a) \subset \mathbb{R}_\pm\}.$$

It is well-known that $AP_\pm(\mathbb{R})$ are closed subalgebras of $AP(\mathbb{R})$ and that $AP_\pm(\mathbb{R}) \cap AP^0(\mathbb{R})$ is dense in $AP_\pm(\mathbb{R})$.

Let now $APW(\mathbb{R})$ stand for the set of all $a \in AP(\mathbb{R})$ which can be written in the form

$$a(x) = \sum_{j=-\infty}^{\infty} a_j e_{\lambda_j}(x), \qquad \sum_{j=-\infty}^{\infty} |a_j| < \infty,$$

and put

$$APW_\pm(\mathbb{R}) = APW(\mathbb{R}) \cap AP_\pm(\mathbb{R}).$$

It is known that $APW(\mathbb{R})$ is a Banach algebra with the norm $\|a\| = \sum_{j=-\infty}^{+\infty} |a_j|$.

Remark 1.1. *Notice that the above mentioned C^*-algebras can be considered on $\mathbb{T}$ by using the isometric isomorphism B_0 from $L^\infty(\mathbb{R})$ onto $L^\infty(\mathbb{T})$, defined by*

$$(B_0 v)(t) := v\left(i\frac{1+t}{1-t}\right), \quad t \in \mathbb{T} \setminus \{1\}. \tag{1.12}$$

Finally, a representation of the form $A = A_- \Lambda A_+$ is called a (*right*) *generalized factorization* (*Wiener-Hopf* [36], Φ*-factorization* [63]) of the invertible matrix-valued function $A \in [L^\infty(\mathbb{T})]^{N\times N}$ in the space $[L^p(\mathbb{T},\sigma)]^N$ if $\Lambda(t) = \operatorname{diag}(t^{\aleph_1}, \dots, t^{\aleph_N})$ with certain integers $\aleph_1 \geq \dots \geq \aleph_N$ and if the factors A_- and A_+ satisfy the following conditions:

(i) $A_- \in [\widetilde{H}^p_-(\mathbb{T},\sigma)]^{N\times N}$, $A_+ \in [H^q_+(\mathbb{T},\sigma^{-1})]^{N\times N}$, $A_-^{-1} \in [\widetilde{H}^q_-(\mathbb{T},\sigma^{-1})]^{N\times N}$, $A_+^{-1} \in [H^p_+(\mathbb{T},\sigma)]^{N\times N}$ ($\frac{1}{p} + \frac{1}{q} = 1$).

(ii) The operator $A_-P_+A_-^{-1}$ is bounded on the space $[L^p(\mathbb{T},\sigma)]^N$.

The integers $\aleph_i$, $i=1,\dots,N$, are called (*right*) *indices* or also *partial indices* of the generalized factorization of the matrix-valued function A. The sum $\aleph_1+\dots+\aleph_N =: \aleph$ is referred to as the *total index* or *sum index* of the matrix function A. In the case when $\aleph_1=\dots=\aleph_N=0$, the factorization is called a *canonical generalized factorization.*

On the other hand, a *left generalized factorization* is a representation of the form:

$$A = A_+\Lambda A_-$$

with the conditions (i) being replaced by

(i*) $A_+ \in [H^p_+(\mathbb{T},\sigma)]^{N\times N}$, $A_+^{-1} \in [H^q_+(\mathbb{T},\sigma^{-1})]^{N\times N}$, $A_- \in [\widetilde{H}^q_-(\mathbb{T},\sigma^{-1})]^{N\times N}$, $A_-^{-1} \in [\widetilde{H}^p_-(\mathbb{T},\sigma)]^{N\times N}$,

and the condition (ii) also being modified appropriately. It is easy to pass from a left factorization to a right factorization and vice versa, for instance, by passing to the inverse, the complex conjugate, or the transpose of the matrix function A.

A factorization where merely condition (i), but not necessarily condition (ii) is fulfilled will is referred to as a *weak factorization* in $[L^p(\mathbb{T})]^N$. Weak factorizations has been studied in detail in the monographs [11, 36, 63].

The connection between the weak factorization in a Banach algebra $\mathfrak{B}^{N\times N}$ and Fredholm properties of Toeplitz operators is given by the Simonenko's fact that if $c \in \mathfrak{B}^{N\times N}$ admits a weak factorization in $\mathfrak{B}^{N\times N}$, then $\mathcal{T}_c$ is a Fredholm operator on $[H^p_+(\mathbb{T},w)]^N$ and the defect number $\alpha(\mathcal{T}_c)$ is given by

$$\dim\ker \mathcal{T}_c = -\sum_{\aleph_k<0} \aleph_k. \tag{1.13}$$

Notice that this statement holds for arbitrary parameters $1<p<\infty$. However, for general functions $c \in [L^\infty(\mathbb{T})]^{N\times N}$, the notion of weak factorization in a Banach algebra is not suitable. Therefore, the generalized factorization notion was introduced, allowing furthermore that the factors c_+, c_- and their inverses to be unbounded functions.

The importance of this definition for Toeplitz operators is revealed by the fundamental fact that for $c \in [L^\infty(\mathbb{T})]^{N\times N}$ the Toeplitz operator $\mathcal{T}_c$ is Fredholm on the space $[H^p_+(\mathbb{T},w)]^N$ if and only if the function c admits a generalized factorization in $[L^p(\mathbb{T},w)]^N$. In this case the defect number is given by the formula (1.13). In particular, the Toeplitz operator $\mathcal{T}_c$ is invertible on $[H^p_+(\mathbb{T},w)]^N$ if and only if it admits a generalized factorization in $[L^p(\mathbb{T},w)]^N$ with all partial indices being equal to zero. This result has been proved first by I. Simonenko in 1968. The notion of generalized factorization has been studied in further detail by K. Clancey and I. Gohberg [29], also by G. Litvinchuk and I.M. Spitkovsky [63] where the same notion has been referred to as Φ-factorization.

Condition (ii) appearing in the definition of a generalized factorization seems in general not easy to verify. However, in the scalar case, it is equivalent to the Hunt-Muckenhoupt-Wheeden condition and in the matrix case ($N > 1$) to a much more complicated generalization of the condition (1.3) (see, [97]).

It has to be emphasized that by using the factorization approach to study the Fredholm properties of Toeplitz operators, one encounters severe difficulties. Namely, in general, it is unavoidable to factor the given matrix-valued function explicitly. Unfortunately, only for certain classes of functions a factorization can be constructed explicitly and the defect numbers can be computed.

There exists a vast and quite heterogeneous literature devoted to explicit factorization techniques. Each new method for explicit factorization represents a huge progress, which can usually be achieved only with enormous efforts. Explicitly factorable matrix-valued functions include, for instance, *rational matrix-valued functions*, *upper triangular matrix-valued functions*, so-called *Daniele-Khrapkov matrix-valued functions* and *piecewise constant matrix-valued functions*. We will not make an attempt here to give an overview but just refer to [29, 63] for some examples, further information and references.

On the other hand, the difficulty of explicit factorization seems at least to some extent to steem from the original Fredholm problem itself. One theoretical explanation for the difficulties can be seen in the fact that the defect numbers (as the partial indices) are instable under small perturbations. But it should also be noted that in many non-trivial cases in which one has been able to compute the defect numbers this has been done by means of factorization.

1.4 Singular integral operators with shift

In this section we are going to present the class of singular integral operators which is the main object of study of this thesis.

First, recall that a *characteristic* (*pure*) *singular integral operator* with essentially bounded coefficients a and b on $L^p(\Gamma, w)$, $1 < p < \infty$, is defined as

$$\mathcal{A} = aI_\Gamma + bS_\Gamma. \tag{1.14}$$

The basic properties of this kind of operators, as boundedness, generalized invertibility, Fredholm property, etc, are already known, even for coefficients a, b belonging to different classes essentially bounded functions. See, e.g., [12, 29, 41, 67, 72, 79, 85]. Even more, in order to apply the existing standard theories for the numerical analysis of the integral equations associated to these operators, characterizations of the stability and (strongly) ellipticity, as well as the properties of one-dimensional splines, Garlekin, finite section and polynomial approximation methods have being studied, see [43, 47, 76, 84, 91].

The fruitful advances using singular integral equations in modeling and solving applied problems in *mechanical engineering, physics, oscillation theory* among others (see, e.g., [39, 69, 93, 96]) stimulate the interest for the study of singular integral equations with shift, boundary value problems with shift and the corresponding singular integral operators with shift. These operators have proved to be useful in several applications to *Theory of infinitesimal bounds of surfaces with positive curvature, Contact theory, Physics, Economy and Medicine.* After the systematic study of singular integral operators initiated by G. Litvinchuk (see the books [55, 64]) a lot of papers and monographs in this subject have been written cf., e.g., the works [5],[27], [28, 34, 38, 40, 41, 42, 48, 49, 50, 51, 52, 53, 54, 56, 57, 58, 79, 80, 81, 94] and references therein. Therefore, much is known about these kind of operators, and even more remains to be known.

1.4.1 Shift functions and shift operators

Let Γ be an oriented Carleson curve. A homeomorphic mapping $\alpha(t)$ of the curve Γ onto itself is called a *shift function*, we shall always suppose that the shift $\alpha(t)$ has a derivative $\alpha'(t)$ which is never zero and satisfies the *Hölder condition* everywhere on Γ. I.e.,

$$|\alpha(t) - \alpha(\tau)| \leq c|t - \tau|^{\mu}, \quad c > 0, \quad 0 < \mu \leq 1.$$

A classification of shifts which is sufficient for our purposes is based on the fact that the function $\alpha(t)$ either preserves the accepted orientation on Γ or changes the orientation on Γ into the opposite one. More refined classifications can be found in [55, 64]. A shift function is called a *forward shift* if it preserves the orientation on Γ and a *backward shift* in case that $\alpha(t)$ changes the orientation on Γ into the oppositive one.

A point $\tau \in \Gamma$ is called a *periodic point* of the shift $\alpha(t)$ with multiplicity $k \geq 1$, if $\alpha_k(\tau) = \tau$ and (for $k > 1$) $\alpha_i(\tau) \neq \tau$ for all $i = 1, 2, \ldots, k-1$, where $\alpha_i(t) = \alpha(\alpha_{i-1}(t))$, and as usual we agree that $\alpha_0(t) \equiv t$. A periodic point with multiplicity one ($k = 1$) is called a *fixed point.* It is of preponderance importance here the so-called *Carleman shifts*, which are shift functions $\alpha(t)$ satisfying, for all $t \in \Gamma$, the involutive condition:

$$\alpha_2(t) = \alpha(\alpha(t)) = t.$$

Lemma 1.6. ([55, Corollary 1.3.1]) *An orientation preserving Carleman shift does not have fixed points.*

Examples 1.1 (Shift functions)**.** We are going to show several examples of shift functions preserving or changing the orientation on Γ.

First, considering Γ to be the real line $\mathbb{R}$ we have the following shift functions:

(1) $\alpha(x) = -x$, reverting orientation.

(2) $\alpha(x) = \frac{\delta x+\beta}{x-\delta}$, $\quad \delta, \beta \in \mathbb{R}$, $\quad \delta^2 + \beta > 0$, reverting orientation.

Moreover, the first two examples satisfy the Carleman condition $\alpha^2(x) \equiv x$. In the unit circle $\mathbb{T}$ we mentioned the following useful shift functions:

(3) $\alpha(t) = -t$, preserving orientation.

(4) $\alpha(t) = \overline{t}$, reverting orientation.

(5) $\alpha(t) = \frac{at+b}{\overline{b}t+\overline{a}}$, $|a|^2 - |b|^2 = \gamma$, where $\gamma = 1$ or $\gamma = -1$ if $\alpha(t)$ preserves or change the orientation on $\mathbb{T}$, respectively.

In this case also only the first two examples are Carleman shift functions.

The *shift operator* induced by a shift function $\alpha(t)$ is defined by

$$(W\varphi)(t) = \varphi(\alpha(t)).$$

The boundedness of this operator on the $L^p(\Gamma)$ spaces is given in the following result.

Theorem 1.7. ([55, Theorem 1.3.4]) *The shift operator W is a linear and bounded continuously invertible operator on the spaces $L^p(\Gamma)$, $1 < p < \infty$.*

Proof. In fact,

$$\begin{aligned}\|W\varphi\|_p^p &= \int_\Gamma |\varphi(\alpha(t))|^p |dt| = \int_\Gamma |\varphi(t)|^p |(\alpha^{-1}(t))'||dt| \\ &\leq \max_{t\in\Gamma} |(\alpha^{-1}(t))'| \int_\Gamma |\varphi(t)|^p |dt| = \max_{t\in\Gamma} |(\alpha^{-1}(t))'| \|\varphi\|_p^p.\end{aligned}$$

Consequently,

$$\|W\|_p \leq (\max |(\alpha^{-1}(t))'|)^{1/p}$$

so the operator W is bounded in $L^p(\Gamma)$, $1 < p < \infty$. $\square$

As a consequence of the theorem above, the natural generalization of the operator W is given by the following isometric shift operator on $L^p(\Gamma)$ (see, [55, Corollary 1.3.3]):

$$(U\varphi) = |\alpha'(t)|^{1/p}\varphi(\alpha(t)).$$

Furthermore, we can define a more general operator, the *weighted shift operator* induced by the shift function $\alpha(t)$ and by a complex-valued function $v(t)$ on Γ:

$$(V\varphi)(t) = v(t)\varphi(\alpha(t)).$$

In that follows, by J we mean a (weighted) shift operator induced by the homeomorphism $\alpha(t)$. An important property relating the Cauchy integral and shift operators is given in the next result. Here, we are only going to give a sketch of a proof.

Theorem 1.8. ([55, Theorem 1.3.5]) *If Γ is a Carleson curve, $\alpha(t)$ is a homeomorphism of Γ onto itself, $\alpha'(t) \neq 0$ satisfying the Hölder condition, then the operator $K = \gamma J S_\Gamma J - S_\Gamma$ is compact in the space $L^p(\Gamma)$, $p \in (1, \infty)$. Here $\gamma = 1$ or $\gamma = -1$ if the shift $\alpha(t)$ preserves or changes the orientation on Γ, respectively.*

Proof. The basis of the proof is the estimate of the kernel function in the integral operator K. Since Γ is a Carleson curve, we are considering the operator K defined on $\mathbb{T}$ through a homeomorphism from Γ onto $\mathbb{T}$. Considering $\gamma = 1$, notice that the operator K has a kernel expressed by the formula

$$\mathcal{K}(\zeta, z) = \frac{\alpha'(z)}{\alpha(z) - \alpha(\zeta)} - \frac{1}{z - \zeta}, \quad |\zeta| = 1, \quad |z| \leq 1.$$

First, letting $z, \zeta \in \mathbb{T}$, $z = e^{i\theta_1}$, $\zeta = e^{\theta_0}$, $u = e^{i\theta}$ ($\theta_0 \leq \theta \leq \theta_1$) and $r = |u - \zeta|$. Then $|du| = d\theta \leq m dr$, $m > 0$. Since Γ is a Carleson curve, the derivative α' satisfies the Hölder condition on Γ with some exponent μ, i.e.,

$$|\alpha'(u) - \alpha'(z)| \leq M r^\mu.$$

We will now write the function $\mathcal{K}(\zeta, z)$ in the form

$$\mathcal{K}(\zeta, z) = \frac{\alpha'(z)}{\alpha(z) - \alpha(\zeta)} - \frac{1}{z - \zeta} = \frac{\alpha'(\zeta)(z - \zeta) - \alpha(z) + \alpha(\zeta)}{(\alpha(z) - \alpha(\zeta))(z - \zeta)}$$

and consider

$$\begin{aligned} |\alpha(z) - \alpha(\zeta) - \alpha'(\zeta)(z - \zeta)| = &\left| \int_\Upsilon [\alpha'(u) - \alpha'(\zeta)] |du| \right| \leq Mm \int_0^{|\zeta - z|} r^\mu dr \\ = &M_1 |z - \zeta|^{\mu+1} \end{aligned} \tag{1.15}$$

where Υ is a circular arc with points z and ζ. Now let $\zeta \in \mathbb{T}$, $|z| < 1$. Let us now take Υ, by supposing $u = \lambda z + (1-\lambda)\zeta$, $0 \le \lambda \le 1$, the line segment connecting the points z and ζ. Reasoning as above, it is clear that we get the same estimate (1.15). Since the condition $\alpha'(\zeta) \neq 0$, $\zeta \in \mathbb{T}$ is fulfilled, we have

$$\left|\frac{\alpha(\zeta) - \alpha(z)}{\zeta - z}\right| \ge M_2 > 0. \tag{1.16}$$

From (1.15) and (1.16), it follows that

$$|\mathcal{K}(\zeta, z)| \le \frac{C}{|\zeta - z|^{\alpha}}, \quad C > 0, \quad 0 < \alpha = 1 - \mu < 1,$$

or, in other words, $\mathcal{K}(\zeta, z)$ is a weakly singular kernel (therefore K is a integral operator with weak singularity). The proof follows by recalling that an integral operator with weak singularity is compact on $L^p(\Gamma, w)$, $1 < p < \infty$ (cf., e.g., [41, 67]).

□

In order to obtain explicit equivalence relations between operators with shift, it is widely used classes of shift operators having a "nice" commutative relation with the Cauchy integral operator. A shift operator J is called *commutative* if it satisfies the relation

$$JS_\Gamma = S_\Gamma J,$$

and *anti-commutative* in case that

$$JS_\Gamma = -S_\Gamma J.$$

Notice that a commutative shift operator is induced, necessarily, by a preserving-orientation shift function. While, in opposition, an anti-commutative shift operator is induced by a reverting-orientation shift.

Examples 1.2 (Shift operators)**.** The reflection shift operator on $L^p(\mathbb{R})$, $p \in (1, \infty)$, defined by

$$(W_{\mathbb{R}}\phi)(x) = \phi(-x) \tag{1.17}$$

is an anti-commutative type Carleman shift operator, whereas the reflection operator on $\mathbb{T}$

$$(J\phi)(t) = \phi(-t) \tag{1.18}$$

is a commutative Carleman shift operator.

Examples of weighted Carleman shift operators of commutative and/or anti-commutative kind are the following:

1. Let $\alpha(t) = \frac{1}{t}$, $t \in \mathbb{T}$. The weighted shift operator on $L^p(\mathbb{T})$, $1 < p < \infty$,
$$(\widetilde{J}\phi)(t) = \frac{1}{t}\phi\left(\frac{1}{t}\right) \tag{1.19}$$
is a commutative Carleman shift operator, which is called the *flip operator*.

2. Let us consider the linear fractional function
$$\alpha(t) = \frac{t-\beta}{\overline{\beta}t - 1}, \quad t \in \mathbb{T}, \quad \beta \in \mathbb{C} \setminus \mathbb{T}. \tag{1.20}$$
This is a Carleman shift function which preserves the orientation on $\mathbb{T}$ if $|\beta| < 1$ and changes the orientation if $|\beta| > 1$. First, we would like to recall that the *winding number*, $\operatorname{wind}\varphi$ of a continuous, non-vanishing function φ on $\mathbb{T}$, is defined as
$$\operatorname{wind}\varphi = \left[\frac{1}{2\pi}\arg\varphi(e^{i\theta})\right]_{\theta=0}^{2\pi},$$
where the argument is chosen in a such way that $\arg\varphi(e^{i\theta})$ is continuous on $[0, 2\pi]$. Thus we have $\operatorname{wind}\alpha = 1$ for a forward shift, and $\operatorname{wind}\alpha = -1$ for a backward shift. In any case, $\alpha(t)$ can be factorized as
$$\alpha = \alpha_+ t^\nu \alpha_-, \quad \nu = \operatorname{wind}\alpha,$$
where
$$\alpha_+(t) = \frac{\lambda}{\overline{\beta}t - 1}, \quad \alpha_-(t) = \frac{t-\beta}{\lambda t}, \quad \text{if } \alpha(t) \text{ is a forward shift,}$$
$$\alpha_+(t) = \frac{t-\beta}{\lambda}, \quad \alpha_-(t) = \frac{\lambda t}{\overline{\beta}t - 1}, \quad \text{if } \alpha(t) \text{ is a backward shift,}$$
and $\lambda = \sqrt{1 - |\beta|^2}$ with the branch so that $\sqrt{-1} = i$. Associated with the shift function (1.20), let us consider the weighted shift operator
$$(Z\psi)(t) = \upsilon(t)\psi(\alpha(t))$$
where $\upsilon(t) = -\alpha_+(t)$ or $\upsilon(t) = t^{-1}\alpha_-(t)$ according to whether $\alpha(t)$ is a forward or backward shift respectively. Besides, with this definition the shift operator Z satisfies the following properties
$$Z^2 = I, \quad ZS_{\mathbb{T}} = \nu S_{\mathbb{T}} Z \quad \text{with} \quad \nu = \operatorname{wind}\alpha.$$

Finally, we can assemble the singular integral operators with shift, whose coefficients a_0, a_1, b_0, b_1 belong to $L^\infty(\Gamma)$, and the extra action of a Carleman shift operator J as:

$$\mathcal{O} = a_0 I_\Gamma + b_0 S_\Gamma + a_1 J + b_1 S_\Gamma J \; : \; L^p(\Gamma, w) \to L^p(\Gamma, w), \quad 1 < p < \infty. \tag{1.21}$$

In comparison, the mathematical fundamental theory of the operator $\mathcal{O}$ is less developed than the corresponding ones of the pure singular integral operator $\mathcal{A}$ given in (1.14). The reason for such a situation is easy to understand since the second kind of operators contains all the structure of the first one and, in addition, the shift actions.

Chapter 2

Singular Integral Operators with Shift and Some Equivalent Operators

This chapter is devoted to display several explicit operators equivalence relations for the singular integral operator with shift $\mathcal{O}$, defined in (1.21), with essentially bounded functions as coefficients. More precisely, we are going to show an explicit similarity relation between the mentioned operator $\mathcal{O}$ with the reflection operator J on the unit circle $\mathbb{T}$ (which is a reverting-orientation Carleman shift operator defined by (1.18)) and a matrix pure singular integral operator as (1.14).

Besides, when in operator $\mathcal{O}$ the shift action is given by the flip operator (1.19), we are going to exhibit two different equivalence relations: First, an equivalence relation after extension with a pure matrix singular integral operator and, secondly, a similarity relation with a matrix Toeplitz plus Hankel operator.

We would like to point out that these explicit equivalence relations will play a central role in the next chapters. Mainly, in order to provide Fredholm criteria and the dimension of the kernel, as well as representations of the (generalized, left, right and Moore-Penrose) inverses of the singular integral operators $\mathcal{O}$ with these mentioned shifts.

2.1 Similarity transformations between SIO's with and without reflection

In this section, we will explicitly construct a similarity transformation between singular integral operators with reflection (defined by a rotation action of π amplitude on the unit circle $\mathbb{T}$), in the weighted Lebesgue space $L^p(\mathbb{T}, w)$, and matrix singular integral operators without reflection, defined in the space $[L^p(\mathbb{T}, w)]^2$. We would like to point out that a corresponding transformation for the particular case of $p = 2$ and $w(t) = 1$, $t \in \mathbb{T}$, was already obtained in [49].

Let us now recall some conditions which are necessary for the boundedness of the operators which we will work with, and then present such operators in a detailed way.

Let us now define

$$A_p^e(\mathbb{T}) := \{w \in A_p(\mathbb{T}) : \ w(-t) = w(t), \ t \in \mathbb{T}\},$$

and suppose that $\varrho \in A_p^e(\mathbb{T})$. Then the reflection operator

$$J : L^p(\mathbb{T}, \varrho) \longrightarrow L^p(\mathbb{T}, \varrho), \quad (Jf)(t) = \widetilde{f}(t) = f(-t)$$

is bounded. Now, let $\chi_{\mathbb{T}_+}$ be the characteristic function of $\mathbb{T}_+ := \{\tau \in \mathbb{T} : 0 < \arg \tau < \pi\}$. The above result implies that the operators

$$S_{\mathbb{T}_+} := \chi_{\mathbb{T}_+} S_{\mathbb{T}} \chi_{\mathbb{T}_+} \quad \text{and} \quad H_{\mathbb{T}_+} := \chi_{\mathbb{T}_+} J S_{\mathbb{T}} \chi_{\mathbb{T}_+}$$

are bounded on $L^p(\mathbb{T}_+, \varrho) := L^p(\mathbb{T}_+, \varrho_{|\mathbb{T}_+})$. These two operators will be (indirectly) used in what follows when constructing the operator relations.

The (bounded) singular integral with reflection which is the main object of the present section is:

$$\mathcal{O} = a_0 I_{\mathbb{T}} + b_0 S_{\mathbb{T}} + a_1 J + b_1 S_{\mathbb{T}} J \ : \ L^p(\mathbb{T}, w) \to L^p(\mathbb{T}, w), \tag{2.1}$$

where $I_{\mathbb{T}}$ denotes the identity operator, J is the above mentioned reflection operator, $1 < p < \infty$, $w \in A_p^e(\mathbb{T})$, and $a_0, a_1, b_0, b_1 \in L^\infty(\mathbb{T})$.

We will now start to apply some transformations to $\mathcal{O}$ (in the form of operator equivalence relations) so that in the end we will reach to a singular integral operator without reflection. In view of this, we will make use of the operator

$$\begin{aligned} & M_{\mathbb{T}_+} : [L^p(\mathbb{T}_+, w)]^2 \longrightarrow L^p(\mathbb{T}, w) \\ & M_{\mathbb{T}_+} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \ell_0 \varphi_1 + J^{-1} \ell_0 \varphi_2, \end{aligned} \tag{2.2}$$

where ℓ_0 denotes the zero extension operator from $\mathbb{T}_+$ to $\mathbb{T}$ (in the corresponding spaces). Thus, it is clear that $\ell_0 \varphi_1$ and $J^{-1} \ell_0 \varphi_2$ belong to $L^p(\mathbb{T}, w)$

if $\varphi_1, \varphi_2 \in L^p(\mathbb{T}_+, w)$. In addition, $M_{\mathbb{T}_+}$ is a linear operator, and $M_{\mathbb{T}_+} \in \mathcal{L}([L^p(\mathbb{T}_+, w)]^2, L^p(\mathbb{T}, w))$. Moreover,

$$M_{\mathbb{T}_+}^{-1}\varphi = \begin{pmatrix} r_{\mathbb{T}_+}\varphi \\ r_{\mathbb{T}_+}J\varphi \end{pmatrix}, \tag{2.3}$$

where $r_{\mathbb{T}_+} : L^p(\mathbb{T}, w) \longrightarrow L^p(\mathbb{T}_+, w)$ denotes the restriction operator $r_{\mathbb{T}_+}\varphi = \varphi_{|\mathbb{T}_+}$. From the fact that $r_{\mathbb{T}_+} : L^p(\mathbb{T}, w) \to L^p(\mathbb{T}_+, w)$ is a bounded operator and $w \in A_p^e(\mathbb{T})$, we have that $r_{\mathbb{T}_+}J : L^p(\mathbb{T}, w) \to L^p(\mathbb{T}_+, w)$ is also bounded. So, $M^{-1} \in \mathcal{L}(L^p(\mathbb{T}, w), [L^p(\mathbb{T}_+, w)]^2)$.

The major utility of operator $M_{\mathbb{T}_+}$ is based on the fact that the composition of $M_{\mathbb{T}_+}^{-1}$ with the multiplication operator by a scalar function, and also the composition with $M_{\mathbb{T}_+}$, performs the same action as a particular diagonal matrix operator. In detail, for $a \in L^\infty(\mathbb{T})$, we have:

$$\begin{aligned} M_{\mathbb{T}_+}^{-1}aM_{\mathbb{T}_+}\varphi =& M_{\mathbb{T}_+}^{-1}aM_{\mathbb{T}_+}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = M_{\mathbb{T}_+}^{-1}(a\ell_0\varphi_1 + aJ^{-1}\ell_0\varphi_2) \\ =& \begin{pmatrix} r_{\mathbb{T}_+}(a\ell_0\varphi_1) + r_{\mathbb{T}_+}(aJ^{-1}\ell_0\varphi_2) \\ r_{\mathbb{T}_+}J(a\ell_0\varphi_1) + r_{\mathbb{T}_+}J(aJ^{-1}\ell_0\varphi_2) \end{pmatrix}. \end{aligned}$$

Since, $r_{\mathbb{T}_+}J\ell_0|_{L^p(\mathbb{T}_+,w)} = 0$, $J^2 = I$ and $J(ab) = (Ja)(Jb)$, we obtain

$$M_{\mathbb{T}_+}^{-1}aM_{\mathbb{T}_+}\varphi = M_{\mathbb{T}_+}^{-1}aM_{\mathbb{T}_+}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} r_{\mathbb{T}_+}(a\ell_0\varphi_1) \\ r_{\mathbb{T}_+}J(a)\ell_0\varphi_2 \end{pmatrix}.$$

Thus,

$$M_{\mathbb{T}_+}^{-1}aM_{\mathbb{T}_+} = \operatorname{diag}(r_{\mathbb{T}_+}a, r_{\mathbb{T}_+}J(a))I_{\mathbb{T}_+}. \tag{2.4}$$

Here, $I_{\mathbb{T}_+}$ is the identity operator on $[L^p(\mathbb{T}_+, w)]^2$.

Remark 2.1. *For the sake of presentation simplification, from now on we will avoid the use of parenthesis when we apply the operators J, ℓ_0 and $r_{\mathbb{T}_+}$.*

It is also important to realize that

$$\begin{aligned} M_{\mathbb{T}_+}^{-1}JM_{\mathbb{T}_+}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} =& M_{\mathbb{T}_+}^{-1}J(\ell_0\varphi_1 + J^{-1}\ell_0\varphi_2) \\ =& M_{\mathbb{T}_+}^{-1}(J\ell_0\varphi_1 + \ell_0\varphi_2) \\ =& \begin{pmatrix} r_{\mathbb{T}_+}J\ell_0\varphi_1 + r_{\mathbb{T}_+}\ell_0\varphi_2 \\ r_{\mathbb{T}_+}J^2\ell_0\varphi_1 + r_{\mathbb{T}_+}J\ell_0\varphi_2 \end{pmatrix}. \end{aligned}$$

Therefore,

$$M_{\mathbb{T}_+}^{-1} J M_{\mathbb{T}_+} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} r_{\mathbb{T}_+} \ell_0 \varphi_2 \\ r_{\mathbb{T}_+} \ell_0 \varphi_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r_{\mathbb{T}_+} \ell_0 \varphi_1 \\ r_{\mathbb{T}_+} \ell_0 \varphi_2 \end{pmatrix} \tag{2.5}$$

and we arrive to the conclusion that $M_{\mathbb{T}_+}^{-1} J M_{\mathbb{T}_+} = V I_{\mathbb{T}_+}$, where

$$V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is clear that $V^2 = I$, and so $V^{-1} = V$.

On the other hand,

$$\begin{aligned} M_{\mathbb{T}_+}^{-1} S_{\mathbb{T}} M_{\mathbb{T}_+} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} =& M_{\mathbb{T}_+}^{-1} S_{\mathbb{T}} (\ell_0 \varphi_1 + J^{-1} \ell_0 \varphi_2) \\ =& M_{\mathbb{T}_+}^{-1} (S_{\mathbb{T}} \ell_0 \varphi_1 + S_{\mathbb{T}} J^{-1} \ell_0 \varphi_2) \\ =& \begin{pmatrix} r_{\mathbb{T}_+} S_{\mathbb{T}} \ell_0 \varphi_1 + r_{\mathbb{T}_+} S_{\mathbb{T}} J^{-1} \ell_0 \varphi_2 \\ r_{\mathbb{T}_+} J S_{\mathbb{T}} \ell_0 \varphi_1 + r_{\mathbb{T}_+} J S_{\mathbb{T}} J^{-1} \ell_0 \varphi_2 \end{pmatrix}. \end{aligned}$$

Let us note that

$$r_{\mathbb{T}_+} J r_{\mathbb{T}_+} S_{\mathbb{T}} J^{-1} \ell_0 \varphi_2 = r_{\mathbb{T}_+} S_{\mathbb{T}} \ell_0 \varphi_2 \,,$$

it follows that

$$\begin{aligned} M_{\mathbb{T}_+}^{-1} S_{\mathbb{T}} M_{\mathbb{T}_+} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} &= \begin{pmatrix} r_{\mathbb{T}_+} S_{\mathbb{T}} \ell_0 \varphi_1 + r_{\mathbb{T}_+} S_{\mathbb{T}} J \ell_0 \varphi_2 \\ r_{\mathbb{T}_+} J S_{\mathbb{T}} \ell_0 \varphi_1 + r_{\mathbb{T}_+} S_{\mathbb{T}} \ell_0 \varphi_2 \end{pmatrix} \\ &= \left[\begin{pmatrix} r_{\mathbb{T}_+} S_{\mathbb{T}} \ell_0 & 0 \\ 0 & r_{\mathbb{T}_+} S_{\mathbb{T}} \ell_0 \end{pmatrix} \right. \\ &\quad \left. + \begin{pmatrix} 0 & r_{\mathbb{T}_+} S_{\mathbb{T}} J \ell_0 \\ r_{\mathbb{T}_+} J S_{\mathbb{T}} \ell_0 & 0 \end{pmatrix} \right] \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}. \end{aligned} \tag{2.6}$$

Therefore,

$$M_{\mathbb{T}_+}^{-1} S_{\mathbb{T}} M_{\mathbb{T}_+} = \sum_{k=0}^{1} V^k U_k$$

where $V^0 = I$ and

$$U_k := r_{\mathbb{T}_+} J^k S_{\mathbb{T}} \ell_0, \qquad U_k \in \mathcal{L}(L^p(\mathbb{T}_+, w)).$$

Note that U_k $(k=0,1)$ can be rewritten in the integral form as

$$\begin{aligned}(U_0 f)(t) =& \frac{1}{\pi i}\int_{\mathbb{T}_+} \frac{f(\tau)}{\tau - t}d\tau\,, \qquad t\in\mathbb{T}_+\\ (U_1 f)(t) =& \frac{1}{\pi i}\int_{\mathbb{T}_+} \frac{f(\tau)}{\tau + t}d\tau\,, \qquad t\in\mathbb{T}_+\,.\end{aligned} \tag{2.7}$$

For the operators $U_k \in \mathcal{L}(L^p(\mathbb{T}_+, w))$ and $\mathrm{diag}(U_k, U_k) \in \mathcal{L}([L^p(\mathbb{T}_+, w)]^2)$ we will use the same notation U_k. Applying the similarity transformation to the operator (2.1), we obtain an operator $A_1 := M_{\mathbb{T}_+}^{-1}\mathcal{O}M_{\mathbb{T}_+}$ with the following form:

$$M_{\mathbb{T}_+}^{-1}\mathcal{O}M_{\mathbb{T}_+} = M_{\mathbb{T}_+}^{-1}a_0M_{\mathbb{T}_+} + M_{\mathbb{T}_+}^{-1}b_0S_{\mathbb{T}}M_{\mathbb{T}_+} + M_{\mathbb{T}_+}^{-1}a_1JM_{\mathbb{T}_+} + M_{\mathbb{T}_+}^{-1}b_1S_{\mathbb{T}}JM_{\mathbb{T}_+}.$$

From (2.2), (2.3) and (2.4) we have,

$$\begin{aligned}M_{\mathbb{T}_+}^{-1}\mathcal{O}M_{\mathbb{T}_+} =& \,\mathrm{diag}(r_{\mathbb{T}_+}a_0, r_{\mathbb{T}_+}Ja_0)I_{\mathbb{T}_+} + M_{\mathbb{T}_+}^{-1}b_0M_{\mathbb{T}_+}M_{\mathbb{T}_+}^{-1}S_{\mathbb{T}}M\\ &+ M_{\mathbb{T}_+}^{-1}a_1M_{\mathbb{T}_+}M_{\mathbb{T}_+}^{-1}JM_{\mathbb{T}_+} + M_{\mathbb{T}_+}^{-1}b_1M_{\mathbb{T}_+}M_{\mathbb{T}_+}^{-1}S_{\mathbb{T}}JM_{\mathbb{T}_+}\\ =& \,\mathrm{diag}(r_{\mathbb{T}_+}a_0, r_{\mathbb{T}_+}Ja_0)I_{\mathbb{T}_+} + \mathrm{diag}(r_{\mathbb{T}_+}b_0, r_{\mathbb{T}_+}Jb_0)\sum_{k=0}^{1}V^kU_k\\ &+ \mathrm{diag}(r_{\mathbb{T}_+}a_1, r_{\mathbb{T}_+}Ja_1)I_{\mathbb{T}_+}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}I_{\mathbb{T}_+}\\ &+ \mathrm{diag}(r_{\mathbb{T}_+}b_1, r_{\mathbb{T}_+}Jb_1)\left[\sum_{k=0}^{1}V^kU_k\right]\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}I_{\mathbb{T}_+}\\ =& \sum_{k=0}^{1}\mathrm{diag}(r_{\mathbb{T}_+}a_k, r_{\mathbb{T}_+}Ja_k)V^k + \sum_{k=0}^{1}\mathrm{diag}(r_{\mathbb{T}_+}b_k, r_{\mathbb{T}_+}Jb_k)\\ &\times V^k\sum_{k=0}^{1}V^kU_k.\end{aligned}$$

Thus,

$$A_1 := M_{\mathbb{T}_+}^{-1}\mathcal{O}M_{\mathbb{T}_+} = u_1I_{\mathbb{T}_+} + v_1\sum_{k=0}^{1}V^kU_k, \qquad A_1 \in \mathcal{L}([L^p(\mathbb{T}_+, w)]^2),$$

where

$$u_1 = \sum_{k=0}^{1}\mathrm{diag}(r_{\mathbb{T}_+}a_k, r_{\mathbb{T}_+}Ja_k)V^k\,, \qquad v_1 = \sum_{k=0}^{1}\mathrm{diag}(r_{\mathbb{T}_+}b_k, r_{\mathbb{T}_+}Jb_k)V^k\,,$$

i.e.,

$$\begin{aligned} u_1(t) &= \begin{pmatrix} r_{\mathbb{T}_+} a_0(t) & 0 \\ 0 & r_{\mathbb{T}_+} Ja_0(t) \end{pmatrix} + \begin{pmatrix} r_{\mathbb{T}_+} a_1(t) & 0 \\ 0 & r_{\mathbb{T}_+} Ja_1(t) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} r_{\mathbb{T}_+} a_0(t) & r_{\mathbb{T}_+} a_1(t) \\ r_{\mathbb{T}_+} Ja_0(t) & r_{\mathbb{T}_+} Ja_1(t) \end{pmatrix} = \begin{pmatrix} r_{\mathbb{T}_+} a_0(t) & r_{\mathbb{T}_+} a_1(t) \\ r_{\mathbb{T}_+} a_0(-t) & r_{\mathbb{T}_+} a_1(-t) \end{pmatrix}, \end{aligned} \tag{2.8}$$

$$v_1(t) = \begin{pmatrix} r_{\mathbb{T}_+} b_0(t) & r_{\mathbb{T}_+} b_1(t) \\ r_{\mathbb{T}_+} Jb_0(t) & r_{\mathbb{T}_+} Jb_1(t) \end{pmatrix} = \begin{pmatrix} r_{\mathbb{T}_+} b_0(t) & r_{\mathbb{T}_+} b_1(t) \\ r_{\mathbb{T}_+} b_0(-t) & r_{\mathbb{T}_+} b_1(-t) \end{pmatrix}. \tag{2.9}$$

In this way, from the scalar operator $\mathcal{O}$ we passed to the matrix operator A_1. In fact, the operator $\mathcal{O}$ acts on $L^p(\mathbb{T}, w)$ while A_1 acts on $[L^p(\mathbb{T}_+, w)]^2$.

On the other hand, for future purposes, it is convenient to have a diagonal matrix instead of V. So, in view of this goal, we start by applying

$$K^{\pm 1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{2.10}$$

to the operator A_1, and therefore obtain the new operator

$$A_2 = K^{-1} A_1 K : [L^p(\mathbb{T}_+, w)]^2 \to [L^p(\mathbb{T}_+, w)]^2 .$$

First note that

$$\begin{aligned} D_{1,-1} :=& K^{-1} V K = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ =& \operatorname{diag}(1, -1) \end{aligned}$$

and so,

$$\begin{aligned} K^{-1} A_1 K =& K^{-1} u_1 K + K^{-1} v_1 K K^{-1} \left(\sum_{k=0}^{1} V^k U_k \right) K \\ =& u_2 + v_2 \left(\sum_{k=0}^{1} K^{-1} V^k U_k \right) K \\ =& u_2 + v_2 \left(\sum_{k=0}^{1} K^{-1} V^k K K^{-1} U_k \right) K \\ =& u_2 + v_2 \left(\sum_{k=0}^{1} D_{1,-1}^k U_k \right) = A_2, \end{aligned}$$

where

$$u_2 = K^{-1}u_1K, \qquad v_2 = K^{-1}v_1K. \tag{2.11}$$

Now, in order to obtain a simplification of the above formulas, let us study the structure of the operator

$$U := \sum_{k=0}^{1} D_{1,-1}^{k} U_k = \begin{pmatrix} U_0 + U_1 & 0 \\ 0 & U_0 - U_1 \end{pmatrix} = G\left(\sum_{k=0}^{1} U_k\right)G^{-1}, \tag{2.12}$$

where

$$G^{\pm 1}(t) = \operatorname{diag}(1, t^{\pm 1}). \tag{2.13}$$

Clearly, the matrix function G is continuous on $\mathbb{T}_+$, does not degenerate, and G and G^{-1} are mutually reciprocal matrices. Using (2.7), and Lemma 1 of [49] for the present conditions, we have:

$$\sum_{k=0}^{1} \frac{1}{\tau - (-1)^k t} = \left(\frac{\tau}{t}\right)^{\gamma(r)} \sum_{k=0}^{1} \frac{(-1)^{rk}}{\tau - (-1)^k t}, \qquad t, \tau \in \mathbb{T}_+, \tag{2.14}$$

where $\gamma(r) = 1 + \operatorname{sign}(r - \frac{1}{2}) - r$, $r = -1, 0, 1, 2$, which leads to

$$\left(\sum_{k=0}^{1} U_k \eta\right)(t) = \frac{1}{\pi i}\int_{\mathbb{T}_+} \frac{2\tau}{\tau^2 - t^2}\eta(\tau)d\tau\,. \tag{2.15}$$

Now, from (2.12) and (2.15), it follows

$$(U\eta)(t) = G(t)\frac{1}{\pi i}\int_{\mathbb{T}_+} \frac{2\tau}{\tau^2 - t^2} G^{-1}(\tau)\eta(\tau)d\tau.$$

Thus, the operator A_2 can be equivalently written in the form

$$(A_2\eta)(t) = u_2(t)\eta(t) + \frac{v_2(t)}{\pi i}G(t)\int_{\mathbb{T}_+} \frac{2\tau}{\tau^2 - t^2}G^{-1}(\tau)\eta(\tau)d\tau\,.$$

We will also make use of the operators

$$N_{\mathbb{T}_+}(\zeta)(t) = \zeta(t^2), \qquad N_{\mathbb{T}_+}^{-1}(\zeta)(t) = \zeta(t^{1/2})\,, \tag{2.16}$$

with $N_{\mathbb{T}_+} \in \mathcal{L}([L^p(\mathbb{T}, w)]^2, [L^p(\mathbb{T}_+, w)]^2)$, $N_{\mathbb{T}_+}^{-1} \in \mathcal{L}([L^p(\mathbb{T}_+, w)]^2, [L^p(\mathbb{T}, w)]^2)$. We construct now the following operator by applying the operator $GN_{\mathbb{T}_+}$ and its inverse to the operator A_2:

$$N_{\mathbb{T}_+}^{-1}G^{-1}u_2(t)GN_{\mathbb{T}_+} + N_{\mathbb{T}_+}^{-1}G^{-1}v_2(t)\frac{G}{\pi i}\left(\int_{\mathbb{T}_+} \frac{2\tau}{\tau^2 - t^2}(\cdot)G^{-1}d\tau\right)GN_{\mathbb{T}_+}$$
$$= \upsilon_{\mathbb{T}} + \vartheta_{\mathbb{T}} S_{\mathbb{T}} =: \mathfrak{D}_{\mathbb{T}}. \tag{2.17}$$

The connection between the coefficients of the operators $\mathcal{O}$ and $\mathfrak{D}_{\mathbb{T}}$ is given by the formulas:

$$\begin{aligned}
&\upsilon_{\mathbb{T}}(t) = N_{\mathbb{T}_+}^{-1} G^{-1} K^{-1}\, u_1(t)\, KGN_{\mathbb{T}_+} = \\
&N_{\mathbb{T}_+}^{-1}\left(\left(\begin{array}{cc} 1 & 0 \\ 0 & t^{-1} \end{array}\right)\frac{1}{\sqrt{2}}\left(\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array}\right) u_1(t) \frac{1}{\sqrt{2}}\left(\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array}\right)\left(\begin{array}{cc} 1 & 0 \\ 0 & t \end{array}\right)\right) N_{\mathbb{T}_+} \\
&= \frac{1}{\sqrt{2}}\left(\begin{array}{cc} 1 & 1 \\ t^{-1/2} & -t^{-1/2} \end{array}\right) u_1(t^{1/2}) \frac{1}{\sqrt{2}}\left(\begin{array}{cc} 1 & t^{1/2} \\ 1 & -t^{1/2} \end{array}\right)
\end{aligned} \tag{2.18}$$

and

$$\vartheta_{\mathbb{T}}(t) = \frac{1}{\sqrt{2}}\left(\begin{array}{cc} 1 & 1 \\ t^{-1/2} & -t^{-1/2} \end{array}\right) v_1(t^{1/2}) \frac{1}{\sqrt{2}}\left(\begin{array}{cc} 1 & t^{1/2} \\ 1 & -t^{1/2} \end{array}\right). \tag{2.19}$$

All these are now assembled in the next theorem.

Theorem 2.1. *The initial singular integral operator with reflection* $(J\varphi)(t) = \varphi(-t)$, $t \in \mathbb{T}$,

$$\mathcal{O} = a_0 I_{\mathbb{T}} + b_0 S_{\mathbb{T}} + a_1 J + b_1 S_{\mathbb{T}} J\,,$$

acting between $L^p(\mathbb{T}, w)$ *spaces* (*with* $w \in A_p^e(\mathbb{T})$), *is equivalent to the matrix singular integral operator* (*without shift*)

$$\mathfrak{D}_{\mathbb{T}} = \upsilon_{\mathbb{T}} I_{\mathbb{T}} + \vartheta_{\mathbb{T}} S_{\mathbb{T}}, \qquad \mathfrak{D}_{\mathbb{T}} \in \mathcal{L}[L^p(\mathbb{T}, w)]^2\,,$$

defined in (2.17). *The operator equivalence relation between* $\mathcal{O}$ *and* $\mathfrak{D}_{\mathbb{T}}$ *is presented in the form of the following similarity transformation*

$$F^{-1}\mathcal{O}F = \mathfrak{D}_{\mathbb{T}},$$

where

$$\begin{aligned}
F &= M_{\mathbb{T}_+} KGN_{\mathbb{T}_+} \in \mathcal{L}([L^p(\mathbb{T}, w)]^2, L^p(\mathbb{T}, w)), \\
F^{-1} &= N_{\mathbb{T}_+}^{-1} G^{-1} K^{-1} M_{\mathbb{T}_+}^{-1} \in \mathcal{L}(L^p(\mathbb{T}, w), [L^p(\mathbb{T}, w)]^2)
\end{aligned}$$

and the explicit forms of the operators $M_{\mathbb{T}_+}^{\pm 1}$, $K^{\pm 1}$, $G^{\pm 1}$, $N_{\mathbb{T}_+}^{\pm 1}$ *are given in* (2.2), (2.3), (2.10), (2.13) *and* (2.16).

Corollary 2.2. *For each* $t \in \mathbb{T}$, *assume that* $a_0(t) \neq b_0(t)$ *or* $a_1(t) \neq b_1(t)$. *The singular integral operator with the reflection*

$$\mathcal{O} = a_0 I_{\mathbb{T}} + b_0 S_{\mathbb{T}} + a_1 J + b_1 S_{\mathbb{T}} J\,,$$

defined in (2.1), *is equivalent to the Toeplitz matrix operator*

$$\mathcal{T}_{\psi_{\mathbb{T}}} = P_+ \psi_{\mathbb{T}|[P_+ L^p(\mathbb{T}, w)]^2} : [H_+^p(\mathbb{T}, w)]^2 \to [H_+^p(\mathbb{T}, w)]^2\,,$$

where $\psi_{\mathbb{T}} := (\upsilon_{\mathbb{T}} - \vartheta_{\mathbb{T}})^{-1}(\upsilon_{\mathbb{T}} + \vartheta_{\mathbb{T}})$, *for* $\upsilon_{\mathbb{T}}$ *and* $\vartheta_{\mathbb{T}}$ *given in* (2.18) *and* (2.19), *respectively.*

Proof. Taking into consideration Theorem 2.1, it is sufficient to show that $\mathfrak{D}_{\mathbb{T}}$ and $\mathcal{T}_{\psi_{\mathbb{T}}}$ are equivalent operators. For this purpose, we start by considering

$$\psi_{\mathbb{T}} := (\upsilon_{\mathbb{T}} - \vartheta_{\mathbb{T}})^{-1}(\upsilon_{\mathbb{T}} + \vartheta_{\mathbb{T}}),$$

and use this element to perform the following explicit computation:

$$\begin{aligned}(\upsilon_{\mathbb{T}} - \vartheta_{\mathbb{T}})^{-1}\mathfrak{D}_{\mathbb{T}} =& (\upsilon_{\mathbb{T}} - \vartheta_{\mathbb{T}})^{-1}\left[\frac{1}{2}(\upsilon_{\mathbb{T}} + \vartheta_{\mathbb{T}})(I_{\mathbb{T}} + S_{\mathbb{T}}) + \frac{1}{2}(\upsilon_{\mathbb{T}} - \vartheta_{\mathbb{T}})(I_{\mathbb{T}} - S_{\mathbb{T}})\right] \\ =& (\upsilon_{\mathbb{T}} - \vartheta_{\mathbb{T}})^{-1}\left[(\upsilon_{\mathbb{T}} + \vartheta_{\mathbb{T}})P_+ + (\upsilon_{\mathbb{T}} - \vartheta_{\mathbb{T}})P_-\right] = \psi_{\mathbb{T}}P_+ + P_- \\ =& (P_+\psi_{\mathbb{T}}P_+ + P_-)(I_{\mathbb{T}} + P_-\psi_{\mathbb{T}}P_+), \end{aligned} \tag{2.20}$$

where the operator $I_{\mathbb{T}} + P_-\psi_{\mathbb{T}}P_+$ is invertible by $I_{\mathbb{T}} - P_-\psi_{\mathbb{T}}P_+$. Note that the existence of the inverse of $\upsilon_{\mathbb{T}} - \vartheta_{\mathbb{T}}$ is guaranteed by the general assumption which ensures that for each $t \in \mathbb{T}$, $a_0(t) \neq b_0(t)$ or $a_1(t) \neq b_1(t)$.

Therefore, in particular, the identity (2.20) shows that $\mathfrak{D}_{\mathbb{T}}$ and $P_+\psi_{\mathbb{T}}P_+ + P_-$ are equivalent operators.

Now, rewriting $P_+\psi_{\mathbb{T}}P_+ + P_-$ in the matrix form

$$\begin{aligned}& P_+\psi_{\mathbb{T}}P_+ + P_- = \begin{pmatrix} \mathcal{T}_{\psi_{\mathbb{T}}} & 0 \\ 0 & I_{\mathbb{T}} \end{pmatrix} \\ & : P_+[L^p(\mathbb{T},w)]^2 \oplus P_-[L^p(\mathbb{T},w)]^2 \to P_+[L^p(\mathbb{T},w)]^2 \oplus P_-[L^p(\mathbb{T},w)]^2\end{aligned}$$

it directly leads us to the desired conclusion. □

2.2 Singular integral operators with flip: Two equivalence relations

As was annunciated in the beginning of the chapter, now we show two explicit operator identities between the operator $\mathcal{O}$ and some extra operators. Namely, one with a matrix pure singular integral operator as in the previous section, and another with a matrix Toeplitz plus Hankel operator.

2.2.1 First equivalence: An equivalence after extension with a pure matrix SIO

The present subsection we consider the singular integral operator

$$\mathcal{O} = a_0 I_{\mathbb{T}} + b_0 S_{\mathbb{T}} + a_1\widetilde{J} + b_1 S_{\mathbb{T}}\widetilde{J}, \tag{2.21}$$

with backward Carleman shift (or *Flip operator*)

$$(\widetilde{J}\varphi)(t) = \frac{1}{t}\varphi\left(\frac{1}{t}\right), \qquad t \in \mathbb{T}$$

where, for the sake of the well definiteness of the operators involved in our equivalence relation, we are going to consider in this case the operator $\mathcal{O}$ defined between weighted Lebesgue spaces $L^p(\mathbb{T},\rho)$, $1 < p < \infty$, $\rho(t) = |t-1|^{1-2/p}$, and where a_0, b_0, a_1, b_1 are essentially bounded functions.

We will construct an operator equivalence after extension relation between the singular integral operator $\mathcal{O} : L^p(\mathbb{T},\rho) \to L^p(\mathbb{T},\rho)$, with $\rho(t) = |t-1|^{1-2/p}$, and a new operator

$$\mathcal{D}_{\mathbb{T}} : [L^p(\mathbb{T},w)]^2 \to [L^p(\mathbb{T},w)]^2, \qquad w(t) = \left|i\frac{1+t}{1-t}\right|^{-1/2p} |1-t|^{1-2/p}, \quad (2.22)$$

which already does not contains any shift. For this purpose, in part of the process, we will use an operator equivalence relation similar to the exhibit in the previous section. Also, in addition, within the procedures presented in previous section, an equivalence relation will be here obtained between the operator $\mathcal{O}$ and a Toeplitz operator $\mathcal{T}_{\phi_{\mathbb{T}}}$.

Let us consider the isometric isomorphism $B : L^p(\mathbb{T},\rho) \to L^p(\mathbb{R})$ (with $\rho(t) = |t-1|^{1-2/p}, t \in \mathbb{T}$), defined by

$$(B\phi)(x) = \frac{2^{1-1/p}}{x+i}\phi\left(\frac{x-i}{x+i}\right), \quad x \in \mathbb{R}, \tag{2.23}$$

and whose inverse is explicitly given by

$$(B^{-1}\psi)(t) = \frac{i2^{1/p}}{1-t}\psi\left(i\frac{1+t}{1-t}\right), \quad t \in \mathbb{T} \setminus \{1\}. \tag{2.24}$$

The following operator identities involving operator B are well-known:

i) $B\widetilde{J}B^{-1} = -W_{\mathbb{R}}$, where $W_{\mathbb{R}}$ is the reflection operator (1.17) on $\mathbb{R}$ defined by $(W_{\mathbb{R}}f)(x) = f(-x)$, $x \in \mathbb{R}$.

ii) $BS_{\mathbb{T}}B^{-1} = S_{\mathbb{R}}$, where $S_{\mathbb{R}}$ is the Cauchy singular integral operator over $\mathbb{R}$ defined by $(S_{\mathbb{R}}\varphi)(x) = \frac{1}{\pi i}\int_{\mathbb{R}} \frac{\varphi(u)}{u-x}du$, $x \in \mathbb{R}$.

iii) $BaB^{-1} = (B_0^{-1}a)I_{\mathbb{R}}$, where for $a \in L^\infty(\mathbb{T})$, $(B_0^{-1}a)(x) = a\left(\frac{x-i}{x+i}\right)$, $x \in \mathbb{R}$, and as in (1.12)

$$(B_0 a)(t) = a\left(i\frac{1+t}{1-t}\right), \quad t \in \mathbb{T} \setminus \{1\}.$$

Due to the above properties, our first step to built the announced operator relation is to apply B and B^{-1} to the operator $\mathcal{O}$, and therefore passing from this operator to the following singular integral operator (acting between the $L^p(\mathbb{R})$ space):

$$\mathcal{B} = B\mathcal{O}B^{-1} = aI_{\mathbb{R}} + bS_{\mathbb{R}} + cW_{\mathbb{R}} + dS_{\mathbb{R}}W_{\mathbb{R}}. \tag{2.25}$$

Note that this new operator $\mathcal{B}$ contains the reflection operator $W_{\mathbb{R}}$ and their coefficients are defined by

$$a = B_0^{-1}a_0, \quad b = B_0^{-1}b_0, \quad c = -B_0^{-1}a_1, \quad d = -B_0^{-1}b_1.$$

Now, we are going to use some of the operators given in the previous section (defining it in the proper space), in order to construct an operator equivalence relation between the singular integral operator $\mathcal{B}$ given in (2.25) and a (vector) pure singular integral operator. Such an equivalence relation can be built with the use of the operator $M_{\mathbb{R}_+} \in \mathcal{L}([L^p(\mathbb{R}_+)]^2, L^p(\mathbb{R}))$ defined by

$$M_{\mathbb{R}_+}\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} = \varphi(x) := \begin{cases} \varphi_1(x), & x \in \mathbb{R}_+ \\ \varphi_2(-x), & x \in \mathbb{R}_-. \end{cases} \tag{2.26}$$

Note that the inverse of $M_{\mathbb{R}_+}$ is the operator

$$M_{\mathbb{R}_+}^{-1}\varphi(x) = \begin{pmatrix} \varphi(x) \\ \varphi(-x) \end{pmatrix}, \qquad x \in \mathbb{R}_+. \tag{2.27}$$

The following idempotent matrix will also have a significant role in the mentioned equivalence relation

$$K^{\pm 1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{2.28}$$

as well as the operator

$$(N_{\mathbb{R}_+}\varphi)(x) = \varphi(x^2), \qquad N_{\mathbb{R}_+} \in \mathcal{L}([L^p(\mathbb{R}_+, |x|^{-1/2p})]^2, [L^p(\mathbb{R}_+)]^2) \tag{2.29}$$

whose inverse is

$$(N_{\mathbb{R}_+}^{-1}\phi)(x) = \phi(\sqrt{x}). \tag{2.30}$$

Finally, it is also needed the operator

$$R_{\mathbb{R}_+} = \begin{pmatrix} S_{\mathbb{R}_+} + U_{1,\mathbb{R}_+} & 0 \\ 0 & I_{\mathbb{R}_+} \end{pmatrix}, \qquad R_{\mathbb{R}_+} \in \mathcal{L}([L^p(\mathbb{R}_+)]^2) \tag{2.31}$$

where $(S_{\mathbb{R}_+}f)(x) = \frac{1}{\pi i}\int_{\mathbb{R}_+}\frac{f(u)}{u-x}du$ and $(U_{1,\mathbb{R}_+}f)(x) = \frac{1}{\pi i}\int_{\mathbb{R}_+}\frac{f(u)}{u+x}du$, $x \in \mathbb{R}_+$. Note that $S_{\mathbb{R}_+} + U_{1,\mathbb{R}_+}$ is an invertible operator and its inverse is given by $S_{\mathbb{R}_+} - U_{1,\mathbb{R}_+}$. Thus, $R_{\mathbb{R}_+}$ is also an invertible operator.

The operator equivalence relation has the explicit form

$$\mathcal{HBF} = D_{\mathbb{R}_+}, \qquad D_{\mathbb{R}_+} \in \mathcal{L}([L^p(\mathbb{R}_+, |x|^{-1/2p})]^2),$$

where $\mathcal{H} = N_{\mathbb{R}_+}^{-1}K^{-1}M_{\mathbb{R}_+}^{-1}$, $\mathcal{F} = M_{\mathbb{R}_+}KR_{\mathbb{R}_+}N_{\mathbb{R}_+}$ and gives rise to the new operator

$$D_{\mathbb{R}_+} = u_{\mathbb{R}_+}I_{\mathbb{R}_+} + v_{\mathbb{R}_+}S_{\mathbb{R}_+}$$

where the relation between the coefficients $u_{\mathbb{R}_+}$ and $v_{\mathbb{R}_+}$ of this operator $D_{\mathbb{R}_+}$ and the coefficients of the operator $\mathcal{O}$ is given by the formulas:

$$u_{\mathbb{R}_+}(x) = \begin{pmatrix} \frac{(a_1(y)+b_1(y))-(a_1(-y)+b_1(-y))}{2} & \frac{(a_0(y)+b_0(y))-(a_0(-y)+b_0(-y))}{2} \\ \frac{(a_1(y)+b_1(y))+(a_1(-y)+b_1(-y))}{2} & \frac{(a_0(y)+b_0(y))+(a_0(-y)+b_0(-y))}{2} \end{pmatrix} \tag{2.32}$$

$$v_{\mathbb{R}_+}(x) = \begin{pmatrix} \frac{(a_0(y)-b_0(y))+(a_0(-y)-b_0(-y))}{2} & \frac{(a_1(y)-b_1(y))+(a_1(-y)-b_1(-y))}{2} \\ \frac{(a_0(y)-b_0(y))-(a_0(-y)-b_0(-y))}{2} & \frac{(a_1(y)-b_1(y))-(a_1(-y)-b_1(-y))}{2} \end{pmatrix} \tag{2.33}$$

where

$$y = \frac{x^{1/2}-i}{x^{1/2}+i}, \qquad x \in \mathbb{R}_+. \tag{2.34}$$

Thus, we have just obtained the following result:

Theorem 2.3. *The singular integral operator* (*with the backward Carleman shift*)

$$\mathcal{O} = a_0I_{\mathbb{T}} + b_0S_{\mathbb{T}} + a_1\widetilde{J} + b_1S_{\mathbb{T}}\widetilde{J}$$

acting on the space $L^p(\mathbb{T},\rho)$ (*where* $\rho(t) = |t-1|^{1-2/p}$) *is equivalent to the matrix singular integral operator* (*without shift*)

$$D_{\mathbb{R}_+} = u_{\mathbb{R}_+}I_{\mathbb{R}_+} + v_{\mathbb{R}_+}S_{\mathbb{R}_+}, \quad D_{\mathbb{R}_+} \in \mathcal{L}([L^p(\mathbb{R}_+, |x|^{-1/2p})]^2). \tag{2.35}$$

The equivalence relation has the following explicit form:

$$\mathcal{GOV} = D_{\mathbb{R}_+},$$

where

$$\begin{aligned} \mathcal{G} =& N_{\mathbb{R}_+}^{-1}K^{-1}M_{\mathbb{R}_+}^{-1}B \in \mathcal{L}(L^p(\mathbb{T},\rho), [L^p(\mathbb{R}_+, |x|^{-1/2p})]^2), \\ \mathcal{V} =& B^{-1}M_{\mathbb{R}_+}KR_{\mathbb{R}_+}N_{\mathbb{R}_+} \in \mathcal{L}([L^p(\mathbb{R}_+, |x|^{-1/2p})]^2, L^p(\mathbb{T},\rho)), \end{aligned}$$

$\rho(t) = |t-1|^{1-2/p}$, *and the explicit form of the operators* $B^{\pm 1}$, $M_{\mathbb{R}_+}^{\pm 1}$, $K^{\pm 1}$, $N_{\mathbb{R}_+}^{\pm 1}$ *and* $R_{\mathbb{R}_+}$ *is given in* (2.23)–(2.24), (2.26)–(2.31), *respectively.*

On the other hand, we will now proceed with the extension by the identity of the operator $D_{\mathbb{R}_+}$ in (2.35) into the $[L^p(\mathbb{R}, |x|^{-1/2p})]^2$ space. This is in fact an equivalence after extension relation ([3]) applied to $D_{\mathbb{R}_+}$ where the resulting operator has the form:

$$D_{\mathbb{R}} := \begin{pmatrix} D_{\mathbb{R}_+} & 0 \\ 0 & I_{[L^p(\mathbb{R}_-, |x|^{-1/2p})]^2} \end{pmatrix} \in \mathcal{L}([L^p(\mathbb{R}, |x|^{-1/2p})]^2. \tag{2.36}$$

Thus, $D_{\mathbb{R}_+} : [L^p(\mathbb{R}_+, |x|^{-1/2p})]^2 \to [L^p(\mathbb{R}_+, |x|^{-1/2p})]^2$ can be viewed as the restriction of $D_{\mathbb{R}}$ to its first component spaces. We will use the following notation for this interpretation:

$$D_{\mathbb{R}_+} = \mathrm{Rest}_{[L^p(\mathbb{R}_+, |x|^{-1/2p})]^2}(D_{\mathbb{R}}).$$

It directly follows from the identity (2.36) that $D_{\mathbb{R}_+}$ and $D_{\mathbb{R}}$ enjoy the same Fredholm and invertibility properties. In addition, the operator $D_{\mathbb{R}}$ can also be written in the form

$$D_{\mathbb{R}} = u_{\mathbb{R}} I_{\mathbb{R}} + v_{\mathbb{R}} S_{\mathbb{R}} \tag{2.37}$$

where

$$u_{\mathbb{R}} = \chi_{\mathbb{R}_-} + \ell_0 u_{\mathbb{R}_+}, \qquad v_{\mathbb{R}} = \ell_0 v_{\mathbb{R}_+}, \tag{2.38}$$

with ℓ_0 being the zero extension operator, and where $\chi_{\mathbb{R}_-}$ is the characteristic function on $\mathbb{R}_-$.

Now we will pass from $\mathcal{D}_{\mathbb{R}}$ to a singular integral operator $\mathcal{D}_{\mathbb{T}}$ using the isometric isomorphism

$$B_2 := \mathrm{diag}(B, B) \tag{2.39}$$

from $[L^p(\mathbb{R}, |x|^{-1/2p})]^2$ onto $[L^p(\mathbb{T}, w)]^2$ with the weight $w(t) = |i\frac{1+t}{1-t}|^{-1/2p}|1-t|^{1-2/p}$. We therefore obtain in the explicit form:

$$\mathcal{D}_{\mathbb{T}} := B_2^{-1} \mathcal{D}_{\mathbb{R}} B_2 = u_{\mathbb{T}} I_{\mathbb{T}} + v_{\mathbb{T}} S_{\mathbb{T}}, \tag{2.40}$$

with

$$u_{\mathbb{T}} I_{\mathbb{T}} = B_2^{-1} u_{\mathbb{R}} B_2, \qquad v_{\mathbb{T}} I_{\mathbb{T}} = B_2^{-1} v_{\mathbb{R}} B_2, \tag{2.41}$$

where $u_{\mathbb{T}} = \mathrm{diag}(B_0, B_0) u_{\mathbb{R}}$ and $v_{\mathbb{T}} = \mathrm{diag}(B_0, B_0) v_{\mathbb{R}}$ in $\mathbb{T}_+$, and $u_{\mathbb{T}} \equiv I_{2\times 2}$, $v_{\mathbb{T}} \equiv 0_{2\times 2}$ in $\mathbb{T}_-$. The explicit form of these matrix functions is given by

$$u_{\mathbb{T}}(t) = \begin{cases} u_{\mathbb{T}_+}(t), & t \in \mathbb{T}_+ \\ I_{2\times 2}, & t \in \mathbb{T}_- \end{cases}, \qquad v_{\mathbb{T}}(t) = \begin{cases} v_{\mathbb{T}_+}(t), & t \in \mathbb{T}_+ \\ 0_{2\times 2}, & t \in \mathbb{T}_- \end{cases} \tag{2.42}$$

where for $t \in \mathbb{T}_+$ we have

$$u_{\mathbb{T}_+}(t) = \frac{1}{2} \begin{pmatrix} \mu_{11}(t) & \mu_{12}(t) \\ \mu_{21}(t) & \mu_{22}(t) \end{pmatrix} \tag{2.43}$$

with

$$\begin{aligned}
\mu_{11}(t) =&(a_1(t^{1/2})+b_1(t^{1/2}))-(a_1(-t^{1/2})+b_1(-t^{1/2}))\\
\mu_{12}(t) =&(a_0(t^{1/2})+b_0(t^{1/2}))-(a_0(-t^{1/2})+b_0(-t^{1/2}))\\
\mu_{21}(t) =&(a_1(t^{1/2})+b_1(t^{1/2}))+(a_1(-t^{1/2})+b_1(-t^{1/2}))\\
\mu_{22}(t) =&(a_0(t^{1/2})+b_0(t^{1/2}))+(a_0(-t^{1/2})+b_0(-t^{1/2}))
\end{aligned}$$

and

$$v_{\mathbb{T}_+}(t)=\frac{1}{2}\begin{pmatrix} \vartheta_{11}(t) & \vartheta_{12}(t)\\ \vartheta_{21}(t) & \vartheta_{22}(t)\end{pmatrix} \tag{2.44}$$

with

$$\begin{aligned}
\vartheta_{11}(t) =&(a_0(t^{1/2})-b_0(t^{1/2}))+(a_0(-t^{1/2})-b_0(-t^{1/2}))\\
\vartheta_{12}(t) =&(a_1(t^{1/2})-b_1(t^{1/2}))+(a_1(-t^{1/2})-b_1(-t^{1/2}))\\
\vartheta_{21}(t) =&(a_0(t^{1/2})-b_0(t^{1/2}))-(a_0(-t^{1/2})-b_0(-t^{1/2}))\\
\vartheta_{22}(t) =&(a_1(t^{1/2})-b_1(t^{1/2}))-(a_1(-t^{1/2})-b_1(-t^{1/2})).
\end{aligned}$$

All the just presented relations are now summarized in the next Theorem.

Theorem 2.4. *The singular integral operator* (*with the backward Carleman shift*)

$$\mathcal{O}=a_0I_{\mathbb{T}}+b_0S_{\mathbb{T}}+a_1\widetilde{J}+b_1S_{\mathbb{T}}\widetilde{J}$$

(*acting on the space* $L^p(\mathbb{T},\rho)$, $\rho(t)=|t-1|^{1-2/p}$) *is equivalent after extension to the matrix singular integral operator*

$$\mathcal{D}_{\mathbb{T}}=u_{\mathbb{T}}I_{\mathbb{T}}+v_{\mathbb{T}}S_{\mathbb{T}},\quad \mathcal{D}_{\mathbb{T}}\in\mathcal{L}([L^p(\mathbb{T},w)]^2),$$

where $w(t)=|i\frac{1+t}{1-t}|^{-1/2p}|1-t|^{1-2/p}$, *and with coefficients* $u_{\mathbb{T}}=\operatorname{diag}(B_0,B_0)u_{\mathbb{R}}$, $v_{\mathbb{T}}=\operatorname{diag}(B_0,B_0)v_{\mathbb{R}}$.

Notice that Corollary 2.2 also holds for this case:

Corollary 2.5. *For each* $t\in\mathbb{T}$, *assume that* $a_0(t)\neq b_0(t)$ *or* $a_1(t)\neq b_1(t)$. *The singular integral operator with shift*

$$\mathcal{O}=a_0I_{\mathbb{T}}+b_0S_{\mathbb{T}}+a_1\widetilde{J}+b_1S_{\mathbb{T}}\widetilde{J},$$

acting on the space $L^p(\mathbb{T},\rho)$, *with* $1<p<\infty$ *and* $\rho(t)=|t-1|^{1-2/p}$, *is equivalent after extension to the Toeplitz matrix operator*

$$\mathcal{T}_{\phi_{\mathbb{T}}}=P_+\phi_{\mathbb{T}|[P_+L^p(\mathbb{T},w)]^2}:[H^p_+(\mathbb{T},w)]^2\to[H^p_+(\mathbb{T},w)]^2\,,$$

where $\phi_{\mathbb{T}}:=(u_{\mathbb{T}}-v_{\mathbb{T}})^{-1}(u_{\mathbb{T}}+v_{\mathbb{T}})$, *and* $w(t)=|i\frac{1+t}{1-t}|^{-1/2p}|1-t|^{1-2/p}$.

2.2.2 Second equivalence: Reduction to a matrix Toeplitz plus Hankel operator

In this subsection, we will reduce the singular integral operator $\mathcal{O}$ with flip given on (2.21) and defined, in this case, on the space $L^p(\mathbb{T})$, $1 < p < \infty$, to some matrix Toeplitz plus Hankel operator acting on $[H^p_+(\mathbb{T})]^2$. Recall that the Hankel operator defined on $[H^p_+(\mathbb{T})]^2$, $1 < p < \infty$, with the generating function $b \in [L^\infty(\mathbb{T})]^{2\times 2}$ is defined by

$$\mathcal{H}_b : f \in [H^p_+(\mathbb{T})]^2 \longmapsto P_+(b(\widetilde{J}f)) \in [H^p_+(\mathbb{T})]^2.$$

In operator theoretic notation this can be rewritten as

$$\mathcal{H}_b = P_+ b \widetilde{J} P_+$$

with the same interpretation of the operators P_+ as in the definition of the Toeplitz operator $\mathcal{T}_a$ in Page 8 and $\widetilde{J}$ is the flip operator (1.19). Thus, the matrix Toeplitz plus Hankel operator with multipliers a and b on $[H^p_+(\mathbb{T})]^2$ is defined by

$$\mathcal{T}_a + \mathcal{H}_b := P_+ a P_+ + P_+ b \widetilde{J} P_+. \tag{2.45}$$

The following result presents such a reduction.

Theorem 2.6. *Let $a_0, a_1, b_0, b_1 \in L^\infty(\mathbb{T})$, $1 < p < \infty$, and consider*

$$a := \begin{pmatrix} x_1 & y_1 \\ \widetilde{J}x_1 & \widetilde{J}y_1 \end{pmatrix}, \qquad b := \begin{pmatrix} x_2 & y_2 \\ \widetilde{J}x_2 & \widetilde{J}y_2 \end{pmatrix} \tag{2.46}$$

where

$$\begin{aligned} x_1 :=& a_0 + b_0, \quad x_2 := a_1 - b_1 \\ y_1 :=& a_1 + b_1, \quad y_2 := a_0 - b_0. \end{aligned}$$

Then, the singular integral operator with flip

$$\mathcal{O} = a_0 I_{\mathbb{T}} + b_0 S_{\mathbb{T}} + a_1 \widetilde{J} + b_1 S_{\mathbb{T}} \widetilde{J}$$

(acting on the space $L^p(\mathbb{T})$) is equivalent to the matrix Toeplitz plus Hankel operator with multipliers a and b, $\mathcal{T}_a + \mathcal{H}_b$ (acting on $[H^p_+(\mathbb{T})]^2$).

Proof. First observe that

$$\begin{aligned} & a_0 I_{\mathbb{T}} + b_0 S_{\mathbb{T}} + a_1 \widetilde{J} + b_1 S_{\mathbb{T}} \widetilde{J} = 2b_0 P_+ + (a_0 - b_0) I_{\mathbb{T}} + 2b_1 P_+ \widetilde{J} + (a_1 - b_1)\widetilde{J} \\ &= 2b_0 P_+ + (a_0 - b_0) I_{\mathbb{T}} + (b_1 + a_1) P_+ \widetilde{J} + (a_1 - b_1)\widetilde{J} - (a_1 - b_1) P_+ \widetilde{J} \\ &= (a_0 + b_0) P_+ + (b_1 + a_1) P_+ \widetilde{J} + (a_1 - b_1) P_- \widetilde{J} + (a_0 - b_0) P_- \\ &= x_1 P_+ + y_1 P_+ \widetilde{J} + x_2 \widetilde{J} P_+ + y_2 \widetilde{J} P_+ \widetilde{J}. \end{aligned}$$

Note that in the last line of the previous computation we put $\widetilde{J}P_+\widetilde{J}$ instead of P_- (which will be used for obtaining the conclusion). On the other hand, observe that the matrix Toeplitz plus Hankel operator on $[H^p_+(\mathbb{T})]^2$, $\mathcal{T}_a + \mathcal{H}_b$, has the following matrix form

$$\mathcal{T}_a + \mathcal{H}_b = \begin{pmatrix} P_+ & P_+\widetilde{J} \\ P_+\widetilde{J} & P_+ \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P_+ & 0 \\ 0 & P_+ \end{pmatrix}$$
$$+ \begin{pmatrix} P_+ & P_+\widetilde{J} \\ P_+\widetilde{J} & P_+ \end{pmatrix} \begin{pmatrix} x_2 & y_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \widetilde{J} & 0 \\ 0 & \widetilde{J} \end{pmatrix} \begin{pmatrix} P_+ & 0 \\ 0 & P_+ \end{pmatrix}.$$

Now, we will use the operators

$$U := (P_+, \widetilde{J}P_+) : [H^p_+(\mathbb{T})]^2 \longrightarrow L^p(\mathbb{T})$$
$$U^{-1} := \begin{pmatrix} P_+ \\ P_+\widetilde{J} \end{pmatrix} : L^p(\mathbb{T}) \longrightarrow [H^p_+(\mathbb{T})]^2,$$

which are inverses to each other and establish a Banach space isomorphism between the spaces $L^p(\mathbb{T})$ and $[H^p_+(\mathbb{T})]^2$. Indeed, we have

$$UU^{-1} = (P_+, \widetilde{J}P_+) \begin{pmatrix} P_+ \\ P_+\widetilde{J} \end{pmatrix} = P_+ + \widetilde{J}P_+\widetilde{J} = I_{\mathbb{T}},$$
$$U^{-1}U = \begin{pmatrix} P_+ \\ P_+\widetilde{J} \end{pmatrix} (P_+, \widetilde{J}P_+) = \begin{pmatrix} P_+ & 0 \\ 0 & P_+ \end{pmatrix} = I_{\mathbb{T}},$$

and moreover

$$U(\mathcal{T}_a + \mathcal{H}_b)U^{-1} = (P_+, \widetilde{J}P_+) \Bigg[\begin{pmatrix} P_+ & P_+\widetilde{J} \\ P_+\widetilde{J} & P_+ \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P_+ & 0 \\ 0 & P_+ \end{pmatrix}$$
$$+ \begin{pmatrix} P_+ & P_+\widetilde{J} \\ P_+\widetilde{J} & P_+ \end{pmatrix} \begin{pmatrix} x_2 & y_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \widetilde{J} & 0 \\ 0 & \widetilde{J} \end{pmatrix} \begin{pmatrix} P_+ & 0 \\ 0 & P_+ \end{pmatrix} \Bigg] \begin{pmatrix} P_+ \\ P_+\widetilde{J} \end{pmatrix}$$
$$=(I, \widetilde{J}) \Bigg[\begin{pmatrix} x_1 & y_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P_+ & 0 \\ 0 & P_+ \end{pmatrix} + \begin{pmatrix} x_2 & y_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \widetilde{J}P_+ & 0 \\ 0 & \widetilde{J}P_+ \end{pmatrix} \Bigg]$$
$$\times \begin{pmatrix} P_+ \\ P_+\widetilde{J} \end{pmatrix}$$
$$=(I, \widetilde{J}) \begin{pmatrix} x_1 & y_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P_+ \\ P_+\widetilde{J} \end{pmatrix} + (I, \widetilde{J}) \begin{pmatrix} x_2 & y_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \widetilde{J} & 0 \\ 0 & \widetilde{J} \end{pmatrix} \begin{pmatrix} P_+ \\ P_+\widetilde{J} \end{pmatrix}$$
$$=(x_1, y_1) \begin{pmatrix} P_+ \\ P_+\widetilde{J} \end{pmatrix} + (x_2, y_2) \begin{pmatrix} \widetilde{J} & 0 \\ 0 & \widetilde{J} \end{pmatrix} \begin{pmatrix} P_+ \\ P_+\widetilde{J} \end{pmatrix}$$

$$=x_1P_+ + y_1P_+\widetilde{J} + (x_2\widetilde{J}, y_2\widetilde{J})\begin{pmatrix} P_+ \\ P_+\widetilde{J} \end{pmatrix} = x_1P_+ + y_1P_+\widetilde{J} + x_2\widetilde{J}P_+ + y_2\widetilde{J}P_+\widetilde{J}.$$

Hence, $\mathcal{O} = U(\mathcal{T}_a + \mathcal{H}_b)U^{-1}$. I.e., $\mathcal{O}$ is equivalent to the matrix Toeplitz plus Hankel operator $\mathcal{T}_a + \mathcal{H}_b$. $\square$

Chapter 3

Fredholm and Invertibility Criteria for SIOS's

In this chapter we are going to use the equivalence operator relations constructed in Chapter 2 in order to extract the Fredholm property characteristics and the conditions for the invertibility of the singular integral operator $\mathcal{O}$ given by (1.21), for the case when in its definition the shift operator is defined by the reflection operator (1.18), and also by the flip operator (1.19). In addition, for the invertibility criterion, we are going to use the generalized factorization of a bounded measurable matrix-valued function (and by using the properties of the factors in such factorization), formulas for the left-sided and right-sided inverses of the initial operator will be obtained.

3.1 Fredholm criteria and index formula

It is known that the Fredholm characteristics of pure singular integral, Toeplitz and Toeplitz plus Hankel operators, depend on the class of essentially bounded functions to which their coefficients belong [8, 9, 10, 11, 12, 13, 36, 67, 88, 89, 100]. Here, we will present the Fredholm property (in the form of necessary and sufficient conditions) for the operator $\mathcal{O}$ with continuous, piecewise continuous and semi-almost-periodic coefficients.

3.1.1 Continuous and piecewise continuous coefficients

In this subsection we will present a characterization for the Fredholm property of the operator $\mathcal{O}$ and a Fredholm index formula (when in the presence of the Fredholm property). The results depend on the specific shift operator and the classes of functions where the elements a_0, a_1, b_0, b_1 in (2.1) and (2.21) belong to (or, equivalently, on the kind of symbols $\psi_{\mathbb{T}}$ and $\phi_{\mathbb{T}}$ of the Toeplitz operators defined in Corollaries 2.2 and 2.5).

For considering the case of $\psi_{\mathbb{T}}$ and $\phi_{\mathbb{T}}$ being piecewise continuous matrix functions we need the following known notions (see, Subsection 1.2.3): Let

$$\mathcal{M}_{p,w} := \bigcup_{t\in\mathbb{T}} (\{t\} \times \mathcal{H}(0,1;\nu_t^-(p,w),\nu_t^+(p,w))),$$

where $\mathcal{H}(0,1;\nu_t^-(p,w),\nu_t^+(p,w))$ is the horn between 0 and 1 determined by $\nu_t^-(p,w)$ and $\nu_t^+(p,w)$ which depend on p and on the weights. Now, put

$$\mathcal{M}^0_{p,w} = \bigcup_{t\in\mathbb{T}} (\{t\} \times \mathbb{A}(0,1;\nu_t^0(p,w))), \quad \nu_t^0(p,w) = \frac{1}{2}(\nu_t^-(p,w)+\nu_t^+(p,w)).$$

Given a 2×2 matrix-valued function $a \in [PC(\mathbb{T})]^{2\times 2}$, it will be useful to consider the functions

$$\begin{aligned} a^{p,w} &: \mathcal{M}_{p,w} \longrightarrow \mathbb{C}^{2\times 2} \\ a^{p,w}(t,\mu) &:= (1-\mu)a(t-0)+\mu a(t+0) \end{aligned}$$

and

$$\begin{aligned} a_0^{p,w} &: \mathcal{M}^0_{p,w} \longrightarrow \mathbb{C}^{2\times 2} \\ a_0^{p,w}(t,\mu) &:= (1-\mu)a(t-0)+\mu a(t+0)\,. \end{aligned}$$

Remark 3.1. *One can think of the range of* $\det a_0^{p,w}$ *(defined on* $\mathcal{M}^0_{p,w}$*) as a closed continuous and naturally oriented curve (induced by the counter-clockwise orientation of* $\mathbb{T}$*) which results from the essential range of* $\det a$ *by joining* $\det a^-(t)$ *to* $\det a^+(t)$ *by the curve*

$$\{\det((1-\mu)a^-(t)+\mu a^+(t)) \ : \ \mu \in \mathbb{A}(0,1;\nu_t^0(p,w))\}$$

whenever $a^-(t) \neq a^+(t)$*. Thus, if* $\det a_0^{p,w}(t,\mu) \neq 0$ *for all* $(t,\mu) \in \mathcal{M}^0_{p,w}$ *then the winding number* $\operatorname{wind}(\det a_0^{p,w})$ *is a well-defined integer.*

Notice that in case when we consider the singular integral operator with the flip shift operator (1.19), we are dealing with the Khvedelidze weight $\varrho(x) = |x|^{-1/2p}$ (up the use of the isomorphism B defined in (2.23)), then in this case $\nu_t^-(p,w) = \nu_t^+(p,w) = \frac{1}{2p}$ (see also, [11, Example 16.19]). So, the horn $\mathcal{H}(0,1;\nu_t^-(p,w),\nu_t^+(p,w))$ is reduced to the arc $\mathbb{A}\left(0,1;\frac{1}{2p}\right)$ and

$$\mathcal{M}_{p,w} = \bigcup_{t\in\mathbb{T}} \left(\{t\} \times \mathbb{A}\left(0,1;\frac{1}{2p}\right)\right).$$

Theorem 3.1. (i) *Let* $\upsilon_{\mathbb{T}}$ *and* $\vartheta_{\mathbb{T}}$ (*resp.* $u_{\mathbb{T}}$ *and* $v_{\mathbb{T}}$) *defined by* (2.18) *and* (2.19), *respectively* (*resp.* (2.41)) *be continuous matrix valued functions. The singular integral operator with the backward Carleman shift* $(J\varphi)(t) = \varphi(-t)$ (*resp.* $(\widetilde{J}\varphi)(t) = \frac{1}{t}\varphi\big(\frac{1}{t}\big)$), $t \in \mathbb{T}$,

$$\mathcal{O} = a_0 I_{\mathbb{T}} + b_0 S_{\mathbb{T}} + a_1 J + b_1 S_{\mathbb{T}} J$$

acting on the space $L^p(\mathbb{T}, \rho)$, *where* $1 < p < \infty$ *and* $\rho \in A_p^e(\mathbb{T})$ (*resp.* $\rho(t) = |t-1|^{1-2/p}$) *is a Fredholm operator if and only if* $\det(\upsilon_{\mathbb{T}}(t) \pm \vartheta_{\mathbb{T}}(t)) \neq 0$, (*resp.* $\det(u_{\mathbb{T}}(t) \pm v_{\mathbb{T}}(t)) \neq 0$) $t \in \mathbb{T}$. *Under the Fredholm property, the Fredholm index of* $\mathcal{O}$ *is given by*

$$\operatorname{Ind} \mathcal{O} = -\text{wind } \det(\psi)$$

where $\psi := (\upsilon_{\mathbb{T}} - \vartheta_{\mathbb{T}})^{-1}(\upsilon_{\mathbb{T}} + \vartheta_{\mathbb{T}})$ (*resp.* $\psi := (u_{\mathbb{T}} - v_{\mathbb{T}})^{-1}(u_{\mathbb{T}} + v_{\mathbb{T}})$) *and* wind $\det(\psi)$ *is the winding number of* $\det(\psi)$.

(ii) *In the case that* $u_{\mathbb{T}}$ *and* $v_{\mathbb{T}}$ (*resp.* $u_{\mathbb{T}}$ *and* $v_{\mathbb{T}}$) *are matrix valued piecewise continuous functions,* $\mathcal{O}$ *is a Fredholm operator if and only if*

$$\det \psi^{p,w}(t,\mu) \neq 0 \quad \textit{for all } (t,\mu) \in \mathcal{M}_{p,w}.$$

Under the Fredholm property, the Fredholm index in this case is given by

$$\operatorname{Ind} \mathcal{O} = -\text{wind } \det(\psi_0^{p,w}).$$

Proof. As a direct consequence of Corollary 2.2 (resp. Corollary 2.5), we already know that the Fredholm properties of $\mathcal{O}$ and $\mathcal{T}_\psi$ coincide. Therefore, it only remains to use two known results for Toeplitz operators: (i) For continuous $\det \psi$ we use the Gohberg-Krein and Douglas Theorems, e.g., Theorem 16.6 from [11]; (ii) In the case of $\psi \in PC(\mathbb{T})$, the above conclusion is obtained by using Theorem 16.21 from [11]. □

Examples

First, we are going to study the Fredholm characteristics of the singular integral operator with reflection $\mathcal{O} = a_0 I_{\mathbb{T}} + b_0 S_{\mathbb{T}} + a_1 J + b_1 S_{\mathbb{T}} J$ acting between $L^p(\mathbb{T}, w)$ spaces, for different values of p, different weights w and a_0, a_1, b_0 and b_1 belonging to some subalgebras of $L^\infty(\mathbb{T})$. In this way, we will use Theorem 2.1 and consider the operator $\mathfrak{D}_{\mathbb{T}} = \upsilon_{\mathbb{T}} I_{\mathbb{T}} + \vartheta_{\mathbb{T}} S_{\mathbb{T}}$ acting on $[L^p(\mathbb{T}, w)]^2$. In addition, we will use Corollary 2.2 and therefore study the corresponding multiplier $\psi_{\mathbb{T}}$ for each of the below examples with particular elements a_0, a_1, b_0 and b_1.

Example 3.1. Let $\mathcal{O} = a_0 I_{\mathbb{T}} + b_0 S_{\mathbb{T}} + a_1 J + b_1 S_{\mathbb{T}} J : L^p(\mathbb{T}, w) \to L^p(\mathbb{T}, w)$, where

$$a_0(t) := \sinh(t), \quad a_1(t) := e^t, \quad b_0(t) := \cosh(t), \quad b_1(t) := \tanh(t)\,,$$

and $1 < p < \infty$, $w \in A_p^e(\mathbb{T})$.

From Theorem 2.1 we know that the operator $\mathcal{O}$ is equivalent to the matrix operator $\mathfrak{D}_{\mathbb{T}} = \upsilon_{\mathbb{T}} I_{\mathbb{T}} + \vartheta_{\mathbb{T}} S_{\mathbb{T}} : [L^p(\mathbb{T}, w)]^2 \to [L^p(\mathbb{T}, w)]^2$, where $\upsilon_{\mathbb{T}}$ and $\vartheta_{\mathbb{T}}$ are obtained as in (2.18) and (2.19):

$$\upsilon_{\mathbb{T}}(t) = \begin{pmatrix} \frac{e^{-t^{1/2}} + e^{t^{1/2}}}{2} & \frac{-t^{1/2}(e^{-t^{1/2}} + e^{t^{1/2}})}{2} \\ \frac{t^{-1/2}(2\sinh(t^{1/2}) + e^{t^{1/2}} - e^{-t^{1/2}})}{2} & \frac{2\sinh(t^{1/2}) - e^{t^{1/2}} + e^{-t^{1/2}}}{2} \end{pmatrix},$$

$$\vartheta_{\mathbb{T}}(t) = \begin{pmatrix} \cosh(t^{1/2}) & t^{1/2}\cosh(t^{1/2}) \\ t^{-1/2}\tanh(t^{1/2}) & -\tanh(t^{1/2}) \end{pmatrix}.$$

Consider $\psi_{\mathbb{T}}$ like in the Corollary 2.2, i.e. $\psi_{\mathbb{T}} = (\upsilon_{\mathbb{T}} - \vartheta_{\mathbb{T}})^{-1}(\upsilon_{\mathbb{T}} + \vartheta_{\mathbb{T}})$. From Theorem 3.1, it is enough to study $\det \psi_{\mathbb{T}}$. Calculating this determinant, we obtain:

$$\begin{aligned} \det \psi_{\mathbb{T}}(t) = &- [2\sinh(t^{1/2}) + \cosh(t^{1/2})e^{t^{1/2}} - \cosh(t^{1/2})e^{t^{1/2}} \\ &- e^{t^{1/2}}\sinh(t^{1/2}) - e^{t^{1/2}}\sinh(t^{1/2})]/\Theta(t) \end{aligned}$$

where

$$\begin{aligned} \Theta(t) = &e^{t^{1/2}}\sinh(t^{1/2}) + e^{t^{1/2}}\sinh(t^{1/2}) + \cosh(t^{1/2})e^{t^{1/2}} - \cosh(t^{1/2})e^{-t^{1/2}} \\ &- 2\sinh(t^{1/2})t^{1/2}\sinh(t^{1/2}) + e^{-t^{1/2}}\sinh(t^{1/2}) + \cosh(t^{1/2})e^{t^{1/2}} \\ &- \cosh(t^{1/2})e^{-t^{1/2}} - 2\sinh(t^{1/2})\,. \end{aligned}$$

The range of $\det \psi_{\mathbb{T}}(t)$ is plotted in Figure 3.1.

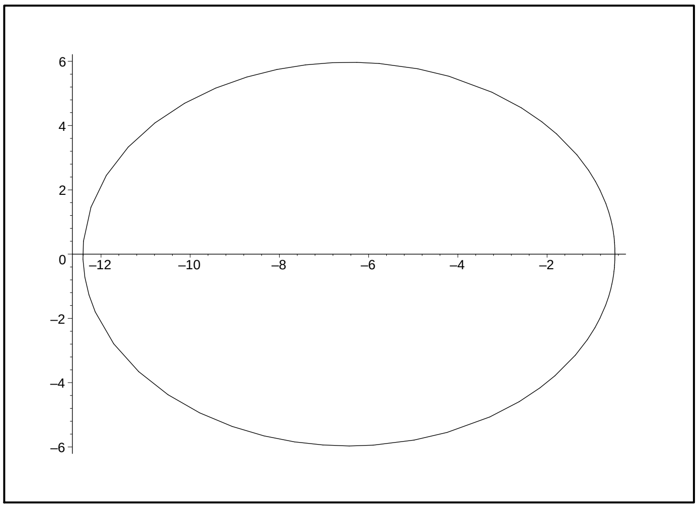


Figure 3.1: Range of $\det \psi_{\mathbb{T}}$

In this way, it turns out that $\mathcal{O}$ is a Fredholm operator in $L^p(\mathbb{T}, w)$, and with $\operatorname{Ind} \mathcal{O} = 0$.

Example 3.2. In this new example we will take a_0, a_1, b_0 and b_1 belonging to the class of piecewise continuous functions, and will analyze four cases of specific integrability parameters p and weight functions w.

Let $\mathcal{O} = a_0 I_{\mathbb{T}} + b_0 S_{\mathbb{T}} + a_1 J + b_1 S_{\mathbb{T}} J$, where

$$a_0(t) := \begin{cases} t, & t \in \mathbb{T}_+ \\ \sinh(t), & t \in \mathbb{T}_- \end{cases}, \quad b_0(t) := \begin{cases} t^2 + t^{-1}, & t \in \mathbb{T}_+ \\ it, & t \in \mathbb{T}_- \end{cases}$$

$$a_1(t) := \begin{cases} 3it, & t \in \mathbb{T}_+ \\ 2t^2, & t \in \mathbb{T}_- \end{cases}, \quad b_1(t) := \begin{cases} \sinh(t), & t \in \mathbb{T}_+ \\ t^{-4} + 1, & t \in \mathbb{T}_-. \end{cases}$$

Again, we know from Theorem 2.1 that the operator $\mathcal{O}$ is equivalent to the matrix operator $\mathfrak{D}_{\mathbb{T}} = \upsilon_{\mathbb{T}} I_{\mathbb{T}} + \vartheta_{\mathbb{T}} S_{\mathbb{T}}$ acting on $[L^p(\mathbb{T}, w)]^2$. In this case the matrix-valued functions $\upsilon_{\mathbb{T}}$ and $\vartheta_{\mathbb{T}}$ have the form

$$\upsilon_{\mathbb{T}}(t) = \begin{cases} \begin{pmatrix} 0 & 0 \\ 1 + 3i & t - 3it^{1/2} \end{pmatrix}, & t \in \mathbb{T}_+ \\ \\ \begin{pmatrix} 0 & 0 \\ \frac{2\sinh(t^{1/2}) + 4t}{2t^{1/2}} & \sinh(t^{1/2}) - t^{-2} \end{pmatrix}, & t \in \mathbb{T}_-, \end{cases}$$

$$\vartheta_{\mathbb{T}}(t) = \begin{cases} \begin{pmatrix} \cosh(t^{1/2}) & \cosh(t^{1/2}) - t^{1/2} \\ t^{1/2} - t^{-1} & t - t^{1/2} \end{pmatrix}, & t \in \mathbb{T}_+ \\ \\ \begin{pmatrix} t^{-2} + 1 & -t^{1/2}(tt^{-2} + 1) \\ i & it^{1/2} \end{pmatrix}, & t \in \mathbb{T}_-. \end{cases}$$

Now, we will calculate $\psi_{\mathbb{T}} = (\upsilon_{\mathbb{T}} - \vartheta_{\mathbb{T}})^{-1}(\upsilon_{\mathbb{T}} + \vartheta_{\mathbb{T}})$:

$$\psi_{\mathbb{T}}(t) = \begin{cases} \begin{pmatrix} \psi_{11}(t) & \psi_{12}(t) \\ \psi_{21}(t) & \psi_{22}(t) \end{pmatrix}, & t \in \mathbb{T}_+ \\ \\ \nabla(t), & t \in \mathbb{T}_- \end{cases}$$

where

$$\nabla(t) = \begin{pmatrix} \frac{1+2it^{5/2}+2t^3}{2t^2\sinh(t^{1/2})-1-2t^{5/2}+2t^3} & -\frac{2t^{1/2}(t^2\sinh(t^{1/2})-1)}{2t^2\sinh(t^{1/2})+1+2it^{5/2}-2t^3} \\ -\frac{2t^{3/2}(\sinh(t^{1/2})+2t)}{-2t^2\sinh(t^{1/2})+1+2it^{5/2}-2t^3} & -\frac{2t^3+2it^{5/2}-1}{-2t^2\sinh(t^{1/2})+1+2it^{5/2}-2t^3} \end{pmatrix},$$

$$\begin{aligned}
\psi_{11}(t) &= \frac{-(3it-1)t\cosh(t^{1/2}) + h(t)\,t^{3/2}(1+3i+t^{1/2}-t^{-1})}{\Delta(t)} \\
\psi_{12}(t) &= -\frac{-(3it-1)t\,h(t) + h(t)\,t^{3/2}(2t-3it^{1/2}-t^{-1})}{\Delta(t)} \\
\psi_{21}(t) &= -\frac{(t+3it-t^{3/2}+1)t^{1/2}\cosh(t^{1/2})}{\Delta(t)} \\
&\quad - \frac{\cosh(t^{1/2})t^{3/2}(1+3i+t^{1/2}-t^{-1})}{\Delta(t)} \\
\psi_{22}(t) &= -\frac{(t+3it-t^{3/2}+1)t^{1/2}\,h(t) + \cosh(t^{1/2})t^{3/2}(2t-3it^{1/2}-t^{1/2})}{\Delta(t)}
\end{aligned}$$

and

$$\begin{aligned}
\Delta(t) :=& 3it^2\cosh(t^{1/2}) - t\cosh(t^{1/2}) + \cosh(t^{1/2})t^{3/2} + 3it^{3/2}\cosh(t^{1/2}) \\
&- \cosh(t^{1/2})t^2 + t^{1/2}\cosh(t^{1/2}) - t^2 - 3it^2 + 5t^{5/2} - t\,, \\
h(t) :=& \cosh(t^{1/2}) - t^{1/2}\,.
\end{aligned}$$

Let us now consider the discontinuity points $\det\psi_{\mathbb{T}}^{\pm}(z)$. For this we parameterize $z \in \mathbb{T}$ in the form $z = e^{i\theta}$, $\theta \in [0, 2\pi]$. The discontinuity points are obtained by calculating the respective limits $\theta \to 0^+$, $\theta \to \pi^{\pm}$ and $\theta \to 2\pi^-$. Therefore,

$$\begin{aligned}
\det\psi_{\mathbb{T}}^{+}(1) &= -1 \\
\det\psi_{\mathbb{T}}^{-}(-1) &= -0.1966030768 + 1.388021197i \\
\det\psi_{\mathbb{T}}^{+}(-1) &= -0.2814102745 - 0.2418689383i \\
\det\psi_{\mathbb{T}}^{-}(1) &= -\frac{5 + 4\sinh(1)^2 + 4\sinh(1)}{(-2\sinh(1) - 1 + 2i)^2}.
\end{aligned}$$

The graph of $\det \psi_{\mathbb{T}}$ is described in Figure 3.2.

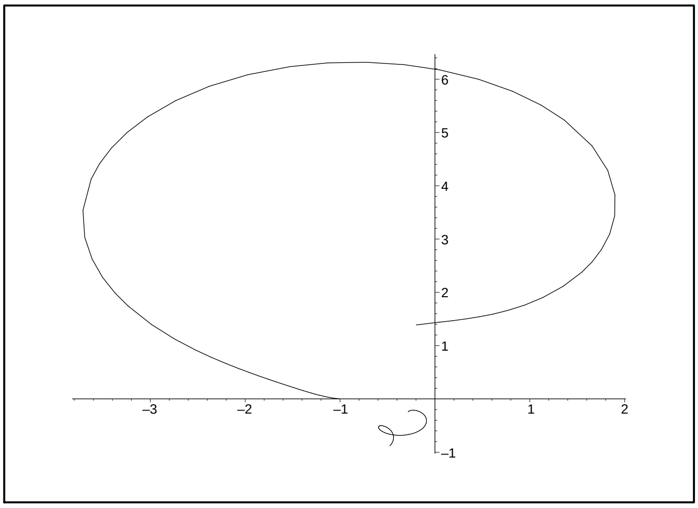


Figure 3.2: Range of $\det \psi_{\mathbb{T}}$, for $\psi_{\mathbb{T}} \in [PC(\mathbb{T})]^{2\times 2}$.

Now, we will construct $\det \psi_0^{p,w}$ for the following different cases.

Case 1: $p = 2$, $w = 1$. In this case, the respective horn is only a segment that joint the discontinuity points of $\det \psi_{\mathbb{T}}$ with the essential range of $\det \psi_{\mathbb{T}}$. Here, the essential range of $\det \psi_{\mathbb{T}}$ is denoted by $\mathcal{R}(\det \psi_{\mathbb{T}})$. Defining the lines

$$\begin{aligned} L_1 :=& \{(1-\mu)\det \psi_{\mathbb{T}}^{+}(1) + \mu \det \psi_{\mathbb{T}}^{-}(1) \ : \ 0 \le \mu \le 1\}, \\ L_2 :=& \{(1-\mu)\det \psi_{\mathbb{T}}^{-}(-1) + \mu \det \psi_{\mathbb{T}}^{+}(-1) \ : \ 0 \le \mu \le 1\}, \end{aligned}$$

we have $\mathrm{Range}(\det \psi_0^{2,1}) = \mathcal{R}(\det \psi_{\mathbb{T}}) \cup L_1 \cup L_2$, where $\det \psi_0^{2,1}$ is defined in $\bigcup_{z\in\mathbb{T}}(\{z\} \times [0,1])$, and its graph is provided in Figure 3.3.

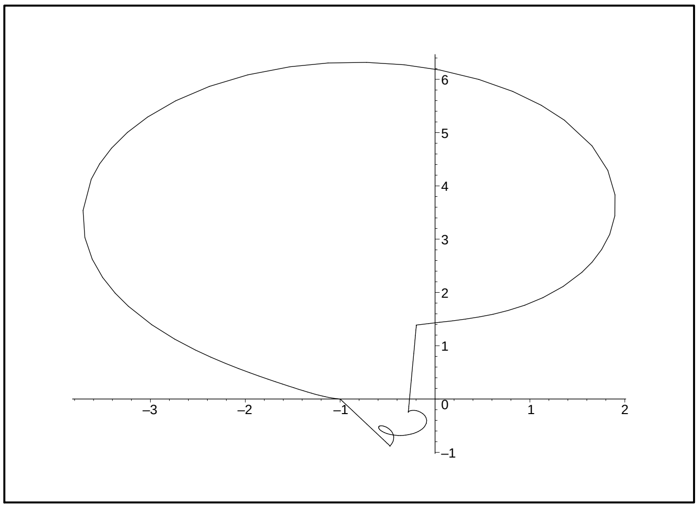


Figure 3.3: Range of $\det \psi_0^{p,w}$, for $p = 2$ and $w = 1$.

In the present case, the operator $\mathcal{O}$ is a Fredholm operator with Fredholm index equal to zero.

Case 2: $p \neq 2$, $w = 1$. In this case the line segments mentioned above go over into circular arcs. It is given by $\mathbb{A}(0, 1; \frac{1}{p})$ (see also [9] or [11], for instance). Thus,

$$\mathcal{M}_{p,1} = \bigcup_{z \in \mathbb{T}} \left(\{z\} \times \mathbb{A}\Big(0, 1; \frac{1}{p}\Big) \right);$$

cf. Figure 4.2, for $p = 1.51$.

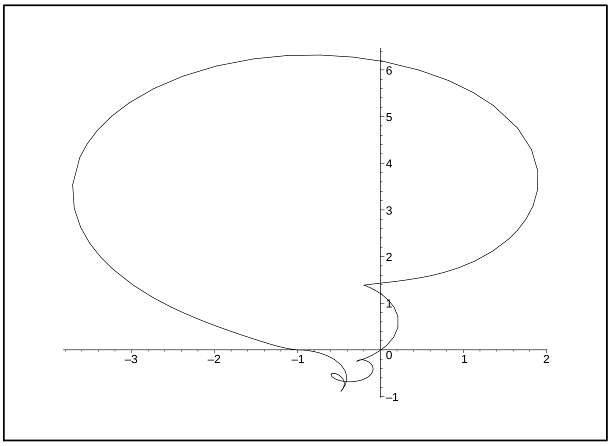


Figure 3.4: Range of $\det \psi_0^{p,w}$, for $p = 1.51$ and $w = 1$.

For $p = 1.51$, the operator $\mathcal{O}$ is not Fredholm (Range($\det \psi_0^{p,w}$) cross the origin).

Case 3: $p \neq 2$, with w being an even power weight. We will illustrate this case for the even power weight $w(t) = |\tau - |t||^{\mu}$, and $p = 3$. For this kind of weights $\nu_t^-(p,w) = \nu_t^+(p,w) = \frac{1}{p} + \mu$, where μ must satisfy the condition $-\frac{1}{p} < \mu < \frac{1}{q}$ (see [9]). Therefore, the corresponding arc is $\mathbb{A}(0, 1, 0.3333 + \mu)$ with $-0.3333 < \mu < 1.5$ (cf. Figure 3.5).

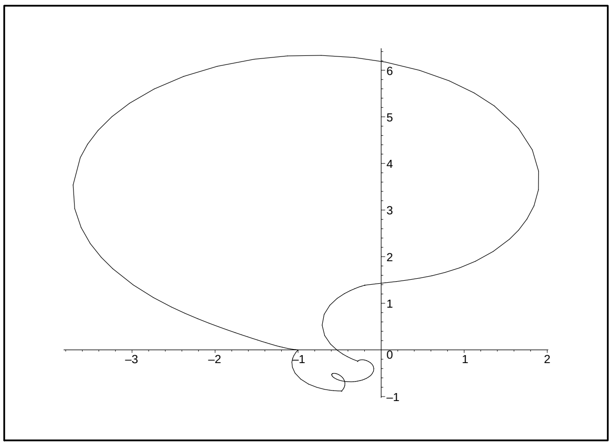


Figure 3.5: Range of $\det \psi_0^{p,w}$, for $p = 3$ and an even weight w.

In such a case, the operator $\mathcal{O}$ is a Fredholm operator with $\operatorname{Ind} \mathcal{O} = 0$.

Case 4: Let $p \neq 2$, and choose w to be the weight defined by

$$w(\tau) = e^{f\left(\log\left(-\log\frac{r}{2}\right)\right)\log\frac{r}{2}},$$

for $0 < r := ||\tau| - t| < 2$ and $f(x) = \mu + \varepsilon \sin(x)$, with

$$-\frac{1}{p} < \mu - |\varepsilon|\sqrt{2} \leq \mu + |\varepsilon|\sqrt{2} < 1 - \frac{1}{p}\,.$$

This last condition ensures that $w \in A_p(\mathbb{T})$. This weight was already used e.g. in [11, Example 16.11].

We will study the two representative sub-cases of $p = 1.3$ and $p = 4$. For $p = 1.3$, by using [11, Proposition 16.12] we derive that $\nu_t^-(p,w) = 0.6442$ and $\nu_t^+(p,w) = 0.8942$ in the case of $\mu = 0$ and $|\varepsilon| = \frac{1}{8\sqrt{2}}$. Therefore, we obtain the following horn for this case: $\mathcal{H}(0, 1;\ 0.6442,\ 0.8942)$. The horn is included in the following Figure 3.6.

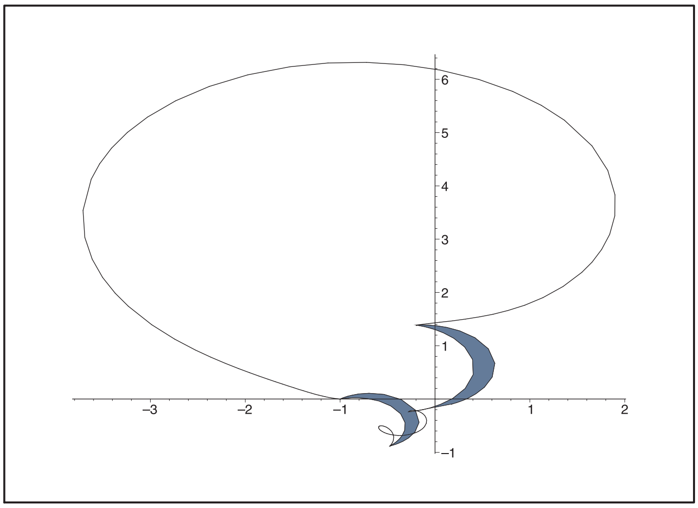


Figure 3.6: Range of $\det \psi_{\mathbb{T}}$, with the horn $\mathcal{H}(0, 1;\ 0.6442,\ 0.8942)$.

In this case it follows that $\mathcal{O}$ is a Fredholm operator with Fredholm index equal to 1.

Now, for $p = 4$, using the above procedure (and the same μ and ε), we obtain $\nu_t^-(p, w) = 0.1250$ and $\nu_t^+(p, w) = 0.3750$. Thus, for this case the horn takes the form $\mathcal{H}(0, 1;\ 0.1250,\ 0.3750)$; see Figure 3.7.

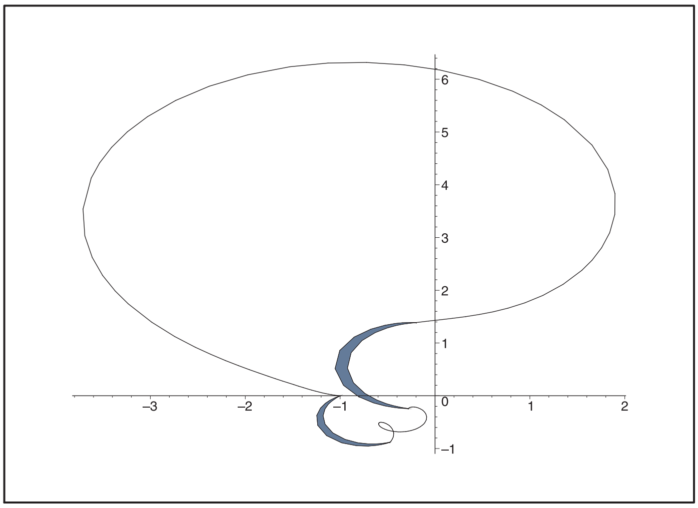


Figure 3.7: Range of $\det \psi_{\mathbb{T}}$, with the horn $\mathcal{H}(0, 1;\ 0.1250,\ 0.3750)$.

In this case, the singular integral operator with reflection $\mathcal{O}$ is a Fredholm operator with zero Fredholm index.

Now, we will illustrate the Fredholm property of the operator $\mathcal{O}$ with the flip operator (1.19), when a_0, a_1, b_0 and b_1 are certain concrete elements from the above subalgebras of $L^\infty(\mathbb{T})$.

Example 3.3. Let $\mathcal{O} = a_0 I_\mathbb{T} + b_0 S_\mathbb{T} + a_1 \widetilde{J} + b_1 S_\mathbb{T} \widetilde{J} : L^p(\mathbb{T},\rho) \to L^p(\mathbb{T},\rho)$, with $\rho(t) = |t-1|^{1-2/p}$ and having coefficients

$$\begin{aligned} a_0(t) &:= -\frac{1}{2}\sin^2\left(\frac{\pi}{2}t^2\right), & a_1(t) &:= \frac{1}{2}\sin\left(\frac{\pi}{2}\mathrm{sign}_\mathbb{T}(t)t^4\right), \\ b_0(t) &:= \frac{1}{2}t^4, & b_1(t) &:= \frac{1}{2}\sin\left(\frac{\pi}{2}\mathrm{sign}_\mathbb{T}(t)t^4\right), \end{aligned}$$

where $\mathrm{sign}_\mathbb{T}$ is defined by the rule

$$\mathrm{sign}_\mathbb{T}(t) := \begin{cases} 1, & \text{if } t \in \mathbb{T}_+ \\ -1, & \text{if } t \in \mathbb{T}_-. \end{cases}$$

From Theorem 2.4 we know that operator $\mathcal{O}$ is equivalent to the matrix operator $\mathcal{D}_\mathbb{T} = u_\mathbb{T} I_\mathbb{T} + v_\mathbb{T} S_\mathbb{T} : [L^p(\mathbb{T},w)]^2 \to [L^p(\mathbb{T},w)]^2$, where

$$w(t) = \left| i\frac{1+t}{1-t} \right|^{-1/2p} |1-t|^{1-2/p}$$

and $u_\mathbb{T}$, $v_\mathbb{T}$ are obtained as in (2.41) and defined on $\mathbb{T}$ as indicated in (2.42), (2.43) and (2.44), with

$$u_{\mathbb{T}_+}(t) = \frac{1}{2}\begin{pmatrix} \mu_{11}(t) & \mu_{12}(t) \\ \mu_{21}(t) & \mu_{22}(t) \end{pmatrix}.$$

In the present case, we have

$$\begin{aligned} \mu_{11}(t) =& \frac{1}{2}\Big[\mathrm{sign}_\mathbb{T}(t^{/2})t^2 + \sin\left(\frac{\pi}{2}\mathrm{sign}_\mathbb{T}(t^{1/2})t^2\right) - \mathrm{sign}_\mathbb{T}(-t^{1/2})t^2 \\ & - \sin\left(\frac{\pi}{2}\mathrm{sign}_\mathbb{T}(-t^{1/2})t^2\right)\Big], \\ \mu_{12}(t) =& 0, \\ \mu_{21}(t) =& \frac{1}{2}\Big[\mathrm{sign}_\mathbb{T}(t^{/2})t^2 + \sin\left(\frac{\pi}{2}\mathrm{sign}_\mathbb{T}(t^{1/2})t^2\right) + \mathrm{sign}_\mathbb{T}(-t^{1/2})t^2 \\ & + \sin\left(\frac{\pi}{2}\mathrm{sign}_\mathbb{T}(-t^{1/2})t^2\right)\Big], \\ \mu_{22}(t) =& \sin^2\left(\frac{\pi}{2}t\right) + t^2. \end{aligned}$$

Note that this yields in particular

$$\begin{aligned}
\mu_{11}(\pm 1) =& \frac{1}{2}[1+\sin\left(\frac{\pi}{2}\right)+1-\sin\left(-\frac{\pi}{2}\right)] = 2,\\
\mu_{12}(\pm 1) =& 0,\\
\mu_{21}(\pm 1) =& \frac{1}{2}[1+\sin\left(\frac{\pi}{2}\right)-1+\sin\left(-\frac{\pi}{2}\right)] = 0,\\
\mu_{22}(\pm 1) =& 2.
\end{aligned}$$

Also for the present case, we have

$$v_{\mathbb{T}_+}(t) = \frac{1}{2}\begin{pmatrix} \vartheta_{11}(t) & \vartheta_{12}(t) \\ \vartheta_{21}(t) & \vartheta_{22}(t) \end{pmatrix}$$

with

$$\begin{aligned}
\vartheta_{11}(t) =& \sin^2\left(\frac{\pi}{2}t\right) - t^2,\\
\vartheta_{12}(t) =& \frac{1}{2}[\mathrm{sign}_{\mathbb{T}}(t^{/2})t^2 - \sin\left(\frac{\pi}{2}\mathrm{sign}_{\mathbb{T}}(t^{1/2})t^2\right) + \mathrm{sign}_{\mathbb{T}}(-t^{1/2})t^2\\
& - \sin\left(\frac{\pi}{2}\mathrm{sign}_{\mathbb{T}}(-t^{1/2})t^2\right)],\\
\vartheta_{21}(t) =& 0,\\
\vartheta_{22}(t) =& \frac{1}{2}[\mathrm{sign}_{\mathbb{T}}(t^{/2})t^2 - \sin\left(\frac{\pi}{2}\mathrm{sign}_{\mathbb{T}}(t^{1/2})t^2\right) - \mathrm{sign}_{\mathbb{T}}(-t^{1/2})t^2\\
& + \sin\left(\frac{\pi}{2}\mathrm{sign}_{\mathbb{T}}(-t^{1/2})t^2\right)],
\end{aligned}$$

satisfying

$$\begin{aligned}
\vartheta_{11}(\pm 1) =& 0,\\
\vartheta_{12}(\pm 1) =& \frac{1}{2}[1-1-(\sin\left(\frac{\pi}{2}\right)+\sin\left(-\frac{\pi}{2}\right))] = 0,\\
\vartheta_{21}(\pm 1) =& 0,\\
\vartheta_{22}(\pm 1) =& \frac{1}{2}\left[1+1-\sin\left(\frac{\pi}{2}\right)+\sin\left(-\frac{\pi}{2}\right)\right] = 0.
\end{aligned}$$

Therefore, the matrix functions $u_{\mathbb{T}}$ and $v_{\mathbb{T}}$ are continuous on the whole $\mathbb{T}$. Considering now $\phi_{\mathbb{T}} = (u_{\mathbb{T}} - v_{\mathbb{T}})^{-1}(u_{\mathbb{T}} + v_{\mathbb{T}})$, it follows (in this case) that

$$\phi_{\mathbb{T}}(t) = \begin{pmatrix} -\frac{2\left(\frac{1}{2}\sin\left(\frac{\pi}{2}t^2\right)+\frac{1}{2}+\frac{1}{2}\sin^2\left(\frac{\pi}{2}t\right)-\frac{1}{2}t^2\right)}{-\sin\left(\frac{\pi}{2}t^2\right)-1+\sin^2\left(\frac{\pi}{2}t\right)-t^2} & 0 \\ 0 & \frac{2\left(\frac{1}{2}\sin^2\left(\frac{\pi}{2}t\right)-\frac{1}{2}\sin\left(\frac{\pi}{2}t^2\right)+t^2\right)}{\sin^2\left(\frac{\pi}{2}t\right)+\sin\left(\frac{\pi}{2}t^2\right)} \end{pmatrix},$$

for $t \in \mathbb{T}_+$, and $\phi_{\mathbb{T}}(t) = I_{2\times 2}$ for $t \in \mathbb{T}_-$. First of all, to consider the eventual Fredholm property of the present operator $\mathcal{O}$ it is enough to study $\det(\phi_{\mathbb{T}})$. Computing such determinant, we have

$$\det(\phi_{\mathbb{T}}(t)) = -\frac{\left(-\sin\left(\frac{\pi}{2}t^2\right) - 1 - \sin^2\left(\frac{\pi}{2}t\right) + t^2\right)\left(\sin^2\left(\frac{\pi}{2}t\right) + 2t^2 - \sin\left(\frac{\pi}{2}t^2\right)\right)}{\left(\sin\left(\frac{\pi}{2}t^2\right) + 1 - \sin^2\left(\frac{\pi}{2}t\right) + t^2\right)\left(\sin^2\left(\frac{\pi}{2}t\right) + \sin\left(\frac{\pi}{2}t^2\right)\right)}$$

for $t \in \mathbb{T}_+$, and $\det(\phi_{\mathbb{T}}(t)) = 1$ in the case of $t \in \mathbb{T}_-$. The range of $\det(\phi_{\mathbb{T}}(t))$ is plotted in Figure 3.8.

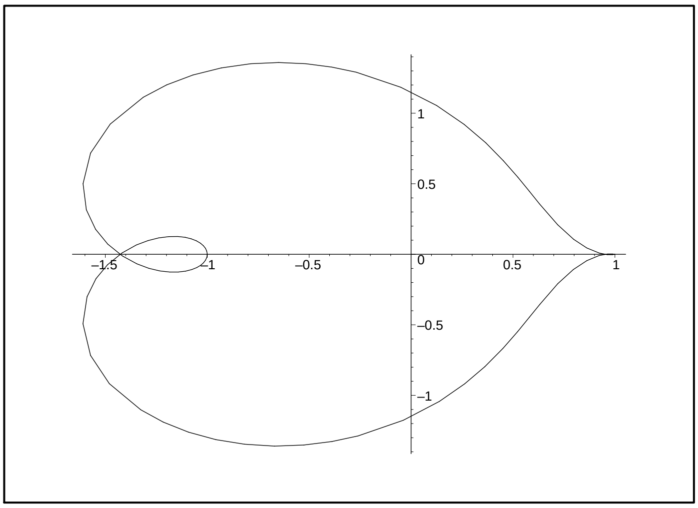


Figure 3.8: The range of $\det(\phi_{\mathbb{T}})$ in the present example.

Since $u_{\mathbb{T}}$ and $v_{\mathbb{T}}$ are continuous matrix-valued functions, the singular integral operator with backward Carleman shift $\mathcal{O} = a_0 I_{\mathbb{T}} + b_0 S_{\mathbb{T}} + a_1 \widetilde{J} + b_1 S_{\mathbb{T}} \widetilde{J}$ acting on the space $L^p(\mathbb{T}, \rho)$, where $1 < p < \infty$ and $\rho(t) = |t-1|^{1-2/p}$, is a Fredholm operator if and only if $\det(u_{\mathbb{T}}(t) \pm v_{\mathbb{T}}(t)) \neq 0$, $t \in \mathbb{T}$. Moreover, under the Fredholm property, the Fredholm index of $\mathcal{O}$ is given by $\operatorname{Ind} \mathcal{O} = -\operatorname{wind} \det(\phi_{\mathbb{T}})$ where $\phi_{\mathbb{T}} := (u_{\mathbb{T}} - v_{\mathbb{T}})^{-1}(u_{\mathbb{T}} + v_{\mathbb{T}})$.

In this way, it turns out that $\mathcal{O}$ is a Fredholm operator in $L^p(\mathbb{T}, w)$ with $\operatorname{Ind} \mathcal{O} = -1$.

Example 3.4. In this last example we will take a_0, a_1, b_0 and b_1 belonging to the class of piecewise continuous functions, leading therefore to 2×2 piecewise matrix-valued functions.

Let $\mathcal{O} = a_0 I_{\mathbb{T}} + b_0 S_{\mathbb{T}} + a_1 \widetilde{J} + b_1 S_{\mathbb{T}} \widetilde{J}$, where

$$a_0(t) := \begin{cases} 3t + i, & \arg t \in (0, \frac{\pi}{2}) \\ 2e^t, & \arg t \in (\frac{\pi}{2}, 2\pi) \end{cases}$$

$$b_0(t) := \begin{cases} 3\cosh(t), & \arg t \in (0, \frac{\pi}{2}) \\ 2t^2, & \arg t \in (\frac{\pi}{2}, 2\pi) \end{cases}$$

$$a_1(t) := \begin{cases} it, & \arg t \in (0, \frac{\pi}{2}) \\ \coth(t), & \arg t \in (\frac{\pi}{2}, 2\pi) \end{cases}$$

$$b_1(t) := \begin{cases} \tanh(t), & \arg t \in (0, \frac{\pi}{2}) \\ it - \ln(t^2 - i), & \arg t \in (\frac{\pi}{2}, 2\pi). \end{cases}$$

In the same way as above, we start by considering the fact that the operator $\mathcal{O}$ is equivalent to the matrix operator $\mathcal{D}_{\mathbb{T}} = u_{\mathbb{T}} I_{\mathbb{T}} + v_{\mathbb{T}} S_{\mathbb{T}} : [L^p(\mathbb{T}, w)]^2 \to [L^p(\mathbb{T}, w)]^2$. In the present case, $u_{\mathbb{T}}$ and $v_{\mathbb{T}}$ are the following matrix valued piecewise continuous functions:

$$u_{\mathbb{T}}(t) = \begin{cases} \Delta_+(t), & \arg t \in (0, \pi/2) \\ \Delta_-(t), & \arg t \in (\pi/2, \pi) \\ I_{2\times 2}, & \arg t \in (\pi, 2\pi). \end{cases}$$

where

$$\Delta_+(t) = \begin{pmatrix} \frac{t^{1/2}+\tanh(t^{1/2})+it^{1/2}-\tanh(-t^{1/2})}{2} & \frac{3t^{1/2}+3t^{1/2}-3\cosh(-t^{1/2})}{2} \\ \frac{t^{1/2}+\tanh(t^{1/2})-it^{1/2}+\tanh(-t^{1/2})}{2} & \frac{3t^{1/2}-3t^{1/2}+3\cosh(-t^{1/2})+2i}{2} \end{pmatrix}$$

$$\Delta_-(t) = \begin{pmatrix} \frac{\coth(t^{1/2})+it^{1/2}-\ln(t-i)-\coth(-t^{1/2})-it^{1/2}+\ln(-t-i)}{2} & \frac{2e^{t^{1/2}}-4t-2e^{-t^{1/2}}}{2} \\ \frac{\coth(t^{1/2})+it^{1/2}-\ln(t-i)+\coth(-t^{1/2})+it^{1/2}-\ln(-t-i)}{2} & \frac{2e^{t^{1/2}}+2e^{-t^{1/2}}}{2} \end{pmatrix},$$

and

$$v_{\mathbb{T}}(t) = \begin{cases} \nabla_+(t), & \arg t \in (0, \pi/2) \\ \nabla_-(t), & \arg t \in (\pi/2, \pi) \\ 0_{2\times 2}, & \arg t \in (\pi, 2\pi). \end{cases}$$

where $\nabla_+(t)$ and $\nabla_-(t)$ are the matrix functions

$$\nabla_+(t) = \begin{pmatrix} \frac{3t^{1/2}+2i-3\cosh(t^{1/2})-3t^{1/2}-3\cosh(-t^{1/2})}{2} & \frac{it^{1/2}-\tanh(t^{1/2})-it^{1/2}-\tanh(-t^{1/2})}{2} \\ \frac{3t^{1/2}-3\cosh(t^{1/2})+3t^{1/2}+3\cosh(-t^{1/2})}{2} & \frac{it^{1/2}-\tanh(t^{1/2})+it^{1/2}+\tanh(-t^{1/2})}{2} \end{pmatrix}$$

$$\nabla_-(t) = \begin{pmatrix} e^{t^{1/2}} + e^{-t^{1/2}} & \frac{\coth(t^{1/2})-t^{1/2}+\ln(t-i)+\coth(-t^{1/2})+it^{1/2}+\ln(-t-i)}{2} \\ e^{t^{1/2}} - 2t - e^{-t^{1/2}} & \frac{\coth(t^{1/2})-t^{1/2}+\ln(t-i)-\coth(-t^{1/2})-it^{1/2}-\ln(-t-i)}{2} \end{pmatrix}.$$

Now, we construct $\phi_{\mathbb{T}} = (u_{\mathbb{T}} - v_{\mathbb{T}})^{-1}(u_{\mathbb{T}} + v_{\mathbb{T}})$ and will consider its determinant. Then, we will look for the discontinuity points of $\det \phi_{\mathbb{T}}$. In view of this purpose, we will parameterize $z \in \mathbb{T}$ in the form $z = e^{i\theta}$, $\theta \in [0, 2\pi]$. The images of the discontinuity points are obtained by calculating the respective limits $\theta \to \frac{\pi}{2}^{\pm}$ and $\theta \to \pi^{-}$. In detail,

$$\begin{aligned} \det \phi_{\mathbb{T}}^{-}(i) &\approx -1.86 - 2.03i \\ \det \phi_{\mathbb{T}}^{+}(i) &\approx 0.1232457739 - 0.2288259030i \\ \det \phi_{\mathbb{T}}^{-}(-1) &\approx 1.824489828 + 3.387463265i. \end{aligned}$$

The range of $\det(\phi_{\mathbb{T}})$ is described in Figure 3.9.

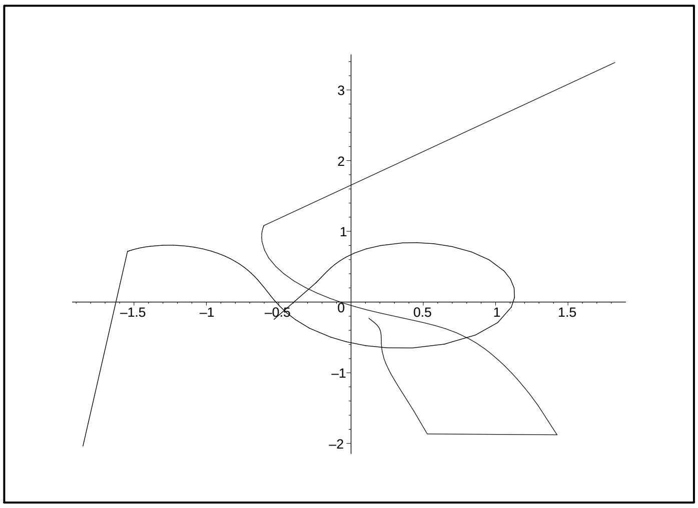
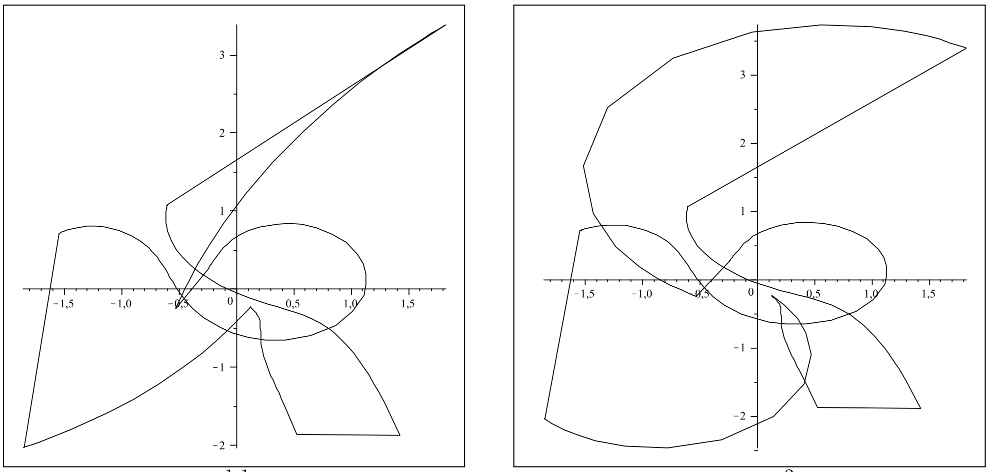


Figure 3.9: The range of $\det(\phi_{\mathbb{T}})$, for the particular $\phi_{\mathbb{T}} \in [PC(\mathbb{T})]^{2\times 2}$.

Now, we will show some graphs of $\det \phi_{\mathbb{T}}^{p,w}$ for different values of p.



$p = 1.1$ $\qquad$ $p = 2$

Figure 3.10: The range of $\det \phi_{\mathbb{T}}^{p,w}$, for the particular values $p = 1.1$ and $p = 2$.

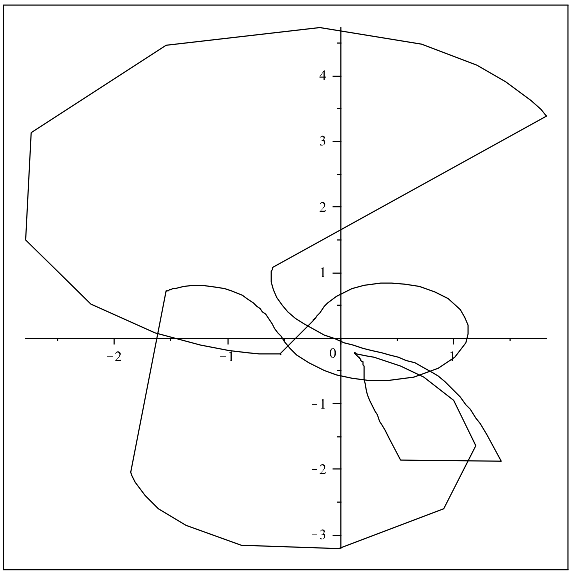


Figure 3.11: The range of $\det \phi_{\mathbb{T}}^{p,w}$, for the particular value $p = 3$.

In all these cases, the operator $\mathcal{O}$ is a Fredholm operator with Fredholm index equal to -1. Notice, from the behavior of the arcs joining the discontinuities of $\det \phi_{\mathbb{T}}$, that the same conclusions hold for every $p \in (1, \infty)$.

3.1.2 A Fredholm criterion for $\mathcal{O}$ with $C(\mathbb{T})$ and $PC(\mathbb{T})$ coefficients via Toeplitz plus Hankel operators

Now, based on Theorem 2.6, we will obtain a Fredholm criterion for the operator $\mathcal{O}$ with coefficients belonging to different classes of functions on $L^\infty(\mathbb{T})$ and the action of the flip operator (1.19). In addition, for some cases, a Fredholm index formula will be also derived.

Due to Theorem 1.1, the Fredholm characteristic of a linear operator is equivalent to the invertibility of its image in the Calkin algebra. R.G. Douglas exploited the studies made in the sixties of past century by P. Halmos and G.K. Pedersen, among other, about C^*-algebras generated by two idempotents with an extra property in the spectrum, and combined it with certain local techniques in order to lead to a symbol calculus for singular integral operators with piecewise continuous coefficients (see e.g., [32]). S.C. Power ([75]) succeeded in applying such ideas to the study of Fredholm properties of Hankel operators and Fourier integral operators with piecewise continuous generating functions. Also, I. Gohberg and N. Krupnik (see, [40, 41, 42]) established a theory for the image in the Calkin algebra of algebras of singular integral operators with piecewise continuous coefficients, which can be viewed as an essential generalization of the usual Gelfand theory for commutative Banach algebras.

In this part, for the singular integral operator $\mathcal{O}$ with the flip operator (1.19) and piecewise continuous coefficients, we are going to apply a homogenization technique which can be viewed as a natural method to associate to each operator $\mathfrak{T}$ a local representative which is just a Mellin convolution operator. To attain such a goal, we will make use of the functions

$$s(x) := \coth\left(\left(x + \frac{i}{p}\right)\pi\right), \qquad n(x) := \sinh^{-1}\left(\left(x + \frac{i}{p}\right)\pi\right), \tag{3.1}$$

for $x \in \overline{\mathbb{R}}$.

Theorem 3.2. *For* $a_0, a_1, b_0, b_1 \in C(\mathbb{T})$ *and* $1 < p < \infty$ *the singular integral operator with flip*

$$\mathcal{O} = a_0 I_{\mathbb{T}} + b_0 S_{\mathbb{T}} + a_1 \widetilde{J} + b_1 S_{\mathbb{T}} \widetilde{J}$$

(*acting on* $L^p(\mathbb{T})$) *is a Fredholm operator if and only if* $\det(a(t)) \neq 0$, $t \in \mathbb{T}$ *where* a *is defined in* (2.46). *Moreover, in this case,*

$$\operatorname{Ind} \mathcal{O} = -\operatorname{wind}\left(\det(a)\right).$$

In the case of $a_0, a_1, b_0, b_1 \in PC(\mathbb{T})$, $\mathcal{O}$ *is a Fredholm operator if and only if*

$$\det(\psi(\tau, x)) \neq 0,$$

for all $x \in \overline{\mathbb{R}}$ *and* $\tau \in \mathbb{T}_+ \cup \{-1, 1\}$, *where*

$$\psi(\tau, x) := (1 + s(x))a^+(\tau) + (1 - s(x))a^-(\tau) + \tau n(x)(b^+(\tau) - b^-(\tau))$$

for $\tau \in \{-1, 1\}$ *and*

$$\begin{aligned}\psi(\tau, x) :=&(1 + s(x)) \begin{pmatrix} a^+(\tau) & 0 \\ 0 & a^+(\overline{\tau}) \end{pmatrix} + (1 - s(x)) \begin{pmatrix} a^-(\tau) & 0 \\ 0 & a^-(\overline{\tau}) \end{pmatrix} \\ &+ n(x) \begin{pmatrix} 0 & b^+(\tau) - b^-(\tau) \\ b^+(\overline{\tau}) - b^-(\overline{\tau}) & 0 \end{pmatrix}\end{aligned}$$

for $\tau \in \mathbb{T}_+$ (*and where* a, b *are defined in* (2.46)).

Proof. From Theorem 2.6 we have that $\mathcal{O}$ and $T_a + H_b$ are equivalent operators. Hence, $\mathcal{O}$ and $T_a + H_b$ have the Fredholm property only at the same time. So, the conclusion will be obtained by using two known results for Toeplitz plus Hankel operators.

For the continuous case we make therefore use of e.g. [36, Proposition 2.7]. In detail: if $T_a + H_b$ is a Fredholm operator, then a is an invertible matrix function, and therefore $\det(a(t)) \neq 0$ for all $t \in \mathbb{T}$.

Conversely, if $\det(a) \neq 0$ then a is invertible. Thus, we can see that $T_{a^{-1}}$ is a Fredholm regularizer of $T_a + H_b$ and the Fredholm conclusion is obtained by using Theorem 1.2.

The index formula is derived from the fact that for a continuous matrix function b, the Hankel operator H_b is compact and therefore from Theorem 1.3 and also by Theorem 2.42 in [13] we get

$$\operatorname{Ind}(T_a + H_b) = \operatorname{Ind}(T_a) = -\operatorname{wind}(\det(a)).$$

For the piecewise continuous case we will make use of e.g. [36, Theorem A.3]. The proof for this case is given by means of a symbol calculus technique and the use of Mellin convolution operators in the following way. Let us start by recalling that the *Mellin Transformation*

$$\mathbf{M} : [L^p(\mathbb{R}_+)]^N \longrightarrow [L^p(\mathbb{R})]^N$$

is defined componentwise by

$$(\mathcal{M}f)(x) := \int_0^\infty \xi^{-1+1/p-ix} f(\xi) d\xi, \quad x \in \mathbb{R}.$$

For $a \in [L^\infty(\mathbb{R})]^{N\times N}$, the corresponding Mellin convolution operator $\mathbf{M^0}(a) \in \mathcal{L}([L^p(\mathbb{R}_+)]^N)$ is given by

$$\mathbf{M^0}(a) = \mathbf{M}^{-1} a \cdot \mathbf{M}.$$

Let $\widehat{S} := S_{\mathbb{R}_+}$ stand for the Cauchy singular integral operator acting on the space $[L^p(\mathbb{R}_+)]^N$,

$$(\widehat{\mathcal{S}}f)(x) = \frac{1}{\pi i}\int_0^\infty \frac{f(y)}{y-x}dy, \quad x \in \mathbb{R}_+,$$

and let $\widehat{N}$ denote the integral operator acting on $[L^p(\mathbb{R}_+)]^N$ by the rule

$$(\widehat{\mathcal{N}}f)(x) = \frac{1}{\pi i}\int_0^\infty \frac{f(y)}{y+x}dy, \quad x \in \mathbb{R}_+.$$

The operators $\widehat{S}$ and $\widehat{N}$ can be identified in the following form of Mellin convolution operators

$$\widehat{S} = \mathbf{M^0}(sI_N), \qquad \widehat{N} = \mathbf{M^0}(nI_N),$$

where the generating functions s and n are the same as in (3.1), and I_N denotes the identity matrix in $\mathbb{C}^{N\times N}$. In addition, let J stand for the following Carleman shift operator

$$(Jf)(t) = f(t^{-1}), \quad t \in \mathbb{T},$$

acting on $[L^p(\mathbb{T})]^N$. Notice that $\widetilde{J} = t^{-1}J$.

At this point, we observe that due to the corresponding actions of P_+, J and the extension by the complimentary projector P_- (of P_+), it follows that the operator $T_a + H_b$ is a Fredholm operator on $[H^p(\mathbb{T})]^2$ if and only if the operator

$$C = P_+(a + bt^{-1}J)P_+ + P_- : [L^p(\mathbb{T})]^2 \to [L^p(\mathbb{T})]^2$$

is Fredholm. In view of this, for obtaining a characterization of the Fredholm property of C (and therefore the one of $T_a + H_b$), we have just to analyze the invertibility of the operator $\Phi_\tau(C)$ for $\tau \in \mathbb{T}_+ \cup \{-1, 1\}$; cf. Theorem A.1 in [36]. Here,

$$\Phi_\tau : \mathcal{Y}_1 \to \mathcal{Y}_2$$

is a Banach algebra homomorphism with $\mathcal{Y}_1$ being the smallest closed subalgebra of $\mathcal{L}([L^p(\mathbb{T})]^2)$ which contains the multiplication operators generated by piecewise continuous 2×2 matrix-valued functions, all the compact operators acting between $[L^p(\mathbb{T})]^2$, the operator J and both projections P_+ and P_-, and $\mathcal{Y}_2 = \mathcal{L}([L^p(\mathbb{R}_+)]^4)$ if $\tau \in \{-1, 1\}$ and $\mathcal{Y}_2 = \mathcal{L}([L^p(\mathbb{R}_+)]^8)$ if $\tau \in \mathbb{T}_+$.

In a more detailed way, letting $\widehat{P}_+$ to be the following operator

$$\widehat{P}_+ := \frac{1}{2}\begin{pmatrix} I_{[L^p(\mathbb{R}_+)]^2} + \widehat{S} & -\widehat{N} \\ \widehat{N} & I_{[L^p(\mathbb{R}_+)]^2} - \widehat{S} \end{pmatrix},$$

where by $I_{[L^p(\mathbb{R}_+)]^2}$ we mean the identity operator on $[L^p(\mathbb{R}_+)]^2$. So we have that

$$\Phi_\tau(P_+) = \begin{cases} \widehat{P}_+ & \text{if} \quad \tau \in \{-1, 1\} \\ \begin{pmatrix} \widehat{P}_+ & 0 \\ 0 & \widehat{P}_+ \end{pmatrix} & \text{if} \quad \tau \in \mathbb{T}_+ . \end{cases}$$

In addition, for $\tau \in \{-1, 1\}$, we have

$$\Phi_\tau(C) = \widehat{P}_+ \begin{pmatrix} a^+(\tau) I_{[L^p(\mathbb{R}_+)]^2} & \tau b^+(\tau) I_{[L^p(\mathbb{R}_+)]^2} \\ \tau b^-(\tau) I_{[L^p(\mathbb{R}_+)]^2} & a^-(\tau) I_{[L^p(\mathbb{R}_+)]^2} \end{pmatrix} \widehat{P}_+ + (I_{[L^p(\mathbb{R}_+)]^4} - \widehat{P}_+).$$

This operator is a Mellin convolution operator $\mathbf{M^0}(c)$ with symbol

$$c(x) = p(x) \begin{pmatrix} a^+(\tau) & \tau b^+(\tau) \\ \tau b^-(\tau) & a^-(\tau) \end{pmatrix} p(x) + (I_4 - p(x)),$$

where

$$p(x) = \frac{1}{2}\begin{pmatrix} (1 + s(x)) I_2 & -n(x) I_2 \\ n(x) I_2 & (1 - s(x)) I_2 \end{pmatrix}.$$

Thus, $\Phi_\tau(C)$ is invertible if and only if $c(x)$ is an invertible matrix for each $x \in \overline{\mathbb{R}}$.

The matrix $p(x)$ is an idempotent matrix for each fixed $x \in \overline{\mathbb{R}}$ and can be written as a product with

$$p_1 = \begin{pmatrix} \alpha I_2 \\ \beta I_2 \end{pmatrix}, \quad p_2 = (\gamma I_2, \delta I_2),$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ are appropriate numbers. So, we obtain

$$p_2 \begin{pmatrix} a^+(\tau) & \tau b^+(\tau) \\ \tau b^-(\tau) & a^-(\tau) \end{pmatrix} p_1 = \varphi(\tau, x).$$

From Lemma A.2 on [36] it follows that $\det c(x) \neq 0$ if and only if $\det \varphi(\tau, x) \neq 0$, and the proof for the case $\tau \in \{-1, 1\}$ is concluded.

For the case $\tau \in \mathbb{T}_+$ the argumentation is similar. In such a case

$$\Phi_\tau(C) = \begin{pmatrix} \widehat{P}_+ & 0 \\ 0 & \widehat{P}_+ \end{pmatrix} \times$$

$$\begin{pmatrix} a^+(\tau)I_{[L^p(\mathbb{R}_+)]^2} & 0 & 0 & \bar{\tau}b^+(\tau)I_{[L^p(\mathbb{R}_+)]^2} \\ 0 & a^-(\tau)I_{[L^p(\mathbb{R}_+)]^2} & \bar{\tau}b^-(\tau)I_{[L^p(\mathbb{R}_+)]^2} & 0 \\ 0 & \tau b^+(\bar{\tau})I_{[L^p(\mathbb{R}_+)]^2} & a^+(\bar{\tau})I_{[L^p(\mathbb{R}_+)]^2} & 0 \\ \tau b^-(\bar{\tau})I_{[L^p(\mathbb{R}_+)]^2} & 0 & 0 & a^-(\bar{\tau})I_{[L^p(\mathbb{R}_+)]^2} \end{pmatrix}$$

$$\times \begin{pmatrix} \widehat{P}_+ & 0 \\ 0 & \widehat{P}_+ \end{pmatrix} + \left(I_{[L^p(\mathbb{R}_+)]^8} - \begin{pmatrix} \widehat{P}_+ & 0 \\ 0 & \widehat{P}_+ \end{pmatrix} \right).$$

This is a Mellin convolution operator with the symbol

$$q(x) \begin{pmatrix} a^+(\tau) & 0 & 0 & \bar{\tau}b^+(\tau) \\ 0 & a^-(\tau) & \bar{\tau}b^-(\tau) & 0 \\ 0 & \tau b^+(\bar{\tau}) & a^+(\bar{\tau}) & 0 \\ \tau b^-(\bar{\tau}) & 0 & 0 & a^-(\bar{\tau}) \end{pmatrix} q(x) + (I_8 - q(x)),$$

where

$$q(x) = \begin{pmatrix} p(x) & 0 \\ 0 & p(x) \end{pmatrix}.$$

Following the above procedure, the desired conclusion is obtained. □

Examples

The applicability of this result is showed in the following examples.

Example 3.5. Let us consider the singular integral operator $\mathcal{O}$ with flip $\widetilde{J}$ and continuous coefficients given by

$$a_0(t) := \tanh(t), \qquad b_0(t) := t^2 + i,$$
$$a_1(t) := \cos(t), \qquad b_1(t) := e^t + 2.$$

From Theorem 3.2, we only need to study the function $a(t)$ as in (2.46). In this case we have

$$a(t) = \begin{pmatrix} \tanh(t) + t^2 + i & \cos(t) + e^t + 2 \\ \frac{1}{t}\tanh\left(\frac{1}{t}\right) + t^{-3} + it^{-1} & \frac{1}{t}\cos\left(\frac{1}{t}\right) + \frac{1}{t}e^{\frac{1}{t}} + \frac{2}{t} \end{pmatrix}.$$

The range of the determinant of $a(t)$ is plotted in the next figure.

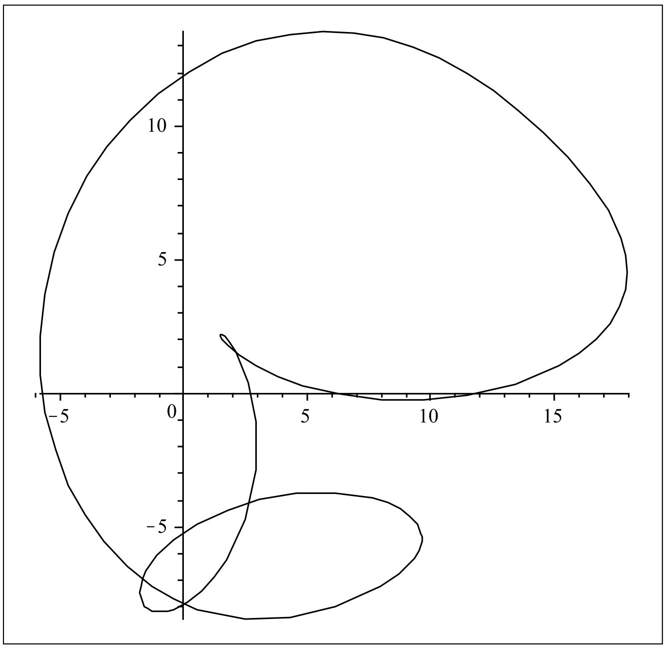


Figure 3.12: The range of $\det a(t)$.

The operator $\mathcal{O}$ in this case is a Fredholm operator with Fredholm index equal to 1.

The piecewise continuous case is analyzed in the next example.

Example 3.6. Let the operator $\mathcal{O}$ with flip and the following piecewise continuous functions as coefficients

$$a_0(t) := \begin{cases} \ln(t),\ t \in \mathbb{T}_+ \\ 2t,\ t \in \mathbb{T}_- \end{cases}, \qquad b_0(t) := \begin{cases} t^3 + i,\ t \in \mathbb{T}_+ \\ t^2 - 3it,\ t \in \mathbb{T}_-. \end{cases}$$

$$a_1(t) := \begin{cases} 3t^2,\ t \in \mathbb{T}_+ \\ t^{-3} - 2,\ t \in \mathbb{T}_- \end{cases}, \qquad b_1(t) := \begin{cases} 5it,\ t \in \mathbb{T}_+ \\ i\pi t^{-4},\ t \in \mathbb{T}_-. \end{cases}$$

The matrices $a(t)$ and $b(t)$, constructed as in (2.46), are given by

$$a(t) = \begin{cases} \begin{pmatrix} \ln(t) + t^3 + i & 3t^2 + 5it \\ \ln(t^{-1/t}) + t^{-4} + it^{-1} & 3t^{-3} + 5it^{-2} \end{pmatrix},\ t \in \mathbb{T}_+ \\ \\ \begin{pmatrix} 2t + t^2 - 3it & t^{-3} + 2 + i\pi t^{-4} \\ 2t^{-2} + t^{-3} - 3it^{-2} & t^2 + 2t^{-1} + i\pi t^3 \end{pmatrix},\ t \in \mathbb{T}_- \end{cases}$$

$$b(t) = \begin{cases} \begin{pmatrix} 3t^2 - 5it & \ln(t) - t^3 - i \\ 3t^{-3} - 5it^{-2} & \ln(t^{-1/t}) - t^{-4} - it^{-1} \end{pmatrix}, & t \in \mathbb{T}_+ \\ \\ \begin{pmatrix} t^{-3} - 2 - i\pi t^{-4} & 2t - t^2 + 3it \\ t^2 - 2t^{-1} - i\pi t^3 & 2t^{-2} - t^{-3} + 3it^{-2} \end{pmatrix}, & t \in \mathbb{T}_-. \end{cases}$$

Now, we construct the functions $\psi(\tau, x)$, for the different values of τ, as is required in Theorem 2.6. The range of these functions are plotted in next figures.

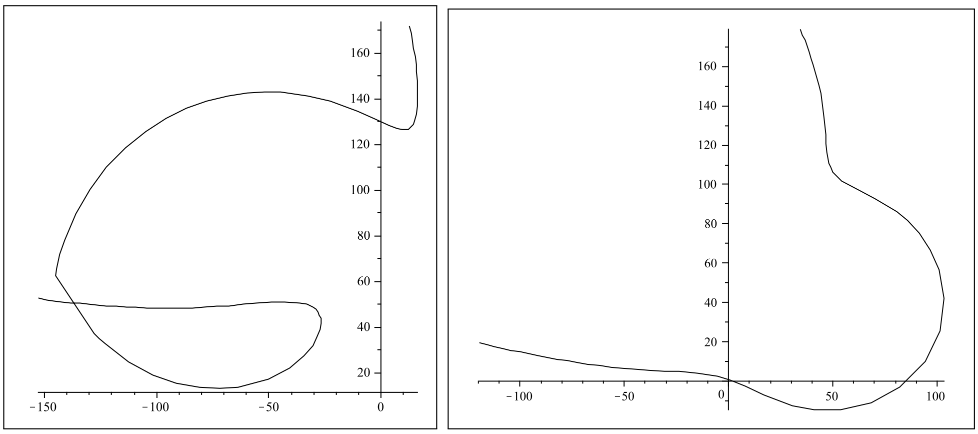


Figure 3.13: The range of $\psi(-1, x)$, when $\mathcal{O}$ is defined in $L^{1.2}(\mathbb{T})$ (left) and $L^{6.3}(\mathbb{T})$ (right).

Notice that $\mathcal{O}$ cannot be a Fredholm operator when it is defined in the space $L^{6.3}(\mathbb{T})$.

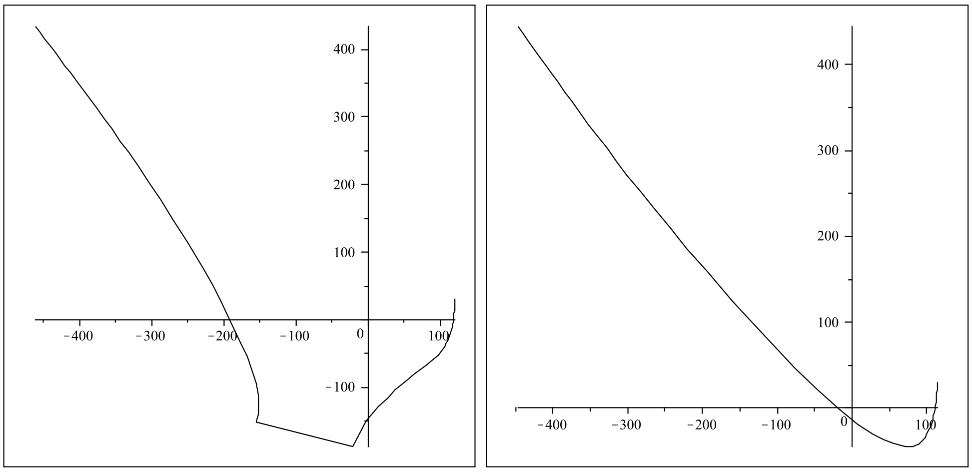


Figure 3.14: The range of $\psi(1, x)$, when $\mathcal{O}$ is defined in $L^{2.83}(\mathbb{T})$ (left) and $L^{6e10}(\mathbb{T})$ (right).

Notice that we have not any extra limitation about the Fredholmness of $\mathcal{O}$ for $p \in (1, \infty)$.

Finally, we study the range of $\psi(\tau, x)$, for $\tau \in \mathbb{T}_+$, which is described in next the figure.

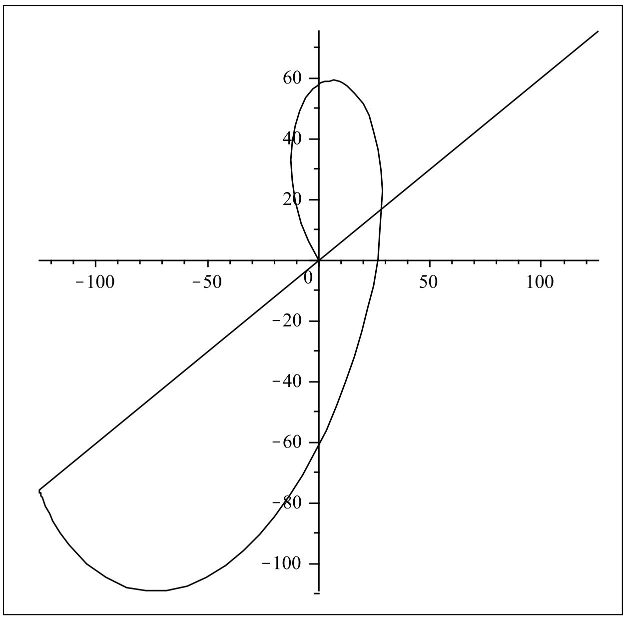


Figure 3.15: The range of $\psi(\tau, x)$, with $\tau \in \mathbb{T}_+$.

From the figure above, we conclude that the related operator $\mathcal{O}$ is not a Fredholm operator on $L^p(\mathbb{T})$, $1 < p < \infty$ (because the range of $\psi(\tau, x)$ cross the origin).

3.1.3 A Fredholm criterion for singular integral operators with flip and semi-almost periodic coefficients

In the spirit of the *Allan-Douglas Localization Principle* (see e.g. [10, 12, 13, 81]), we present in this subsection a Fredholm criterion for $\mathcal{O}$ with the flip operator (1.19) on $L^2(\mathbb{T})$ when with semi-almost periodic coefficients. Our criterion may be seen as an adaptation of Theorem 3.1 on [12] for the operator $\mathcal{O}$.

The local principle of Allan and Douglas may be regarded as a generalization of the Gelfand theory for commutative Banach algebras to the non-commutative case. Let A be a Banach algebra with identity e. A closed subalgebra Z of A is called a *central subalgebra* if $az = za$ for all $a \in A$ and all $z \in Z$. Let Z be a central subalgebra of A which contains e. Then Z is a unital commutative Banach algebra, and we denote by M the maximal ideal space of Z. Given a maximal ideal element $m \in M$, J_m denotes the smallest closed two-sided ideal of A containing the ideal m ($\subset Z \subset A$). One can show (see, e.g., [8, Proposition 8.6]) that

$$J_m = \{cz \;:\; c \in A,\; z \in m\} = \{zc \;:\; z \in m,\; c \in A\}.$$

Theorem 3.3 (Local principle of Allan and Douglas)**.** *Let $a \in A$. Then a is invertible in A if and only if $a + J_m$ is invertible in the quotient algebra A/J_m for all $m \in M$.*

For C^*-algebras, Theorem 3.3 can be supplemented by the following.

Theorem 3.4. *If A is a unitial C^*-algebra and Z is a central C^*-subalgebra of A which contains the identity of A, then the map*

$$A \longrightarrow \bigoplus_{m \in M} A/J_m, \quad a \longmapsto \{a + J_m\}_{m \in M}$$

is an injective (and thus isometric) C^*-algebra homomorphism.

By using these results (see, [40, 41, 42] and references therein) was constructed a set of matrices, which are the corresponding local representatives at the point ∞ (also called *symbols*) of the elements in the C^*-algebra generated by the singular integral operator $S_{\mathbb{R}}$ and the multiplication operator by functions on $[C(\overline{\mathbb{R}})]^{N\times N}$, $\mathfrak{B} = \operatorname{alg}(S_{\mathbb{R}}, [C(\overline{\mathbb{R}})]^{N\times N})$, in the Calkin algebra $\mathfrak{B}/\mathcal{K}(L^2(\mathbb{R}))$, as it will shown in the proof of Theorem 3.5.

We would like to point out that we are going to use the Sarason decomposition form of a semi-almost periodic function a as in (1.11), with the almost periodic representatives having a canonical generalized right AP factorization, where a *canonical (right) APW factorization* of a matrix function

$$a \in APW^{N\times N}(\mathbb{R})$$

is a representation

$$a = a_- a_+$$

with $a_\pm \in \mathcal{G}APW_\pm^{N\times N}(\mathbb{R})$. If $c \in \mathcal{G}APW^{N\times N}(\mathbb{R})$ has a canonical right APW factorization $c = c_- c_+$, the so-called *geometric mean* is uniquely defined by

$$d(c) := M(c_-)M(c_+).$$

In order to generalize the concept of APW factorization of an AP-matrix function, we need some results from harmonic analysis on locally abelian groups.

The *Besicovitch space* B^2 is defined as the completion of $AP^0(\mathbb{R})$ with respect to the norm

$$\|f\|_{B^2} := \left(\sum_\lambda |f_\lambda|^2\right)^{1/2} = (M(|f|^2))^{1/2},$$

where $f = \sum_\lambda f_\lambda e_\lambda \in AP^0(\mathbb{R})$. Let $\mathbb{R}_B$ denote the Bohr compactification of $\mathbb{R}$ and $d\mu$ the normalized Haar measure on $\mathbb{R}_B$. It is known that $AP(\mathbb{R})$ may be identify with $C(\mathbb{R}_B)$ and that one can identify B^2 with $L^2(\mathbb{R}_B)$. Thus, B^2 is a (nonseparable) Hilbert space, and the inner product in $B^2 = L^2(\mathbb{R}_B)$ is given by

$$(f,g) := \int_{\mathbb{R}_B} f(\xi)\overline{g(\xi)}d\mu(\xi).$$

Since $\mu(\mathbb{R}_B)$ is finite, $AP(\mathbb{R})$ is contained in B^2. Moreover, $AP(\mathbb{R})$ is a dense subset of B^2. The Cauchy-Schwartz inequality shows that the mean value

$$M(f) := \int_{\mathbb{R}_B} f(\xi)d\mu(\xi)$$

exists and is finite for every $f \in B^2$. Thus, the Bohr-Fourier spectrum $\Omega(f)$ of a function f in B^2 is well-defined and from the inner product above, one can prove that for every $f \in B^2$

$$\|f\|_{B^2}^2 = \sum_{\lambda\in\Omega(f)} |M(fe_{-\lambda})|^2.$$

Let $l^2(\mathbb{R})$ denote the collection of all functions $f : \mathbb{R} \longrightarrow \mathbb{C}$ for which the set $\{\lambda \in \mathbb{R} \ : \ f(\lambda) \neq 0\}$ is at most countable and

$$\|f\|^2_{l^2(\mathbb{R})} := \sum |f(\lambda)|^2 < \infty.$$

Further, $l^\infty(\mathbb{R})$ is defined as the set of all functions $f : \mathbb{R} \longrightarrow \mathbb{C}$ such that

$$\|f\|_{l^\infty(\mathbb{R})} := \sup_{\lambda\in\mathbb{R}} |f(\lambda)| < \infty.$$

The map $F_B : l^2(\mathbb{R}) \longrightarrow B^2$ which transforms a function $f \in l^2(\mathbb{R})$ with a finite support into the function

$$(F_B f)(x) = \sum_{\lambda\in\mathbb{R}} f(\lambda)e_\lambda(x), \quad x \in \mathbb{R},$$

can be extended by continuity to all $l^2(\mathbb{R})$. It is referred to as the *Bohr-Fourier transform*. The operator F_B is an isometric isomorphism, and the inverse Bohr-Fourier transformation acts by the rule

$$F_B^{-1} : B^2 \longrightarrow l^2(\mathbb{R}), \qquad (F_B^{-1})(\lambda) = M(fe_{-\lambda}), \quad \lambda \in \mathbb{R}.$$

If $a \in l^\infty(\mathbb{R})$, then the operator $\psi(a) := F_B a \cdot F_B^{-1}$ is bounded.

Now, let $B^2_\pm$ be the Hilbert spaces consisting of the functions in B^2 with the Bohr-Fourier spectra in $\mathbb{R}_\pm$. A *canonical generalized right AP factorization* of a matrix function $a \in \mathcal{G}AP^{N\times N}(\mathbb{R})$ is a representation

$$a = a_- a_+, \tag{3.2}$$

where

$$a_- \in \mathcal{G}[B^2_-]^{N\times N}, \quad a_+ \in \mathcal{G}[B^2_+]^{N\times N}, \quad a_-\widetilde{P}a_-^{-1}I \in \mathcal{L}(B^2_N). \tag{3.3}$$

Here $\widetilde{P}$ is the projection $\widetilde{P} := F_B\chi_+F_B^{-1} \in \mathcal{L}(B^2_N)$, with χ_+ being the characteristic function of $\mathbb{R}_+$. Moreover, although $B^2_\pm$ is not a Banach algebra, we denote by $\mathcal{G}[B^2_\pm]^{N\times N}$ the class of matrix functions that belong to $[B^2_\pm]^{N\times N}$ and have inverse also in $[B^2_\pm]^{N\times N}$.

Recall that the above mentioned C^*-algebras can be considered on $\mathbb{T}$ by using the isometric isomorphism B_0 from $L^\infty(\mathbb{R})$ onto $L^\infty(\mathbb{T})$ (see, Remark 1.1).

Remark 3.2. *We would like to point out that pure matrix singular integral operators of the form* (1.14) *with coefficients* $a, b \in [SAPW(\mathbb{R})]^{N\times N}$ *were studied in* [4] *using the APW factorization of the local APW representatives* $G_\pm := (b^{-1}a)_\pm \in [APW(\mathbb{R})]^{N\times N}$ *at* $\pm\infty$ *of the matrix function*

$G = b^{-1}a \in [SAPW(\mathbb{R})]^{N\times N}$. *However, the method of* [4] *is not applicable to the operators* $\mathcal{A}$ *in case* $a, b \in [SAP(\mathbb{R})]^{N\times N} \setminus [SAPW(\mathbb{R})]^{N\times N}$. *Thus by Theorems* 2.1 *and* 2.4*, we can transfer those Fredholm characteristics to the operator* $\mathcal{O}$ *for that class of coefficients. On the other hand, the final results of the Fredholm theory of block Toeplitz operators with symbols in* $[SAP(\mathbb{R})]^{N\times N}$ *were presented with full proofs in* [11]. *The approach of* [11] *is based on exploitation of generalized AP factorization.*

The factorization approach was powerful enough to work also in the Banach space case $[H^p_+(\mathbb{R})]^N$, $p \in (1,\infty)$. *So, using Corollaries* 2.2 *and* 2.5*, these results are available for the operator* $\mathcal{O}$ *with reflection shift operator* (1.18) *and the flip operator* (1.19) (*up to the use of the isometric isomorphisms* (1.12) *and* (2.23)).

Here, however, the strategy will be based on using the Allan-Douglas local principle and an appropriate isomorphism theorem for C^**-dynamical systems. First, we will present the result for the* C^**-algebra* $\mathfrak{B}$ *and then we extend it to a more general algebra including the corresponding symbol of the shift operator* $\widetilde{J}$.

Theorem 3.5. *Let* $a_0, a_1, b_0, b_1 \in SAP(\mathbb{T})$. *The operator*

$$\mathcal{O} = a_0 I_{\mathbb{T}} + b_0 S_{\mathbb{T}} + a_1 \widetilde{J} + b_1 S_{\mathbb{T}} \widetilde{J}$$

is a Fredholm operator on the space $L^2(\mathbb{T})$ *if and only if the following three conditions hold:*

a) $a, b \in \mathcal{G}SAP^{2\times 2}(\mathbb{T})$, *where* a *and* b *are given in* (2.46);

b) the almost-periodic representatives a_l, a_r, b_l, b_r *have canonical generalized right AP factorizations* $a_l = a_l^- d(a_l) a_l^+$, $a_r = a_r^- d(a_r) a_r^+$, $b_l = b_l^- d(b_l) b_l^+$ *and* $b_r = b_r^- d(b_r) b_r^+$, *where* $M(a_l^\pm) = M(a_r^\pm) = M(b_l^\pm) = M(b_r^\pm) = I_2$;

c) $\det(O(\mu)) \neq 0$ *for* $\mu \in [0,1]$, *where*

$$\begin{aligned} O(\mu) =& P_+(1,\mu) \begin{pmatrix} d(a_l) & 0 \\ 0 & d(a_r) \end{pmatrix} P_+(1,\mu) \\ &+ P_+(1,\mu) \begin{pmatrix} d(b_l) & 0 \\ 0 & d(b_r) \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} P_+(1,\mu), \end{aligned}$$

with $P_+(1,\mu) = I_4 - S(1,\mu)$, *and*

$$S(1,\mu) := \begin{pmatrix} 2\mu - 1 & 2\sqrt{\mu(1-\mu)} \\ 2\sqrt{\mu(1-\mu)} & 1 - 2\mu \end{pmatrix} \otimes I_2.$$

Proof. In virtue of Theorem 2.6 we need to study the Fredholm theory for the Toeplitz plus Hankel operator $T_a + H_b$ on $[H^2(\mathbb{T})]^2$.

On the other hand, Theorem 3.1 on [12] is based on the local principle of Allan and Douglas (Theorem 3.3) and a symbol calculus at infinity. To achieve this goal, it is used the C^*-algebra

$$\mathcal{B} := \mathrm{alg}\,(S_{\mathbb{R}}, SAP(\mathbb{R})) \subset \mathcal{L}(L^2(\mathbb{R}))$$

generated by the multiplication operators $aI_{L^2(\mathbb{R})}$ with $a \in SAP(\mathbb{R})$ and the Cauchy singular integral operator $S_{\mathbb{R}}$. The first step is to study the C^*-algebra

$$\mathcal{U} := \mathrm{alg}\,(PC(\mathbb{R}), W^0(PC(\mathbb{R}))) \subset \mathcal{L}(L^2(\mathbb{R}))$$

generated by the operators $aW^0(b)$ with $a, b \in PC(\mathbb{R})$, where

$$W^0(b) = \mathcal{F}^{-1} b \cdot \mathcal{F},$$

with $\mathcal{F}$ being the Fourier transformation given by

$$(\mathcal{F}f)(x) := \int_{\mathbb{R}} f(t) e^{ixt} dt, \quad x \in \mathbb{R}.$$

This algebra $\mathcal{U}$ contains the C^*-algebra

$$Z := \mathrm{alg}\,(C(\dot{\mathbb{R}}), W^0(C(\dot{\mathbb{R}})))$$

which also contains the set $\mathcal{K}(L^2(\mathbb{R}))$ of all compact operators on $L^2(\mathbb{R})$. Therefore, by means of a Duduchava theorem (cf. [12, Lemma 5.1], or [10, Theorem B]), the corresponding localization techniques can be used and in this case $Z^\pi := Z/\mathcal{K}$ is a central C^*-subalgebra of $\mathcal{U}^\pi = \mathcal{U}/\mathcal{K}$.

At this point, in virtue of [55, Theorem 5], we can substitute the C^*-algebra $\mathcal{B}$ by the C^*-algebra

$$\mathcal{C} := \mathrm{alg}\,(S_{\mathbb{R}}, W_{\mathbb{R}}, SAP(\mathbb{R})),$$

where $W_{\mathbb{R}}$ is the reflection operator in $\mathbb{R}$, $(W_{\mathbb{R}})(x) = f(-x)$, and as in [12] the symbol $A(t, x, \mu)$, for $A = aW^0(b)$ $(a, b \in PC(\mathbb{R}))$ and

$$(t, x, \mu) \in (\mathbb{R} \times \{0,1\} \times [0,1]) \cup (\{\infty\} \times \mathbb{R} \times [0,1]) \cup ((\infty, \infty) \times \{0,1\}),$$

is given by

$$A(t, x, \mu) = \begin{pmatrix} A_{11}(t, x, \mu) & A_{12}(t, x, \mu) \\ A_{21}(t, x, \mu) & A_{22}(t, x, \mu) \end{pmatrix}$$

with

$$\begin{aligned}
A_{11}(t,x,\mu) =& a(t+0)(b(x+0)\mu + b(x-0)(1-\mu))\\
A_{12}(t,x,\mu) =& a(t+0)(b(x+0) - b(x-0))\sqrt{\mu(1-\mu)}\\
A_{21}(t,x,\mu) =& a(t-0)(b(x+0) - b(x-0))\sqrt{\mu(1-\mu)}\\
A_{22}(t,x,\mu) =& a(t-0)(b(x-0)\mu + b(x+0)(1-\mu))
\end{aligned}$$

(where by convention $a(\infty \pm 0) = a(\mp\infty)$, $b(0 \pm 0) = b(\mp\infty)$, and $\sqrt{\mu(1-\mu)}$ denotes any function $\varrho : [0,1] \longrightarrow \mathbb{R}$ such that $\varrho^2(\mu) = \mu(1-\mu)$).

The matrix $A(t,x,\mu)$ can be simplified at the points $t \in \mathbb{R}$ for the generating operators $A = aW^0(b)$ ($a \in C(\overline{\mathbb{R}})$, $b \in PC(\mathbb{R})$),

$$A(t,\infty,\mu) = \begin{pmatrix} a(t)b(-\infty) & 0 \\ 0 & a(t)b(+\infty) \end{pmatrix}.$$

As in [12], we set

$$S(\infty,\mu) = \begin{pmatrix} 2\mu - 1 & 2\sqrt{\mu(1-\mu)} \\ 2\sqrt{\mu(1-\mu)} & 1 - 2\mu \end{pmatrix} \otimes I_2.$$

In this way, we have that all the results in [12] hold also for the present case (i.e., also with the inclusion of the reflection operator $W_{\mathbb{R}}$). To conclude the proof, we pass from $\mathbb{R}$ to $\mathbb{T}$ as usual, using the isometric isomorphism of $L^2(\mathbb{T})$ onto $L^2(\mathbb{R})$ given by $(B\varphi)(x) = \frac{\sqrt{2}}{x+i}\varphi\Big(\frac{x-i}{x+i}\Big)$, $x \in \mathbb{R}$, with inverse $(B^{-1}\psi)(t) = \frac{i\sqrt{2}}{1-t}\varphi\Big(i\frac{1+t}{1-t}\Big)$, $t \in \mathbb{T} \setminus \{1\}$, recall that

$$P_+ = B^{-1}PB, \qquad -B^{-1}W_{\mathbb{R}}B = \widetilde{J},$$

here $P = \frac{1}{2}(I_{L^2(\mathbb{R})} + S_{\mathbb{R}})$ and, like in [79], the symbol of $\widetilde{J}$ is

$$\widetilde{J}(1,\mu) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

Now, we can rewrite condition (c) of Theorem 3.1 on [12] which is:

$$\begin{aligned}
A(\mu) =& P(\infty,\mu)\begin{pmatrix} d(a_l) & 0 \\ 0 & d(a_r) \end{pmatrix} P(\infty,\mu) + Q(\infty,\mu)\\
&+ Q(\infty,\mu)\begin{pmatrix} M(c_l) & 0 \\ 0 & M(c_r) \end{pmatrix} H_+(\infty,\mu)\\
&+ H_-(\infty,\mu)\begin{pmatrix} M(d_l) & 0 \\ 0 & M(d_r) \end{pmatrix} H_+(\infty,\mu)
\end{aligned}$$

$$
\begin{aligned}
&+ P(\infty,\mu)\begin{pmatrix} M((a_l^-)^{-1}c_l) & 0 \\ 0 & M((a_r^+)^{-1}c_r) \end{pmatrix} H_+(\infty,\mu) \\
&+ H_-(\infty,\mu)\begin{pmatrix} M(d_l(a_l^+)^{-1}) & 0 \\ 0 & M(d_r(a_r^+)^{-1}) \end{pmatrix} P(\infty,\mu) \\
&- H_-(\infty,\mu)\begin{pmatrix} M(\eta_l) & 0 \\ 0 & M(\eta_r) \end{pmatrix} H_+(\infty,\mu),
\end{aligned}
$$

for the operator $T_a + H_b$ acting on $[H^2(\mathbb{T})]^2$ satisfying conditions (a) and (b) in the mentioned theorem, as:

$$
\begin{aligned}
O(\mu) =&P_+(1,\mu)\begin{pmatrix} d(a_l) & 0 \\ 0 & d(a_r) \end{pmatrix} P_+(1,\mu) \\
&+ P_+(1,\mu)\begin{pmatrix} d(b_l) & 0 \\ 0 & d(b_r) \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} P_+(1,\mu)
\end{aligned}
$$

where the point ∞ is transformed into $1^+ \in \mathbb{T}$ in the usual way – which concludes the proof. □

3.2 Generalized inverses

As it was stated in the introduction (and as a consequence of the operator relations presented in the Chapter 2), we will obtain in the present section an invertibility criterion for the operator $\mathcal{O}$ and the form of its inverse/lateral inverse (under the conditions which ensure such invertibility).

In the next result we extract the invertibility conditions of a pure singular integral operator, as operator $\mathcal{A}$ defined in (1.14), and transfer it to a singular integral operator with shift as the operator $\mathcal{O}$ given in (1.21). This transference is possible in view of the equivalence operator relations presented in Theorems 2.1 and 2.4. Therefore, the conclusions of the following theorem are maintained for the singular integral operator with reflection (2.1) as well as for the operator with flip (2.21). However, we are going to present it only for the case of operator (2.21), stating that the form of the inverses for the case of operator (2.1) are obvious.

Theorem 3.6. *Let $u_\mathbb{T}, v_\mathbb{T} \in [L^\infty(\mathbb{T})]^{2\times 2}$ be the matrix functions given by* (2.41) *such that* $\det(u_\mathbb{T}(t)\pm v_\mathbb{T}(t)) \neq 0$*, and assume that* $\phi_\mathbb{T} = (u_\mathbb{T} - v_\mathbb{T})^{-1}(u_\mathbb{T} + v_\mathbb{T})$ *admits a generalized right factorization*

$$\phi_\mathbb{T}(t) = \phi_{\mathbb{T}-}(t)\Lambda(t)\phi_{\mathbb{T}+}(t).$$

Then the operator $\mathcal{O}$ is generalized invertible on the space $L^p(\mathbb{T}, |t-1|^{1-2/p})$, $1 < p < \infty$. A generalized inverse of $\mathcal{O}$ is given by

$$\begin{aligned}\mathcal{O}^- =& B^{-1} M_{\mathbb{R}_+} K R_{\mathbb{R}_+} N_{\mathbb{R}_+} \mathrm{Rest}_{[L^p(\mathbb{R}_+,|x|^{-1/2p})]^2} \left(B_2 (\phi_{\mathbb{T}+}^{-1} P_+ + \phi_{\mathbb{T}-} P_-) \right. \\ & \left. (\Lambda^{-1} P_+ + P_-) \phi_{\mathbb{T}-}^{-1} (u_\mathbb{T} I_\mathbb{T} - v_\mathbb{T} I_\mathbb{T})^{-1} B_2^{-1} \right) N_{\mathbb{R}_+}^{-1} K^{-1} M_{\mathbb{R}_+}^{-1} B, \end{aligned} \tag{3.4}$$

where $B^{\pm 1}, M_{\mathbb{R}}^{\pm 1}, K, R_{\mathbb{R}_+}, N_{\mathbb{R}}^{\pm 1}$ and $B_2^{\pm 1}$ are given in (2.23)–(2.24), (2.26)–(2.31) *and* (2.39) *respectively.*

The operator $\mathcal{O}$ is invertible (*left-sided invertible, right-sided invertible*) *if and only if all indices of the matrix function $\phi_\mathbb{T}$ are zero* (*non-negative, non-positive*). *In such a case, the inverse* (*left inverse, right inverse*) *is also given by* (3.4) (*where in each case some simplifications occur in the formula*).

Proof. From Theorem 2.4 we have that the operator $\mathcal{O}$ is equivalent after extension to the operator $\mathcal{D}_\mathbb{T}$ given by (2.40). Now, we rewrite the operator $\mathcal{D}_\mathbb{T}$ in terms of the Riesz projections P_+ and P_-, e.g.

$$\mathcal{D}_\mathbb{T} = u_\mathbb{T} I_\mathbb{T} + v_\mathbb{T} S_\mathbb{T} = (u_\mathbb{T} I_\mathbb{T} + v_\mathbb{T} I_\mathbb{T}) P_+ + (u_\mathbb{T} I_\mathbb{T} - v_\mathbb{T} I_\mathbb{T}) P_-.$$

The invertibility conclusions for the operator $\mathcal{D}_\mathbb{T}$ are obtained from the well-known Simonenko's Theorem; see for instance Theorem 4.2 on [67] for continuous matrix-valued functions. This can be generalized for bounded measurable matrix-valued functions as follows (see, [67, Chapter V, §5]): Under the assumption that the matrix-valued function $\phi_\mathbb{T}$ admits a generalized factorization in the space $[L^p(\mathbb{T}, w)]^2$, say $\phi_\mathbb{T} = \phi_{\mathbb{T}-} \Lambda \phi_{\mathbb{T}+}$, then the operator $\mathcal{D}_\mathbb{T}$ is generalized invertible on the space $[L^p(\mathbb{T}, w)]^2$, $1 < p < \infty$ and $w(t) = |i \frac{1+t}{1-t}|^{-1/2p} |1-t|^{1-2/p}$ with a generalized inverse given by

$$\mathcal{D}_\mathbb{T}^- = (\phi_{\mathbb{T}+}^{-1} P_+ + \phi_{\mathbb{T}-} P_-)(\Lambda^{-1} P_+ + P_-) \phi_{\mathbb{T}-}^{-1} (u_\mathbb{T} I_\mathbb{T} - v_\mathbb{T} I_\mathbb{T})^{-1}. \tag{3.5}$$

In the case of all right partial indices of the matrix function $\phi_\mathbb{T}$ being zero (non-negative, non-positive) then $\mathcal{D}_\mathbb{T}$ is invertible (left-sided invertible, right-sided invertible) and the inverse (left-sided inverse, right-sided inverse) is also given by (3.5).

Finally, we will use the explicit equivalence relation exhibited in Theorem 2.4 to obtain a generalized inverse (inverse, left inverse, right inverse) of the operator $\mathcal{O}$:

$$\mathcal{O}^- = \mathcal{V} \, \mathrm{Rest}_{[L^p(\mathbb{R}_+,|x|^{-1/2p})]^2} (B_2 \mathcal{D}_\mathbb{T}^- B_2^{-1}) \, \mathcal{G}, \tag{3.6}$$

where the operators $B_2^{\pm 1}$, $\mathcal{G}$ and $\mathcal{V}$ are given in (2.39) and Theorem 2.3. Putting equality (3.5) into the equality (3.6), and writing the explicit form of $\mathcal{G}$ and $\mathcal{V}$ we obtain the conclusion. □

3.2.1 Examples

We end this chapter by considering concrete examples of operators $\mathcal{O}$ in order to derive a corresponding conclusion about its invertibility.

Example 3.7. Let us consider the operator $\mathcal{O}$ as in (2.1) with reflection operator J defined in (1.18) and coefficients given by

$$\begin{aligned}
a_0(t) =& \frac{1}{2}[t^{2(s-1)} + t^{-2} + t^{-2s}],\\
a_1(t) =& \frac{1}{2}[-t^{2(s-1)} - t^{-2} + t^{-2s}],\\
b_0(t) =& \frac{t^{-2s}}{2t^{2m}+1}\left(\frac{1}{2}(2t^{2m}-1) + \frac{2t^{2m}+3t^{2(-n-\alpha-1/2)}}{3t^{-2n}+1}\right)\\
&+ \frac{1}{2}\frac{3t^{-2n}-1}{3t^{-2n}+1}(t^{2(s-1)}+t^{-2}),\\
b_1(t) =& \frac{t^{-2s}}{2t^{2m}+1}\left(\frac{1}{2}(2t^{2m}-1) - \frac{2t^{2m}+3t^{2(-n-\alpha-1/2)}}{3t^{-2n}+1}\right)\\
&- \frac{1}{2}\frac{3t^{-2n}-1}{3t^{-2n}+1}(t^{2(s-1)}+t^{-2}),
\end{aligned}$$

with $m, n, s \in 2\mathbb{Z}$ and $\alpha = \frac{4k-1}{2}$, $k \in \mathbb{Z}$. From Theorem 2.1 we have that $\mathcal{O}$ is equivalent to the operator $\mathfrak{D}_{\mathbb{T}}$ with coefficients $\upsilon_{\mathbb{T}}$ and $\vartheta_{\mathbb{T}}$ given by

$$\upsilon_{\mathbb{T}}(t) = \begin{pmatrix} t^{-s} & 0 \\ 0 & t^{s-1}+t^{-1} \end{pmatrix} \text{ and } \vartheta_{\mathbb{T}}(t) = \begin{pmatrix} t^{-s}\frac{2t^{m}-1}{2t^{m}+1} & \frac{t^{-s}(4t^{m+1/2}+6t^{-n-\alpha})}{(2t^{m}+1)(3t^{-n}+1)} \\ 0 & \frac{3t^{-n}-1}{3t^{-n}+1}(t^{s-1}+t^{-1}) \end{pmatrix}.$$

From Corollary 2.2 we know that $\mathcal{O}$ is equivalent to the Toeplitz operator $\mathcal{T}_{\psi_{\mathbb{T}}}$ with

$$\psi_{\mathbb{T}}(t) = (\upsilon_{\mathbb{T}}(t) - \vartheta_{\mathbb{T}}(t))^{-1}(\upsilon_{\mathbb{T}}(t) + \vartheta_{\mathbb{T}}(t)) = \begin{pmatrix} 2t^{m} & 2t^{m+1/2}+3t^{-n-\alpha} \\ 0 & 3t^{-n} \end{pmatrix}.$$

If $\alpha > 0$, $\psi_{\mathbb{T}}$ admits a (right) generalized factorization

$$\psi_{\mathbb{T}}(t) = \begin{pmatrix} 2 & t^{-\alpha} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t^{m} & 0 \\ 0 & t^{-n} \end{pmatrix}\begin{pmatrix} 1 & t^{1/2} \\ 0 & 3 \end{pmatrix}.$$

The operator $\mathcal{O}$ is invertible (left-sided invertible, right-sided invertible) if and only if all indices $m, -n$ of the matrix function $\psi_{\mathbb{T}}$ are zero (non-negative, non-positive), with (a generalized) inverse given by

$$\mathcal{O}^{-} = F(\psi_{+}^{-1}P_{+} + \psi_{-}P_{-})(\Lambda^{-1}P_{+} + P_{-})\psi_{-}^{-1}(u_{\mathbb{T}}I_{\mathbb{T}} - \upsilon_{\mathbb{T}}I_{\mathbb{T}})^{-1}F^{-1},$$

where $F^{\pm 1}$ are given in Theorem 2.1.

Example 3.8. Here we consider the operator $\mathcal{O}$ as in (2.21) with flip operator $\widetilde{J}$ defined in (1.19), acting on the space $L^p(\mathbb{T},\rho)$, with $1<p<\infty$, $\rho(t)=|t-1|^{1-2/p}$ and coefficients given is this case by

$$a_0(t):=3t^8,\quad a_1(t):=5t^4,\quad b_0(t):=3t^8,\quad b_1(t):=2t^8.$$

From Theorem 3.6, we need to study the matrix function $\phi_{\mathbb{T}}=(u_{\mathbb{T}}-v_{\mathbb{T}})^{-1}(u_{\mathbb{T}}+v_{\mathbb{T}})$, where $u_{\mathbb{T}},v_{\mathbb{T}}\in[L^\infty(\mathbb{T})]^{2\times 2}$ be the matrix-valued functions given by (2.42) such that $\det(u_{\mathbb{T}}(t)\pm v_{\mathbb{T}}(t))\neq 0$. Thus we get that

$$\phi_{\mathbb{T}}(t)=\begin{cases}\begin{pmatrix}1 & 6-\frac{30}{5+2t^2}\\ 0 & -1\end{pmatrix}, & t\in\mathbb{T}_+\\ \operatorname{diag}(I_{\mathbb{T}},I_{\mathbb{T}}), & t\in\mathbb{T}_-.\end{cases}$$

On the other hand, notice that if the (generalized) inverse of $\mathcal{O}$ exists, then it has the form

$$\mathcal{O}^-=\mathcal{V}\,\mathrm{Rest}_{[L^p(\mathbb{R}_+,|x|^{-1/2p})]^2}(B_2\mathcal{D}_{\mathbb{T}}^-B_2^{-1})\,\mathcal{G},$$

where the operators $\mathcal{G}$ and $\mathcal{V}$ are given in Theorem 2.3 and $B_2^{\pm 1}$ are defined in (2.39). The fact that we apply the restriction operator, $\mathrm{Rest}_{[L^p(\mathbb{R}_+,|x|^{-1/2p})]^2}$, in the formula of the inverse of $\mathcal{O}$, tell us that we must consider only the function $\phi_{\mathbb{T}}$ for $t\in\mathbb{T}_+$. I.e.,

$$\phi_{\mathbb{T}}(t)=\begin{pmatrix}1 & 6-\frac{30}{5+2t^2}\\ 0 & -1\end{pmatrix}.$$

This matrix-valued function $\phi_{\mathbb{T}}$ has the following canonical factorization representation:

$$\phi_{\mathbb{T}}(t)=\begin{pmatrix}1 & -\frac{30}{5+t^2}\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 6\\ 0 & 1\end{pmatrix},$$

therefore, the operator $\mathcal{O}$ in this case is invertible with inverse given by

$$\mathcal{O}^{-1}=\mathcal{V}B_2(\phi_{\mathbb{T}+}^{-1}P_++\phi_{\mathbb{T}-}P_-)\phi_{\mathbb{T}-}^{-1}(u_{\mathbb{T}}I_{\mathbb{T}}-v_{\mathbb{T}}I_{\mathbb{T}})^{-1}B_2^{-1}\,\mathcal{G},$$

with the operators $B_2^{\pm 1}$, $\mathcal{G}$ and $\mathcal{V}$ given in (2.39) and Theorem 2.3.

Chapter 4

On the Kernel Dimensions of Singular Integral Operators with Shift

By using the equivalent operator relations given in Chapter 2 and a polynomial collocation method, in the first part of this chapter we are going to compute the kernel dimensions of singular integral operators $\mathcal{O}$ with reflection and flip shift operators, having piecewise continuous functions as coefficients. In addition, the used strategy will allow us to compute the Moore-Penrose inverses of these singular integral operators (subjected to appropriate conditions). In the second part, upper bounds for the kernel dimension of singular integral operators with orientation-preserving weighted Carleman shift and continuous coefficients are obtained. This is implemented by using some estimations which are derived with the help of certain explicit operator relations. In particular, the interplay between classes of operators with and without Carleman shifts has a preponderant importance to achieve the mentioned bounds.

4.1 Kernel dimensions of $\mathcal{O}$ with reflection and flip shift operators

Under the assumption that the operator $\mathcal{O}$ given by (1.21) (and with shift operator defined by (1.18) or (1.19)) is a Fredholm operator (see Theorem 3.1 for such a criteria), we will compute their kernel dimensions by means of a *polynomial collocation method for singular integral operators* proposed by A. Rogozhin and B. Silbermann in [85].

We would like to point out that we can compute those dimensions using the *modified finite section method for Toeplitz operators* proposed by B. Silbermann in [91] and [90] (because of the equivalence relation between the operator $\mathcal{O}$ and a Toeplitz operator; see Corollary 2.2, for the case where we are dealing with the reflection operator (1.18), and Corollary 2.5 for the case where we are considering the flip operator (1.19)). Even more, the estimates for the convergence speed of the k-th singular values to zero (for smooth coefficients) in both methods are the same. Therefore, to compute the kernel dimensions of a Fredholm Toeplitz operator and/or of a Fredholm singular integral operator one can use the collocation method instead of the finite section methods (see [85]).

4.1.1 General framework

Approximation numbers

Let F be a finite dimensional Banach space with $\dim F = m$. The k-th *approximation number* ($k \in \{0, 1, \ldots, m\}$) of an operator $A \in \mathcal{L}(F)$ is defined as

$$s_k(A) = \mathrm{dist}(A, \mathcal{F}_{m-k}) := \inf\{\|A - F\| \ : \ F \in \mathcal{F}_{m-k}\},$$

where $\mathcal{F}_{n-k}$ denotes the collection of all operators (or matrices from $\mathbb{C}^{n\times n}$) having the dimension of the range equal to at most $n - k$. It is clear that

$$0 \leq s_1(A) \leq \cdots \leq s_m(A) = \|A\|_{\mathcal{L}(F)}.$$

Notice that the approximation numbers can be also defined as the singular values of a square matrix $A_n \in \mathbb{C}^{nN\times nN}$ which are the square roots of the spectral points of $A_n^* A_n$, where A_n^* means the adjoint matrix of A_n.

Definition 4.1. A sequence (A_n) of matrices $nN \times nN$ is said to have the k-splitting property if there is an integer $k \geq 0$ such that

$$\lim_{n\to\infty} s_k(A_n) = 0 \quad \text{and} \quad \liminf_{n\to\infty} s_{k+1}(A_n) > 0.$$

The number k is called the splitting number. Alternatively, we say the singular values Λ_n (computed via $A_n^* A_n$) of a sequence (A_n) of $k(n) \times l(n)$ matrices A_n have the splitting property if there exist a sequence $c_n \to 0$ ($c_n \geq 0$) and a number $d > 0$ such that

$$\Lambda_n \subset [0, c_n] \cup [d, \infty) \quad \text{for all } n,$$

and the singular values of A_n are said to meet the k-splitting property if, in addition, for all sufficiently large n exactly k singular values of A_n lie in $[0, c_n]$.

Approximation methods

For the sake of self-contained global presentation we will describe here the approximation method in the scope of Banach spaces. Afterwards, we will show the natural adaptation to our cases. More information about this method can be found, for instance, in [43, 84, 85].

Let X be a Banach space. Given a bounded linear operator A on X, $A \in \mathcal{L}(X)$, and an element f of X, consider the operator equation

$$A\varphi = f. \tag{4.1}$$

For the approximate solution of this equation, we choose to approximate closed subspaces X_n in which the approximate solutions φ_n of (4.1) will be sought. In practice, the X_n spaces usually have finite dimensions but we will not require this assumption. We will assume that X_n are ranges of certain projection operators $L_n : X \longrightarrow X_n$ so that these projections converge strongly to the identity operator: $s - \lim_{n\to\infty} L_n = I$. This strong convergence implies that $\cup_{n=1}^{\infty} X_n$ is dense in X.

Having fixed subspaces X_n, we choose convenient linear operators $A_n : X_n \longrightarrow X_n$ and consider in place of (4.1) the equations

$$A_n\varphi_n = L_n f, \quad n = 1, 2, \ldots \tag{4.2}$$

with their solutions sought in $X_n = \operatorname{Im} L_n$.

A sequence (A_n) of operators $A_n \in \mathcal{L}(\operatorname{Im} L_n)$ is an *approximation method* for $A \in \mathcal{L}(X)$ if $A_n L_n$ converge strongly to A as $n \to \infty$.

Note that even if (A_n) is an approximation method for A, we do not yet know anything about the solvability of the equations (4.2) and about the relations between (eventual) solutions φ_n of (4.2) and the (possible) solution φ of (4.1).

The approximation method (A_n) for A is *applicable* if there exists a number n_0 such that the equations (4.2) possess unique solutions φ_n for every $n \geq n_0$ and every right-hand side $f \in X$, and if these solutions converge in the norm of X to a solution of (4.1). An equivalent characterization of applicable approximation methods is the notion of *stability*, where a sequence (A_n) of operators $A_n \in \mathcal{L}(\operatorname{Im} L_n)$ is *stable* if there exists a number n_0 such that the operators A_n are invertible for every $n \geq n_0$ and if the norms of their inverses are uniformly bounded:

$$\sup_{n \geq n_0} \|A_n^{-1} L_n\| < \infty.$$

These notions are connected by the Polski's Theorem.

Theorem 4.1 (Polski; see [43, Theorem 1.4]). *Let (L_n) be a sequence of projections which converges strongly to the identity operator, and let (A_n) with $A_n \in \mathcal{L}(\operatorname{Im} L_n)$ be an approximation method for the operator $A \in \mathcal{L}(X)$. This method is applicable if and only if the operator A is invertible and the sequence (A_n) is stable.*

Projection methods and the algebraization of stability

Let A be a bounded linear operator on X and (L_n) a sequence of projections converging strongly to the identity $I \in \mathcal{L}(X)$. The idea of any projection method for the approximate solution of (4.1) is to choose a further sequence (R_n) of projections which also converge strongly to the identity and which satisfy $\operatorname{Im} R_n = \operatorname{Im} L_n$. Thus, we choose $A_n = R_n A L_n : \operatorname{Im} L_n \longrightarrow \operatorname{Im} L_n$ as the approximate operators of A. In fact, Lemma 1.5 in [43] proves that $(R_n A L_n)$ is indeed an approximate method for A.

Let X be an infinite dimensional Banach space and let (X_n) be a sequence of finite dimensional subspaces of X. Moreover, we assume that there is a sequence (L_n) of projections from X onto X_n with strong limit $I \in X$ as $n \to \infty$. Let $\mathcal{F}$ refer to the set of all sequences $(A_n)_{n=0}^{\infty}$ of operators $A_n \in \mathcal{L}(\operatorname{Im} L_n)$ which are uniformly bounded: $\sup_{n\geq 0} \|A_n L_n\| < \infty$. The "algebraization" of $\mathcal{F}$ is given by the natural operations

$$\lambda_1(A_n) + \lambda_2(B_n) := (\lambda_1 A_n + \lambda_2 B_n), \quad (A_n)(B_n) := (A_n B_n) \tag{4.3}$$

and

$$\|(A_n)\|_{\mathcal{F}} := \sup\{\|A_n L_n\| \ : \ n > 0\}$$

which make $\mathcal{F}$ to be an initial Banach algebra with identity $(I_{|\operatorname{Im} L_n})$. The set $\mathcal{G}$ of all sequences (G_n) in $\mathcal{F}$ with $\lim_{n\to\infty} \|G_n L_n\| = 0$ is a closed two sided ideal in $\mathcal{F}$. The Kozak's Theorem (Theorem 1.5 in [43]) establishes that a sequence $(A_n) \in \mathcal{F}$ is stable if and only if its coset $(A_n) + \mathcal{G}$ is invertible in the quotient algebra $\mathcal{F}/\mathcal{G}$.

If instead of a Banach space X we consider a Hilbert space $\mathcal{H}$ and L_n are the orthogonal projections P_n from $\mathcal{H}$ onto $\mathcal{H}_n$, then $(A_n)^* = (A_n^*)$ defines an involution in $\mathcal{F}$ which makes $\mathcal{F}$ a C^*-algebra. Note that in this case the approximation numbers of an operator $A_n \in \mathcal{L}(\mathcal{H}_n)$ are just the singular values of A_n.

Let further T be a (possible infinite) index set and suppose that, for every $t \in T$, we are given an infinite dimensional Hilbert space $\mathcal{H}^t$ with identity operator I^t as well as a sequence (E_n^t) of partial isometries $E_n^t : \mathcal{H}^t \longrightarrow \mathcal{H}$ such that the initial projections P_n^t of E_n^t converge strongly to I^t as $n \to \infty$, the range projection of E_n^t is P_n and the separation condition

$$(E_n^s)^* E_n^t \longrightarrow 0 \quad \text{weakly as } n \to \infty \tag{4.4}$$

holds for every $s, t \in T$ with $s \neq t$. Recall that an operator $E : \mathcal{H}' \longrightarrow \mathcal{H}''$ is a partial isometry if $EE^*E = E$ and that E^*E and EE^* are orthogonal projections, which are called the initial and the range projections of E, respectively. The restriction of E to $\operatorname{Im}(E^*E)$ is an isometry from $\operatorname{Im}(E^*E)$ onto $\operatorname{Im}(EE^*) = \operatorname{Im} E$. We write E^t_{-n} instead of $(E^t_n)^*$, and set $\mathcal{H}_n := \operatorname{Im} P_n$ and $\mathcal{H}^t_n := \operatorname{Im} P^t_n$.

Let $\mathcal{F}^T$ stand for the set of all sequences $(A_n) \in \mathcal{F}$ for which the strong limits

$$s - \lim_{n\to\infty} E^t_{-n} A_n E^t_n \quad \text{and} \quad s - \lim_{n\to\infty} (E^t_{-n} A_n E^t_n)^*$$

exist for every $t \in T$, and define mappings $W^t : \mathcal{F}^T \longrightarrow \mathcal{L}(\mathcal{H}^t)$ by

$$W^t(A_n) := s - \lim_{n\to\infty} E^t_{-n} A_n E^t_n.$$

The algebra $\mathcal{F}^T$ is a C^*-subalgebra of $\mathcal{F}$ which contains the identity, and W^t are $*$-homomorphisms. Moreover, $\mathcal{F}^T$ is a *standard* algebra, this means that any sequence $(A_n) \in \mathcal{F}^T$ is stable if and only if all the operators $W^t(A_n)$ are invertible.

The separation condition (4.4) ensures that, for every $t \in T$ and every compact operator $K^t \in \mathcal{K}(\mathcal{H}^t)$, the sequence $(E^t_n K^t E^t_{-n})$ belongs to the algebra $\mathcal{F}^T$, and for all $s \in T$

$$W^s(E^t_n K^t E^t_{-n}) = \begin{cases} K^t & \text{if } s = t \\ 0 & \text{if } s \neq t. \end{cases} \tag{4.5}$$

Conversely, the above equality implies the separation condition (4.4). Moreover, the ideal $\mathcal{G}$ belongs to $\mathcal{F}^T$. So we can introduce the smallest closed ideal $\mathcal{J}^T$ of $\mathcal{F}^T$ which contains all sequences $(E^t_n K^t E^t_{-n})$ with $t \in T$ and $K^t \in \mathcal{K}(\mathcal{H}^t)$, as well as all sequences $(G_n) \in \mathcal{G}$.

Corresponding to the ideal $\mathcal{J}^T$ we introduce a class of Fredholm sequences by calling a sequence $(A_n) \in \mathcal{F}^T$ Fredholm if the coset $(A_n) + \mathcal{J}^T$ is invertible in the quotient algebra $\mathcal{F}^T/\mathcal{J}^T$. It is also known (see [43]) that if $(A_n) \in \mathcal{F}^T$ is Fredholm, then all operators $W^t(A_n)$ are Fredholm on $\mathcal{H}^t$, and the number of the non-invertible operators among the $W^t(A_n)$ is finite.

The main result concerning standard algebras reads as follows:

Theorem 4.2 (see, [43]). *Let (A_n) be a sequence from the standard C^*-algebra $\mathcal{F}^T$.*

(i) *If the coset $(A_n) + \mathcal{J}^T$ is invertible in the quotient algebra $\mathcal{F}^T/\mathcal{J}^T$, then all operators $W^t(A_n)$ are Fredholm on $\mathcal{H}^t$, the number of the non-invertible operators among the $W^t(A_n)$ is finite, and the singular values* *of A_n have the k-splitting property with*

$$k(A_n) = \sum_{t \in T} \dim \ker W^t(A_n).$$

(ii) *If $W^t(A_n)$ is not Fredholm for at least one $t \in T$, then for every integer $k \geq 0$*

$$s_k(A_n) \longrightarrow 0, \quad \textit{as} \quad n \longrightarrow \infty.$$

4.1.2 The collocation method for singular integral operators on $[L^2(\mathbb{T}, w)]^2$

In this part we will consider pure (matrix) singular integral operators defined on weighted Lebesgue spaces $[L^2(\mathbb{T}, w)]^2$, where the weight w belongs to $A_2(\mathbb{T})$.

In addition, let us consider the following singular integral equation on $[L^2(\mathbb{T}, w)]^2$:

$$(aI_{\mathbb{T}} + bS_{\mathbb{T}})u = f. \tag{4.6}$$

In view of obtaining an approximate solution of (4.6) by the collocation method, we seek to polynomials u_n by solving the linear $(2n+1) \times (2n+1)$-system

$$a(z_j)u_n(z_j) + b(z_j)(S_{\mathbb{T}})u_n(z_j) = f(z_j), \quad j \in \{-n, \ldots, n\}$$

which can be written equivalently in the form

$$L_n(aI_{\mathbb{T}} + bS_{\mathbb{T}})P_n u_n = L_n f$$

and our objective is to examine the stability of the sequence $(L_n(aI_{\mathbb{T}} + bS_{\mathbb{T}})P_n)$.

The algebraization of the stability in this case runs as follows: Let the Fourier projection $P_n \in \mathcal{L}([L^2(\mathbb{T}, w)]^2)$ that in terms of the Fourier coefficients of a function $\psi \in [L^2(\mathbb{T}, w)]^2$, acts componentwise according to the rule

$$\psi = \sum_{k \in \mathbb{Z}} \psi_k t^k \longmapsto \sum_{k=-n}^{n} \psi_k t^k, \qquad n \in \mathbb{N},$$

and the Lagrange interpolation operator L_n (which is bounded in $[L^2(\mathbb{T}, w)]^2$, see for instance [101]) associated to the points

$$t_j = \exp\left(\frac{2\pi i j}{2n+1}\right), \quad j = 0, 1, \ldots, 2n$$

(that is L_n assigns to a function ψ its Lagrange interpolation polynomial $L_n\psi \in \operatorname{Im} P_n$, uniquely determined, on each component, by the conditions $(L_n\psi)(t_j) = \psi(t_j)$, $j = 0, 1, \ldots, 2n$). One can show that $\|P_n\psi - \psi\|_{2,w} \longrightarrow 0$ as $n \longrightarrow \infty$ for every $\psi \in [L^2(\mathbb{T}, w)]^2$ and in [70] it was proved (for the scalar case) that $\|L_n\psi - \psi\|_{2,w} \longrightarrow 0$, $n \longrightarrow \infty$.

For $r \in \mathbb{Z}_+$ given, consider the operators:

$$A_{n,r} := L_n(aI_{\mathbb{T}} + bS_{\mathbb{T}})P_n(P_n - W_nP_{r-1}W_n), \quad n \in \mathbb{Z}_+, \tag{4.7}$$

where the operators $W_n \in \mathcal{L}([L^2(\mathbb{T}, w)]^2)$, which componentwise is the discrete version of the flip operator (1.19), act by the rule

$$W_n\psi = \sum_{k=0}^{n} \psi_{n-k}t^k + \sum_{k=-n}^{-1} \psi_{-n-k-1}t^k.$$

Note that if $r = 0$, then we get a polynomial collocation method A_n for the solution of the singular integral equation (4.6).

First, note that the operators W_n and P_n are related as follows:

$$W_n^2 = P_n, \quad W_nP_n = P_nW_n = W_n. \tag{4.8}$$

On the other hand, in [43, 47, 76] it was shown that:

$$L_naI_{\mathbb{T}} = L_naL_n, \quad S_{\mathbb{T}}P_n = P_nS_{\mathbb{T}}P_n, \quad W_nL_naW_n = L_n\widetilde{a}P_n \tag{4.9}$$

$$(L_naP_n)^* = L_n\overline{a}P_n, \quad (P_nS_{\mathbb{T}}P_n)^* = P_nS_{\mathbb{T}}P_n \tag{4.10}$$

where for $a \in L^\infty(\mathbb{T})$,

$$\widetilde{a}(t) = a\left(\frac{1}{t}\right), \quad t \in \mathbb{T}.$$

We denote by T_2 the index set $\{1, 2\}$ and by $\mathcal{F}^{T_2}$ the C^*-algebra of all operator sequences (A_n), with $A_n \in \mathcal{L}(\operatorname{Im} P_n)$, for which there exist operators ($*$-homomorphisms) $W^1(A_n)$, $W^2(A_n) \in \mathcal{L}([L^2(\mathbb{T}, w)]^2)$ such that

$$s - \lim_{n\to\infty} P_nA_nP_n = W^1(A_n) \quad \text{and} \quad s - \lim_{n\to\infty} W_nA_nW_n = W^2(A_n)$$

$$s - \lim_{n\to\infty} (P_nA_nP_n)^* = W^1(A_n)^* \quad \text{and} \quad s - \lim_{n\to\infty} (W_nA_nW_n)^* = W^2(A_n)^*.$$

Furthermore, let us introduce the subsets $\mathcal{J}^1$ and $\mathcal{J}^2$ of the C^*-algebra $\mathcal{F}^{T_2}$:

$$\begin{aligned}\mathcal{J}^1 =&\{(P_nKP_n) + (G_n) \ : \ K \in \mathcal{K}([L^2(\mathbb{T}, w)]^2), \ \|G_n\| \to \infty\}\\ \mathcal{J}^2 =&\{(W_nLW_n) + (G_n) \ : \ L \in \mathcal{K}([L^2(\mathbb{T}, w)]^2), \ \|G_n\| \to \infty\}.\end{aligned}$$

Again, $\mathcal{J}^{T_2}$ is the smallest closed two-sided ideal of $\mathcal{F}^{T_2}$ which contains all sequences (J_n) such that J_n belongs to one of the ideals $\mathcal{J}^t$, $t = 1, 2$.

Theorem 4.3. *Let $a, b \in [PC(\mathbb{T})]^{2\times 2}$ and consider the operators*

$$A_{n,r} := L_n(aI_{\mathbb{T}} + bS_{\mathbb{T}})P_n(P_n - W_nP_{r-1}W_n),\ n \in \mathbb{Z}_+.$$

(1) *The sequence $(A_{n,r})$ belongs to the C^*-algebra $\mathcal{F}^{T_2}$. In particular*

$$W^1(A_{n,r}) = aI_{\mathbb{T}} + bS_{\mathbb{T}}, \quad \text{and} \quad W^2(A_{n,r}) = (\widetilde{a}I_{\mathbb{T}} + \widetilde{b}S_{\mathbb{T}})Q_{r-1}$$

where $Q_{r-1} = I - P_{r-1}$.

(2) *The coset $(A_{n,r}) + \mathcal{J}^{T_2}$ is invertible in $\mathcal{F}^{T_2}/\mathcal{J}^{T_2}$ if and only if the operator $W^1(A_{n,r}) = aI_{\mathbb{T}} + bS_{\mathbb{T}}$ is Fredholm.*

(3) *If the operators $W^1(A_{n,r})$ and $W^2(A_{n,r})$ are Fredholm on $[L^2(\mathbb{T}, w)]^2$, then the approximation numbers of $A_{n,r}$ have the k-splitting property with*

$$k(A_{n,r}) = \dim\ker(aI_{\mathbb{T}} + bS_{\mathbb{T}}) + \dim\ker((\widetilde{a}I_{\mathbb{T}} + \widetilde{b}S_{\mathbb{T}})Q_{r-1}).$$

(4) *Otherwise, $s_l(A_{n,r}) \longrightarrow 0$ for each $l \in \mathbb{N}$.*

Proof. We are going to compute $W^1(A_{n,r})$ and $W^2(A_{n,r})$. Having this goal in mind, we will use the relations (4.8) and (4.9). First note that for each $r \in \mathbb{N}$ the sequence $(W_nP_{r-1}W_n)$ belongs to $\mathcal{J}^2$. So, from (4.5) we have that $W^1(P_n - W_nP_{r-1}W_n) = I$ and $W^2(P_n - W_nP_{r-1}W_n) = I - P_{r-1}$. Since W^t, $t \in T_2$, are $*$-homomorphisms, then it only remains to compute

$$\begin{aligned} W^1(L_n(aI_{\mathbb{T}} + bS_{\mathbb{T}})P_n) =& s - \lim_{n\to\infty} L_n(aI_{\mathbb{T}} + bS_{\mathbb{T}})P_nP_n \\ =& \lim_{n\to\infty} L_n(aI_{\mathbb{T}} + bS_{\mathbb{T}})P_n \\ =& aI_{\mathbb{T}} + bS_{\mathbb{T}} \end{aligned}$$

and

$$\begin{aligned} W^2(L_n(aI_{\mathbb{T}} + bS_{\mathbb{T}})P_n) =& s - \lim_{n\to\infty} W_n(L_n(aI_{\mathbb{T}} + bS_{\mathbb{T}})P_n)W_n \\ =& \lim_{n\to\infty} W_n(L_n(aI_{\mathbb{T}} + bS_{\mathbb{T}})P_n) \\ =& \lim_{n\to\infty} L_n(\widetilde{a}I_{\mathbb{T}} + \widetilde{b}S_{\mathbb{T}})P_n \\ =& \widetilde{a}I_{\mathbb{T}} + \widetilde{b}S_{\mathbb{T}}. \end{aligned}$$

Therefore, $W^1(A_{n,r}) = aI_{\mathbb{T}} + bS_{\mathbb{T}}$ and $W^2(A_{n,r}) = (\widetilde{a}I_{\mathbb{T}} + \widetilde{b}S_{\mathbb{T}})Q_{r-1}$. Similarly, using the mentioned properties (4.8) and (4.9), as well as properties (4.10), we are able to compute $W^1(A_{n,r})^*$ and $W^2(A_{n,r})^*$, which proves proposition (1).

On the other hand, from the previous part we have that $W^1(A_{n,r}) = aI_{\mathbb{T}} + bS_{\mathbb{T}}$ and $W^2(A_{n,r}) = (\widetilde{a}I_{\mathbb{T}} + \widetilde{b}S_{\mathbb{T}})Q_{r-1}$. Then parts (2), (3) and (4) follow from Theorem 4.2. □

4.1.3 The kernel dimension of the operator $\mathcal{O}$

Now, we are in conditions to compute the kernel dimension of the operator $\mathcal{O}$ given in (1.21) for both cases of the shift operator J defined by (1.18) and (1.19).

Theorem 4.4. *If the singular integral operator $\mathcal{O}$ is Fredholm, then the singular values of the operators $A_{n,r}$ defined in* (4.7) *have the k-splitting property with*

$$k = k(A_{n,r}) = \dim\ker\mathcal{O} + \dim\ker(\widetilde{u}_{\mathbb{T}}I_{\mathbb{T}} + \widetilde{b}_{\mathbb{T}}S_{\mathbb{T}})Q_{r-1}$$

where $Q_{r-1} := I - P_{r-1}$.

Proof. From Theorems 2.1 and 2.4 we know that the operator $\mathcal{O}$ is equivalent to a matrix singular integral operator of the form

$$\mathcal{D}_{\mathbb{T}} = u_{\mathbb{T}}I_{\mathbb{T}} + v_{\mathbb{T}}S_{\mathbb{T}}, \tag{4.11}$$

where for the case when the operator $\mathcal{O}$ has in its definition the reflection shift operator (1.18), we have $\mathcal{D}_{\mathbb{T}} \in \mathcal{L}([L^2(\mathbb{T},w)]^2)$, $w \in A_2^e(\mathbb{T})$. The coefficients are given by (2.18) and (2.19). If we are considering the flip shift operator (1.19), then the operator $\mathcal{D}_{\mathbb{T}} \in \mathcal{L}([L^2(\mathbb{T},\gamma)]^2)$ with the weight

$$\gamma(t) = \left| i\frac{1+t}{1-t} \right|^{-1/4}$$

and, in this case, the coefficients $u_{\mathbb{T}}$ and $v_{\mathbb{T}}$ are defined in (2.42).

The conclusion is obtained from Theorem 4.3 (3), taking into account that $W^1(A_{n,r}) = \mathcal{D}_{\mathbb{T}}$, and the fact that two equivalent (or equivalent after extension) operators have the same kernel dimensions. □

Lemma 3.7 in [84] implies that if r is large enough then the kernel dimension of the operator $\widetilde{u}_{\mathbb{T}}I_{\mathbb{T}} + \widetilde{v}_{\mathbb{T}}S_{\mathbb{T}}$ is equal to the rank of the projection P_{r-1}, that is $2(2r-1)$. Observe that if r is replaced by $r+1$ and the number of singular values increases exactly by 2, then a correct r is found. I.e., $k(A_{n,r+1}) = k(A_{n,r}) = 4$ (see [91] for a more detailed explanation). Moreover, we would to know the number $\dim\ker(\mathcal{O})$ provided that we would be able to compute $\Lambda_n \cap [0, c_n]$ where Λ_n is the set of the singular values of $(A_{n,r})$.

4.1.4 Order of convergence of $s_k(A_{n,k})$

In order to compute $\dim\ker\mathcal{O}$, we have to determinate the number of singular values of $A_{n,r}$ tending to zero. This suggests us to investigate the convergence speed of $s_k(A_{n,k})$ to zero. To this end, by use the equivalence relations given in Theorems 2.1 and 2.4, also by Theorem 4.3, we can adapt the results of Section 4 on [85] as follows:

Lemma 4.5. *Let $a_0, a_1, b_0, b_1 \in PC(\mathbb{T})$. If the singular integral operator $\mathcal{O}$ is Fredholm, then*

$$s_k(A_{n,r}) \leq C \max(\|A_{n,r}\varphi_1\|, \ldots, \|A_{n,r}\varphi_l\|, \|W_n A_{n,r} W_n \psi_1\|, \ldots, \|W_n A_{n,r} W_n \psi_m\|)$$

with $k = \dim\ker(\mathcal{O}) + \dim\ker(\widetilde{u}_{\mathbb{T}} I_{\mathbb{T}} + \widetilde{v}_{\mathbb{T}} S_{\mathbb{T}})Q_{r-1}$, where the constant C does not depend on n, and $\{\varphi_i\}_{i=1}^{l}$ and $\{\psi_i\}_{i=1}^{m}$ are some orthonormal bases of $\ker(u_{\mathbb{T}} I_{\mathbb{T}} + v_{\mathbb{T}} S_{\mathbb{T}})$ *and* $\ker(\widetilde{u}_{\mathbb{T}} I_{\mathbb{T}} + \widetilde{v}_{\mathbb{T}} S_{\mathbb{T}})Q_{r-1}$ *respectively.*

Thus, we have to estimate the norms $\|A_{n,r}\varphi\|$ and $\|W_n A_{n,r} W_n \varphi\|$, where $\varphi \in \ker(u_{\mathbb{T}} I_{\mathbb{T}} + v_{\mathbb{T}} S_{\mathbb{T}})$, $\psi \in \ker(\widetilde{u}_{\mathbb{T}} I_{\mathbb{T}} + \widetilde{v}_{\mathbb{T}} S_{\mathbb{T}})Q_{r-1}$, $\|\varphi\| = \|\psi\| = 1$. Such estimates are given in [85]. Here, for the sake of the presentation, we are going to write it. First, we will deal with smooth coefficients $u_{\mathbb{T}}$ and $v_{\mathbb{T}}$. By $\mathcal{H}^s(\mathbb{T}) \subset C(\mathbb{T})$ we denote the *Hölder-Zygmund* space and by $\mathcal{R}(\mathbb{T}) \subset C(\mathbb{T})$ the *algebra of all rational functions* on $\mathbb{T}$. For each continuous function $f \in [C(\mathbb{T})]^{2\times 2}$ we put

$$E_n(f) := \inf_{p \in [\mathcal{R}^n(\mathbb{T})]^{2\times 2}} \|f - p\|_\infty, \quad n \in \mathbb{Z}_+,$$

where $[\mathcal{R}^n(\mathbb{T})]^{2\times 2}$ is the set of all *matrix trigonometric polynomials p on $\mathbb{T}$ of the form* $p(t) = \sum_{k=-n}^{n} p_k t^k$, with $p_n \in \mathbb{C}^{2\times 2}$. Recall that for any $f \in [C(\mathbb{T})]^{2\times 2}$ and $n \in \mathbb{Z}_+$, there is a polynomial $p_n(f) \in [\mathcal{R}^n(\mathbb{T})]^{2\times 2}$ such that $E_n(f) = \|f - p_n(f)\|_\infty$.

In that follows, by $[\alpha n]$ we denotes the *integer part* of αn, $n \in \mathbb{Z}_+$.

Lemma 4.6. *Let $a_0, a_1, b_0, b_1 \in PC(\mathbb{T})$ and let $\alpha \in (0,1)$. If the singular integral operator $\mathcal{O}$ is Fredholm, then*

$$s_k(A_{n,r}) \leq C \max(E_{[\alpha n]}(u_{\mathbb{T}}), E_{[\alpha n]}(v_{\mathbb{T}}), \|Q_{n-[\alpha n]}\varphi_1\|, \ldots, \|Q_{n-[\alpha n]}\varphi_l\|, \|Q_{n-[\alpha n]}\psi_1\|, \ldots, \|Q_{n-[\alpha n]}\psi_m\|)$$

for $\alpha \in (0,1)$ with $k = \dim\ker(\mathcal{O}) + \dim\ker(\widetilde{u}_{\mathbb{T}} I_{\mathbb{T}} + \widetilde{v}_{\mathbb{T}} S_{\mathbb{T}})Q_{r-1}$, where the constant C does not depend on n, and $\{\varphi_i\}_{i=1}^{l}$ and $\{\psi_i\}_{i=1}^{m}$ are some orthonormal bases of $\ker(u_{\mathbb{T}} I_{\mathbb{T}} + v_{\mathbb{T}} S_{\mathbb{T}})$ *and* $\ker(\widetilde{u}_{\mathbb{T}} I_{\mathbb{T}} + \widetilde{v}_{\mathbb{T}} S_{\mathbb{T}})Q_{r-1}$ *respectively.*

Inequality above can be used in order to estimate the convergence speed for a_0, a_1, b_0 and b_1 smooth functions.

Proposition 4.7. *Let $a_0, a_1, b_0, b_1 \in C(\mathbb{T})$ and let the singular integral operator*

$$\mathcal{O} = a_0 I_{\mathbb{T}} + b_0 S_{\mathbb{T}} + a_1 J + b_1 S_{\mathbb{T}} J,$$

be Fredholm. If the functions $u_{\mathbb{T}}, v_{\mathbb{T}}$ given by (2.18) and (2.19) for J as in (1.18), or defined by (2.42) for J as in (1.19), belong to $[\mathcal{H}^s(\mathbb{T})]^{2\times 2}$ for some $s > 0$, then

$$s_k(A_{n,r}) = O(n^{-s}), \quad \text{as } n \to \infty. \tag{4.12}$$

On the other hand, if the functions a_0, a_1, b_0 and b_1 belong to $\mathcal{R}(\mathbb{T})$, then there is a $\rho > 0$ such that

$$s_k(A_{n,r}) = O(e^{-\rho n}), \quad \text{as } n \to \infty. \tag{4.13}$$

For the more general case where non smoothness conditions are imposed over the coefficients a_0, a_1, b_0 and b_1, similar estimates to (4.12) and (4.13) can be obtained. For this case, the equivalence relation between the operator $\mathcal{O}$ and the Toeplitz operator $\mathcal{T}_{\psi_{\mathbb{T}}}$, with $\psi_{\mathbb{T}} = (u_{\mathbb{T}} - v_{\mathbb{T}})^{-1}(u_{\mathbb{T}} + v_{\mathbb{T}})$ (see, Corollaries 2.2 and 2.5 for the corresponding cases) allows us to use the results of Section 2 on [85], in particular Theorem 2.2, which gives the estimates (4.12) and (4.13) for truncate Toeplitz matrices $A_{n,r} := \mathcal{T}_{n,r}(\psi_{\mathbb{T}})$.

4.1.5 Example

Example 4.1. Here we present an example illustrating, for smooth coefficients, the applicability of Theorem 4.4. Let us consider the Example 3.7. I.e., let the operator $\mathcal{O}$ as in (1.21) with reflection operator J defined in (1.18) and coefficients given by

$$\begin{aligned}
a_0(t) =& \frac{1}{2}[t^{2(s-1)} + t^{-2} + t^{-2s}],\\
a_1(t) =& \frac{1}{2}[-t^{2(s-1)} - t^{-2} + t^{-2s}],\\
b_0(t) =& \frac{t^{-2s}}{2t^{2m}+1}\left(\frac{1}{2}(2t^{2m}-1) + \frac{2t^{2m} + 3t^{2(-n-\alpha-1/2)}}{3t^{-2n}+1}\right)\\
&+ \frac{1}{2}\frac{3t^{-2n}-1}{3t^{-2n}+1}(t^{2(s-1)} + t^{-2}),\\
b_1(t) =& \frac{t^{-2s}}{2t^{2m}+1}\left(\frac{1}{2}(2t^{2m}-1) - \frac{2t^{2m} + 3t^{2(-n-\alpha-1/2)}}{3t^{-2n}+1}\right)\\
&- \frac{1}{2}\frac{3t^{-2n}-1}{3t^{-2n}+1}(t^{2(s-1)} + t^{-2}),
\end{aligned}$$

with $m, n, s \in 2\mathbb{Z}$ and $\alpha = \frac{4k-1}{2}$, $k \in \mathbb{Z}$. From Theorem 2.1 we have that $\mathcal{O}$ is equivalent to the operator $\mathfrak{D}_{\mathbb{T}}$ with coefficients $\upsilon_{\mathbb{T}}$ and $\vartheta_{\mathbb{T}}$ given by

$$\upsilon_{\mathbb{T}}(t) = \begin{pmatrix} t^{-s} & 0 \\ 0 & t^{s-1} + t^{-1} \end{pmatrix} \text{ and } \vartheta_{\mathbb{T}}(t) = \begin{pmatrix} t^{-s}\frac{2t^m-1}{2t^m+1} & \frac{t^{-s}(4t^{m+1/2}+6t^{-n-\alpha})}{(2t^m+1)(3t^{-n}+1)} \\ 0 & \frac{3t^{-n}-1}{3t^{-n}+1}(t^{s-1}+t^{-1}) \end{pmatrix}.$$

To perform our computations, as in [85, 90], instead of the operators $A_{n,r}$ defined in (4.7) we are going to consider the following operators which have the same singular values that $A_{n,r}$:

$$B_{n,r} := F_{2n+1}A_{n,r}F_{2n+1}^{-1} = (u_{\mathbb{T}}(t_j)\delta_{j,k})_{j,k=0}^{2n} + (\upsilon_{\mathbb{T}}(t_j)\delta_{j,k})_{j,k=0}^{2n}F_{2n+1}Q_{n,r}F_{2n+1}^{-1}$$

where $\delta_{j,k}$ is the Kronecker symbol and F_{2n+1} (with inverses F_{2n+1}^{-1}) are the $2(2n+1) \times 2(2n+1)$ matrices (I_2 is the identity 2×2 matrix)

$$F_{2n+1} := \left(\frac{1}{\sqrt{2n+1}}e^{\frac{2i\pi ij}{2n+1}}I_2\right)_{i,j=0}^{2n}, \quad F_{2n+1}^{-1} := \left(\frac{1}{\sqrt{2n+1}}e^{-\frac{2i\pi ij}{2n+1}}I_2\right)_{i,j=0}^{2n}.$$

With these matrices we rewrite $A_{n,r}$ with respect to the standard basis $\operatorname{Im} P_n$ as

$$A_{n,r} = F_{2n+1}^{-1}(u_{\mathbb{T}}(t_j)\delta_{j,k})_{j,k=0}^{2n}F_{2n+1} + F_{2n+1}^{-1}(\upsilon_{\mathbb{T}}(t_j)\delta_{j,k})_{j,k=0}^{2n}F_{2n+1}Q_{n,r},$$

here

$$Q_{n,r} = \operatorname{diag}(\underbrace{0I_2,\ldots,0I_2}_{n+1}, \underbrace{I_2,\ldots,I_2}_{n-\max(0,r-1)}, \underbrace{0I_2,\ldots,0I_2}_{\max(0,r-1)}).$$

From Corollary 2.2 we know that $\mathcal{O}$ is equivalent to the Toeplitz operator $\mathcal{T}_{\psi_{\mathbb{T}}}$ with

$$\psi_{\mathbb{T}}(t) = (\upsilon_{\mathbb{T}}(t) - \vartheta_{\mathbb{T}}(t))^{-1}(\upsilon_{\mathbb{T}}(t) + \vartheta_{\mathbb{T}}(t)) = \begin{pmatrix} 2t^m & 2t^{m+1/2} + 3t^{-n-\alpha} \\ 0 & 3t^{-n} \end{pmatrix}.$$

If $\alpha > 0$, $\psi_{\mathbb{T}}$ admits a (right) generalized factorization in L^p

$$\psi_{\mathbb{T}}(t) = \begin{pmatrix} 2 & t^{-\alpha} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t^m & 0 \\ 0 & t^{-n} \end{pmatrix}\begin{pmatrix} 1 & t^{1/2} \\ 0 & 3 \end{pmatrix}.$$

From the Simonenko's Theorem we get that

$$\dim\ker\mathcal{T}_{\psi} = \sum_{j\in\{m,-n\}} \max(0,-j).$$

Notice that for $m, n \geq 0$, $\widetilde{\psi}_{\mathbb{T}}(t) = \psi_{\mathbb{T}}\left(\frac{1}{t}\right)$ admits a right Wiener-Hopf factorization

$$\widetilde{\psi}_{\mathbb{T}}(t) =$$
$$\begin{pmatrix} 2t^{-m} & 2t^{-m-1/2} + 3t^{n+\alpha} \\ 0 & 3t^{n} \end{pmatrix} = \begin{pmatrix} 2 & \frac{2}{3}t^{h} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-m} & 0 \\ 0 & t^{n} \end{pmatrix} \begin{pmatrix} 1 & \frac{3}{2}t^{g} \\ 0 & 3 \end{pmatrix}$$

with $g = m+n+\alpha$ and $h = -m-n-1/2$. Therefore, $\dim\ker(\widetilde{u}_{\mathbb{T}} I_{\mathbb{T}} + \widetilde{v}_{\mathbb{T}} S_{\mathbb{T}}) = \dim\ker \mathcal{T}_{\widetilde{\psi}_{\mathbb{T}}} = m$. Thus, these facts give us the value of $k(A_{n,r})$ on Theorem 4.4, which is $k = m + n$. By considering $m = 2$, $n = 0$ and $\alpha = \frac{7}{2}$, the following figures show that in fact, $A_{n,r}$ have the 2-splitting property.

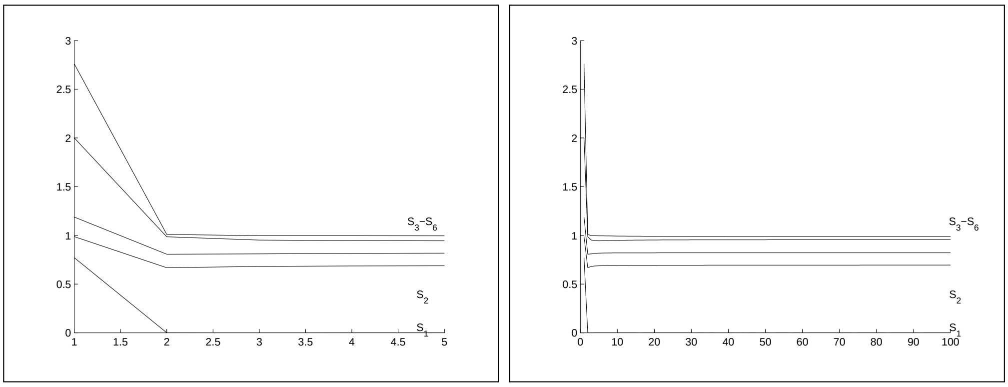


Figure 4.1: The behavior of the first 6 singular values of $A_{n,0}$ ($n = 5$ and $n = 100$).

4.2 The Moore-Penrose invertibility of $\mathcal{O}$

In the case when equation (4.1) is solvable, in general it is not uniquely solvable (also for our operator $\mathcal{O}$). In Hilbert spaces a distinguished generalized solution of (4.1) –the *least square solution*– can be obtained as follows: among all x in a Hilbert space $\mathcal{H}$ which minimize $\|Ax - y\|$ choose that one with minimal $\|x\|$. The Moore-Penrose inverse A^+ of A is such that the least square solution of $Ax = y$ is given by $x = A^+ y$.

In more detail, an operator $A \in \mathcal{L}(\mathcal{H})$ is said to be *Moore-Penrose invertible* if there is an operator $B \in \mathcal{L}(\mathcal{H})$ such that

$$ABA = A, \quad BAB = B, \quad (AB)^* = AB, \quad (BA)^* = BA.$$

If such an operator B exists, then it is unique and we denote it by A^+. It is also well known that an operator is Moore-Penrose invertible if and only if its range is closed (normally solvable). In addition, note that if A is invertible then A^{-1} coincides with A^+.

The following results about Moore-Penrose invertibility are well-known.

Proposition 4.8 (see, [91]). *Let $P_M^{\mathcal{H}}$ denote the orthogonal projection onto the closed subspace $M \subset \mathcal{H}$. The following statements are equivalent:*

(i) *The operator $A \in \mathcal{L}(\mathcal{H})$ is Moore-Penrose invertible.*

(ii) *The operator $AA^* + P_{\ker A}^{\mathcal{H}}$ is invertible.*

(iii) *The operator $A^*A + P_{\ker A^*}^{\mathcal{H}}$ is invertible.*

Moreover, if one of the above conditions is fulfilled then

$$A^+ = (A^*A + P_{\ker A}^{\mathcal{H}})^{-1}A^* = A^*(AA^* + P_{\ker A^*}^{\mathcal{H}})^{-1}.$$

Moore-Penrose invertibility can be defined for elements in a C^*-algebra.

Proposition 4.9 (see, [43, 91]). (i) *An element A of a C^*-algebra with identity is Moore-Penrose invertible if and only if the element AA^* is invertible or if 0 is an isolated point of the spectrum (denoted by sp) of A^*A. If this condition is fulfilled, then $\|A^+\| = \min\{\mathrm{sp}(AA^* \setminus \{0\})\}$.*

(ii) *C^*-subalgebras of C^*-algebras with identity are inverse closed with respect to Moore-Penrose invertibility.*

A sequence of operators (A_n) satisfying $A_nP_n \longrightarrow A$ and $A_n^*P_n \longrightarrow A^*$ is said to be Moore-Penrose stable if

$$\sup_{n\geq 1} \|A_n^+\| < \infty.$$

Recall that A_n^+ exists for all n because $\dim \mathrm{Im}\, P_n < \infty$. Theorem 2.12 in [43] states that if (A_n) is Moore-Penrose stable, then A is Moore-Penrose invertible and $A_n^+ \longrightarrow A^+$, strongly as $n \longrightarrow \infty$.

We will apply these results to the C^*-algebra $\mathcal{F}^{T_2}$ given in Subsection 4.1.2 and to some C^*-subalgebras of it. In particular, we are going to study the Moore-Penrose stability of the Fredholm sequence $(A_n) := (A_{n,0})$ of the operators in (4.7) to the case where $r = 0$. I.e.,

$$A_n = L_n(aI_{\mathbb{T}} + bS_{\mathbb{T}})P_n.$$

Proposition 4.10 (Cf. Proposition 6.9 in [43]). *Let $a, b \in [PC(\mathbb{T})]^{2\times 2}$ and suppose that $(L_n(aI_{\mathbb{T}}+bS_{\mathbb{T}})P_n)$ is a Fredholm sequence (equivalently, suppose $aI_{\mathbb{T}}+bS_{\mathbb{T}}$ and $\widetilde{a}I_{\mathbb{T}}+\widetilde{b}S_{\mathbb{T}}$ to be Fredholm operators). If $\ker(aI_{\mathbb{T}}+bS_{\mathbb{T}}) \subseteq \operatorname{Im} P_{n_0}$ and $\ker(\widetilde{a}I_{\mathbb{T}}+\widetilde{b}S_{\mathbb{T}}) \subseteq \operatorname{Im} P_{n_0}$ for a certain n_0, then*

$$P^{\operatorname{Im} P_n}_{\ker(L_n(aI_{\mathbb{T}}+bS_{\mathbb{T}})P_n)} = P^{[L^2(\mathbb{T},w)]^2}_{\ker(aI_{\mathbb{T}}+bS_{\mathbb{T}})} + W_n P^{[L^2(\mathbb{T},w)]^2}_{\ker(\widetilde{a}I_{\mathbb{T}}+\widetilde{b}S_{\mathbb{T}})} W_n$$

for all sufficiently large n.

From this result, the connection between the k-splitting property and the Moore-Penrose stability is clear:

$$\dim\ker A_n = \dim\ker W^1(A_n) + \dim\ker W^2(A_n).$$

The above proposition implies that (A_n) is a Moore-Penrose stable sequence, and from Proposition 6.5 in [43] we have that the sequence

$$A_n^* A_n + P^{\operatorname{Im} P_n}_{\ker(L_n(aI_{\mathbb{T}}+bS_{\mathbb{T}})P_n)}$$

is stable and the sequence

$$B_n := \left(A_n A_n^* + P^{\operatorname{Im} P_n}_{\ker(L_n(aI_{\mathbb{T}}+bS_{\mathbb{T}})P_n)}\right)^{-1} A_n^*, \quad \text{for all sufficiently large } n,$$

is the Moore-Penrose inverse A_n.

We are now in conditions to provide the explicit Moore-Penrose inverse of the operator $\mathcal{O}$ defined on (1.21) with the Carleman shift operator J as in (1.18) or in (1.19).

Theorem 4.11. *Let us suppose $\mathcal{O}$ to be Fredholm. Moreover, assume that for a certain n_0, $\ker(\mathcal{D}_{\mathbb{T}}) \subseteq \operatorname{Im} P_{n_0}$ and $\ker(\widetilde{\mathcal{D}}_{\mathbb{T}}) \subseteq \operatorname{Im} P_{n_0}$, where the operator $\mathcal{D}_{\mathbb{T}}$ is given as in Theorem* 2.1 *in the case of J to be the shift operator* (1.18) *and as in Theorem* 2.4 *for J in* (1.19) *with, in each case, $\widetilde{\mathcal{D}}_{\mathbb{T}} = \widetilde{u}_{\mathbb{T}}I_{\mathbb{T}} + \widetilde{v}_{\mathbb{T}}S_{\mathbb{T}}$, where for a function $a \in [PC(\mathbb{T})]^{2\times 2}$ we have $\widetilde{a}(t) = a\left(\frac{1}{t}\right)$, $t \in \mathbb{T}$. Then, the operator $\mathcal{O}$ is Moore-Penrose invertible by $\mathcal{O}^+$, where:*

(1) *For the shift operator $(J\varphi)(t) = \varphi(-t)$,*

$$\mathcal{O}^+ = MKGN\left[\left(\mathfrak{D}_{\mathbb{T}}^*\mathfrak{D}_{\mathbb{T}} + P^{[L^2(\mathbb{T},w)]^2}_{\ker\mathfrak{D}_{\mathbb{T}}}\right)^{-1}\mathfrak{D}_{\mathbb{T}}^*\right] N^{-1}G^{-1}K^{-1}M^{-1},$$

with $\mathcal{O}^+ \in \mathcal{L}(L^2(\mathbb{T},w))$ and $w \in A_2^e(\mathbb{T})$. We recall that the explicit form of the operators $M^{\pm 1}$, K, $G^{\pm 1}, N^{\pm 1}$ and $\mathfrak{D}_{\mathbb{T}}$ are given in (2.2), (2.3), (2.10), (2.13), (2.16) *and* (2.17)*;*

(2) *In the case of the shift operator* $(J\varphi)(t) = \frac{1}{t}\varphi\left(\frac{1}{t}\right)$*, we have*

$$\mathcal{O}^{+} = B^{-1}M_{\mathbb{R}_+}KR_{\mathbb{R}_+}N_{\mathbb{R}_+}\,\mathrm{Rest}_{|_{[L^2(\mathbb{R}_+,|x|^{-1/4})]}}\,B_2\left[\left(\mathcal{D}_{\mathbb{T}}^{*}\mathcal{D}_{\mathbb{T}} + P_{\ker\mathcal{D}_{\mathbb{T}}}^{[L^2(\mathbb{T},\gamma)]^2}\right)^{-1}\right.$$
$$\left.\mathcal{D}_{\mathbb{T}}^{*}\right]B_2^{-1}N_{\mathbb{R}_+}^{-1}K^{-1}M_{\mathbb{R}_+}^{-1}B,$$

with $\mathcal{O}^{+} \in \mathcal{L}(L^2(\mathbb{T}))$ *and* γ *is the weight* $\gamma(t) = \left|i\frac{1+t}{1-t}\right|^{-1/4}$*. The explicit form of the operators* $B^{\pm 1}, M_{\mathbb{R}_+}^{\pm 1}, N_{\mathbb{R}_+}^{\pm 1}, K, R_{\mathbb{R}_+},\ B_2^{\pm 1}$ *and* $\mathcal{D}_{\mathbb{T}}$ *are given in* (2.23)–(2.24)*,* (2.26)–(2.31)*,* (2.39) *and* (2.40)*.*

Proof. Since $\mathcal{O}$ is a Fredholm operator, then $\mathcal{O}$ is a Moore-Penrose invertible operator. Also, from Theorem 2.1 (Theorem 2.4) we have that $\mathcal{O}$ is equivalent (equivalent after extension) to the operator $\mathcal{D}_{\mathbb{T}}$ in (4.11) with coefficients $u_{\mathbb{T}}$ and $v_{\mathbb{T}}$ depending of the shift J.

On the other hand, we know that $(A_n) = (L_n(u_{\mathbb{T}}I_{\mathbb{T}} + v_{\mathbb{T}}S_{\mathbb{T}})P_n)$ converge strongly to $\mathcal{D}_{\mathbb{T}}$, as $n \longrightarrow \infty$. Also, the hypothesis that $\ker(\mathcal{D}_{\mathbb{T}}) \subseteq \mathrm{Im}\, P_{n_0}$ and $\ker(\widetilde{\mathcal{D}}_{\mathbb{T}}) \subseteq \mathrm{Im}\, P_{n_0}$ for a certain n_0, allow us to apply Proposition 4.10. Thus, we have that (A_n) is a Moore-Penrose stable sequence with Moore-Penrose inverse $B_n := (A_nA_n^{*} + P_{\ker(L_n(aI_{\mathbb{T}}+bS_{\mathbb{T}})P_n)}^{\mathrm{Im}\, P_n})^{-1}A_n^{*}$. Moreover, $B_n \longrightarrow \mathcal{D}_{\mathbb{T}}^{+}$.

The explicit form of $\mathcal{O}^{+}$ is obtained from B_n, as $n \longrightarrow \infty$, and from the operators given in Theorem 2.1, for the shift operator J given on (1.18), and Theorem 2.4 for J defined on (1.19) (being fundamental in this last step to have in a complete explicit form the corresponding operator relations). □

4.3 Bounds for the kernel dimensions of $\mathcal{O}$ with preserving orientation shift

In the existing literature some estimates for the kernel dimension of certain singular integral operators with preserving orientation non-Carleman shift [5, 56, 57] have already been presented. In addition, the Fredholm property of these operators is typically based on certain invertibility criteria of corresponding functional operators which are *associated* to the initial operators [55]. In particular, this leads to the fact that such estimates depend on that criteria. Moreover, the fixed points of the shift also play a central role in obtaining the mentioned criteria. In view of this, the Carleman shift case cannot be considered directly (since orientation-preserving Carleman shifts do not have fixed points; see Lemma 1.6).

In the present section we establish analogous results to those given in [5] and [56] but in view of obtaining an estimate to the kernel dimensions of a class of singular integral operators with orientation-preserving Carleman shifts.

In more detail, we are going to consider the following class of singular integral operators with shift

$$\mathcal{O} = a_0 I_{\mathbb{T}} + a_1 S_{\mathbb{T}} + b_0 J + b_1 S_{\mathbb{T}} J \tag{4.14}$$

defined in the classic Lebesgue space $L^p(\mathbb{T})$. The coefficients a_0, a_1, b_0, b_1 belong to $C(\mathbb{T})$ and J is a preserving orientation and commutative weighted Carleman shift operator

$$(J\phi)(t) = \vartheta(t)\phi(\alpha(t)), \quad t \in \mathbb{T},$$

induced by α and a complex-valued function ϑ on $\mathbb{T}$.

4.3.1 Equivalence after extension procedure

Considering the plan briefly mentioned, we will construct two operators which will be related to the operator (4.14) in the just presented sense, and from which we will compute estimates for their kernels. Let us define

$$x_1 = b_1 + b_0\,, \quad x_2 = a_0 + a_1\,, \quad y_1 = b_0 - b_1\,, \quad y_2 = a_0 - a_1. \tag{4.15}$$

Using these, the operator $\mathcal{O}$ given in (4.14) is written in the form

$$\mathcal{O} = \sum_{i=1}^{2} x_i J^i P_+ + \sum_{i=1}^{2} y_i J^i P_- : L^p(\mathbb{T}) \longrightarrow L^p(\mathbb{T}),$$

where $P_\pm$ are the Riesz projections.

Moreover, assuming that $y_1 \neq y_2$, it turns out that the operator $\mathcal{O}$ is equivalent to the new operator

$$\mathcal{T} = \frac{1}{y_2^2 - y_1^2}(y_2 I - y_1 J)\mathcal{O} = -\sum_{i=1}^{2} z_i J^i P_+ + P_- : L^p(\mathbb{T}) \longrightarrow L^p(\mathbb{T}),$$

where

$$z_1 := \frac{y_1 x_2 - y_2 x_1}{y_2^2 - y_1^2}\,, \quad z_2 := \frac{y_1 x_1 - y_2 x_2}{y_2^2 - y_1^2} \tag{4.16}$$

are well defined due to the above assumption which in particular implies that $(y_2 I - y_1 J)/(y_2^2 - y_1^2) : L^p(\mathbb{T}) \longrightarrow L^p(\mathbb{T})$ is an invertible operator with inverse being given by $y_2 I + y_1 J : L^p(\mathbb{T}) \longrightarrow L^p(\mathbb{T})$.

Consider now the matrix operator

$$\mathrm{T} : \mathcal{U}P_+ + P_- : [L^p(\mathbb{T})]^2 \longrightarrow [L^p(\mathbb{T})]^2 \tag{4.17}$$

with $\mathcal{U} = I_{\mathbb{T}} - aJ$, and

$$a(t) := \begin{pmatrix} z_1(t) & z_2(t) + 1 \\ 1 & 0 \end{pmatrix}. \tag{4.18}$$

Note that the operator T has the following matricial form

$$\mathrm{T} = \begin{pmatrix} (I_{\mathbb{T}} - z_1 J)P_+ + P_- & -(z_2 + I_{\mathbb{T}})JP_+ \\ -JP_+ & I_{\mathbb{T}} \end{pmatrix}.$$

Proposition 4.12. *The operator $\mathcal{T}$ is equivalent after extension to* T. *Therefore, the operator $\mathcal{T}$ is a Fredholm operator in $L^p(\mathbb{T})$ if and only if the operator* T *given by* (4.17) *is a Fredholm operator in $[L^p(\mathbb{T})]^2$. Moreover, it holds* $\dim\ker\mathcal{T} = \dim\ker\mathrm{T}$ *and* $\dim\operatorname{coker}\mathcal{T} = \dim\operatorname{coker}\mathrm{T}$.

Proof. Bearing in mind that the Fredholm property of a bounded linear operator T is preserved under the multiplication by invertible operators, as well as the defect numbers $\dim\ker\mathrm{T}$ and $\dim\operatorname{coker}\mathrm{T}$, we will multiply T on the right by the invertible operator

$$F = \begin{pmatrix} 0 & I_{\mathbb{T}} \\ I_{\mathbb{T}} & JP_+ \end{pmatrix}.$$

Using the fact that $JP_+ = P_+J$, this leads us to the following identity

$$\begin{aligned} &\mathrm{T}F = \\ &\begin{pmatrix} (I_{\mathbb{T}} - z_1 J)P_+ + P_- & -(z_2 + I_{\mathbb{T}})JP_+ \\ -JP_+ & I_{\mathbb{T}} \end{pmatrix} \begin{pmatrix} 0 & I_{\mathbb{T}} \\ I_{\mathbb{T}} & JP_+ \end{pmatrix} = \\ &\begin{pmatrix} -(z_2 + I_{\mathbb{T}})JP_+ & \mathcal{T} \\ I_{\mathbb{T}} & 0 \end{pmatrix} = \begin{pmatrix} I_{\mathbb{T}} & -(z_2 + I_{\mathbb{T}})JP_+ \\ 0 & I_{\mathbb{T}} \end{pmatrix} \begin{pmatrix} \mathcal{T} & 0 \\ 0 & I_{\mathbb{T}} \end{pmatrix} \begin{pmatrix} 0 & I_{\mathbb{T}} \\ I_{\mathbb{T}} & 0 \end{pmatrix}. \end{aligned}$$

Thus, the just obtained identity shows an equivalence after extension relation between $\mathcal{T}$ and T. Consequently, the operator T is Fredholm if and only if the operator $\mathcal{T}$ is Fredholm and their defect numbers coincide. □

4.3.2 Auxiliary polynomials and operators

To achieve our main goal we will first obtain an estimate to the kernel dimension of the operator T given by (4.17). The question about the Fredholm property of the operator T is narrowed down to the question of the continuous invertibility of the functional operator $\mathcal{U}$ given above. In the scalar case, the necessary and sufficient condition for the operator $I_{\mathbb{T}} - aJ$ to be invertible is that $1 - a(t)a(\alpha(t)) \neq 0$ (see pp. 42 in [55] for the abstract scheme) or equivalently $\inf_{t\in\mathbb{T}} |1 - a(t)a(\alpha(t))| > 0$ (cf. Theorem 2.1.3 in [55]). In the matricial case it turns into the condition

$$\|a(t)a(\alpha(t))\| < 1 \quad \text{or} \quad \|a(t)a(\alpha(t))\| > 1. \tag{4.19}$$

In what follows, wherever v arises it will denote a polynomial matrix satisfying the conditions

$$P_+ v^{\pm 1} P_+ = v^{\pm 1} P_+. \tag{4.20}$$

The following result is a version of Lemma 2.1 in [5], for the case of a continuous function c and an orientation-preserving Carleman shift α.

Lemma 4.13. *Let α be an orientation-preserving Carleman shift on the unit circle $\mathbb{T}$. For any function $c \in C(\mathbb{T})$, there is a polynomial*

$$v(t) = \prod_{k=1}^{n} (t - \lambda_k), \quad |\lambda_k| > 1, \quad k = \overline{1:n} \tag{4.21}$$

such that the condition

$$\left| \frac{c(t)v(\alpha(t))}{v(t)} \right| < 1 \tag{4.22}$$

holds for any $t \in \mathbb{T}$.

Proof. We will only consider the case $\|c(t)\|_{C(\mathbb{T})} \geq 1$ (since the remaining case is trivial). As in the proof of Lemma 2.1 in [5], we represent the function c in the form $c(t) = c_0(t)b(t)$ with $c_0 \in C(\mathbb{T})$, $\|c_0\|_{C(\mathbb{T})} = \gamma < 1$, and b in this case is a real-valued function on $\mathbb{T}$ such that $\inf_{t\in\mathbb{T}} b(t) > 2$. We will construct a real-valued continuous function f such that

$$f(\alpha(t)) \geq f(t)b(t). \tag{4.23}$$

Let $f(t) = b(\alpha(t)) - b(t)b(\alpha(t))$. Since $\inf_{t\in\mathbb{T}} b(t) > 2$, then $b(\alpha(t))b(t)[2 - b(t)] - b((t) \leq 0$ which is equivalent to condition (4.23).

In addition, let us consider the continuous function χ in $\mathbb{T}$, analytic for $|z| < 1$ and defined by

$$\chi(z) = \exp \frac{1}{2\pi} \int_0^{2\pi} \frac{\tau + z}{\tau - z} \ln(|f(z)|)|d\tau|\,,$$

notice that $\chi(z)$ is well defined since $f(t) < 0$, $t \in \mathbb{T}$, and satisfies the properties:

(a) $|\chi(t)| = |f(t)|$, $t \in \mathbb{T}$;

(b) $|\chi(z)| \neq 0$, $|z| < 1$;

(c) χ can be uniformly approximated on $\mathbb{T}$ by a polynomial of a finite degree with any prescribed exactness ε and all the zeros of this polynomial lie outside of the unit circle $\mathbb{T}$ (see [5] and references therein).

Let υ be a polynomial of the above mentioned type and

$$|\chi(t) - \upsilon(t)| < \varepsilon. \tag{4.24}$$

Then,

$$\|c_0(t)b(t)\| \leq \|c_0(t)\|\|b(t)\| = \gamma\|b(t)\|. \tag{4.25}$$

Taking into account (4.25), (4.23), (4.24) and the condition (a) above, we can estimate the norm of the function $[c(t)\upsilon(\alpha(t))]/\upsilon(t)$ in the space $C(\mathbb{T})$. Indeed, we get

$$\begin{aligned}\left\|\frac{c(t)\upsilon(\alpha(t))}{\upsilon(t)}\right\| \leq &\gamma\left\|\frac{b(t)\upsilon(\alpha(t))}{\upsilon(t)}\right\| \leq \gamma\left\|\frac{b(t)f(t)}{\upsilon(t)}\right\|\left\|\frac{\upsilon(\alpha(t))}{f(t)}\right\| \\ \leq &\gamma\left\|\frac{f(\alpha(t))}{\upsilon(t)}\right\|\left\|\frac{\upsilon(\alpha(t))}{f(t)}\right\| = \gamma\left\|\frac{f(t)}{\upsilon(t)}\right\|\left\|\frac{\upsilon(\alpha(t))}{f(t)}\right\|. \end{aligned} \tag{4.26}$$

Moreover, we have

$$\left\|\frac{f(t)}{\upsilon(t)}\right\| = \left\|\frac{\chi(t)}{\upsilon(t)}\right\| = \left\|\frac{\chi(t)-\upsilon(t)}{\upsilon(t)}+1\right\| \leq \frac{\varepsilon}{\|\upsilon(t)\|}+1 < \frac{\varepsilon}{\widetilde{f}-\varepsilon}+1 = \frac{\widetilde{f}}{\widetilde{f}-\varepsilon}, \tag{4.27}$$

and

$$\left\|\frac{\upsilon(t)}{f(t)}\right\| = \left\|\frac{\upsilon(t)-\chi(t)}{\chi(t)}+1\right\| < \frac{\varepsilon+\widetilde{f}}{\widetilde{f}} \tag{4.28}$$

where the notation $\widetilde{f} = \inf(|f|)$ is being used.

Putting together (4.27), (4.28) and (4.26), we derive

$$\left\|\frac{c(t)\upsilon(\alpha(t))}{\upsilon(t)}\right\| \leq \gamma\frac{\widetilde{f}+\varepsilon}{\widetilde{f}-\varepsilon}.$$

In order to obtain inequality (4.22), we must show that the right hand part of inequality above is less to 1. This is obtained by considering $\varepsilon < \widetilde{f}\frac{1-\gamma}{1+\gamma}$. $\square$

Proposition 4.14. *For any continuous matrix function $a \in [C(\mathbb{T})]^{2\times 2}$ and an orientation-preserving Carleman shift α satisfying condition (4.19), there is an induced matrix norm $\|\cdot\|_0$ and a polynomial matrix υ satisfying the condition*

$$\max_{t\in\mathbb{T}} \|\upsilon^{-1}(t)a(t)\upsilon(\alpha(t))\|_0 < 1. \tag{4.29}$$

Proof. The proof of this proposition follows in a similarly way to the proof of Lemma 4.13, by considering $\upsilon = rI_2$, where r is as in (4.21) and I_2 is the identity 2×2 matrix. □

The next result corresponds to Proposition 2.2 in [56] for the case where J is commutative preserving-orientation weighted Carleman shift operator. Here, for the sake of the completeness, we will exhibit its proof.

Proposition 4.15. *Let*

$$\begin{aligned} N =& I - cJP_+ \\ R =& \upsilon P_- + (I - cJ)P_+ \\ M =& I - \upsilon P_+ \upsilon^{-1} P_- N^{-1} \end{aligned}$$

where $c \in [C(\mathbb{T})]^{2\times 2}$, N is an invertible operator, υ is a polynomial 2×2 matrix satisfying the condition (4.20) and

$$l := \sum_{i=1}^{m} \max_{j=\overline{1:m}} l_{i,j},$$

with $l_{i,j}$ being the degree of the element υ_{ij} of the polynomial matrix υ. Then, the relations

$$\dim\ker R = \dim\ker M \le l \tag{4.30}$$

hold.

Proof. Let us consider the invertible operator $K = (P_+ + P_-\upsilon^{-1}P_-)N^{-1}$. Note that

$$\begin{aligned} RK =& (\upsilon P_- + (I - cJ)P_+)(P_+ + P_-\upsilon^{-1}P_-)N^{-1} \\ =& (\upsilon P_-\upsilon^{-1}P_- + P_+ - cJP_+)N^{-1} \\ =& (-\upsilon P_+\upsilon^{-1}P_- + I - cJP_+)N^{-1} = M. \end{aligned}$$

Thus, $\dim\ker R = \dim\ker M$. Further, we have

$$\begin{aligned} p \in \ker M \Longleftrightarrow & p = \upsilon P_+\upsilon^{-1}P_-N^{-1}p \\ \Longleftrightarrow & p = (P_+\upsilon - P_+\upsilon P_-)\upsilon^{-1}P_-N^{-1}p \\ \Longleftrightarrow & p = -P_+\upsilon P_-(\upsilon^{-1}P_-N^{-1}p) \end{aligned}$$

which means that p belongs to the image of the finite dimension operator $P_+ \upsilon P_-$. Since υ is a polynomial matrix with degree at most l, then $\dim \operatorname{Im} P_+ \upsilon P_- \leq l$. So $\dim \ker M \leq l$ and therefore (4.30) holds. □

4.3.3 An estimate for the kernel dimension of the operator $\mathcal{O}$

The following result presents an upper bound for $\dim \ker \mathcal{O}$.

Theorem 4.16. *For the initial singular integral operator $\mathcal{O} = a_0 I_{\mathbb{T}} + a_1 S_{\mathbb{T}} + b_0 J + b_1 S_{\mathbb{T}} J$ with orientation-preserving Carleman shift, acting between $L^p(\mathbb{T})$ spaces, and with coefficients $a_0, b_0, a_1, b_1 \in C(\mathbb{T})$ such that the matrix a is defined by* (4.18) *with entries identified in* (4.16) *and satisfying condition* (4.19)*, it holds the estimate*

$$\dim \ker \mathcal{O} \leq l \tag{4.31}$$

where l is defined in Proposition 4.15 (*and depends on the* 2×2 *polynomial matrix υ which is satisfying conditions* (4.20) *and* (4.29)).

Proof. From the equivalence relation between the operators $\mathcal{O}$ and $\mathcal{T}$ we have that $\dim \ker \mathcal{O} = \dim \ker \mathcal{T}$. On the other hand, Proposition 4.12 gives us that $\dim \ker \mathcal{T} = \dim \ker \mathrm{T}$. So, we will estimate the dimension of the kernel of the operator T in view to obtain the claimed estimate.

According to Proposition 4.14, there is an induced matrix norm $\|\cdot\|_0$ and a polynomial matrix υ such that $c(t) = \upsilon^{-1}(t) a(t) \upsilon(\alpha(t))$ satisfies condition (4.29). We will now write the operator T as a product of operators: $\mathrm{T} = I_{\mathbb{T}} - aJP_+ = \upsilon^{-1}(\upsilon P_- + (I_{\mathbb{T}} - aJ)P_+)(P_- + \upsilon P_+)$. Both operators $\upsilon^{-1} I_{\mathbb{T}}$ and $P_- + \upsilon P_+$ are continuously invertible, and so T and $\upsilon P_- + (I_{\mathbb{T}} - aJ)P_+$ are equivalent operators. Therefore, $\dim \ker \mathrm{T} = \dim \ker(\upsilon P_- + (I_{\mathbb{T}} - aJ)P_+) = \dim \ker R$ (for $c = a$). Applying Proposition 4.15, the estimate (4.31) is obtained. □

4.3.4 Example

In this section we would like to exemplify the applicability of the previous results with an example.

Example 4.2. Let us consider the singular integral operator

$$\mathcal{O} = e^{(\cdot)} I_{\mathbb{T}} + \frac{1}{(\cdot) - 3} S_{\mathbb{T}} + (2 - \sqrt{5})((\cdot)^2 + 2)J + (2 - \sqrt{5}) \sinh(\cdot) S_{\mathbb{T}} J,$$

where the orientation-preserving Carleman shift operator J is induced by

$$\alpha(t) = \frac{(2t-1)}{(t-2)} \quad \text{and} \quad \vartheta(t) = -\frac{\sqrt{3}}{(t-2)}.$$

We will identify the coefficients of the operator $\mathcal{O}$ as

$$\begin{aligned} a_0(t) &= e^t, & a_1(t) &= (t-3)^{-1} \\ b_0(t) &= (2-\sqrt{5})(t^2+2), & b_1(t) &= (2-\sqrt{5})\sinh(t). \end{aligned}$$

Now, with these functions we will construct the matrix function a as in equality (4.18). More precisely, using (4.16) and (4.15), we have:

$$a(t) = \begin{pmatrix} z_1(t) & z_2(t)+1 \\ 1 & 0 \end{pmatrix},$$

with

$$\begin{aligned} z_1(t) =& \frac{1}{4}\frac{\frac{2t^2}{t-3} - 2(\sinh(t)+2)e^t}{(e^t - \frac{1}{t-3})^2 - (t^2 - \sinh(t)+2)^2}, \\ z_2(t) =& \frac{1}{4}\frac{t^4 - (\sinh(t)+2)^2 - e^{2t} + \frac{1}{(t-3)^2}}{(e^t - \frac{1}{t-3})^2 - (t^2 - \sinh(t)+2)^2}. \end{aligned}$$

The next step is to verify that condition (4.19) is satisfied. Computing $a(t)a(\alpha(t))$, we explicitly obtain a matrix function

$$a(t)a(\alpha(t)) = \begin{pmatrix} f_1(t) & f_2(t) \\ f_3(t) & f_4(t) \end{pmatrix},$$

where the ranges of the functions f_i $(i = 1, \ldots, 4)$ can be seen in Figure 4.2 and Figure 4.3.

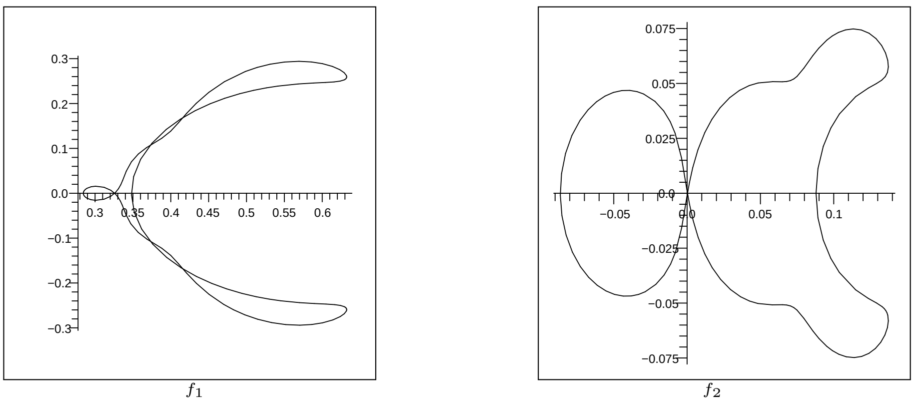


Figure 4.2: The range of the functions f_1 and f_2

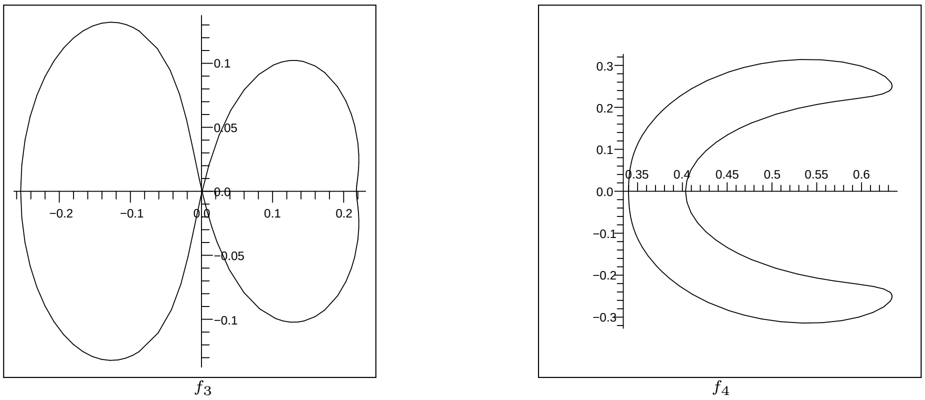


Figure 4.3: The range of the functions f_3 and f_4.

Since we are considering $a(t)a(\alpha(t))$ in the C^*-algebra of the continuous matrix functions on $\mathbb{T}$ (endowed with the maximum norm), then we have that $\|a(t)a(\alpha(t))\|_\infty = 0.6664082833$, i.e., condition (4.19) is verified.

On the other hand, let us introduce the polynomial matrix

$$r(t) := \begin{pmatrix} 1 & \frac{1}{20}t \\ 0 & \frac{11}{10} \end{pmatrix} \tag{4.32}$$

which satisfies condition (4.20). Let $\|\cdot\|_0$ be the induced maximum norm, which allows us compute $\|r^{-1}(t)a(t)r(\alpha(t))\|_0$. As

$$\wp(t) := r^{-1}(t)a(t)r(\alpha(t)) = \begin{pmatrix} z_1(t) - \frac{t}{22} & \frac{z_1(t)(2t-1)}{20(t-2)} - \frac{t(2t-1)}{440(t-2)} + \frac{11(z_2(t)+1)}{10} \\ \frac{10}{11} & \frac{1}{22}\frac{2t-1}{t-2} \end{pmatrix}$$

where the ranges of the functions $\wp_{ij}$ $(i,j = 1,2)$ on the entries of the matrix-valued function $\wp$ can be seen in next the figures.

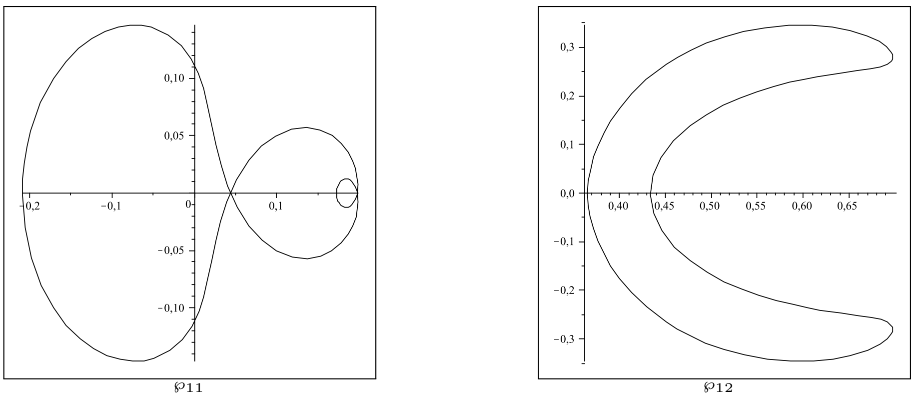


Figure 4.4: The range of the functions $\wp_{11}$ and $\wp_{12}$

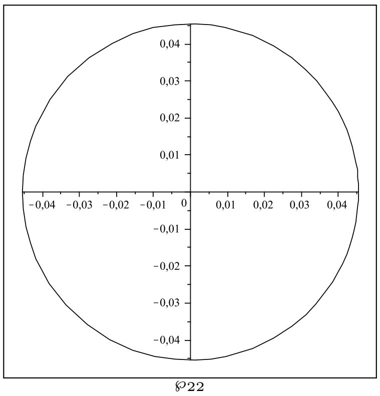


Figure 4.5: The range of the function $\wp_{22}$.

We have that $\|r^{-1}(t)a(t)r(\alpha(t))\|_0 < 1$. Thus, the polynomial matrix r given in (4.32) satisfies condition (4.29). Finally, from Theorem 4.16 we have that $\dim \ker \mathcal{O} \leq 1$.

Chapter 5

A Simple Representation for Singular Integral Operators with Shift

The boundary integral equation methods in the analytic theory of bounded values problems goes back to C. Newmann's pioneering work in 1870. However, only in the last two decades, numerical analysis techniques for the solution of this kind of problems (which become a rather powerful and popular technique in the engineering computations of boundary values problems arising from different fields of applications) had a rapid development. According to S. Prossdorf and B. Silbermann [76] the reason for this delay lies on the fact that boundary integral operators are, in general, neither integral operators of the form of identity plus a compact operator nor of the form of identity plus an operator with small norm, so that the existing standard theories for the numerical analysis of second kind integral equations cannot be applied.

A crucial assumption for modeling applications using generalized boundary integral and integro-differential operators is the so-called strong ellipticity. However, when considering certain approximation methods for solving the equation

$$Ax = y,$$

it is often useful to have a more general concept of strong ellipticity. In particular, the representation

$$A = BDC + K, \tag{5.1}$$

where B, C being, in a sense, simple operators and K is compact. In this chapter, we are going to give conditions in order to obtain a representation similar to representation (5.1) for singular integral operators with shift of the form

$$\text{Đ} := aP_+ + bP_- + cP_+J + dP_-J, \tag{5.2}$$

acting on $L^2(\mathbb{T})$, with coefficients $a, b, c, d \in PC(\mathbb{T})$ and furthermore with the extra action of an anti-commutative weighted Carleman shift operator J. We are always going to assume in this chapter that $a = k_1 c$ and $b = k_2 d$ where $k_1, k_2 \in \mathbb{C} \setminus \{0, 1\}$.

Our main goal is to prove the following representation theorem:

Theorem 5.1. *Let us consider the operator* $\text{Đ} = aP_+ + bP_- + cPJ + dP_-J$ *defined on* $L^2(\mathbb{T})$ *with coefficients* $a, b, c, d \in PC(\mathbb{T})$ *satisfying*

$$\tfrac{a(t+0)}{a(t-0)}\mu + \tfrac{b(t+0)}{b(t-0)}(1-\mu) \notin \mathbb{R}_- := (-\infty, 0], \quad \textit{and}$$

$$\tfrac{c(t+0)}{c(t-0)}\mu + \tfrac{d(t+0)}{d(t-0)}(1-\mu) \notin \mathbb{R}_-, \quad \textit{for} \quad 0 \le \mu \le 1, \quad t \in \mathbb{T}.$$

Then, the operator Đ *admits a representation*

$$\text{Đ} = D(a_1 P + b_1 Q + c_1 PJ + d_1 QJ) + K,$$

where $K \in \mathcal{K}(L^2(\mathbb{T}))$, $\|I - D\| < 1$ *and the functions* $a_1, b_1, c_1, d_1 \in C(\mathbb{T})$ *do not vanish on* $\mathbb{T}$.

To reach our goals, first, in Section 5.1 we will prove a representation theorem for Banach algebras generated by two idempotents and one flip. Afterwards, in Section 5.2, we will present a symbol calculus associated to the operator Đ given in (5.2) which finally will allow us to prove Theorem 5.1.

5.1 Symbol calculus for Banach algebras generated by two idempotents and one flip

For a Banach algebra A with identity e, we will denote by $\operatorname{alg}(a_1, \ldots, a_r)$ the algebra of all finite sums of products of $a_1, \ldots, a_r$ which is dense in A. We will consider the Banach algebra $\mathfrak{A} = \operatorname{alg}(e, p, q, j)$ with the unit e and where

$$p^2 = p, \quad q^2 = q, \quad j^2 = e, \quad jpj = e - p, \quad jqj = e - q. \tag{5.3}$$

In the late sixties, C^*-algebras generated by two idempotents were studied extensively from an operator theory point of view. These results, combined with certain local techniques, lead to a symbol calculus for singular integral operators with piecewise continuous coefficients. The corresponding version for Banach algebras $\mathrm{alg}\,(e,p,q,j)$ with two idempotents p,q and a shift j relating the invertibility in the algebra and used for the study of convolution, Toeplitz, Hankel and singular integral operators with shift was analyzed in the eighties; see for instance, [40, 42, 58, 79, 80, 89]. In order to prove our representation theorem, the main results about algebras generated by two idempotents and a flip are the following ones.

Let $M(\mathfrak{A})$ be the set of the two-sided maximal ideals of $\mathfrak{A}$ and $M_l(\mathfrak{A})$ $(l=1,\ldots,r)$ stand for the set of all maximal ideals M of $\mathfrak{A}$ with $\mathfrak{A}/M \cong \mathbb{C}^{l\times l}$. Considering η_M to be the canonical homomorphism $\mathfrak{A} \longrightarrow \mathfrak{A}/M$, if ζ_M is the isomorphism $\mathfrak{A}/M \longrightarrow \mathbb{C}^{l\times l}$ and if $\nu_M = \zeta_M\eta_M$, then $x \in \mathfrak{A}$ is invertible in $\mathfrak{A}$ (i.e. $x \in \mathcal{G}\mathfrak{A}$) if and only if $\det \nu_M(x) \neq 0$ for all $M \in M(\mathfrak{A})$.

Theorem 5.2. ([79, Theorem 7]) $M(\mathfrak{A}) = M_2(\mathfrak{A})$, *and for each* $M \in M_2(\mathfrak{A})$ *there is an invertible matrix* $E \in \mathbb{C}^{2\times 2}$ *and a complex number* z *both depending on* M *such that*

$$(\mathrm{smb}\, e)(M) = E^{-1}\nu_M(e)E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.4}$$

$$(\mathrm{smb}\, p)(M) = E^{-1}\nu_M(p)E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{5.5}$$

$$(\mathrm{smb}\, q)(M) = E^{-1}\nu_M(q)E = \begin{pmatrix} z & \sqrt{z(1-z)} \\ \sqrt{z(1-z)} & 1-z \end{pmatrix} \tag{5.6}$$

and either

$$(\mathrm{smb}\, j)(M) = E^{-1}\nu_M(j)E = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

or

$$(\mathrm{smb}\, j)(M) = E^{-1}\nu_M(j)E = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Since axioms (5.3) do not determine a unique symbol, we need additional information to overcome this difficulty.

Let $\mathfrak{B} = \mathrm{alg}\,(e,p,q)$, with p and q idempotents and

$$\begin{aligned}
\mathfrak{B}_p &:= \mathrm{alg}\,(p, pqp) \\
\mathfrak{B}_q &:= \mathrm{alg}\,(q, qpq) \\
\mathfrak{B}_{e-p} &:= \mathrm{alg}\,(e-p, (e-p)(e-q)(e-p)) \\
\mathfrak{B}_{p,e-p} &:= \mathrm{alg}\,(e, pqp + (e-p)(e-q)(e-p)).
\end{aligned}$$

From Theorem 8 on [79], the maximal ideal space $M(\mathfrak{A})$ is homeomorphic to $M(\mathfrak{B}_{p,e-p}) \cong \sigma_{\mathfrak{B}_p}(pqp)$ (here we denote by $\sigma_{\mathfrak{B}_p}(pqp)$ the spectrum of pqp in the algebra $\mathfrak{B}_p$) and the symbol $\operatorname{smb} a$ (for $a \in \mathfrak{A}$) is given at $x \in \sigma_{\mathfrak{B}_p}(pqp)$ as (5.4), (5.5), (5.6), and in this case either

$$(\operatorname{smb} j)(x) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \text{or} \quad (\operatorname{smb} j)(x) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Since

$$\operatorname{smb}(ipqjp)(x) = \begin{pmatrix} \pm\sqrt{x(1-x)} & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{i.e.,}$$

the spectrum of $ipqjp$ in $\mathfrak{A}$ is equals either

$$\{y \in \mathbb{C} \,:\, y = \sqrt{x(1-x)},\ x \in \sigma_{\mathfrak{B}_p}(pqp)\}$$

or

$$\{y \in \mathbb{C} \,:\, y = -\sqrt{x(1-x)},\ x \in \sigma_{\mathfrak{B}_p}(pqp)\}$$

where $\sqrt{\ }$ refers to the main branch. If the spectrum $\sigma_{\mathfrak{B}_p}(pqp)$ is given, then the knowledge of only one suitable point of $\sigma_{\mathfrak{A}}(ipqjp)$ would allow to decided which sign (+ or -) is valid and so to make the symbol unique.

As a consequence of the above, the corresponding version for C^*-algebras is given as follows:

Theorem 5.3. ([79, Corollary 3]) *Let $\bar{e}, \bar{p}, \bar{q}, \bar{j} \in C([0,1], \mathbb{C}^{2\times 2})$ be defined by*

$$\bar{e}(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \bar{p}(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{q}(x) = \begin{pmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{pmatrix}$$

and $\bar{j}(x) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, *and let e, p, q, j be self adjoint elements of a certain C^*-algebra $\mathfrak{A}$ which fulfil* (5.3) *and*

$$ipqjp \geq 0 \quad \text{and} \quad \sigma_{\mathfrak{A}}(pqp) = [0,1]. \tag{5.7}$$

Then, the C^*-algebras $alg(e,p,q,j)$ and $alg(\bar{e}, \bar{p}, \bar{q}, \bar{j}) = C([0,1], \mathbb{C}^{2\times 2})$ are isometrically isomorphic and the isomorphism transforms e, p, q, j into $\bar{e}, \bar{p}, \bar{q}, \bar{j}$, respectively. The norm of an element a in the C^*-algebra $C([0,1], \mathbb{C}^{2\times 2})$ is given by the formula

$$\|a\| := \max_{x\in[0,1]} (\max_{i=1,2} |\lambda_i(x)|^{1/2}),$$

where $\lambda_i(x)$ are the eigenvalues of $a(x)a^*(x)$. Here $a^*(t)$ means the adjoint matrix of $a(x)$.

For the C^*-algebra $\mathfrak{A} = \mathrm{alg}\,(e, p, q, j)$ with identity e fulfilling (5.3) and (5.7) we denote by Λ the C^*-subalgebra generated by q and e. It is readily seen that every element $a \in \Lambda$ has the form $a = (a_+ - a_-)q + a_-e$, where a_+ and a_- are complex numbers. Moreover, for such an element $a = (a_+ - a_-)q + a_-e$, we have the equality

$$\|a\| = \max(|a_+|, |a_-|),$$

because a is normal (i.e., $aa^* = a^*a$) and the spectrum of a consists of the points a_+ and a_- only.

Theorem 5.4. *Let $A = ap + b(e-p) + cpj + d(e-p)j \in \mathfrak{A}$ be an element with coefficients $a, b, c, d \in \Lambda$ i.e.,*

$$\begin{array}{ll} a = (a_+ - a_-)q + a_-e, & b = (b_+ - b_-)q + b_-e, \\ c = (c_+ - c_-)q + c_-e, & d = (d_+ - d_-)q + d_-e. \end{array} \tag{5.8}$$

If the coefficients a, b, c and d satisfy

$$\begin{gathered} \left|\arg \tfrac{a_+}{a_-}\right| < \pi, \quad \left|\arg \tfrac{b_+}{b_-}\right| < \pi, \quad \left|\arg \tfrac{c_+}{c_-}\right| < \pi, \quad \left|\arg \tfrac{d_+}{d_-}\right| < \pi, \\ \left|\arg \tfrac{a_+}{a_-} - \arg \tfrac{b_+}{b_-}\right| < \pi, \quad \left|\arg \tfrac{c_+}{c_-} - \arg \tfrac{d_+}{d_-}\right| < \pi \end{gathered} \tag{5.9}$$

or equivalently,

$$\begin{gathered} \tfrac{a_+}{a_-}\mu + \tfrac{b_+}{b_-}(1-\mu) \notin \mathbb{R}_-, \quad \textit{and} \\ \tfrac{c_+}{c_-}\mu + \tfrac{d_+}{d_-}(1-\mu) \notin \mathbb{R}_-, \quad \textit{for} \quad 0 \le \mu \le 1, \end{gathered}$$

then this element can be represented as

$$A = (e+T)B,$$

where $\|T\| < 1$ and $B = \alpha p + \beta(e-p) + \delta pj + \gamma(e-p)j$ is an invertible element with $\alpha, \beta, \delta, \gamma \in \mathbb{C}$.

Proof. First we are going to show that there are complex numbers $\lambda, \kappa, \vartheta, \nu$ such that

$$\|\lambda a - e\| < 1, \quad \|\kappa b - e\| < 1, \quad \|\nu c - e\| < 1, \quad \|\vartheta d - e\| < 1 \tag{5.10}$$

$$(\lambda a_+ - 1)(\kappa b_- - 1)(\nu c_+ - 1)(\vartheta d_- - 1) = (\lambda a_- - 1)(\kappa b_+ - 1)(\nu c_- - 1)(\vartheta d_+ - 1). \tag{5.11}$$

Indeed, if $a_+ = a_- \ (= a)$ then, put $\lambda = a^{-1}$. Obviously $\lambda a - 1 = 0$ and equality (5.11) holds. Since $|\arg \frac{b_+}{b_-}| < \pi$, there is a $\kappa_0 \in \mathbb{C}$ such that $\Re e(\kappa_0 b_\pm) > 0$. Clearly, for sufficiently small $t > 0$ we have $\|(t\kappa_0)b - e\| < 1$. We can see that the same argument is valid to proof that $\|\nu c - e\| < 1$. The cases $b_+ = b_-$, $c_+ = c_-$ and $d_+ = d_-$ can be treated analogously.

Now, let $a_+ \neq a_-$, $b_+ \neq b_-$, $c_+ \neq c_-$ and $d_+ \neq d_-$. For $t \in \mathbb{R}$, putting

$$\begin{aligned}
\lambda^{-1} &= \frac{it(a_+ - a_-) + a_+ + a_-}{2} \\
\kappa^{-1} &= \frac{it(b_+ - b_-) + b_+ + b_-}{2} \\
\nu^{-1} &= \frac{it(c_+ - c_-) + c_+ + c_-}{2} \\
\vartheta^{-1} &= \frac{it(d_+ - d_-) + d_+ + d_-}{2}
\end{aligned}$$

we can check that (5.11) is fulfilled. Some computations show that the inequalities (5.10) will follow once we have proved that

$$|t - i| < \left|t - i\tfrac{a_+ a_-}{a_+ - a_-}\right|, \quad |t - i| < \left|t - i\tfrac{b_+ b_-}{b_+ - b_-}\right|$$

$$|t - i| < \left|t - i\tfrac{c_+ c_-}{c_+ - c_-}\right|, \quad |t - i| < \left|t - i\tfrac{d_+ d_-}{d_+ - d_-}\right|$$

or, what is essentially the same

$$2t\Re e(z_i) < |z_i|^2 - 1, \quad i = 1, \ldots, 4 \tag{5.12}$$

where:

$$z_1 = i\frac{a_+ + a_-}{a_+ - a_-}, \quad z_2 = i\frac{b_+ + b_-}{b_+ - b_-}, \quad z_3 = i\frac{c_+ + c_-}{c_+ - c_-}, \quad z_4 = i\frac{d_+ + d_-}{d_+ - d_-}.$$

From conditions (5.9) we have that $\frac{a_+}{a_-} = \rho_1 e^{2\pi i\alpha_1}$, $\frac{b_+}{b_-} = \rho_2 e^{2\pi i\alpha_2}$, $\frac{c_+}{c_-} = \rho_3 e^{2\pi i\alpha_3}$, $\frac{d_+}{d_-} = \rho_4 e^{2\pi i\alpha_4}$ with $\rho_i > 0$, $|\alpha_1 - \alpha_2| < \frac{1}{2}$, $|\alpha_3 - \alpha_4| < \frac{1}{2}$ for $-\frac{1}{2} < \alpha_i < \frac{1}{2}$ $(i = 1, \ldots, 4)$.

Using the following identities,

$$2\Re e(z_i) = \frac{4\rho_i \sin(2\pi\alpha_i)}{\rho_i^2 - 2\rho_i\cos(2\pi\alpha_i) + 1}, \quad |z_i|^2 - 1 = \frac{4\rho_i\cos(2\pi\alpha_i)}{\rho_i^2 - 2\rho_i\cos(2\pi\alpha_i) + 1}$$

we get the inequalities

$$t\sin(2\pi\alpha_i) < \cos(2\pi\alpha_i), \qquad i = 1, \ldots, 4 \tag{5.13}$$

which are equivalent to those in (5.12). If $\alpha_1\alpha_2\alpha_3\alpha_4 \geq 0$, then (5.13) has clearly a solution. Let $\alpha_1\alpha_2\alpha_3\alpha_4 < 0$. Without loss of generality we can assume that

$$\tfrac{1}{4} < \alpha_1 < \tfrac{1}{2}, \quad -\tfrac{1}{2} < \alpha_2 < -\tfrac{1}{4}, \quad \tfrac{1}{8} < \alpha_3 < \tfrac{1}{4}, \quad 0 < \alpha_4 < \tfrac{1}{8} \tag{5.14}$$

which implies $\frac{1}{2} < \alpha_1 - \alpha_2 < 1, \quad 0 < \alpha_3 - \alpha_4 < \frac{1}{4}$ and

$$0 < \frac{\sin(2\pi(\alpha_1 - \alpha_2))}{\sin(2\pi\alpha_1)\sin(2\pi\alpha_2)} + \frac{\sin(2\pi(\alpha_3 - \alpha_4))}{\sin(2\pi\alpha_3)\sin(2\pi\alpha_4)}.$$

Notice that the inequality above holds because each summand is positive for the values of α_i in the corresponding intervals (5.14). Thus, in this case there exists also a value $t \in \mathbb{R}$ satisfying (5.13). Hence, the relations (5.10) and (5.11) are proved.

Now, we are going to prove the estimate:

$$\|ap + b(e-p) + cpj + d(e-p)j\| \quad \leq \quad 6\max(\|a\|, \|b\|, \|c\|, \|d\|) \tag{5.15}$$

for coefficients given by (5.8).

According to Theorem 5.3, the norm of $ap + b(e-p) + cpj + d(e-p)j$ can be computed throughout its matrix symbol as

$$\|ap + b(e-p) + cpj + d(e-p)j\|^2 = \max_{x\in[0,1]}(\max(\xi_1(x), \xi_2(x)))$$

where $\xi_1(x)$ and $\xi_2(x)$ are the roots of the polynomial

$$\det(B(x)B^*(x) - \xi I_{2\times 2});$$

here $I_{2\times 2}$ is the identity matrix and

$$B(x) = \begin{pmatrix} \Xi_{1,1}(x) & \Xi_{1,2}(x) \\ \Xi_{2,1}(x) & \Xi_{2,2}(x) \end{pmatrix},$$

whose entries are:

$$\begin{aligned}
\Xi_{1,1}(x) &:= a_+x + a_-(1-x) - (d_+ - d_-)\sqrt{x(1-x)}i \\
\Xi_{1,2}(x) &:= (b_+ - b_-)\sqrt{x(1-x)} + (c_+x + c_-(1-x))i \\
\Xi_{2,1}(x) &:= (a_+ - a_-)\sqrt{x(1-x)} - (d_+x + d_-(1-x))i \\
\Xi_{2,2}(x) &:= b_+x + b_-(1-x) + (c_+ - c_-)\sqrt{x(1-x)}i.
\end{aligned}$$

Put $\xi(x) = \max(\xi_1(x), \xi_2(x))$. Since for a complex valued 2×2 matrix A, the following equality holds,

$$\det(A - \lambda I) = \det(A) + \lambda^2 - \lambda\,\mathrm{tr}(A),$$

where $\operatorname{tr}(A)$ is the trace of the matrix A, then some computations give us that

$$\xi(t) = \frac{1}{2}\left(t(x) + \sqrt{t^2(x) - 4d(x)}\right).$$

Here, $t(x) := \operatorname{tr}(B(x)B^*(x))$ and $d(x) := \det(B(x)B^*(x))$. Note that we can rewrite $d(x)$ and $t(x)$ as

$$\begin{aligned}
d(x) =& |\Xi_{1,1}(x)\Xi_{2,2}(x) - \Xi_{1,2}(x)\Xi_{2,1}(x)|^2 \\
t(x) =& |\Xi_{1,1}(x)|^2 + |\Xi_{1,2}(x)|^2 + |\Xi_{2,1}(x)|^2 + |\Xi_{2,2}(x)|^2.
\end{aligned}$$

Or, explicitly as

$$\begin{aligned}
t(x) &= (|a_-|^2 + |b_-|^2 + |c_-^2| + |d_-^2|)(1-x) + (|a_+^2| + |b_+^2| + |c_+^2| + |d_+^2|)x \\
&+ 4(|a_-a_+| + |b_-b_+| + |c_-c_+| + |d_-d_+|)x(1-x) + 2\{2((|a_+d_+| + |b_+c_+|)x \\
&+ (|a_-d_-| + |b_-c_-|)(1-x)) + |a_+d_-| + |a_-d_+| + |b_+c_-| + |b_-c_+|\}\sqrt{x(1-x)}
\end{aligned}$$

and

$$\begin{aligned}
d(t) =& |(2x^2 - x)a_+b_+ - (2x^2 - x)a_+b_- - (x^2 - x)a_-b_+ + (2x^2 - 3x)a_-b_- \\
&- (2x^2 - x)d_+c_+ + (2x^2 - x)d_+c_- + (2x^2 - x)d_-c_+ - (2x^2 - 3x)d_-c_- \\
&+ a_-b_- - d_-c_- + (-a_+c_- + a_-c_+ - d_+b_- + d_-b_+)\sqrt{x(1-x)}i|^2.
\end{aligned}$$

Now, we are going to find an upper bound for $|\xi(x)|$. Assuming that $|a_-|^2 + |b_-|^2 + |c_-|^2 + |d_-|^2 \geq |a_+|^2 + |b_+|^2 + |c_+|^2 + |d_+|^2$ and $|a_-d_-| + |b_-c_-| \geq |a_+d_+| + |b_+c_+|$, we obtain

$$\begin{aligned}
t(x) \leq& |a_-|^2 + |b_-|^2 + |c_-|^2 + |d_-|^2 + 2(|a_+d_-| + |a_-d_+| + |b_-c_+| + |b_+c_-|) \\
&+ 4(|a_-a_+| + |b_-b_+| + |c_-c_+| + |d_-d_+| + |a_-d_-| + |b_-c_-|) \\
\leq& 36(\max(\|a\|^2, \|b\|^2, \|c\|^2, \|d\|^2)). \qquad (5.16)
\end{aligned}$$

Analogously, the same holds true for the remaining cases. On the other hand,

$$\begin{aligned}
d(x) &\geq (|(2x^2 - x)a_+b_+ + (2x^2 - 3x)a_-b_- + a_-b_- + (2x^2 - x)d_+c_- \\
&+ (2x^2 - x)d_-c_+ + (a_-c_+ + d_-b_+)\sqrt{x(1-x)}| - |(2x^2 - x)a_+b_- + d_-c_- \\
&+ (x^2 - x)a_-b_+ + (2x^2 - x)d_+c_+ + (a_+c_- + d_+b_-)\sqrt{x(1-x)}i \\
&+ (2x^2 - 3x)d_-c_-|)^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&- 4d(x) \leq 8|(2x^2 - x)a_+b_+ + (2x^2 - 3x)a_-b_- + a_-b_- + (2x^2 - x)d_+c_- \\
&+ (2x^2 - x)d_-c_+ + (a_-c_+ + d_-b_+)\sqrt{x(1-x)}| \times |(2x^2 - x)a_+b_- + d_-c_- \\
&+ (x^2 - x)a_-b_+ + (2x^2 - x)d_+c_+ + (a_+c_- + d_+b_-)\sqrt{x(1-x)}i \\
&+ (2x^2 - 3x)d_-c_-|.
\end{aligned}$$

Simplifying the last inequality we obtain the following estimate:

$$\begin{aligned}
&-4d(x) \le \frac{1}{2}\,|a_+b_+ + 7a_-b_- + d_+c_- + d_-c_+ + 2(a_-c_+ + d_-b_+)i| \\
&\times |a_+b_- + d_+c_+ + 7d_-c_- + 2(a_-b_+ + (a_+c_- + d_+b_-)i)| \\
&\le 105(\max(\|a\|^2, \|b\|^2, \|c\|^2, \|d\|^2))^2. \qquad (5.17)
\end{aligned}$$

From the estimates (5.16) and (5.17) we have

$$(tr(x))^2 - 4d(x) \le 1401(\max(\|a\|^2, \|b\|^2, \|c\|^2, \|d\|^2))^2,$$

which proves inequality (5.15).

Let us finally consider the element $A = ap + b(e-p) + cpj + d(e-p)j$ with coefficients a, b, c, d satisfying (5.8) and (5.9). We will also assume that $c = \frac{a\nu\kappa^{-1}}{2}$ and $d = \frac{b\lambda^{-1}\vartheta}{2}$. Therefore,

$$\begin{aligned}
&ap + b(e-p) + cpj + d(e-p)j = \left(\frac{\lambda a}{2}p + \frac{\kappa b}{2}(e-p) + \frac{\nu a}{2}pj + \frac{\vartheta b}{2}(e-p)j\right) \\
&\times \left(\lambda^{-1}p + \kappa^{-1}(e-p) + \vartheta^{-1}pj + \nu^{-1}(e-p)j\right) \\
&= 3\left(T + \frac{e}{6}\right)\left(\lambda^{-1}p + \kappa^{-1}(e-p) + \vartheta^{-1}pj + \nu^{-1}(e-p)j\right),
\end{aligned}$$

where $T = \frac{\lambda a - e}{6}p + \frac{\kappa b - e}{6}(e-p) + \frac{\nu a - e}{6}pj + \frac{\vartheta b - e}{6}(e-p)j + \frac{1}{6}j$. The estimate $\|T\| < 1$ is a consequence of (5.10), (5.11), (5.15) and the fact $\|j\| = 1$. Therefore, the theorem is proved.

□

5.2 A symbol calculus for $\mathrm{alg}\,(P_+, J, I_{\mathbb{T}}, PC(\mathbb{T}))$

Since J is an anti-commutative Carleman shift, without loss of generality we will assume that it has 1 and -1 as its fixed points (remember that an anti-commutative Carleman shift is necessarily a reverting orientation shift operator, therefore it has two fixed points; see [55]).

This section is devoted to exhibit the image symbol of the elements on $\mathfrak{C} = \mathrm{alg}\,(P_+, J, I_{\mathbb{T}}, PC(\mathbb{T}))$ in the Calkin algebra $\mathfrak{C}^\pi := \mathrm{alg}\,(P_+, J, PC(\mathbb{T}))/\mathcal{K}(L^2(\mathbb{T}))$. Such images are already known even in a more general setting (see e.g., [79, 81]). Here we will rewrite these results to the algebra $\mathfrak{C}$.

Let $\mathfrak{D} = \mathrm{alg}\,(P_+, \chi_{\mathbb{T}_+}, J, I_{\mathbb{T}})$ the algebra generated by $P_+, \chi_{\mathbb{T}_+}$ and J with identity $I_{\mathbb{T}}$. Note that for this algebra we can check directly that conditions (5.3) holds.

Proposition 5.5. ([79, Proposition 7])

(a) $\sigma_{\mathfrak{D}}(P_+\chi_{\mathbb{T}_+}P_+) = [0,1]$

(b) $iP_+\chi_{\mathbb{T}_+}JP_+ \geq 0$.

Corollary 5.6. ([79, Corollary 4]) *The maximal ideal space of* $\mathfrak{D}$ *is homeomorphic to* $[0,1]$ *and a symbol for the invertibility in* $\mathfrak{D}$ *is given by*

$$(\operatorname{smb} P_+)(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (\operatorname{smb} \chi_{\mathbb{T}_+})(x) = \begin{pmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{pmatrix}$$

$$(\operatorname{smb} J)(x) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \textit{and} \quad (\operatorname{smb} I_{\mathbb{T}})(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now, we are going to extend these results to a symbol calculus of operators belonging to the algebra in $L^2(\mathbb{T})$ generated by $I_{\mathbb{T}}$, P_+, J and piecewise continuous functions: $\operatorname{alg}(P_+, J, I_{\mathbb{T}}, PC(\mathbb{T}))$.

Proposition 5.7. ([76, Proposition 6.13]) *For each* $z \in \mathbb{T}$, *there exists a homomorphism from* $\operatorname{alg}(P_+, PC(\mathbb{T}))$ *onto* $C([0,1], \mathbb{C}^{2\times 2})$ *such that*

(a) *for* $a \in PC(\mathbb{T})$,

$$(\operatorname{smb} a(z)I_{\mathbb{T}})(x) =$$
$$\begin{pmatrix} a(z+0)x + a(z-0)(1-x) & (a(z+0)-a(z-0))\sqrt{x(1-x)} \\ (a(z+0)-a(z-0))\sqrt{x(1-x)} & a(z-0)x + a(z+0)(1-x) \end{pmatrix}.$$

(b) $A \in \operatorname{alg}(P_+, PC(\mathbb{T}))$ *is compact if and only if* $(\operatorname{smb} A)(x,z) = 0$, *for all* $z \in \mathbb{T}$.

Let us denote by π the canonical homomorphism from $L^2(\mathbb{T})$ onto the Calkin algebra $L^2(\mathbb{T})/\mathcal{K}(L^2(\mathbb{T}))$ and $\mathfrak{C}^\pi$ for the quotient algebra $\mathfrak{C}/\mathcal{K}(L^2(\mathbb{T}))$. The maximal ideal space of the center of $\mathfrak{C}^\pi$ (where the center of a non-commutative Banach algebra is the set of all elements that commute with all remains others) is homeomorphic to $\mathbb{T}_+$. For given $x \in \mathbb{T}_+$, denote by J_x the smallest closed two-sided ideal of $\mathfrak{C}^\pi$ containing x, put $\mathfrak{C}^\pi_x := \mathfrak{C}^\pi/J_x$, and write Φ^π_x for the canonical homomorphism from $\mathfrak{C}^\pi$ onto $\mathfrak{C}^\pi_x$.

Theorem 5.8. (Theorem 9, [79]) *Let* $a \in \mathfrak{C}$. *Then*

$$\sigma_{\mathfrak{C}^\pi}(\pi(A)) = \bigcup_{x\in\mathbb{T}_+} \sigma_{\mathfrak{C}^\pi_x}(\Phi^\pi_x(A)).$$

The symbols for $\mathfrak{C}_x^\pi$ depend on the values of x as it is showen in Propositions 4 and 5 in [79]. These results and the Allan-Douglas localization principle (see e.g., [13, 81]) give a symbol for $\operatorname{alg}(I_{\mathbb{T}}, P_+, J, PC(\mathbb{T}))/\mathcal{K}(L^2(\mathbb{T}))$.

Theorem 5.9. ([79, Theorem 11]) *A symbol calculus for elements on* $\mathfrak{C}^\pi = \operatorname{alg}(I_{\mathbb{T}}, P_+, J, PC(\mathbb{T}))/\mathcal{K}(L^2(\mathbb{T}))$ *is given for* $(x,t) \in (\mathbb{T}_+ \cup \{-1,1\}) \times [0,1]$

(a) *if* $x = \pm 1$ *by*

$$(\operatorname{smb} P_+)(x,t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (\operatorname{smb} J)(x) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \tag{5.18}$$

$$(\operatorname{smb} a)(x,t) = \begin{pmatrix} a(x+0)t + a(x-0)(1-t) & (a(x+0)-a(x-0))\sqrt{t(1-t)} \\ (a(x+0)-a(x-0))\sqrt{t(1-t)} & a(x-0)t + a(x+0)(1-t) \end{pmatrix}, \tag{5.19}$$

(b) *if* $\Im m(x) > 0$ *by*

$$(\operatorname{smb} P_+)(x,t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\operatorname{smb} J)(x,t) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$(\operatorname{smb} a)(x,t) = \begin{pmatrix} X & 0 \\ 0 & \widetilde{X} \end{pmatrix} \qquad \textit{with} \tag{5.20}$$

$$\begin{aligned} X &= a(x+0)\begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix} \\ &\quad + a(x-0)\begin{pmatrix} 1-t & -\sqrt{t(1-t)} \\ -\sqrt{t(1-t)} & t \end{pmatrix}, \\ \widetilde{X} &= a(x^{-1}-0)\begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix} \\ &\quad + a(x^{-1}+0)\begin{pmatrix} 1-t & -\sqrt{t(1-t)} \\ -\sqrt{t(1-t)} & t \end{pmatrix}. \end{aligned}$$

Remark 5.1. *The results above remain valid when* $\mathbb{T}$ *is replaced by a system of piecewise Lyapunov curves* Γ *and* J *by a Carleman shift operator changing the orientation on* Γ.

5.3 Proof of Theorem 5.1

Proof. First of all, we identify $\mathbf{Đ} = aP_+ + bP_- + cP_+J + dP_-J \in \mathfrak{C}$ with its symbols given in Theorem 5.9. Since the symbol calculus depends on the values of $x \in \mathbb{T} \cup \{-1, 1\}$, then we will consider the two cases separately. For the case $x = \pm 1$ the symbols are given by formulas (5.18) and (5.19). Applying Theorem 5.4 to the algebra $\mathfrak{C}$, to its subalgebra $\operatorname{alg}(P_+, PC(\mathbb{T}))$ and taking into account that $C(\mathbb{T}) \cong \mathbb{C}$, we have that

$$\|(\operatorname{smb} \mathbf{Đ})(x,t)(\operatorname{smb} B)^{-1}(x,t) - e\| < 1,$$

where $(\operatorname{smb} B)(x,t)$ is given by formulas (5.18) and having as coefficients the numbers $\alpha, \beta, \delta, \gamma \in \mathbb{C} \setminus \{0\}$. On the other hand, Proposition 5.7 (b) guarantees the existence of an operator $K_1 \in \mathcal{K}(L^2(\mathbb{T}))$ such that $\|\mathbf{Đ}B^{-1} - I - K_1\| < 1$. Setting $D := \mathbf{Đ}B^{-1} - K_1$, we obtain $\|D - I\| < 1$ as well as $\mathbf{Đ} = DB + K$ with $K := K_1 B \in \mathcal{K}(L^2(\mathbb{T}))$ which prove the theorem for this case.

For the case $x \in \mathbb{T}_+$, it is sufficient to prove that the conclusion of Theorem 5.4 remains valid for elements on $\mathfrak{A} = \operatorname{alg}(e, p, q, j)$, whose symbols are given by the formulas (5.20). To reach such a goal, it is enough to show that inequality (5.15) is satisfied also in this case, which is our next step.

Let $A = ap + b(e-p) + cpj + d(e-p)j$ with coefficients $a = (a_+ - a_-)q + a_-e$, $b = (b_+ - b_-)q + b_-e$, $c = (c_+ - c_-)q + c_-e$ and $d = (d_+ - d_-)q + d_-e$. The symbol for this element A is given by

$$(\operatorname{smb} A)(x) = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}, \tag{5.21}$$

where

$$X = \begin{pmatrix} a_+x + a_-(1-x) & -(b_+ - b_-)\sqrt{x(1-x)} \\ (a_+ - a_-)\sqrt{x(1-x)} & -(b_+(1-x) + b_-x) \end{pmatrix}, \tag{5.22}$$

$$Y = \begin{pmatrix} c_+x + c_-(1-x) & -(d_+ - d_-)\sqrt{x(1-x)} \\ (c_+ - c_-)\sqrt{x(1-x)} & -(d_-(1-x) + d_+x) \end{pmatrix}, \tag{5.23}$$

$$Z = \begin{pmatrix} -(d_-x + d_+(1-x)) & (c_- - c_+)\sqrt{x(1-x)} \\ -(d_- - d_+)\sqrt{x(1-x)} & c_+(1-x) + c_-x \end{pmatrix}, \tag{5.24}$$

$$W = \begin{pmatrix} -(b_-x + b_+(1-x)) & (a_- - a_+)\sqrt{x(1-x)} \\ -(b_- - b_+)\sqrt{x(1-x)} & a_-(1-x) + a_+x \end{pmatrix}. \tag{5.25}$$

With this representation, we know that

$$\|ap + b(e-p) + cpj + d(e-p)j\|^2 = \max_{x\in[0,1]}\big(\max_{i=1,\ldots,4}(\lambda_i(x))\big),$$

where $\lambda_i(x)$ (for $i = 1, \ldots, 4$) are the roots of the polynomial

$$\det(\mathrm{smb}(A)(x)\,\mathrm{smb}(A)^*(x) - \lambda I_{4\times 4}).$$

From Theorem 1.6.5, Chapter III in [68] we have the following bound for the characteristic values of $\mathrm{smb}(A)(x)\,\mathrm{smb}(A)^*(x)$:

$$|\lambda_i(x)| \le \min(R, T), \qquad i = 1, \ldots, 4,$$

where $R = \max_k(R_k)$, $T = \max_j(T_j)$ with R_k, T_j $(k, j = 1, \ldots, 4)$ being the sum of absolute values of the entries of $\mathrm{smb}(A)(x)\,\mathrm{smb}(A)^*(x)$ in the kth row and the sum of the absolute values of the entries in the jth column, respectively.

On the other hand, for a 4×4 complex valued matrix

$$M := \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}$$

some computations give us that

$$MM^* = \begin{pmatrix} mm^*_{11} & r & s & t \\ \bar{r} & mm^*_{22} & u & v \\ \bar{s} & \bar{u} & mm^*_{33} & w \\ \bar{t} & \bar{v} & \bar{w} & mm^*_{44} \end{pmatrix},$$

whose entries are given by:

$$\begin{aligned} mm^*_{11} =& |a|^2 + |b|^2 + |c|^2 + |d|^2 \\ mm^*_{22} =& |e|^2 + |f|^2 + |g|^2 + |h|^2 \\ mm^*_{33} =& |i|^2 + |j|^2 + |k|^2 + |l|^2 \\ mm^*_{44} =& |m|^2 + |n|^2 + |o|^2 + |p|^2 \end{aligned}$$

and

$$\begin{aligned} r =& a\bar{e} + b\bar{f} + c\bar{k} + d\bar{h} \\ s =& a\bar{i} + b\bar{j} + c\bar{k} + d\bar{l} \\ t =& a\bar{m} + b\bar{n} + c\bar{o} + d\bar{p} \\ u =& e\bar{i} + f\bar{j} + g\bar{k} + h\bar{l} \\ v =& e\bar{m} + f\bar{n} + g\bar{o} + h\bar{p} \\ w =& i\bar{m} + j\bar{n} + k\bar{o} + l\bar{p}. \end{aligned}$$

So, we get

$$R = \max \begin{cases} mm_{11}^* + |r| + |s| + |t| \\ mm_{22}^* + |\bar{r}| + |u| + |v| \\ mm_{33}^* + |\bar{s}| + |\bar{u}| + |w| \\ mm_{44}^* + |\bar{t}| + |\bar{v}| + |\bar{w}| \end{cases} \qquad \text{and} \quad T = \overline{R}.$$

Without loss of generality we are going to suppose that

$$|\lambda_i(x)| \leq T, \qquad i = 1, \ldots, 4$$

also, we will assume that $T = |a|^2 + |b|^2 + |c|^2 + |d|^2 + |\bar{r}| + |\bar{s}| + |\bar{t}|$. Under these assumptions, if we return to our case i.e., when $M = \mathrm{smb}(A)(x)$, we have

$$|a|^2 = |d|^2 = |a_+ x + a_-(1-x)|^2, \quad |b|^2 = |c|^2 = |b_+ x + b_-(1-x)|^2,$$

and in this case,

$$\begin{aligned} |\bar{r}| \leq & |(\bar{a}_+ x + \bar{a}_-(1-x))(a_+ - a_-)\sqrt{x(1-x)}| \\ & + |(b_+(1-x) + b_- x)(\bar{b}_+ - \bar{b}_-)\sqrt{x(1-x)}| \\ & + |(\bar{c}_+ x + \bar{c}_-(1-x))(c_+ - c_-)\sqrt{x(1-x)}| \\ & + |(d_+(1-x) + d_- x)(\bar{d}_+ - \bar{d}_-)\sqrt{x(1-x)}|, \end{aligned}$$

$$\begin{aligned} |\bar{s}| \leq & |(\bar{a}_+ x + \bar{a}_-(1-x))(d_- t + d_+(1-t))| \\ & + |(c_- - c_+)\sqrt{x(1-x)}(\bar{b}_+ - \bar{b}_-)\sqrt{x(1-x)}| \\ & + |(\bar{c}_+ x + \bar{c}_-(1-x))(b_- t + b_+(1-t))| \\ & + |(a_- - a_+)\sqrt{x(1-x)}(\bar{d}_+ - \bar{d}_-)\sqrt{x(1-x)}|, \end{aligned}$$

$$\begin{aligned} |\bar{t}| \leq & |(\bar{a}_+ x + \bar{a}_-(1-x))(d_- - d_+)\sqrt{x(1-x)}| \\ & + |(c_+(1-x) + c_- x)(\bar{b}_+ - \bar{b}_-)\sqrt{x(1-x)}| \\ & + |(\bar{c}_+ x + \bar{c}_-(1-x))(b_- - b_+)\sqrt{x(1-x)}| \\ & + |(a_-(1-x) + a_+ x)(\bar{d}_+ - \bar{d}_-)\sqrt{x(1-x)}|. \end{aligned}$$

If we assume for instance (without loss of generality):

$$|a_-| \geq |a_+|, \quad |b_-| \geq |b_+|, \quad |c_-| \geq |c_+|, \quad |d_-| \geq |d_+|$$

then, we get

$$|a|^2 = |d|^2 \leq |a_-|^2, \quad |b|^2 = |c|^2 \leq |b_-|^2$$

as well as the inequalities,

$$\begin{aligned}|\bar{r}| \leq & |\bar{a}_-(a_+ - a_-)\sqrt{x(1-x)}| + |b_-(\bar{b}_+ - \bar{b}_-)\sqrt{x(1-x)}| \\ & + |\bar{c}_-(c_+ - c_-)\sqrt{x(1-x)}| + |d_-(\bar{d}_+ - \bar{d}_-)\sqrt{x(1-x)}|,\end{aligned}$$

$$\begin{aligned}|\bar{s}| \leq & |\bar{a}_-||d_-| + |(\bar{b}_+ - \bar{b}_-)(c_+ - c_-)|t(1-t) + |\bar{c}_-||b_-| \\ & + |(\bar{d}_+ - \bar{d}_-)(a_- - a_+)|t(1-t),\end{aligned}$$

$$\begin{aligned}|\bar{t}| \leq & |\bar{a}_-(d_- - d_+)\sqrt{t(1-t)}| + |c_+(\bar{b}_+ - \bar{b}_-)\sqrt{t(1-t)}| \\ & + |\bar{c}_-(b_- - b_+)\sqrt{t(1-t)}| + |a_+(\bar{d}_+ - \bar{d}_-)\sqrt{t(1-t)}|\end{aligned}$$

or, improving these bounds,

$$\begin{aligned}|\bar{r}| \leq & \frac{1}{2}\left(|\bar{a}_-(a_+ - a_-)| + |b_-(\bar{b}_+ - \bar{b}_-)| + |\bar{c}_-(c_+ - c_-)| + |d_-(\bar{d}_+ - \bar{d}_-)|\right) \\ |\bar{s}| \leq & |\bar{a}_-||d_-| + |\bar{c}_-||b_-| + \frac{1}{4}(|(\bar{b}_+ - \bar{b}_-)(c_+ - c_-)| + |(\bar{d}_+ - \bar{d}_-)(a_+ - a_-)|) \\ |\bar{t}| \leq & \frac{1}{2}\left(|\bar{a}_-(d_+ - d_-)| + |c_+(\bar{b}_+ - \bar{b}_-)| + |\bar{c}_-(b_+ - b_-)| + |a_-(\bar{d}_+ - \bar{d}_-)|\right).\end{aligned}$$

Therefore, the following estimate was obtained:

$$|\lambda_i(x)| \leq 16 \max(\|a\|^2, \|b\|^2, \|c\|^2, \|d\|^2)$$

which certainly proves the estimate (5.15) for 4×4 matricial symbols. $\square$

We would like to point out that the representation (5.1) is equivalent to the concept of locally strongly ellipticity, which is a crucial assumption for modeling applications using generalized boundary integral and integro-differential operators (see, [76, Proposition 6.19]). Here we were not able to prove conditions that ensure the strong ellipticity of the operator **Đ**. However, we think that the representation given in Theorem 5.1 can be useful in order to apply approximation methods for this kind of operators. Also, we would like to remark that conditions for the locally strong ellipticity of pure singular integral operators as operator $\mathcal{A}$ defined in (1.14) are already known, remaining open the corresponding ones to singular integral operators with shift.

Chapter 6

On the Solvability of SIE's with Carleman Shift

In this chapter, we will reduce a class of singular integral equations with shift on a weighted Lebesgue space $L^p(\mathbb{T}, w)$ $(1 < p < \infty)$ to a system of SIE's by using some operator identities and projections, which allow us to study the solutions of the initial equation throughout a Riemann boundary value problem. The projection method which we have in mind was also used by Nguyen Minh Tuan *et al* [28, 27, 94] for the case of linear fractional Carleman shift on the space of Hölder-Zygmund continuous functions $H^\mu(\mathbb{T})$ $(0 < \mu < 1)$.

We would like to remark that the *classical method* for the reduction of a singular integral operator with shift (SIOS) into singular integral equations without a shift is based on a procedure which requires the "duplication of the size of the space" in which the operators are defined. As a consequence, it is obtained a pure vector singular integral operator which has the same Fredholm properties as the initial one but with a "double" symbol matrix. In much of the cases, the so-called *Gohberg-Krupnik-Litvichuk identity* (see e.g., [48, 51, 54]) and other explicit *equivalence relations* (e.g., [49, 50, 51]) are main ingredients for such analysis. In this way, the solvability of a (scalar) SIE associated with the SIO is equivalently formulated as a matrix factorization problem for corresponding matrices (which are built based on the new matrix coefficients); for these and other methods see, for instance, [29, 52, 54, 63].

The techniques of the present chapter avoid the use of the just mentioned (independent) matrix singular integral operators by relating the solutions of the SIES to the solutions of a pure system of two SIE which presents some dependencies between both equations. This allows a direct construction of the corresponding solutions by using an appropriate substituting *ansatz* which revels here to be a fundamental piece in the full process of finding the solutions to the initial problem.

First of all, we are going to establish an extra notation: As usual, we shall say that a function ϕ analytic in the unit disk $\mathbb{D} := \{z \in \mathbb{C} \,:\, |z| < 1\}$ is an element of the *Smirnov class* $E^p(\mathbb{D})$, $1 \leq p < \infty$, if it possible to find an expanding sequence of domains $\mathbb{D}_k$ with rectificable boundaries $\mathbb{T}_k$ such that:

(i) $\mathbb{D}_k \cup \mathbb{T}_k \subset \mathbb{D}$;

(ii) $\bigcup_k \mathbb{D}_k = \mathbb{D}$;

(iii) $\sup_k \int_{\mathbb{T}_k} |f(t)|^p \, dt < \infty$.

We will also consider the *weighted Smirnov class* defined by $E^p(\mathbb{D}, \rho) := \{\phi \in E^1(\mathbb{D}) : \phi_{|\mathbb{T}} \in L^p(\mathbb{T}, \rho)\}$, where $\phi_{|\mathbb{T}}$ denotes the non-tangential limit of ϕ a.e. on $\mathbb{T}$.

Other remarkable sets associated with $E^p(\mathbb{D}, \rho)$ are the set of *analytic functions on* $\mathbb{D}$, the set of *analytic functions on* $\mathbb{C} \setminus \overline{\mathbb{D}}$ and the set of *analytic functions on* $\mathbb{C} \setminus \overline{\mathbb{D}}$ *vanishing at infinity* – here denoted by $E^p_+(\mathbb{D}, \rho)$, $E^p_-(\mathbb{D}, \rho)$ and $\overset{\bullet}{E}{}^p_-(\mathbb{D}, \rho)$, respectively. These sets allow us to define the Hardy spaces in the following form: A function f belongs to $H^p_+(\mathbb{T}, \rho)$ ($\widetilde{H}^p_-(\mathbb{T}, \rho)$, $H^p_-(\mathbb{T}, \rho)$), $1 \leq p \leq \infty$, if there exists a function of class $E^p_+(\mathbb{D}, \rho)$ ($E^p_-(\mathbb{D}, \rho)$, $\overset{\bullet}{E}{}^p_-(\mathbb{D}, \rho)$) for which their boundary values coincide with f at almost all $t \in \mathbb{T}$.

Now we are going to describe in more detail the aims of this chapter: In the first part of the chapter we will consider a SIES defined on *weighted Lebesgue spaces* $L^p(\mathbb{T}, \rho)$, $p \in (1, \infty)$, with

$$\varrho(t) := |t - t_0|^{\beta(t_0)} \left| \frac{1}{t} - \frac{1}{t_0} \right|^{\beta(t_0)} \widetilde{\varrho}(t),$$

where $t_0 \in \mathbb{T}_+$ and $\widetilde{\varrho}$ is a continuous function at t_0 and such that $\widetilde{\varrho}(t_0) \neq 0$ (and the exponents $\beta(t_0)$ are then subjected to the fact that ϱ belongs to the Muckenhoupt class).

Our main purpose is the solvability of the following singular integral equation, which cannot be reduced to a two-term boundary value problem (see [64]), defined on the space $L^p(\mathbb{T}, \varrho)$, $p \in (1, \infty)$:

$$a(t)\varphi(t) + \frac{b(t)}{2} \sum_{k=0}^{1} (-1)^{2-k} \frac{1}{\pi i} \int_{\mathbb{T}} \frac{\varphi(\tau)}{\tau - \alpha_k(t)} d\tau + \sum_{j=1}^{m} \frac{a_j(t)}{\pi i} \int_{\mathbb{T}} b_j(\tau)\varphi(\tau) d\tau = f(t) \tag{6.1}$$

where $a, b, a_1, \ldots, a_m \in L^\infty(\mathbb{T})$ (and later on are required to satisfy some extra conditions; see (6.28)–(6.32)), $b_1, \ldots, b_m$ are given functions satisfying $\int_{\mathbb{T}} b_j(\tau)\varphi(\tau)d\tau < \infty$, and $\alpha_k(t) = \alpha(\alpha_{k-1}(t))$ with $\alpha_0(t) = t$, $\alpha(t) = 1/t$.

In the second part of the chapter, we will rewrite the results of the first part to the case of a more general equation than (6.1) defined on the classic Lebesgue space $L^p(\mathbb{T})$, $p \in (1, \infty)$:

$$a(t)\varphi(t) + \frac{b(t)}{2}\sum_{k=0}^{1}(-1)^{2-k}(v(t))^k \frac{1}{\pi i}\int_{\mathbb{T}} \frac{\varphi(\tau)}{\tau - \theta_k(t)}d\tau + \sum_{j=1}^{m} \frac{a_j(t)}{\pi i}\int_{\mathbb{T}} b_j(\tau)\varphi(\tau)d\tau = f(t) \tag{6.2}$$

with the coefficients $a, b, a_1, \ldots, a_m$ as above. Here, $v(t)$ is a complex-valued function on $\mathbb{T}$ and θ is a Carleman shift function on $\mathbb{T}$ for which we assume that both functions $v(t)$ and $\alpha(t)$ induce a bounded, commutative or anti-commutative, weighted Carleman shift operator.

In the final part, we consider the solvability of one kind of this last type of integral equations, i.e., we will consider the following integral equation in the Lebesgue space $L^p(\Gamma)$ $(1 < p < \infty)$:

$$f(t)\varphi(t) + g(t)(S_\Gamma\varphi)(t) + g(t)v(t)(S_\Gamma\varphi)(\alpha(t)) = h(t), \tag{6.3}$$

where Γ is a Carleson curve dividing the complex plane into the interior part D^+ $(0 \in D^+)$ and exterior part D^- $(\infty \in D^-)$. The elements $f(t)$ and $g(t)$ are complex-valued continuous functions on Γ, satisfying an extra condition (see (6.134) and (6.135)), $v(t)$ is a complex-valued function and $\alpha(t)$ is a Carleman shift function from Γ onto itself (which may preserve or change the orientation of Γ), we assume that both functions $v(t)$ and $\alpha(t)$ induce a bounded commutative (anti-commutative) weighted Carleman shift operator.

6.1 Projections and singular integral operators with reflection

Let us consider the following complementary projections

$$P_1 := \frac{1}{2}(I_{\mathbb{T}} - J) \quad \text{and} \quad P_2 := \frac{1}{2}(I_{\mathbb{T}} + J) \tag{6.4}$$

where J is the shift operator $(J\varphi)(t) = \varphi(1/t)$, $t \in \mathbb{T}$.

Note that $J^k = \sum_{j=1}^{2}(-1)^{kj}P_j,\ \ k = 1,2$, and

$$P_k = \frac{1}{2}\sum_{j=1}^{2}(-1)^{k(1-j)}J^{j+1}, \quad k = 1,2. \tag{6.5}$$

In that follows, we will denote the multiplication operator of a function $\varphi \in L^p(\mathbb{T},\varrho)$ by a function $a \in L^\infty(\mathbb{T})$, as $(K_a\varphi)(t) = a(t)\varphi(t)$.

The next proposition presents some of the dependencies between the projections P_k $(k = 1,2)$ and the multiplication operator K_a.

Proposition 6.1. *Let $a \in L^\infty(\mathbb{T})$ be fixed. Then for every (j,k), with $k,j = \{1,2\}$, there exists an element $b \in L^\infty(\mathbb{T})$ such that $K_bX_j \subset X_k$ and*

$$P_kK_aP_j = K_bP_j,$$

where $X_k = P_k(L^p(\mathbb{T},\varrho))$. The function b will be denoted by a_{kj} and determined as follows

$$a_{kj}(t) := \frac{1}{2}\sum_{m=1}^{2}(-1)^{(j-k)(m+1)}a(\alpha_{m+1}(t)), \qquad t \in \mathbb{T}. \tag{6.6}$$

Proof. Based on the properties of the projections P_k given in (6.5), we directly obtain:

$$\begin{aligned}
P_kK_aP_j =& \frac{1}{2}\sum_{m=1}^{2}(-1)^{k(1-m)}J^{m+1}K_aP_j\\
=& \frac{1}{2}\sum_{m=1}^{2}(-1)^{k(1-m)}a(\alpha_{m+1}(\cdot))J^{m+1}P_j\\
=& \frac{1}{2}\sum_{m=1}^{2}(-1)^{k(1-m)}a(\alpha_{m+1}(\cdot))\sum_{s=1}^{2}(-1)^{s(m+1)}P_sP_j\\
=& \frac{1}{2}\sum_{m=1}^{2}(-1)^{k(1-m)}a(\alpha_{m+1}(\cdot))(-1)^{j(m+1)}P_j\\
=& a_{kj}(\cdot)P_j = K_bP_j.
\end{aligned}$$

□

A more direct relation between the projections P_k and the multiplication operator with symbol b is now exhibited in the next result.

Proposition 6.2. *Let $a \in L^\infty(\mathbb{T})$ be fixed. Then for any $k, j \in \{1, 2\}$, we have*

$$P_k K_{a_{kj}} = K_{a_{kj}} P_j$$

where a_{kj} is determined by (6.6).

Proof. For any $\varphi \in L^p(\mathbb{T}, \varrho)$, we have

$$\begin{aligned}
(P_k K_{a_{kj}}\varphi)(t) &= P_k(a_{kj}(t)\varphi(t)) \\
&= \frac{1}{2}\sum_{m=1}^{2}(-1)^{k(1-m)}J^{m+1}(a_{kj}(t)\varphi(t)) \\
&= \frac{1}{2}\sum_{m=1}^{2}(-1)^{k(1-m)}J^{m+1}\left(\frac{1}{2}\left[\sum_{n=1}^{2}(-1)^{(j-k)(n+1)}a(\alpha_{n+1}(t))\right]\varphi(t)\right) \\
&= \frac{1}{2}\sum_{m=1}^{2}\left[\frac{1}{2}\sum_{n=1}^{2}(-1)^{(j-k)(n+1)}a(\alpha_{n+1+m+1}(t))\right](-1)^{k(1-m)}\varphi(\alpha_{m+1}(t)).
\end{aligned}$$

Notice that for $m = 1$ we get $a(\alpha_{n+1+m+1}(t)) = a(\alpha_{n+3}(t))= a(\alpha_{n+1}(t))$ and, for $m = 2$, $a(\alpha_{n+1+m+1}(t)) = a(\alpha_{n+4}(t)) = a(\alpha_n(t))$. Thus

$$\begin{aligned}
(P_k K_{a_{kj}}\varphi)(t) =&\frac{1}{2}\left[\frac{1}{2}\sum_{n=1}^{2}(-1)^{(n+1)(j-k)}a(\alpha_{n+1}(t))\varphi(t)+\right. \\
&\left.(-1)^k\left(\frac{1}{2}\sum_{n=1}^{2}(-1)^{(n+1)(j-k)}a(\alpha_n(t))\right)\varphi(\alpha(t))\right] \\
=&\frac{1}{2}\left[\frac{1}{2}\sum_{n=1}^{2}(-1)^{(n+1)(j-k)}a(\alpha_{n+1}(t))\varphi(t)\right. \\
&\left.+(-1)^j\left(\frac{1}{2}\sum_{n=1}^{2}(-1)^{n(j-k)}a(\alpha_n(t))\right)\varphi(\alpha(t))\right] \\
=&a_{kj}(t)(P_j\varphi)(t) = (K_{a_{kj}}P_j\varphi)(t).
\end{aligned}$$

Therefore, $P_k K_{a_{kj}} \equiv K_{a_{kj}} P_j$. □

Proposition 6.3. *Let $\varphi \in L^p(\mathbb{T}, \varrho)$. Then, for $z \in \mathbb{C} \setminus \{0\}$, we have*

(1) $(S_\mathbb{T} J\varphi)(z) = (S_\mathbb{T}\varphi)(0) - (JS_\mathbb{T}\varphi)(z)$.

(2) $(P_k S_\mathbb{T}\varphi)(z) = (S_\mathbb{T} P_j\varphi)(z) + \frac{(-1)^k}{2}(S_\mathbb{T}\varphi)(0), \quad k, j = 1, 2, \quad k \neq j$.

Proof. We start by recalling that in our case $\varphi \in L^p(\mathbb{T}, \varrho) \subset L^1(\mathbb{T})$ and therefore, based on the Cauchy integral of φ, we may consider the corresponding analytic functions in the unitary disk or its exterior. We will have this in mind in the following calculations.

(i) Let

$$(S_{\mathbb{T}} J f)(z) = \frac{1}{\pi i} \int_{\mathbb{T}} \frac{f(\frac{1}{\tau})}{\tau - z} d\tau.$$

Putting $\tau = \frac{1}{x}$, $d\tau = -\frac{1}{x^2} dx$ we get

$$\begin{aligned}
(S_{\mathbb{T}} J f)(z) =& -\frac{1}{\pi i} \int_{\mathbb{T}} \frac{f(x)}{\frac{1}{x} - z} \left(-\frac{1}{x^2} \right) dx = -\frac{1}{\pi i} \int_{\mathbb{T}} \frac{f(x)}{zx - 1} \left(\frac{1}{x} \right) dx \\
=& -\frac{1}{\pi i} \int_{\mathbb{T}} \left(-\frac{1}{x} + \frac{1}{x - \frac{1}{z}} \right) f(x) dx \\
=& \frac{1}{\pi i} \int_{\mathbb{T}} \frac{f(x)}{x} dx - \frac{1}{\pi i} \int_{\mathbb{T}} \frac{f(x)}{x - \frac{1}{z}} dx \\
=& (S_{\mathbb{T}} f)(0) - (J S_{\mathbb{T}} f)(z).
\end{aligned}$$

Therefore, the proposition (1) is obtained.

(ii) To carry out the second part, we perform the following computations:

$$\begin{aligned}
(P_k S_{\mathbb{T}} f)(z) =& \frac{1}{2} \sum_{j=1}^{2} (-1)^{k(1-j)} (J^{j+1} S_{\mathbb{T}} f)(z) \\
=& \frac{1}{2} \left\{ (S_{\mathbb{T}} f)(z) + (-1)^k (J S_{\mathbb{T}} f)(z)) \right\} \\
=& \frac{1}{2} \left\{ (S_{\mathbb{T}} f)(z) + (-1)^k [-(S_{\mathbb{T}} J f)(z) + (S_{\mathbb{T}} f)(0)] \right\}.
\end{aligned}$$

From here we conclude that

$$(P_k S_{\mathbb{T}} f)(z) = (S_{\mathbb{T}} P_j f)(z) + \frac{(-1)^k}{2} (S_{\mathbb{T}} f)(0), \quad k, j = 1, 2, \quad k \neq j.$$

□

6.2 The reduction of equation (6.1) to a system of pure singular integral equations

In this section we will relate the solutions of the SIES (6.1) with the solutions of a pure system of SIE. First, with the help of projection P_1 given in the previous section we rewrite equation (6.1) as follows

$$a(t)\varphi(t) + b(t)(P_1 S_{\mathbb{T}}\varphi)(t) + \sum_{j=1}^{m} a_j(t)\frac{1}{\pi i}\int_{\mathbb{T}} b_j(\tau)\varphi(\tau)d\tau = f(t). \qquad (6.7)$$

Additionally, suppose that $a(t)$ is a non-vanishing function on $\mathbb{T}$. Denoting by M_{b_j}, $j = 1, \ldots, m$, the linear functional on $L^p(\mathbb{T}, \varrho)$ defined as

$$M_{b_j}(\varphi) := \frac{1}{\pi i}\int_{\mathbb{T}} b_j(\tau)\varphi(\tau)d\tau,$$

and putting

$$M_{b_j}(\varphi) = \lambda_j, \qquad j = 1, \ldots, m, \qquad (6.8)$$

then (6.7) can be rewritten in the form

$$a(t)\varphi(t) + b(t)(P_1 S_{\mathbb{T}}\varphi)(t) = f(t) - \sum_{j=1}^{m} \lambda_j a_j(t). \qquad (6.9)$$

Lemma 6.4. *Let $\varphi \in L^p(\mathbb{T}, \varrho)$. Then φ is a solution of* (6.9) *if and only if $\{\varphi_k = P_k\varphi,\ k = 1, 2\}$ is a solution of the following system*

$$a_\alpha(t)\varphi_k(t) + [ab]_k(t)[(S_{\mathbb{T}}\varphi_2)(t) - (S_{\mathbb{T}}\varphi_2)(1)] = [af]_k(t), \quad k = 1, 2, \quad (6.10)$$

where

$$\begin{aligned} a_\alpha(t) =& a(t)a(\alpha(t)) \\ [ab]_k(t) =& \frac{1}{2}\sum_{j=1}^{2}(-1)^{(j+1)(1-k)} a(\alpha_j(t))b(\alpha_{j+1}(t)) \\ [af]_k(t) =& \frac{1}{2}\sum_{j=1}^{2}(-1)^{k(1-j)}[f(\alpha_{j+1}(t)) - \sum_{v=1}^{m}\lambda_v a_v(\alpha_{j+1}(t))]a(\alpha_j(t)). \end{aligned} \qquad (6.11)$$

Proof. Suppose that $\varphi \in L^p(\mathbb{T}, \varrho)$ is a solution of (6.9). Then, multiplying by $a(\alpha(t))$, applying the projections P_k ($k = 1, 2$) to both sides of such equation and using Propositions 6.1 and 6.2, we have

$$\begin{aligned} P_k(a(t)a(\alpha(t))\varphi(t)) &+ \frac{1}{2}\sum_{j=1}^{2}(-1)^{(j+1)(1-k)} a(\alpha_{j+2}(t))b(\alpha_{j+1}(t))(P_1 S_{\mathbb{T}}\varphi)(t) \\ &= \frac{1}{2}\sum_{j=1}^{2}(-1)^{k(1-j)}[f(\alpha_{j+1}(t)) - \sum_{v=1}^{m}\lambda_v a_v(\alpha_{j+1}(t))]a(\alpha_{j+2}(t)). \end{aligned}$$

Since $a(\alpha_{j+2}(t)) = a(\alpha_j(t))$, it follows that the equation above is equivalent to the following system

$$a_\alpha(t)(P_k\varphi)(t) + [ab]_k(t)(P_1 S_\mathbb{T}\varphi)(t) = [af]_k(t), \qquad k = 1, 2. \tag{6.12}$$

Using Proposition 6.3, we are able to rewrite the system (6.12) in the form

$$a_\alpha(t)(P_k\varphi)(t) + [ab]_k(t)[(S_\mathbb{T}P_2\varphi)(t) - \frac{1}{2}(S_\mathbb{T}\varphi)(0)] = [af]_k(t), \quad k = 1, 2.$$

Evaluating $z = 1$ in the equality $(P_1 S_\mathbb{T}\varphi)(z) = (S_\mathbb{T}P_2\varphi)(z) - \frac{1}{2}(S_\mathbb{T}\varphi)(0)$, we obtain

$$\begin{aligned}
(P_1 S_\mathbb{T}\varphi)(1) =& (S_\mathbb{T}P_2\varphi)(1) - \frac{1}{2}(S_\mathbb{T}\varphi)(0) \\
\frac{1}{2}[(S_\mathbb{T}\varphi)(1) - (JS_\mathbb{T}\varphi)(1)] =& (S_\mathbb{T}P_2\varphi)(1) - \frac{1}{2}(S_\mathbb{T}\varphi)(0) \\
0 =& (S_\mathbb{T}P_2\varphi)(1) - \frac{1}{2}(S_\mathbb{T}\varphi)(0).
\end{aligned}$$

Thus, $(P_1\varphi, P_2\varphi)$ is a solution of (6.10).

Conversely, suppose that there exists $\varphi \in L^p(\mathbb{T}, \varrho)$ such that $(P_1\varphi, P_2\varphi)$ is a solution of (6.10). Summing from 1 to 2, we obtain

$$\sum_{k=1}^{2}[a_\alpha(t)(P_k\varphi)(t) + [ab]_k(t)\left((S_\mathbb{T}P_2\varphi)(t) - (S_\mathbb{T}\varphi_2)(1)\right)] = \sum_{k=1}^{2}[af]_k(t). \tag{6.13}$$

Now, note that

$$\begin{aligned}
\sum_{k=1}^{2}[ab]_k(t) =& \frac{1}{2}[a(\alpha(t))b(t) + a(t)b(\alpha(t)) + a(\alpha(t))b(t) - a(t)b(\alpha(t))] \\
=& a(\alpha(t))b(t).
\end{aligned} \tag{6.14}$$

Similarly,

$$\begin{aligned}
\sum_{k=1}^{2}[af]_k(t) =& \sum_{k=1}^{2}\left[\frac{1}{2}\sum_{j=1}^{2}(-1)^{k(1-j)}[f(\alpha_{j+1}(t)) - \sum_{v=1}^{m}\lambda_v a_v(\alpha_{j+1}(t))]a(\alpha_j(t))\right] \\
=& \sum_{k=1}^{2}\frac{1}{2}\Bigg\{[f(t) - \sum_{v=1}^{m}\lambda_v a_v(t)]a(\alpha(t)) + \\
& (-1)^k[f(\alpha(t)) - \sum_{v=1}^{m}\lambda_v a_v(\alpha(t))]a(t)\Bigg\} \\
=& \left[f(t) - \sum_{v=1}^{m}\lambda_v a_v(t)\right]a(\alpha(t)).
\end{aligned} \tag{6.15}$$

Thus, (6.13) is equivalent to the following equality

$$a_\alpha(t)\varphi(t)+b(t)a(\alpha(t))[(S_{\mathbb{T}}P_2\varphi)(t)-(S_{\mathbb{T}}\varphi_2)(1)] = \left[f(t) - \sum_{v=1}^{m} \lambda_v a_v(t)\right] a(\alpha(t)).$$

By Proposition 6.3, this implies the desired form (6.9). □

Lemma 6.5. *If* (ψ_1, ψ_2) *is a solution of system* (6.10)*, then* $(P_1\psi_1, P_2\psi_2)$ *is also a solution of* (6.10)*.*

Proof. Suppose that (ψ_1, ψ_2) is a solution of the system (6.10). Applying the projections P_k to both sides of the k-th equation of (6.10), we get

$$a_\alpha(t)(P_k\psi_k)(t) + P_k[[ab]_k(t)((S_{\mathbb{T}}\psi_2)(t) - (S_{\mathbb{T}}\psi_2)(1))] = P_k([af]_k(t)). \quad (6.16)$$

Now, we claim that

$$P_k([ab]_k)I = [ab]_k(t)P_1 \quad \text{and} \quad P_k([af]_k)(t) = [af]_k(t). \quad (6.17)$$

In fact,

$$\begin{aligned}
&P_k([ab]_k(t)f)(t) = \\
&P_k\left[\frac{1}{2}\left(a(\alpha(t))b(t) + (-1)^{3(1-k)}a(t)b(\alpha(t))\right)f(t)\right] \\
=&\frac{1}{2}\sum_{j=1}^{2}(-1)^{k(1-j)}J^{j+1}\left(\frac{1}{2}[a(\alpha(t))b(t) + (-1)^{3(1-k)}a(t)b(\alpha(t))]f(t)\right) \\
=&\frac{1}{4}\{[a(\alpha(t))b(t) + (-1)^{3(1-k)}a(t)b(\alpha(t))]f(t)+ \\
&(-1)^k[a(t)b(\alpha(t)) + (-1)^{3(1-k)}a(\alpha(t))b(t)]f(\alpha(t))\} \\
=&\frac{1}{4}\{[a(\alpha(t))b(t) + (-1)^{3(1-k)}a(t)b(\alpha(t))]f(t)+ \\
&[(-1)^k a(t)b(\alpha(t)) - a(\alpha(t))b(t)]f(\alpha(t))\} \\
=&\frac{1}{4}\left\{a(\alpha(t))b(t)(f(t) - f(\alpha(t)))\right. \\
&\left.+a(t)b(\alpha(t))[(-1)^{3(1-k)}f(t) + (-1)^k f(\alpha(t))]\right\} \\
=&\frac{1}{4}\left\{a(\alpha(t))b(t)(f(t) - f(\alpha(t))) + (-1)^{3(1-k)}a(t)b(\alpha(t))[f(t) - f(\alpha(t))]\right\} \\
=&[ab]_k(t)(P_1f)(t).
\end{aligned}$$

On the other hand,

$$P_k([af]_k(t)) = \frac{1}{2}\sum_{j=1}^{2}(-1)^{k(1-j)}J^{j+1}([af]_k(t)) = \frac{1}{2}\sum_{j=1}^{2}(-1)^{k(1-j)}$$

$$J^{j+1}\left(\frac{1}{2}\sum_{j=1}^{2}(-1)^{k(1-j)}[f(\alpha_{j+1}(t)) - \sum_{v=1}^{m}\lambda_v a_v(\alpha_{j+1}(t))]a(\alpha_j(t))\right)$$

$$=\frac{1}{4}\sum_{j=1}^{2}(-1)^{k(1-j)}J^{j+1}\left([f(t) - \sum_{v=1}^{m}\lambda_v a_v(t)]a(\alpha(t))\right.$$

$$\left.+(-1)^k[f(\alpha(t)) - \sum_{v=1}^{m}\lambda_v a_v(\alpha(t))]a(t)\right)$$

$$=\frac{1}{4}\{([f(t) - \sum_{v=1}^{m}\lambda_v a_v(t)]a(\alpha(t)) + (-1)^k[f(\alpha(t)) - \sum_{v=1}^{m}\lambda_v a_v(\alpha(t))]a(t))$$

$$+ (-1)^k([f(\alpha(t)) - \sum_{v=1}^{m}\lambda_v a_v(\alpha(t))]a(t) + (-1)^k[f(t) - \sum_{v=1}^{m}\lambda_v a_v(t)]a(\alpha(t)))\}$$

$$=\frac{1}{4}\{[f(t) - \sum_{v=1}^{m}\lambda_v a_v(t)]a(\alpha(t)) + (-1)^k[f(\alpha(t)) - \sum_{v=1}^{m}\lambda_v a_v(\alpha(t))]a(t)$$

$$+ (-1)^k[f(\alpha(t)) - \sum_{v=1}^{m}\lambda_v a_v(\alpha(t))]a(t) + [f(t) - \sum_{v=1}^{m}\lambda_v a_v(t)]a(\alpha(t))\}.$$

Therefore, we have

$$P_k([af]_k(t))$$

$$=\frac{1}{4}\{2[f(t) - \sum_{v=1}^{m}\lambda_v a_v(t)]a(\alpha(t)) + 2(-1)^k[f(\alpha(t)) - \sum_{v=1}^{m}\lambda_v a_v(\alpha(t))]a(t)\}$$

$$=[af]_k(t).$$

Now, substituting (6.17) into (6.16), we obtain

$$a_\alpha(t)(P_k\psi_k)(t) + [ab]_k(t)P_1[(S_{\mathbb{T}}\psi_2)(t) - (S_{\mathbb{T}}\psi_2)(1)] = [af]_k(t). \tag{6.18}$$

Using Proposition 6.3, we have that (6.18) is equivalent to the following equation:

$$a_\alpha(t)(P_k\psi_k)(t) + [ab]_k(t)[(S_{\mathbb{T}}P_2\psi_2)(t) - \frac{1}{2}(S_{\mathbb{T}}\psi_2)(0) - P_1(S_{\mathbb{T}}\psi_2)(1)] = [af]_k(t),$$

for $k = 1, 2$. Also, from Proposition 6.3 we have that

$$\frac{1}{2}(S_{\mathbb{T}}\psi_2)(0) + P_1(S_{\mathbb{T}}\psi_2)(1) = (S_{\mathbb{T}}P_2\psi_2)(1).$$

Thus, we obtain that $(P_1\psi_1, P_2\psi_2)$ is a solution of (6.10). $\square$

Theorem 6.6. *The equation* (6.9) *has solutions in* $L^p(\mathbb{T}, \varrho)$ *if and only if the following equation*

$$a_\alpha(t)\varphi_2(t) + [ab]_2(t)(S_\mathbb{T}\varphi_2)(t) - [ab]_2(t)(S_\mathbb{T}\varphi_2)(1) = [af]_2(t) \tag{6.19}$$

has solutions. Moreover, if φ_2 *is a solution of equation* (6.19)*, then equation* (6.9) *has a solution given by formula*

$$\varphi(t) = \frac{f(t) - \sum_{j=1}^m \lambda_j a_j(t) - b(t)(P_1 S_\mathbb{T}\varphi_2)(t)}{a(t)}. \tag{6.20}$$

Proof. Suppose that $\varphi \in L^p(\mathbb{T}, \varrho)$ is a solution of equation (6.9). By Lemma 6.4 we know that $(P_1\varphi, P_2\varphi)$ is a solution of system (6.10). Hence $P_2\varphi$ is a solution of (6.19).

Conversely, suppose that φ_2 is a solution of (6.19). In this case (6.10) has a solution determined by the formula

$$\varphi_1(t) = \frac{[af]_1(t) - [ab]_1(t)[(S_\mathbb{T}\varphi_2)(t) - (S_\mathbb{T}\varphi_2)(1)]}{a_\alpha(t)}. \tag{6.21}$$

By Lemma 6.5 we know that $(P_1\varphi_1, P_2\varphi_2)$ is also a solution of (6.10). Put

$$\varphi = \sum_{k=1}^{2} P_k\varphi_k. \tag{6.22}$$

It is clear that $P_k\varphi = P_k\varphi_k$. This means that $(P_1\varphi, P_2\varphi)$ is a solution of (6.10). From Lemma 6.4 it follows that φ is a solution of (6.9). Moreover, from (6.17), (6.21) and (6.22), we get

$$\begin{aligned}\varphi(t) =& \sum_{k=1}^{2} P_k\varphi_k(t) = \sum_{k=1}^{2} P_k\left[\frac{[af]_k(t) - [ab]_k(t)[(S_\mathbb{T}\varphi_2)(t) - (S_\mathbb{T}\varphi_2)(1)]}{a_\alpha(t)}\right]\\ =&\frac{1}{a_\alpha(t)}\sum_{k=1}^{2}\{[af]_k(t) - [ab]_k(t)[(P_1 S_\mathbb{T}\varphi_2)(t) - P_1(S_\mathbb{T}\varphi_2)(1)]\}. \end{aligned}\tag{6.23}$$

Substituting (6.14) and (6.15) into (6.23), we obtain (6.20). Thus the proof is completed. $\square$

6.3 The solutions of equation (6.19) by means of the associate BVP

In this section we are going to obtain the explicit solutions of equation (6.19). In view of this goal, we will reduce that equation to a Riemann boundary value problem. In order to establish the solutions of this problem we shall use the following weak factorization notion (cf. [63, 67]): A *factorization* of an element $\psi \in L^\infty(\mathbb{T})$ in the space $L^p(\mathbb{T},\rho)$ $(1 < p < \infty)$ is a representation of the form

$$\psi(t) = \psi_+(t)t^{\aleph}\psi_-(t), \quad t \in \mathbb{T},$$

where

$$\begin{aligned}\psi_+ &\in H_+^p(\mathbb{T},\varrho), \quad \psi_+^{-1} \in H_+^q(\mathbb{T},\varrho^{-1}),\\ \psi_- &\in \widetilde{H}_-^q(\mathbb{T},\varrho^{-1}), \quad \psi_-^{-1} \in \widetilde{H}_-^p(\mathbb{T},\varrho),\end{aligned}$$

$\aleph$ is an integer which is called the p-index of ψ, and $q := p/(p-1)$ is the conjugate exponent of $p \in (1,\infty)$.

Let us consider the equation (6.19), and define

$$\Phi_2(z) := \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\varphi_2(\tau)}{\tau - z}d\tau, \qquad z \in \mathbb{C}\setminus\mathbb{T}. \tag{6.24}$$

According to *Sokhotski-Plemelij formula*, we have:

$$\varphi_2(t) = \Phi_2^+(t) - \Phi_2^-(t) \tag{6.25}$$

$$(S_{\mathbb{T}}\varphi_2)(t) = \Phi_2^+(t) + \Phi_2^-(t), \tag{6.26}$$

where $\Phi_2^\pm(t)$ denote the usual nontangential limits of $\Phi_2(z)$ for elements $z \in \mathbb{D}$ and $z \in \mathbb{C}\backslash\overline{\mathbb{D}}$, respectively. These instruments allow us to equivalently reduce equation (6.19) to the following boundary problem: Find a function $\Phi_2(z)$ sectionally analytic in the corresponding domains ($\Phi_2(z) = \Phi_2^+(z)$ for $z \in \mathbb{D}$ and $\Phi_2(z) = \Phi_2^-(z)$ for $z \in \mathbb{C}\setminus\overline{\mathbb{D}}$), vanishing at infinity and

$$\Phi_2^+(t) + \psi(t)\Phi_2^-(t) = g(t) \tag{6.27}$$

imposed on their boundary values on $\mathbb{T}$, where

$$\psi(t) := \frac{[ab]_2(t) - a_\alpha(t)}{[ab]_2(t) + a_\alpha(t)}, \quad \text{and} \quad g(t) := \frac{[af]_2(t) + 2\lambda_0[ab]_2(t)}{[ab]_2(t) + a_\alpha(t)} \tag{6.28}$$

with $\psi(t)$ away from zero on $\mathbb{T}$ and $\lambda_0 := \frac{1}{2}(S_{\mathbb{T}}\varphi_2)(1)$. The problem is considered on $L^p(\mathbb{T},\varrho)$, $1 < p < \infty$. This means that the function g belongs to $L^p(\mathbb{T},\varrho)$ and $\Phi_2^{\pm}$ must belong to the classes $E_+^p(\mathbb{T},\varrho)$ and $\overset{\bullet}{E}{}_-^p(\mathbb{T},\varrho)$, respectively.

As about additional conditions, we will assume that the bounded and measurable function ψ, defined in (6.28), admits a factorization

$$\psi(t) = \psi_+(t)\, t^{\aleph}\, \psi_-(t) \tag{6.29}$$

in $L^p(\mathbb{T},\varrho)$. Moreover, we are also assuming that:

$$\psi_+^{-1} g \in \mathcal{L}^1(\mathbb{T}) := H_+^1(\mathbb{T}) \oplus H_-^1(\mathbb{T}), \tag{6.30}$$

$$\varphi_0^+ := \psi_+ P_+ \psi_+^{-1} g \in H_+^p(\mathbb{T},\varrho), \tag{6.31}$$

$$\varphi_0^- := \psi_-^{-1} t^{-\aleph} P_- \psi_+^{-1} g \in H_-^p(\mathbb{T},\varrho). \tag{6.32}$$

Adapting the methods of [63, §3.1] to the present situation of spaces with weights, it follows that the general solution of problem (6.27) is of the form

$$\Phi_2^+ = \varphi_0^+ + \psi_+ p_{\aleph-1}, \qquad \Phi_2^- = \varphi_0^- + \psi_-^{-1} t^{-\aleph} p_{\aleph-1}, \tag{6.33}$$

where

$$p_{\aleph-1}(z) = p_1 + p_2 z + \cdots + p_{\aleph} z^{\aleph-1}, \qquad \text{if } \aleph \geq 1, \tag{6.34}$$

is a polynomial of degree not greater than $\aleph - 1$ in case that $\aleph > 0$, and equal to zero if $\aleph \leq 0$. The solutions can be then written in the following form:

$$\Phi_2^+(z) = \psi_+(z)[A(z) + \lambda_0 B(z) + p_{\aleph-1}(z)] \tag{6.35}$$

$$\Phi_2^-(z) = \psi_-^{-1}(z) z^{-\aleph}[C(z) + \lambda_0 D(z) + p_{\aleph-1}(z)]. \tag{6.36}$$

Here,

$$A(z) = P_+\left(\psi_+^{-1}(\cdot)\frac{[af]_2(\cdot)}{[ab]_2(\cdot) + a_\alpha(\cdot)}\right)(z), \tag{6.37}$$

$$B(z) = P_+\left(2\psi_+^{-1}(\cdot)\frac{[ab]_2(\cdot)}{[ab]_2(\cdot) + a_\alpha(\cdot)}\right)(z), \tag{6.38}$$

$$C(z) = P_-\left(\psi_+^{-1}(\cdot)\frac{[af]_2(\cdot)}{[ab]_2(\cdot) + a_\alpha(\cdot)}\right)(z), \tag{6.39}$$

$$D(z) = P_-\left(2\psi_+^{-1}(\cdot)\frac{[ab]_2(\cdot)}{[ab]_2(\cdot) + a_\alpha(\cdot)}\right)(z). \tag{6.40}$$

Notice that in the right-hand side of the last four identities (in view of not increasing the notation), we are using the same notation which is used for the Cauchy projections (on $\mathbb{T}$) although these right-hand sides should be read as the existing extensions to $E^p_\pm(\mathbb{D},\varrho)$ or $\overset{\bullet}{E}{}^p_-(\mathbb{D},\varrho)$ of the corresponding functions on $H^p_\pm(\mathbb{T},\varrho)$ or $\widetilde{H}^p_-(\mathbb{T},\varrho)$. The same notation choice is consistent in the remaining corresponding parts.

The function $\Phi_2 = \frac{\Phi_2^+ + \Phi_2^-}{2}$ is a solution of (6.27) if $\Phi_2(1) = \frac{\Phi_2^+(1)+\Phi_2^-(1)}{2} = \lambda_0$ holds. I.e.,

$$\begin{aligned}\frac{\Phi_2^+(1)+\Phi_2^-(1)}{2} =& \frac{\psi_+(1)[A(1)+\lambda_0 B(1)+\sum_{j=1}^{\aleph} p_j]}{2}\\ &+\frac{\psi_-^{-1}(1)[C(1)+\lambda_0 D(1)+\sum_{j=1}^{\aleph} p_j]}{2} = \lambda_0.\end{aligned}$$

Since the representation of the solutions depends on the p-index $\aleph$, we divide the analysis in the following different cases.

Case $\aleph \geq 0$. We start by recalling that $p_{\aleph-1}(1) = 0$ in case $\aleph = 0$. Moreover,

$$\psi_+(1)\left[A(1)+\lambda_0 B(1)+\sum_{j=1}^{\aleph} p_j\right] + \psi_-^{-1}(1)\left[C(1)+\lambda_0 D(1)+\sum_{j=1}^{\aleph} p_j\right] = 2\lambda_0$$

implies that

$$\begin{aligned}&\psi_+(1)\left[A(1)+\sum_{j=1}^{\aleph} p_j\right] + \psi_-^{-1}(1)\left[C(1)+\sum_{j=1}^{\aleph} p_j\right]\\ &\qquad = \lambda_0[2-\psi_+(1)B(1)-\psi_-^{-1}(1)D(1)].\end{aligned} \tag{6.41}$$

From here we need to consider the following two sub-cases:

i) $2-\psi_+(1)B(1)-\psi_-^{-1}(1)D(1) \neq 0$. From equation (6.41) we get

$$\lambda_0 = \frac{\psi_+(1)A(1)+\psi_-^{-1}(1)C(1)+(\psi_+(1)+\psi_-^{-1}(1))\sum_{j=1}^{\aleph} p_j}{2-\psi_+(1)B(1)-\psi_-^{-1}(1)D(1)}. \tag{6.42}$$

In this case the general solutions of problem (6.27) are given by the formulas

$$\begin{aligned}&\Phi_2^+(z) = \psi_+(z)\,[A(z)\\ &\quad+\frac{\psi_+(1)A(1)+\psi_-^{-1}(1)C(1)+(\psi_+(1)+\psi_-^{-1}(1))\sum_{j=1}^{\aleph} p_j}{2-\psi_+(1)B(1)-\psi_-^{-1}(1)D(1)}B(z)\\ &\quad+p_{\aleph-1}(z)]\,,\end{aligned} \tag{6.43}$$

$$\Phi_2^-(z) = \psi_-^{-1}(z)z^{-\aleph}\left[C(z) + \frac{\psi_+(1)A(1) + \psi_-^{-1}(1)C(1) + (\psi_+(1) + \psi_-^{-1}(1))\sum_{j=1}^{\aleph} p_j}{2 - \psi_+(1)B(1) - \psi_-^{-1}(1)D(1)}D(z) + p_{\aleph-1}(z)\right], \tag{6.44}$$

where $\psi_\pm^{\pm 1}$ are the outer factors in the factorization of the function ψ given in (6.28), A, B, C and D are the functions defined in (6.37)–(6.40) and $p_{\aleph-1}$ is a polynomial of degree less than or equal to $\aleph - 1$.

ii) $2 - \psi_+(1)B(1) - \psi_-^{-1}(1)D(1) = 0$. In this case we have from (6.41) that:

$$\psi_+(1)A(1) + \psi_-^{-1}(1)C(1) + (\psi_+(1) + \psi_-^{-1}(1))\sum_{j=1}^{\aleph} p_j = 0. \tag{6.45}$$

Then, the general solutions of problem (6.27) are given by the formulas:

$$\Phi_2^+(z) = \psi_+(z)[A(z) + \lambda_0 B(z) + p_{\aleph-1}(z)] \tag{6.46}$$

$$\Phi_2^-(z) = \psi_-^{-1}(z)z^{-\aleph}[C(z) + \lambda_0 D(z) + p_{\aleph-1}(z)], \tag{6.47}$$

where $\psi_\pm^{\pm 1}$ are the outer factors in the factorization of the function ψ given in (6.28), A, B, C and D are the functions defined in (6.37)–(6.40), λ_0 is arbitrary and $p_{\aleph-1}(z)$ is a polynomial of degree less than or equal to $\aleph - 1$ with complex coefficients satisfying condition (6.45).

Case $\aleph < 0$. The necessary condition for the problem (6.27) to be solvable is that (see [63])

$$\int_{\mathbb{T}} \psi_+^{-1}(\tau)g(\tau)\tau^k d\tau = 0, \qquad k = 0, \ldots, -(\aleph - 1).$$

This condition can be written as follows:

$$\int_{\mathbb{T}} \frac{\psi_+^{-1}(\tau)[af]_2(\tau)\tau^k}{[ab]_2(\tau) + a_\alpha(\tau)}d\tau = -2\lambda_0 \int_{\mathbb{T}} \frac{\psi_+^{-1}(\tau)[ab]_2(\tau)\tau^k}{[ab]_2(\tau) + a_\alpha(\tau)}d\tau. \tag{6.48}$$

In this case $p_{\aleph-1}(z) \equiv 0$. So, we receive:

i) $2 - \psi_+(1)B(1) - \psi_-^{-1}(1)D(1) \neq 0$. In this case, by means of equation (6.41), we get

$$\lambda_0 = \frac{\psi_+(1)A(1) + \psi_-^{-1}(1)C(1)}{2 - \psi_+(1)B(1) - \psi_-^{-1}(1)D(1)}.$$

Hence, (6.48) becomes into the following condition

$$\int_{\mathbb{T}} \frac{\psi_+^{-1}(\tau)[af]_2(\tau)\tau^k}{[ab]_2(\tau)+a_\alpha(\tau)} d\tau = -2\frac{\psi_+(1)A(1)+\psi_-^{-1}(1)C(1)}{2-\psi_+(1)B(1)-\psi_-^{-1}(1)D(1)} \times \int_{\mathbb{T}} \frac{\psi_+^{-1}(\tau)[ab]_2(\tau)\tau^k}{[ab]_2(\tau)+a_\alpha(\tau)} d\tau. \tag{6.49}$$

If condition (6.49) is satisfied, then the solution of the problem (6.27) is given by the following formulas

$$\begin{aligned}\Phi_2^+(z) =& \psi_+(z)\left[A(z)+\frac{\psi_+(1)A(1)+\psi_-^{-1}(1)C(1)}{2-\psi_+(1)B(1)-\psi_-^{-1}(1)D(1)}B(z)\right]\\ \Phi_2^-(z) =& \psi_-^{-1}(z)z^{-\aleph}\left[C(z)+\frac{\psi_+(1)A(1)+\psi_-^{-1}(1)C(1)}{2-\psi_+(1)B(1)-\psi_-^{-1}(1)D(1)}D(z)\right].\end{aligned}$$

The elements $\psi_\pm^{\pm 1}$ are the outer factors in the factorization of the function ψ given in (6.28), A, B, C and D are the functions defined in (6.37)–(6.40).

ii) $2-\psi_+(1)B(1)-\psi_-^{-1}(1)D(1)=0$. From (6.45), we obtain

$$\psi_+(1)A(1)+\psi_-^{-1}(1)C(1)=0. \tag{6.50}$$

If condition (6.48) and (6.50) are satisfied, then the solution of the problem (6.27) is given by

$$\begin{aligned}\Phi_2^+(z) =& \psi_+(z)[A(z)+\lambda_0 B(z)]\\ \Phi_2^-(z) =& \psi_-^{-1}(z)z^{-\aleph}[C(z)+\lambda_0 D(z)],\end{aligned}$$

where, as before, $\psi_\pm^{\pm 1}$ are the outer factors in the factorization of the function ψ given in (6.28), A, B, C and D are the functions defined in (6.37)–(6.40), λ_0 is determined from condition (6.48).

In the next theorem we give the explicit representation of the solutions of equation (6.19).

Theorem 6.7. *Let us suppose that the functions $[ab]_2(t)\pm a_\alpha(t)$ do not vanish on $\mathbb{T}$ and that the function $\psi=\frac{[ab]_2-a_\alpha}{[ab]_2+a_\alpha}$ admits a factorization in $L^p(\mathbb{T},\varrho)$, say $\psi(t)=\psi_+(t)t^{\aleph}\psi_-(t)$.*

(1.) *If* $2-\psi_+(1)B(1)-\psi_-^{-1}(1)D(1)\neq 0$ *and* $\aleph\geq 0$, *then equation* (6.19) *has solutions* φ_2 *satisfying the following formula*

$$\begin{aligned}(S_{\mathbb{T}}\varphi_2)(t) &= \psi_+(t)[A(t)\\ &+\frac{\psi_+(1)A(1)+\psi_-^{-1}(1)C(1)+(\psi_+(1)+\psi_-^{-1}(1))\sum_{j=1}^{\aleph}p_j}{2-\psi_+(1)B(1)-\psi_-^{-1}(1)D(1)}B(t)\\ &+p_{\aleph-1}(t)]+\psi_-^{-1}(t)t^{-\aleph}[C(t)\\ &+\frac{\psi_+(1)A(1)+\psi_-^{-1}(1)C(1)+(\psi_+(1)+\psi_-^{-1}(1))\sum_{j=1}^{\aleph}p_j}{2-\psi_+(1)B(1)-\psi_-^{-1}(1)D(1)}D(t)\\ &+p_{\aleph-1}(t)].\end{aligned} \tag{6.51}$$

Here $\psi_\pm^{\pm 1}$ *are the outer factors in the factorization of the function* ψ, A, B, C *and* D *are the functions defined in* (6.37)–(6.40), *and* $p_{\aleph-1}$ *is a polynomial of degree less than or equal to* $\aleph-1$.

(2.) *If* $2-\psi_+(1)B(1)-\psi_-^{-1}(1)D(1)\neq 0$ *and* $\aleph<0$, *then the equation* (6.19) *is solvable if the condition* (6.49) *is satisfied. In this case, equation* (6.19) *has a unique solution which satisfies the formula* (6.51) *where* $p_{\aleph-1}(t)\equiv 0$.

(3.) *If* $2-\psi_+(1)B(1)-\psi_-^{-1}(1)D(1)=0$ *and* $\aleph\geq 0$, *then the equation* (6.19) *has solutions* φ_2 *satisfying the following formula:*

$$\begin{aligned}&(S_{\mathbb{T}}\varphi_2)(t)=\\ &\psi_+(t)[A(z)+\lambda_0 B(t)+p_{\aleph-1}(t)]+\psi_-^{-1}t^{-\aleph}[C(t)+\lambda_0 D(t)+p_{\aleph-1}(t)]\end{aligned} \tag{6.52}$$

where $\psi_\pm^{\pm 1}$ *are the outer factors in the factorization of the function* ψ, A, B, C *and* D *are the functions defined in* (6.37)–(6.40), λ_0 *is arbitrary and* $p_{\aleph-1}$ *is a polynomial of degree less than or equal to* $\aleph-1$ *with complex coefficients satisfying condition* (6.45).

(4.) *If* $2-\psi_+(1)B(1)-\psi_-^{-1}(1)D(1)=0$ *and* $\aleph<0$, *then the equation* (6.19) *is solvable if the conditions* (6.48) *and* (6.50) *are satisfied. In this case, the equation* (6.19) *has a unique solution which satisfies the formula* (6.52), *where* $p_{\aleph-1}(t)\equiv 0$ *and* λ_0 *is determined from the condition* (6.48).

Proof. It is known that under conditions (6.30)–(6.32) the boundary value problem (6.27) defined on $L^p(\mathbb{T},\varrho)$ has solutions given by (6.33) (see [63]). On the other hand, from the Sokhotski-Plemelij formulas (6.25) and (6.26), we have that equation (6.19) has a solution φ_2 determined by

$$\varphi_2(t)=\Phi_2^+(t)-\Phi_2^-(t).$$

The conclusions are obtained from the equality $(S_{\mathbb{T}}\varphi_2)(t) = \Phi_2^+(t) + \Phi_2^-(t)$, applying (for each case) the conditions required. In this way, equation (6.51) is obtained adding equations (6.43) and (6.44), and equation (6.52) from the sum of equations (6.46) and (6.47). $\square$

6.4 The solutions of equation (6.7) satisfying condition (6.8)

As it was shown in the previous sections, Theorems 6.6 and 6.7 prove that if $[ab]_2(t) \pm a_\alpha(t) \neq 0$ on $\mathbb{T}$ and ψ defined in (6.28) admits a factorization in $L^p(\mathbb{T}, \varrho)$ (6.29), then equation (6.9) is solvable in closed form. In this section, we are going to study the solutions of (6.9) (considering (6.8)). As distinguished above, we consider the following cases:

(1.) $2 - \psi_+(1)B(1) - \psi_-^{-1}(1)D(1) \neq 0$, $\aleph \geq 0$. From Theorems 6.6 and 6.7 we have that the solutions of (6.9) are given by the following formula

$$\varphi(t) = \frac{f(t) - \sum_{j=1}^{m} \lambda_j a_j(t) - b(t)(P_1 S_{\mathbb{T}}\varphi_2)(t)}{a(t)}, \tag{6.53}$$

where $(S_{\mathbb{T}}\varphi_2)(t)$ is determined by (6.51). From (6.11) we rewrite (6.37)–(6.40) as

$$\begin{aligned} A(z) =& P_+\Big(\psi_+^{-1}(\cdot)\frac{\frac{1}{2}\sum_{j=1}^{2}[f(\alpha_{j+1}(\cdot)) - \sum_{v=1}^{m}\lambda_v a_v(\alpha_{j+1}(\cdot))]}{[ab]_2(\cdot) + a_\alpha(\cdot)} \\ & \times a(\alpha_j(\cdot))\Big)(z), \\ C(z) =& P_-\Big(\psi_+^{-1}(\cdot)\frac{\frac{1}{2}\sum_{j=1}^{2}[f(\alpha_{j+1}(\cdot)) - \sum_{v=1}^{m}\lambda_v a_v(\alpha_{j+1}(\cdot))]}{[ab]_2(\cdot) + a_\alpha(\cdot)} \\ & \times a(\alpha_j(\cdot))\Big)(z). \end{aligned}$$

Or, equivalently

$$A(z) = \Theta_1(z) - \sum_{v=1}^{m} \lambda_v \Xi_{1v}(z), \tag{6.54}$$

$$C(z) = \Theta_2(z) - \sum_{v=1}^{m} \lambda_v \Xi_{2v}(z), \tag{6.55}$$

where

$$\Theta_1(z) = P_+\left(\frac{\psi_+^{-1}(\cdot)\frac{1}{2}\sum_{j=1}^{2} f(\alpha_{j+1}(\cdot))a(\alpha_j(\cdot))}{[ab]_2(\cdot)+a_\alpha(\cdot)}\right)(z), \tag{6.56}$$

$$\Xi_{1v}(z) = P_+\left(\frac{\psi_+^{-1}(\cdot)\frac{1}{2}\sum_{j=1}^{2} a_v(\alpha_{j+1}(\cdot))a(\alpha_j(\cdot))}{[ab]_2(\cdot)+a_\alpha(\cdot)}\right)(z), \tag{6.57}$$

$$\Theta_2(z) = P_-\left(\frac{\psi_+^{-1}(\cdot)\frac{1}{2}\sum_{j=1}^{2} f(\alpha_{j+1}(\cdot))a(\alpha_j(\cdot))}{[ab]_2(\cdot)+a_\alpha(\cdot)}\right)(z), \tag{6.58}$$

$$\Xi_{2v}(z) = P_-\left(\frac{\psi_+^{-1}(\cdot)\frac{1}{2}\sum_{j=1}^{2} a_v(\alpha_{j+1}(\cdot))a(\alpha_j(\cdot))}{[ab]_2(\cdot)+a_\alpha(\cdot)}\right)(z). \tag{6.59}$$

Substituting (6.34), (6.54) and (6.55) into (6.51), we have

$$\begin{aligned}(S_{\mathbb{T}}\varphi_2)(t) =&\psi_+(t)\Theta_1(t) + \psi_-^{-1}(t)t^{-\aleph}\Theta_2(t)\\
&+ \frac{\psi_+(1)A(1) + \psi_-^{-1}(1)C(1) + (\psi_+(1)+\psi_-^{-1}(1))\sum_{j=1}^{\aleph} p_j}{2-\psi_+(1)B(1)-\psi_-^{-1}(1)D(1)}\\
&\times(\psi_+(t)B(t)+\psi_-^{-1}(t)t^{-\aleph}D(t))\\
&-\sum_{v=1}^{m}\lambda_v[\Xi_{1v}(t)\psi_+(t)+\psi_-^{-1}(t)t^{-\aleph}\Xi_{2v}(t)]\\
&+(\psi_+(t)+\psi_-^{-1}(t)t^{-\aleph})\sum_{j=1}^{\aleph}p_j t^{j-1}.\end{aligned}$$

Then, (6.53) can be rewritten in the following form:

$$\begin{aligned}\varphi(t) =& \frac{f(t)-b(t)P_1[\psi_+(t)\Theta_1(t)+\psi_-^{-1}(t)t^{-\aleph}\Theta_2(t)]}{a(t)}\\
&-\left\{\frac{\psi_+(1)(\Theta_1(1)-\sum_{j=1}^{m}\lambda_j\Xi_{1j}(1))+\psi_-^{-1}(1)(\Theta_2(1)-\sum_{j=1}^{m}\lambda_j\Xi_{2j}(1))}{2-\psi_+(1)B(1)-\psi_-^{-1}(1)D(1)}\right.\\
&\left.+\frac{(\psi_+(1)+\psi_-^{-1}(1))\sum_{j=1}^{\aleph}p_j}{2-\psi_+(1)B(1)-\psi_-^{-1}(1)D(1)}\right\}\frac{b(t)P_1[\psi_+(t)B(t)+\psi_-^{-1}(t)t^{-\aleph}D(t)]}{a(t)}\\
&-\sum_{j=1}^{m}\lambda_j\frac{a_j(t)-b(t)P_1[\Xi_{1j}(t)\psi_+(t)+\Xi_{2j}(t)\psi_-^{-1}(t)t^{-\aleph}]}{a(t)}\\
&-\sum_{j=1}^{\aleph}p_j\frac{b(t)P_1[(\psi_+(t)+\psi_-^{-1}(t)t^{-\aleph})t^{j-1}]}{a(t)},\end{aligned} \tag{6.60}$$

with $\psi_\pm^{\pm1}$, $B, D, \Theta_1, \Xi_{1j}, \Theta_2, \Xi_{2j}$ $(j = 1, \ldots, m)$ determined by (6.29), (6.38)–(6.40), (6.56), (6.57), (6.58) and (6.59), respectively, and $p_1, \ldots, p_\aleph$ are arbitrary. The function φ is a solution of the equation (6.7) if it satisfies condition (6.8), that is:

$$M_{b_j}(\varphi) = \lambda_j, \qquad j = 1, \ldots, m.$$

Substituting (6.60) into the last condition, we obtain

$$\begin{aligned}
\lambda_k =& d_k - [\psi_+(1)\Theta_1(1) + \psi_-^{-1}(1)\Theta_2(1) - \sum_{j=1}^{m} \lambda_j[\psi_+(1)\Xi_{1j}(1) \\
&+ \psi_-^{-1}(1)\Xi_{2j}(1)] + \sum_{j=1}^{\aleph} p_j(\psi_+(1) + \psi_-^{-1}(1))]f_k - \sum_{j=1}^{m} e_{kj}\lambda_j \\
&- \sum_{j=1}^{\aleph} p_j g_{kj} \\
=&[d_k - (\psi_+(1)\Theta_1(1) + \psi_-^{-1}(1)\Theta_2(1))f_k] \\
&- \sum_{j=1}^{m} \lambda_j[e_{kj} - (\psi_+(1)\Xi_{1j}(1) + \psi_-^{-1}(1)\Xi_{2j}(1))f_k] \\
&- \sum_{j=1}^{\aleph} p_j[g_{kj} + (\psi_+(1) + \psi_-^{-1}(1))f_k], \quad k = 1, \ldots, m
\end{aligned} \tag{6.61}$$

where

$$\begin{aligned}
d_k(t) :=& M_{b_k}\left(\frac{f(t) - b(t)P_1[\psi_+(t)\Theta_1(t) + \psi_-^{-1}(t)t^{-\aleph}\Theta_2(t)]}{a(t)}\right), \\
e_{kj}(t) :=& M_{b_k}\left(\frac{a_j(t) - b(t)P_1[\Xi_{1j}(t)\psi_+(t) + \Xi_{2j}(t)\psi_-^{-1}(t)t^{-\aleph}]}{a(t)}\right), \\
f_k(t) :=& M_{b_k}\left(\frac{b(t)P_1[\psi_+(t)B(t) + \psi_-^{-1}(t)t^{-\aleph}D(t)]}{a(t)(2 - \psi_+(1)B(1) - \psi_-^{-1}(1)D(1))}\right), \\
g_{kj}(t) :=& M_{b_k}\left(\frac{b(t)P_1\left(t^{j-1}(\psi_+(t) + \psi_-^{-1}(t)t^{-\aleph})\right)}{a(t)}\right).
\end{aligned} \tag{6.62}$$

Putting

$$\lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix}_{m\times 1}, \qquad E = \Big(e_{ij} - (\psi_+(1)\Xi_{1j} + \psi_-^{-1}(1)\Xi_{2j})f_i\Big)_{i,j=1}^{m}$$

$$P = \begin{pmatrix} p_1 \\ \vdots \\ p_\aleph \end{pmatrix}_{\aleph\times 1}, \; D = \begin{pmatrix} d_1 - (\psi_+(1)\Theta_1(1) + \psi_-^{-1}(1)\Theta_2(1))f_1 \\ \vdots \\ d_m - (\psi_+(1)\Theta_1(1) + \psi_-^{-1}(1)\Theta_2(1))f_m \end{pmatrix}_{m\times 1}$$

$$G = \begin{pmatrix} g_{11} + (\psi_+(1) + \psi_-^{-1})f_1 & \dots & g_{1\aleph} + (\psi_+(1) + \psi_-^{-1})f_1 \\ \vdots & \ddots & \vdots \\ g_{m1} + (\psi_+(1) + \psi_-^{-1})f_m & \dots & g_{m\aleph} + (\psi_+(1) + \psi_-^{-1})f_m \end{pmatrix}_{m\times\aleph} \tag{6.63}$$

we write (6.61) in matricial form

$$(I_{m\times m} + E)\lambda = D - GP. \tag{6.64}$$

Here, $I_{m\times m}$ is the $(m\times m)$-identity matrix. So we can formulate that the function determined by (6.60) is a solution of (6.7) if and only if $(\lambda_1, \dots, \lambda_m)$ satisfy the condition (6.64).

(2.) $2 - \psi_+(1)B(1) - \psi_-^{-1}(1)D(1) \neq 0$, $\aleph < 0$. From Theorems 6.6 and 6.7 it follows that equation (6.9) has solutions if and only if the condition (6.49) is satisfied. If this is the case, then $p_{\aleph-1} \equiv 0$ and the solutions of (6.9) are given as follows:

$$\begin{aligned} \varphi(t) = {} & \frac{f(t) - b(t)P_1[\psi_+(t)\Theta_1(t) + \psi_-^{-1}(t)t^{-\aleph}\Theta_2(t)]}{a(t)} \\ & - \frac{\psi_+(1)(\Theta_1(1) - \sum_{j=1}^m \lambda_j\Xi_{1j}(1)) + \psi_-^{-1}(1)(\Theta_2(1) - \sum_{j=1}^m \lambda_j\Xi_{2j}(1))}{2 - \psi_+(1)B(1) - \psi_-^{-1}(1)D(1)} \\ & \times \frac{b(t)P_1[\psi_+(t)B(t) + \psi_-^{-1}(t)t^{-\aleph}D(t)]}{a(t)} \\ & - \sum_{j=1}^m \lambda_j \frac{a_j(t) - b(t)P_1[\Xi_{1j}(t)\psi_+(t) + \Xi_{2j}(t)\psi_-^{-1}(t)t^{-\aleph}]}{a(t)}. \end{aligned} \tag{6.65}$$

Therefore, the function φ determined by (6.65) is a solution of the equation (6.7) if and only if $(\lambda_1, \dots, \lambda_m)$ satisfies the following matricial condition

$$(I_{m\times m} + E)\lambda = D, \tag{6.66}$$

where E and D are determined by (6.63). On the other hand, substituting (6.11), (6.54) and (6.55) into (6.49), we obtain

$$d'_k - \sum_{v=1}^{m} e'_{kv}\lambda_v = -\left[\psi_+(1)\Theta_1(1) + \psi_-^{-1}(1)\Theta_2(1) - \sum_{v=1}^{m}\lambda_m[\psi_+(1)\Xi_{1v}(1) \right.$$
$$\left. + \psi_-^{-1}(1)\Xi_{2v}(1)]\right] f'_k, \quad k = 1, \cdots - \aleph, \tag{6.67}$$

where

$$d'_k := \int_{\mathbb{T}} \frac{\psi_+^{-1}(\tau)\frac{1}{2}\sum_{j=1}^{2} f(\alpha_{j+1}(\tau))a(\alpha_j(\tau))}{[ab]_2(\tau) + a_\alpha(\tau)}\tau^k d\tau,$$
$$e'_{kv} := \int_{\mathbb{T}} \frac{\psi_+^{-1}(\tau)\frac{1}{2}\sum_{j=1}^{2} a_v(\alpha_{j+1}(\tau))a(\alpha_j(\tau))}{[ab]_2(\tau) + a_\alpha(\tau)}\tau^k d\tau, \tag{6.68}$$
$$f'_k := \int_{\mathbb{T}} \frac{\sum_{j=1}^{2}(-1)^{j+1}a(\alpha_j(\tau))b(\alpha_{j+1}(\tau))\psi_+^{-1}(\tau)}{[ab]_2(\tau) + a_\alpha(\tau)}\tau^k d\tau$$
$$\times \frac{1}{2 - \psi_+(1)B(1) - \psi_-^{-1}(1)D(1)}.$$

Defining

$$D' := \begin{pmatrix} d'_1 + (\psi_+(1)\Theta_1(1) + \psi_-^{-1}(1)\Theta_2(1))f'_1 \\ \vdots \\ d'_{-\aleph} + (\psi_+(1)\Theta_1(1) + \psi_-^{-1}(1)\Theta_2(1))f'_{-\aleph} \end{pmatrix}_{-\aleph\times 1},$$
$$E' := \Big(e'_{ij} + (\psi_+(1)\Xi_{1j} + \psi_-^{-1}(1)\Xi_{2j})f'_i\Big)_{i=1,\dots,-\aleph}^{j=1,\dots,m}, \tag{6.69}$$

we rewrite (6.67) in the matricial form:

$$E'\lambda = D'. \tag{6.70}$$

Combining (6.66) and (6.70) we conclude that the function φ determined by (6.65) is a solution of (6.7) if and only if $(\lambda_1, \dots, \lambda_m)$ satisfies the following matricial identity

$$\begin{pmatrix} I_{m\times m} + E \\ E' \end{pmatrix}_{(m-\aleph)\times m} \lambda = \begin{pmatrix} D \\ D' \end{pmatrix}_{(m-\aleph)\times 1}. \tag{6.71}$$

(3.) $2 - \psi_+(1)B(1) - \psi_-^{-1}(1)D(1) = 0$, $\aleph \geq 0$. In such a case, the solution of the equation (6.9) is given by the formula

$$\begin{aligned}\varphi(t) =& \frac{f(t) - b(t)P_1[\psi_+(t)\Theta_1(t) + \psi_-^{-1}(t)t^{-\aleph}\Theta_2(t)]}{a(t)} \\ &- \lambda_0 \frac{b(t)P_1[\psi_+(t)B(t) + \psi_-^{-1}(t)t^{-\aleph}D(t)]}{a(t)} \\ &- \sum_{j=1}^{m} \lambda_j \frac{a_j(t) - b(t)P_1[\Xi_{1j}(t)\psi_+(t) + \Xi_{2j}(t)\psi_-^{-1}(t)t^{-\aleph}]}{a(t)} \\ &- \sum_{j=1}^{\aleph} p_j \frac{b(t)P_1[(\psi_+(t) + \psi_-^{-1}(t)t^{-\aleph})t^{j-1}]}{a(t)}, \end{aligned} \tag{6.72}$$

where $\psi_\pm^{\pm 1}$, $B, D, \Theta_1, \Xi_{1j}, \Theta_2, \Xi_{2j}$ $(j = 1, \ldots, m)$ are determined by (6.29), (6.38)–(6.40), (6.56), (6.57), (6.58) and (6.59) respectively, λ_0 is an arbitrary complex number and $p_1, \ldots, p_\aleph$ satisfy the condition (6.45). Substituting (6.54) and (6.55) into (6.45), we obtain

$$\begin{aligned}&\psi_+(1)\Theta_1(1) + \psi_-^{-1}(1)\Theta_2(1) - \sum_{v=1}^{m} \lambda_v(\psi_+(1)\Xi_{1v}(1) + \psi_-^{-1}(1)\Xi_{2v}(1)) \\ &+ (\psi_+(1) + \psi_-^{-1}(1)) \sum_{j=1}^{\aleph} p_j = 0. \end{aligned} \tag{6.73}$$

The function φ is a solution of the equation (6.7) if it satisfies the condition (6.8). Substituting (6.72) into (6.8) we have

$$\lambda_k = d_k - \lambda_0 h_k - \sum_{j=1}^{\aleph} p_j g_{kj} - \sum_{j=1}^{m} \lambda_j e_{kj}, \quad k = 1, 2, \ldots, m, \tag{6.74}$$

where d_k, e_{kj} and g_{kj} are determined by (6.62) and

$$h_k(t) := M_{b_k}\left(\frac{b(t)P_1\left[\psi_+(t)B(t) - \psi_-^{-1}(t)t^{-\aleph}D(t)\right]}{a(t)} \right).$$

Let

$$\lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix}_{m\times 1}, \quad \mathcal{E} = \begin{pmatrix} e_{11} & \dots & e_{1m} \\ \vdots & \ddots & \vdots \\ e_{m1} & \dots & e_{mm} \end{pmatrix}_{m\times m}$$

$$\mathcal{D} = \begin{pmatrix} d_1 \\ \vdots \\ d_m \end{pmatrix}_{m\times 1}, \quad P = \begin{pmatrix} p_1 \\ \vdots \\ p_\aleph \end{pmatrix}_{\aleph\times 1} \tag{6.75}$$

$$\mathcal{G} = \begin{pmatrix} g_{11} & \dots & g_{1\aleph} \\ \vdots & \ddots & \vdots \\ g_{m1} & \dots & g_{m\aleph} \end{pmatrix}_{m\times \aleph}, \quad H = \begin{pmatrix} h_1 \\ \vdots \\ h_m \end{pmatrix}_{m\times 1}.$$

Then, we rewrite (6.74) in the form

$$(I_{m\times m} + \mathcal{E})\lambda = \mathcal{D} - \lambda_0 H - \mathcal{G}P. \tag{6.76}$$

Combining (6.73) and (6.76), we conclude that the function φ determined by (6.72) is a solution of (6.7) if and only if $(\lambda_1, \dots, \lambda_m)$ satisfies the following matricial condition:

$$\overline{I_{m\times m} + \mathcal{E}}\lambda = \overline{\mathcal{D}} - \lambda_0 \overline{H} - \overline{\mathcal{G}}P. \tag{6.77}$$

Here,

$$\overline{I_{m\times m} + \mathcal{E}}_{(m+1)\times m} = \begin{pmatrix} I_{m\times m} + \mathcal{E} \\ \psi_+(1)\Xi_{11}(1) + \psi_-^{-1}(1)\Xi_{21}(1), \dots, \psi_+(1)\Xi_{1m}(1) + \psi_-^{-1}(1)\Xi_{2m}(1) \end{pmatrix}$$
$$\overline{H} = \begin{pmatrix} H \\ 0 \end{pmatrix}_{(m+1)\times 1}, \quad \overline{\mathcal{D}} = \begin{pmatrix} \mathcal{D} \\ \psi_+(1)\Theta_1(1) + \psi_-^{-1}(1)\Theta_2(1) \end{pmatrix}_{(m+1)\times 1}$$

$$\overline{\mathcal{G}} = \begin{pmatrix} \mathcal{G} \\ -(\psi_+(1) + \psi_-^{-1})p_1, \dots, -(\psi_+(1) + \psi_-^{-1})p_\aleph \end{pmatrix}_{(m+1)\times \aleph}. \tag{6.78}$$

(4.) $2 - \psi_+(1)B(1) - \psi_-^{-1}(1)D(1) = 0$, $\aleph < 0$. Again, Theorems 6.6 and 6.7 give us that equation (6.9) has solutions if the conditions (6.48) and (6.50) are satisfied. Since $p_{\aleph-1} \equiv 0$, then the solutions of (6.9) are expressed by:

$$\begin{aligned}\varphi(t)=&\frac{f(t)-b(t)P_1[\psi_+(t)\Theta_1(t)+\psi_-^{-1}(t)t^{-\aleph}\Theta_2(t)]}{a(t)}\\&-\lambda_0\frac{b(t)P_1[\psi_+(t)B(t)+\psi_-^{-1}(t)t^{-\aleph}D(t)]}{a(t)}\\&-\sum_{j=1}^{m}\lambda_j\frac{a_j(t)-b(t)P_1[\Xi_{1j}(t)\psi_+(t)+\Xi_{2j}(t)\psi_-^{-1}(t)t^{-\aleph}]}{a(t)}.\end{aligned}\tag{6.79}$$

The function φ determined by (6.79) is a solution of the equation (6.7) if and only if $(\lambda_1,\ldots,\lambda_m)$ satisfies the identity

$$\overline{I_{m\times m}+\mathcal{E}}\lambda=\overline{\mathcal{D}}-\lambda_0\overline{H},\tag{6.80}$$

where $\overline{I_{m\times m}+\mathcal{E}}$, $\overline{\mathcal{D}}$ and $\overline{H}$ are given by (6.78). On the other hand, (6.48) is equivalent to the condition

$$d'_k-\sum_{j=1}^{m}e'_{kj}\lambda_j=\lambda_0h'_k,\quad k=1,\cdots-\aleph\tag{6.81}$$

with d'_k, e'_{kj} determined by (6.68) and

$$h'_k=-\int_{\mathbb{T}}\frac{\sum_{j=1}^{2}(-1)^{j+1}a(\alpha_j(\tau))b(\alpha_{j+1}(\tau))\psi_+^{-1}(\tau)}{[ab]_2(\tau)+a_\alpha(\tau)}\tau^k d\tau.$$

Putting

$$\begin{aligned}&\mathcal{D}'=\begin{pmatrix}d'_1\\\vdots\\d'_{-\aleph}\end{pmatrix}_{-\aleph\times1},\quad H'=\begin{pmatrix}h'_1\\\vdots\\h'_{-\aleph}\end{pmatrix}_{-\aleph\times1}\\&\mathcal{E}'=\begin{pmatrix}e'_{11}&\ldots&e'_{1m}\\\vdots&\ddots&\vdots\\e'_{-\aleph1}&\ldots&e'_{-\aleph m}\end{pmatrix}_{-\aleph\times m},\end{aligned}\tag{6.82}$$

we have that (6.81) can be rewritten in the matricial form

$$\mathcal{E}'\lambda=\mathcal{D}'-\lambda_0H'.\tag{6.83}$$

Combining (6.80) and (6.83) we can say that the function φ determined by (6.79) is a solution of (6.7) if and only if $(\lambda_1,\ldots,\lambda_m)$ satisfies the following matricial identity:

$$\begin{pmatrix}\overline{I_{m\times m}+\mathcal{E}}\\\mathcal{E}'\end{pmatrix}_{(m+1-\aleph)\times m}\lambda=\begin{pmatrix}\overline{\mathcal{D}}\\\mathcal{D}'\end{pmatrix}_{(m+1-\aleph)\times1}-\lambda_0\begin{pmatrix}\overline{H}\\H'\end{pmatrix}_{(m+1-\aleph)\times1}.\tag{6.84}$$

Theorem 6.8. *Let us suppose that the functions* $[ab]_2(t) \pm a_\alpha(t)$ *do not vanish on* $\mathbb{T}$, *and that the function* $\psi = \frac{[ab]_2 - a_\alpha}{[ab]_2 + a_\alpha}$ *admits a factorization in* $L^p(\mathbb{T}, \varrho)$, *say* $\psi(t) = \psi_+(t) t^{\aleph} \psi_-(t)$.

(1.) *If* $2 - \psi_+(1)B(1) - \psi_-^{-1}(1)D(1) \neq 0$, $\aleph \geq 0$, *set*

$$r = \operatorname{rank}((I_{m\times m} + E) \quad G)_{m\times(m+\aleph)},$$

where E, G *are determined by* (6.63). *Then, the equation* (6.9) *is solvable if and only if the matrix* D *determined by* (6.63) *satisfies the condition*

$$\operatorname{rank}((I_{m\times m} + E) \quad G \quad D)_{m\times(m+\aleph+1)} = r.$$

If this is the case, then the solutions of the equation (6.9) *are given by the formula* (6.60), *where* $(\lambda_1, \ldots, \lambda_m, p_1, \ldots, p_\aleph)$ *satisfies* (6.64). *Moreover, we can choose* $m+\aleph-r$ *coefficients in* $\{\lambda_1, \ldots, \lambda_m, p_1, \ldots, p_\aleph\}$ *which are arbitrary up to the circumstance of* φ *being uniquely determined by these coefficients. In particular, if* $r = m$ *then the equation* (6.9) *is solvable for any function* f.

(2.) $2 - \psi_+(1)B(1) - \psi_-^{-1}(1)D(1) \neq 0$, $\aleph < 0$. *Put*

$$r = \operatorname{rank}\begin{pmatrix} I_{m\times m} + E \\ E' \end{pmatrix}_{(m-\aleph)\times m},$$

where E *and* E' *are determined by* (6.63) *and* (6.69), *respectively. The equation* (6.9) *is solvable if and only if the function* f *determines* D *and* D' *by the formulas* (6.63) *and* (6.69) *which satisfy the following matricial condition*

$$\operatorname{rank}\begin{pmatrix} I_{m\times m} + E & D \\ E' & D' \end{pmatrix}_{(m-\aleph)\times(m+1)} = r. \tag{6.85}$$

If this is the case, then the solutions of the equation (6.9) *are given by the formula* (6.65), *where* $(\lambda_1 \ldots, \lambda_m)$ *satisfies* (6.71). *In particular, if* $r = m$ *and the condition* (6.85) *is satisfied, then the equation* (6.9) *has an unique solution.*

(3.) $2 - \psi_+(1)B(1) - \psi_-^{-1}(1)D(1) = 0$, $\aleph \geq 0$. *Choose*

$$r = \operatorname{rank}(\overline{I_{m\times m} + \mathcal{E}} \quad \overline{H} \quad \overline{\mathcal{G}})_{(m+1)\times(m+1+\aleph)},$$

with $\overline{I_{m\times m} + \mathcal{E}}$, $\overline{H}$, $\overline{\mathcal{G}}$ *determined by* (6.78). *The equation* (6.9) *is solvable if and only if the matrix* $\overline{\mathcal{D}}$ *determined by* (6.78) *satisfies the condition*

$$\operatorname{rank}\left(\overline{I_{m\times m} + \mathcal{E}} \quad \overline{H} \quad \overline{\mathcal{G}} \quad \overline{\mathcal{D}}\right)_{(m+1)\times(m+2+\aleph)} = r.$$

If the above condition is satisfied, then the solutions of equation (6.9) *are given by the formula* (6.72), *where* $(\lambda_0, \ldots, \lambda_m, p_1, \ldots, p_\aleph)$ *satisfies* (6.77). *Moreover, we can choose* $m+1+\aleph-r$ *coefficients in* $\{\lambda_0, \ldots, \lambda_m, p_1, \ldots, p_\aleph\}$ *which are arbitrary so that* φ *is uniquely determined by these coefficients. In particular, if* $r = m+1$ *then the equation* (6.9) *is solvable for any function* f.

(4.) $2 - \psi_+(1)B(1) - \psi_-^{-1}(1)D(1) = 0$, $\aleph < 0$. *Put*

$$r = \operatorname{rank}\begin{pmatrix} \overline{I_{m\times m} + \mathcal{E}} & \overline{H} \\ \mathcal{E}' & H' \end{pmatrix},$$

where $\overline{I_{m\times m} + \mathcal{E}}$, $\overline{H}$, $\mathcal{E}'$, H' *are determined by* (6.78) *and* (6.82). *Then, the equation* (6.9) *is solvable if and only if the function* f *determines* $\overline{\mathcal{D}}$ *and* $\mathcal{D}'$ *by the formulas* (6.78) *and* (6.82) *which satisfy the condition*

$$\operatorname{rank}\begin{pmatrix} \overline{I_{m\times m} + \mathcal{E}} & \overline{H} & \overline{\mathcal{D}} \\ \mathcal{E}' & H' & \mathcal{D}' \end{pmatrix}_{(m+1-\aleph)\times(m+2)} = r. \tag{6.86}$$

If the condition (6.86) *is satisfied, then the solutions of the equation* (6.9) *are given by the formula* (6.79), *where* $(\lambda_0, \ldots, \lambda_m)$ *satisfies* (6.84). *In particular, if* $r = m+1$ *and the condition* (6.86) *is satisfied, then the equation* (6.9) *has a unique solution.*

Proof. (1.) From the assumption it follows that the equation (6.9) has solutions if and only if there exist $(\lambda_1, \ldots, \lambda_m)$ and $(p_1, \ldots, p_\aleph)$ which satisfy (6.64). We can rewrite (6.64) in the form

$$((I_{m\times m} + E) \quad G)_{m\times(m+\aleph)} \begin{pmatrix} \lambda \\ P \end{pmatrix}_{(m+\aleph)\times 1} = D.$$

Therefore, $\begin{pmatrix} \lambda \\ P \end{pmatrix}$ is a solution of the following equation

$$((I_{m\times m} + E) \quad G)X = D. \tag{6.87}$$

It follows that the necessary and sufficient condition for which the equation (6.9) has solutions, is that the equation (6.87) has solutions in $\mathbb{C}^{m+\aleph}$. Since

$$\operatorname{rank}((I_{m\times m} + E) \quad G \quad D) = \operatorname{rank}((I_{m\times m} + E) \quad G) = r,$$

then using (6.87) we can express r coefficients in $\{\lambda_1, \ldots, \lambda_m, p_1, \ldots, p_\aleph\}$ by $m + \aleph - r$ remaining ones. In particular, if $r = m$ then the equation (6.87) has solutions with any D. Therefore the equation (6.9) is solvable with any f. The cases (2.), (3.) and (4.) are proved in a similar way. $\square$

6.5 The solvability of equation (6.2) for commutative or anti-commutative Carleman shift

In this section we are going to study the solvability of the equation (6.2) in the cases when the Carleman shift function θ is of commutative or anti-commutative type. We will rewrite the corresponding results of the previous sections for each one of these cases.

6.5.1 Properties of the solutions of equation (6.2)

Let us introduce the weighted Carleman shift operator, induced by $v(t)$ and $\theta(t)$, on $L^p(\mathbb{T})$

$$(W\varphi)(t) = v(t)\varphi(\theta(t)), \quad t \in \mathbb{T}.$$

We are going to assume henceforth that W is of commutative or anti-commutative type. With this operator we define the complementary projections

$$\mathcal{P}_1 := \frac{1}{2}(I_{\mathbb{T}} - W) \quad \text{and} \quad \mathcal{P}_2 := \frac{1}{2}(I_{\mathbb{T}} + W) \tag{6.88}$$

satisfying $W^k = \sum_{j=1}^{2}(-1)^{kj}\mathcal{P}_j, \ k = 1, 2$, and

$$\mathcal{P}_k = \frac{1}{2}\sum_{j=1}^{2}(-1)^{k(1-j)}W^{j+1}, \quad k = 1, 2. \tag{6.89}$$

Notice that Propositions 6.1 and 6.2 hold for these projections $\mathcal{P}_k$, $k = 1, 2$. Now, the corresponding result to Proposition 6.3 has the following form:

Proposition 6.9. *Let $\psi \in L^p(\mathbb{T})$. Then, for $z \in \mathbb{C}$, we have*

$$(\mathcal{P}_k S_{\mathbb{T}}\psi)(z) = \begin{cases} (S_{\mathbb{T}}\mathcal{P}_k\psi)(z), & \text{if } WS_{\mathbb{T}} = S_{\mathbb{T}}W \\ (S_{\mathbb{T}}\mathcal{P}_{3-k}\psi)(z), & \text{if } WS_{\mathbb{T}} = -S_{\mathbb{T}}W. \end{cases} \tag{6.90}$$

Proof. We have directly

$$\begin{aligned}(\mathcal{P}_k S_{\mathbb{T}}\psi)(z) =& \frac{1}{2}\{(S_{\mathbb{T}}\varphi)(z) + (-1)^k W(S_{\mathbb{T}}\varphi)(z)\} \\ =& \frac{1}{2}\{(S_{\mathbb{T}}\varphi)(z) \pm (-1)^k (S_{\mathbb{T}}W\varphi)(z)\} \\ =& \frac{1}{2}S_{\mathbb{T}}(\varphi \pm (-1)^k W\varphi)(z)\end{aligned}$$

in $S_{\mathbb{T}}W = WS_{\mathbb{T}}$ and $S_{\mathbb{T}}W = -WS_{\mathbb{T}}$ cases, respectively. From here, equality (6.90) follows. □

Now with projection $\mathcal{P}_1$, equation (6.2) is rewritten as

$$a(t)\varphi(t) + b(t)(\mathcal{P}_1 S_{\mathbb{T}}\varphi)(t) + \sum_{j=1}^{m} a_j(t)\frac{1}{\pi i}\int_{\mathbb{T}} b_j(\tau)\varphi(\tau)d\tau = f(t). \qquad (6.91)$$

As in the previous sections we assume that $a(t)$ is a non-vanishing function on $\mathbb{T}$. We denote by $\mathcal{M}_{b_j}$, $j = 1, \dots, m$ the linear functional on $L^p(\mathbb{T})$ defined as

$$\mathcal{M}_{b_j}(\varphi) := \frac{1}{\pi i}\int_{\mathbb{T}} b_j(\tau)\varphi(\tau)d\tau,$$

putting

$$\mathcal{M}_{b_j}(\varphi) = \lambda_j \qquad j = 1, \dots m \qquad (6.92)$$

then (6.91) appears with the form

$$a(t)\varphi(t) + b(t)(\mathcal{P}_1 S_{\mathbb{T}}\varphi)(t) = f(t) - \sum_{j=1}^{m} \lambda_j a_j(t). \qquad (6.93)$$

A corresponding result to the former Lemma 6.4, appears now in the present case as follows:

Proposition 6.10. *Let $\varphi \in L^p(\mathbb{T})$. Then φ is a solution of* (6.93) *if and only if $\{\varphi_k := \mathcal{P}_k\varphi, \quad k = 1, 2\}$ is a solution of the following system*

$$\begin{array}{l} a_\theta(t)\varphi_k(t) + [ab]^*_{3-k}(t)(S_{\mathbb{T}}\varphi_1)(t) = [af]_k(t), \quad \textit{if } S_{\mathbb{T}}W = WS_{\mathbb{T}} \\ \textit{or} \\ a_\theta(t)\varphi_k(t) + [ab]^*_{3-k}(t)(S_{\mathbb{T}}\varphi_2)(t) = [af]_k(t), \quad \textit{if } S_{\mathbb{T}}W = -WS_{\mathbb{T}} \end{array} \qquad (6.94)$$

where, for $k = 1, 2$

$$\begin{aligned} a_\theta(t) &= a(\theta(t))a(t) \\ [ab]^*_{3-k}(t) &= \frac{1}{2}\sum_{j=1}^{2}(-1)^{(3-k)(1-j)}a(\theta_j(t))b(\theta_{j+1}(t)) \qquad (6.95) \\ [af]_k(t) &= \mathcal{P}_k([f(t) - \sum_{j=1}^{m}\lambda_j a_j(t)]a(\theta(t))). \end{aligned}$$

Proof. Suppose that $\varphi \in L^p(\mathbb{T})$ is a solution of (6.93). Multiplying by $a(\theta(t))$ and applying the projections $\mathcal{P}_k$ $(k = 1, 2)$ we have

$$\mathcal{P}_k(a(\theta(t))a(t)\varphi(t) + a(\theta(t))b(t)(\mathcal{P}_1 S_{\mathbb{T}}\varphi)(t)) = \mathcal{P}_k([f(t) - \sum_{j=1}^{m}\lambda_j a_j(t)]a(\theta(t))). \qquad (6.96)$$

By using (6.89), we can verify that

$$\begin{aligned}\mathcal{P}_k[a(\theta(t))a(t)\varphi(t)](t) =& a(\theta(t))a(t)(\mathcal{P}_k\varphi)(t)\\ \mathcal{P}_k[a(\theta(t))b(t)(\mathcal{P}_1 S_{\mathbb{T}}\varphi)](t) =& [ab]^*_{3-k}(t)(\mathcal{P}_1 S_{\mathbb{T}}\varphi)(t).\end{aligned}$$

Therefore, we are able rewrite (6.96) as

$$\begin{aligned}a(\theta(t))a(t)(\mathcal{P}_k\varphi(t)) + [ab]^*_{3-k}(t)(\mathcal{P}_1 S_{\mathbb{T}}\varphi)(t) =\\ \mathcal{P}_k([f(t) - \sum_{j=1}^{m}\lambda_j a_j(t)]a(\theta(t))).\end{aligned}$$

Now, by Proposition 6.9 we have that $\mathcal{P}_1 S_{\mathbb{T}} = S_{\mathbb{T}}\mathcal{P}_1$ for the $S_{\mathbb{T}}W = WS_{\mathbb{T}}$ case and $\mathcal{P}_1 S_{\mathbb{T}} = S_{\mathbb{T}}\mathcal{P}_2$ for the case of $S_{\mathbb{T}}W = -WS_{\mathbb{T}}$. Thus $(\mathcal{P}_1\varphi, \mathcal{P}_2\varphi)$ is a solution of (6.94).

Conversely, suppose that there exists φ such that $(\mathcal{P}_1\varphi, \mathcal{P}_2\varphi)$ is a solution of (6.94). Summing k from 1 to 2, we directly obtain that

$$\begin{aligned}&\sum_{k=1}^{2}\left[a(\theta(t))a(t)\varphi_k(t) + [ab]^*_{3-k}(t)(S_{\mathbb{T}}\varphi_i)(t)\right]\\ &= \sum_{k=1}^{2}\mathcal{P}_k([f(t) - \sum_{j=1}^{m}\lambda_j a_j(t)]a(\theta(t))), \qquad i = 1, 2,\end{aligned}$$

is equivalent to

$$a(\theta(t))a(t)\varphi(t) + a(\theta(t))b(t)(S_{\mathbb{T}}\varphi_i)(t) = [f(t) - \sum_{j=1}^{m}\lambda_j a_j(t)]a(\theta(t)), \quad i = 1, 2$$

and this implies that

$$a(t)\varphi(t) + b(t)\mathcal{P}_1(S_{\mathbb{T}}\varphi)(t) = f(t) - \sum_{j=1}^{m}\lambda_j a_j(t).$$

□

Proposition 6.11. *If* (ϕ_1, ϕ_2) *is a solution of the system* (6.94)*, then* $(\mathcal{P}_1\phi_1, \mathcal{P}_2\phi_2)$ *is also a solution of* (6.94)*.*

Proof. Let (ϕ_1, ϕ_2) be a solution of the system (6.94). Applying the projections $\mathcal{P}_k$ to both sides of (6.94), we have

$$\mathcal{P}_k\left(a_\theta(t)\phi_k(t) + [ab]^*_{3-k}(t)(S_{\mathbb{T}}\phi_i)(t)\right) = \mathcal{P}_k([af]_k(t)), \quad k, i = 1, 2. \tag{6.97}$$

Notice that $\mathcal{P}_k[a_\theta(t)\phi_k](t) = a_\theta(t)\mathcal{P}_k\phi_k(t)$ and

$$\begin{aligned}\mathcal{P}_k([ab]^*_{3-k}(t)(S_\mathbb{T}\phi_i))(t) =& \frac{1}{2}\left\{[ab]^*_{3-k}(t)(S_\mathbb{T}\phi_i)(t)\right.\\ &\left.+(-1)^k[ab]^*_{3-k}(\theta(t))W(S_\mathbb{T}\phi_i)(t)\right\}\\ =&[ab]^*_{3-k}(t)\frac{1}{2}\{(S_\mathbb{T}\phi_i)(t) - W(S_\mathbb{T}\phi_i)(t)\}. \qquad (6.98)\end{aligned}$$

Equality (6.98) holds because $[ab]^*_{3-k}(t) = (-1)^{3-k}[ab]^*_{3-k}(\theta(t))$. Thus the right-hand side of equality (6.98) can be rewritten as $[ab]^*_{3-k}(t)\mathcal{P}_1(S_\mathbb{T}\phi_i)(t)$. From (6.94), the value of the index i depends on the commuting property of the shift operator with $S_\mathbb{T}$. Therefore,

$$\mathcal{P}_k([ab]^*_{3-k}(t)(S_\mathbb{T}\phi_i))(t) = [ab]^*_{3-k}(t)(S_\mathbb{T}\mathcal{P}_i\phi_i)(t).$$

Finally, note that $\mathcal{P}_k[af]_k(t) = [af]_k(t)$. Therefore, $(\mathcal{P}_1\phi_1, \mathcal{P}_2\phi_2)$ is a solution of (6.94). □

A corresponding result to the previous Theorem 6.6 also holds in the present case, and takes the following form.

Theorem 6.12. *The equation* (6.93) *has solutions in* $L^p(\mathbb{T})$ *if and only if the following equation*

$$\begin{aligned}&a_\theta(t)\varphi_1(t) + [ab]^*_2(t)(S_\mathbb{T}\varphi_1)(t) = [af]_1(t), \quad \text{if } S_\mathbb{T}W = WS_\mathbb{T}\\ &\text{or}\\ &a_\theta(t)\varphi_2(t) + [ab]^*_1(t)(S_\mathbb{T}\varphi_2)(t) = [af]_2(t), \quad \text{if } S_\mathbb{T}W = -WS_\mathbb{T}\end{aligned} \qquad (6.99)$$

has solutions. Moreover, if $\varphi_k(t)$ $(k = 1, 2)$ *is a solution of equation* (6.99)*, then equation* (6.93) *has a solution given by formula*

$$\varphi(t) = \begin{cases} \frac{f(t)-\sum_{j=1}^m \lambda_j a_j(t) - b(t)(\mathcal{P}_1 S_\mathbb{T}\varphi_1)(t)}{a(t)}, & \text{if } S_\mathbb{T}W = WS_\mathbb{T}\\ \frac{f(t)-\sum_{j=1}^m \lambda_j a_j(t) - b(t)(\mathcal{P}_1 S_\mathbb{T}\varphi_2)(t)}{a(t)}, & \text{if } S_\mathbb{T}W = -WS_\mathbb{T}.\end{cases} \qquad (6.100)$$

Proof. Suppose that $\varphi \in L^p(\mathbb{T})$ is a solution of equation (6.93). By Proposition 6.10 we know that $(\mathcal{P}_1\varphi, \mathcal{P}_2\varphi)$ is a solution of system (6.94). Hence, for the $S_\mathbb{T}W = WS_\mathbb{T}$ case, $\mathcal{P}_1\varphi$ is a solution of (6.99) and $\mathcal{P}_2\varphi$ is the corresponding solution for the $S_\mathbb{T}W = -WS_\mathbb{T}$ case.

Conversely, suppose that φ_1 is a solution of (6.99). Without loss of generality, we assume now that $S_\mathbb{T}W = WS_\mathbb{T}$ (since the situation of $S_\mathbb{T}W = -WS_\mathbb{T}$ is dealt similarly). In this case, the system (6.94) has a solution (φ_1, φ_2) determined by

$$\varphi_2(t) = \frac{[af]_2(t) - [ab]^*_1(t)(S_\mathbb{T}\varphi_1)(t)}{a_\theta(t)}. \qquad (6.101)$$

By Proposition 6.11 we have that $(\mathcal{P}_1\varphi_1, \mathcal{P}_2\varphi_2)$ is also a solution of (6.94). Set $\varphi = \mathcal{P}_1\varphi_1 + \mathcal{P}_2\varphi_2$. It is clear that $\mathcal{P}_k\varphi = \mathcal{P}_k\varphi_k$. This means that $(\mathcal{P}_1\varphi, \mathcal{P}_2\varphi)$ is a solution of (6.96). From Proposition 6.10 it follows that φ is a solution of (6.94). Moreover, from (6.101), we obtain

$$\varphi(t) = \sum_{k=1}^{2} \mathcal{P}_k \left[\frac{[af]_k(t) - [ab]^*_{3-k}(S_{\mathbb{T}}\varphi_1)(t)}{a_\theta(t)} \right]. \tag{6.102}$$

As before, we can see that

$$\begin{aligned} &\sum_{k=1}^{2} \mathcal{P}_k[af]_k(t) = [f(t) - \sum_{j=1}^{m} \lambda_j a_j(t)]a(\theta(t)), \\ &\sum_{k=1}^{2} \mathcal{P}_k([ab]^*_{3-k}(S_{\mathbb{T}}\varphi_1)(t)) = a(\theta(t))b(t)(\mathcal{P}_1 S_{\mathbb{T}}\varphi_1)(t). \end{aligned}$$

Thus, substituting these in (6.102), we have

$$\varphi(t) = \frac{f(t) - \sum_{j=1}^{m} \lambda_j a_j(t) - b(t)(\mathcal{P}_1 S_{\mathbb{T}}\varphi_1)(t)}{a(t)}.$$

□

6.5.2 The BVP associate to the equation (6.99)

Equation (6.99) can be solved as in sections 6.3 and 6.4. I.e., by means of an associated Riemann boundary value problem.

Letting

$$\varpi(t) = \begin{cases} \dfrac{[ab]^*_2(t) - a_\theta(t)}{[ab]^*_2(t) + a_\theta(t)}, & \text{if } S_{\mathbb{T}}W = WS_{\mathbb{T}} \\[2ex] \dfrac{[ab]^*_1(t) - a_\theta(t)}{[ab]^*_1(t) + a_\theta(t)}, & \text{if } S_{\mathbb{T}}W = -WS_{\mathbb{T}}, \end{cases}$$

$$h(t) = \begin{cases} \dfrac{[af]_1(t)}{[ab]^*_2(t) + a_\theta(t)}, & \text{if } S_{\mathbb{T}}W = WS_{\mathbb{T}} \\[2ex] \dfrac{[af]_2(t)}{[ab]^*_1(t) + a_\theta(t)}, & \text{if } S_{\mathbb{T}}W = -WS_{\mathbb{T}}, \end{cases}$$

$$\varphi_k(t) = \frac{1}{2}(\varphi(t) + (-1)^k v(t)\varphi(\theta(t))), \quad k = 1, 2$$

and

$$\Pi_k(t) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\varphi_k(\tau)}{\tau - z}d\tau, \quad z \in \mathbb{C}\setminus\mathbb{T}.$$

We get that equation (6.99) is reduced to the problem

$$\Pi_k^+(t) + \varpi(t)\Pi_k^-(t) = h(t) \tag{6.103}$$

imposed on their boundary values on $\mathbb{T}$, where $k = 1$ in case of W being a commutative weighted Carleman shift operator, and $k = 2$ in case W being a weighted Carleman shift operator of anti-commutative type.

In order to solve this problem we assume that the functions ϖ admit a factorization in $L^p(\mathbb{T})$ (as the function ψ in Section 6.3). I.e.,

$$\varpi(t) = \varpi_+(t)t^{\aleph}\varpi_-(t). \tag{6.104}$$

Furthermore we assume that:

$$\varpi_+^{-1}h \in \mathcal{L}^1(\mathbb{T}) := (H_+^1(\mathbb{T}) \oplus H_-^1(\mathbb{T})), \tag{6.105}$$

$$\upsilon_0^+ := \varpi_+P_+\varpi_+^{-1}h \in H_+^p(\mathbb{T}), \tag{6.106}$$

$$\upsilon_0^- := \varpi_-^{-1}t^{-\aleph}P_-\varpi_+^{-1}h \in H_-^p(\mathbb{T}). \tag{6.107}$$

The general solutions of problem (6.103) (with $k = 1$ if $S_{\mathbb{T}}W = WS_{\mathbb{T}}$ and $k = 2$ in case $S_{\mathbb{T}}W = -WS_{\mathbb{T}}$) have the form

$$\Pi_k^+ = \upsilon_0^+ + \varpi_+p_{\aleph-1}, \qquad \Pi_k^- = \upsilon_0^- + \varpi_-^{-1}t^{-\aleph}p_{\aleph-1}$$

where

$$p_{\aleph-1}(z) = p_1 + p_2z + \cdots + p_{\aleph}z^{\aleph-1}, \qquad \text{if } \aleph \geq 1$$

is a polynomial of degree less than or equal to $\aleph - 1$ if $\aleph > 0$, and equal to zero if $\aleph \leq 0$. The representation of the solutions can be rewritten in the following form:

$$\Pi_k^+(z) = \varpi_+(z)[\mathcal{A}(z) + p_{\aleph-1}(z)] \tag{6.108}$$

$$\Pi_k^-(z) = \varpi_-^{-1}(z)z^{-\aleph}[\mathcal{C}(z) + p_{\aleph-1}(z)], \tag{6.109}$$

where the functions $\mathcal{A}$ and $\mathcal{C}$ are given by

$$\mathcal{A}(z) = \begin{cases} P_+\left(\varpi_+^{-1}(\cdot)\frac{[af]_1(\cdot)}{[ab]_2^*(\cdot)+a_\theta(\cdot)}\right)(z), & \text{if } WS_{\mathbb{T}} = S_{\mathbb{T}}W \\[2ex] P_+\left(\varpi_+^{-1}(\cdot)\frac{[af]_2(\cdot)}{[ab]_1^*(\cdot)+a_\theta(\cdot)}\right)(z), & \text{if } WS_{\mathbb{T}} = -S_{\mathbb{T}}W, \end{cases} \tag{6.110}$$

$$\mathcal{C}(z) = \begin{cases} P_-\left(\varpi_+^{-1}(\cdot)\frac{[af]_1(\cdot)}{[ab]_2^*(\cdot)+a_\theta(\cdot)}\right)(z), & \text{if } WS_{\mathbb{T}} = S_{\mathbb{T}}W \\ \\ P_-\left(\varpi_+^{-1}(\cdot)\frac{[af]_2(\cdot)}{[ab]_1^*(\cdot)+a_\theta(\cdot)}\right)(z), & \text{if } WS_{\mathbb{T}} = -S_{\mathbb{T}}W. \end{cases} \tag{6.111}$$

In addition, for the case of $\aleph < 0$, the problem (6.103) is solvable if the following conditions holds:

$$\int_{\mathbb{T}} \varpi_+^{-1}(\tau)h(\tau)\tau^k d\tau = 0, \qquad k = 0, \dots, -(\aleph - 1).$$

This condition can be rewritten as follows:

$$\begin{cases} \displaystyle\int_{\mathbb{T}} \frac{\varpi_+^{-1}(\tau)[af]_1(\tau)}{[ab]_2^*(\tau)+a_\theta(\tau)}\tau^k d\tau = 0, & \text{if } WS_{\mathbb{T}} = S_{\mathbb{T}}W \\ \\ \displaystyle\int_{\mathbb{T}} \frac{\varpi_+^{-1}(\tau)[af]_2(\tau)}{[ab]_1^*(\tau)+a_\theta(\tau)}\tau^k d\tau = 0, & \text{if } WS_{\mathbb{T}} = -S_{\mathbb{T}}W. \end{cases} \tag{6.112}$$

The following results summarize all the above mentioned.

Theorem 6.13. *Suppose that the functions $[ab]_k^*(t) \pm a_\theta(t)$ $(k = 1, 2)$ do not vanish on $\mathbb{T}$ and that the functions $\varpi = \frac{[ab]_k^*-a_\theta}{[ab]_k^*+a_\theta}$ $(k = 1, 2)$ admit a factorization in $L^p(\mathbb{T})$, say $\varpi(t) = \varpi_+(t)t^{\aleph}\varpi_-(t)$. If the p-index $\aleph$ is greater or equal to zero, then equation* (6.99) *has solutions φ_k $(k = 1, 2)$ which satisfy the following formula*

$$(S_{\mathbb{T}}\varphi_k)(t) = \varpi_+(t)[\mathcal{A}(z) + p_{\aleph-1}(t)] + \varpi_-^{-1}t^{-\aleph}[\mathcal{C}(t) + p_{\aleph-1}(t)], \tag{6.113}$$

where $\varpi_\pm^{\pm1}$ are the outer factors (and their inverses) in the factorization of the functions ϖ. $\mathcal{A}$ and $\mathcal{C}$ are the functions defined in (6.110) *and* (6.111), *and $p_{\aleph-1}$ is a polynomial of degree less than or equals to $\aleph - 1$ which is identically equal to zero if $\aleph = 0$. In the case that $\aleph < 0$, the equation* (6.99) *is solvable if the condition* (6.112) *is satisfied. In this case equation* (6.99) *has a unique solution which satisfies the formula* (6.113), *where $p_{\aleph-1}(t) \equiv 0$.*

Proof. We know that under conditions (6.105)–(6.107) the boundary value problem (6.103) defined on $L^p(\mathbb{T})$ has a solution given by (6.108) and (6.109). On the other hand, from the Sokhotski-Plemelij formulas we have that equation (6.99) has solutions φ_k (for $k = 1, 2$ depending on the commutative property of the shift operator W) determined by

$$\varpi(t) = \Pi_k^+(t) - \Pi_k^-(t).$$

The conclusions are obtained from $(S_{\mathbb{T}}\varphi_k)(t) = \Pi_k^+(t) + \Pi_k^-(t)$, applying the required conditions. □

6.5.3 The explicit solutions of equation (6.2) conditioned to (6.92)

In this part, we will exhibit the explicit representation of the solutions of equation (6.2) satisfying condition (6.92). We are going to use the solutions given in Theorem 6.13 for such a goal.

Since the representation of the solutions depends on the sign of the p-index $\aleph$ of the factorization (6.104), we will consider the next two different cases:

Case $\aleph \geq 0$ From Theorem 6.12 we know that the solutions of equation (6.93) are given by

$$\varphi(t) = \begin{cases} \frac{f(t)-\sum_{j=1}^{m}\lambda_j a_j(t)-b(t)(\mathcal{P}_1 S_{\mathbb{T}}\varphi_1)(t)}{a(t)}, & \text{if } S_{\mathbb{T}}W = WS_{\mathbb{T}} \\ \frac{f(t)-\sum_{j=1}^{m}\lambda_j a_j(t)-b(t)(\mathcal{P}_1 S_{\mathbb{T}}\varphi_2)(t)}{a(t)}, & \text{if } S_{\mathbb{T}}W = -WS_{\mathbb{T}}, \end{cases} \tag{6.114}$$

moreover, by Theorem 6.13 we know that

$$(S_{\mathbb{T}}\varphi_k)(t) = \varpi_+(t)[\mathcal{A}(z) + p_{\aleph-1}(t)] + \varpi_-^{-1}t^{-\aleph}[\mathcal{C}(t) + p_{\aleph-1}(t)] \tag{6.115}$$

where, as in (6.54) and (6.55), $\mathcal{A}$ and $\mathcal{C}$ have the form

$$\mathcal{A}(z) = \Theta_1(z) - \sum_{v=1}^{m}\lambda_v \Xi_{1v}(z) \tag{6.116}$$

$$\mathcal{C}(z) = \Theta_2(z) - \sum_{v=1}^{m}\lambda_v \Xi_{2v}(z), \tag{6.117}$$

with $\Theta_1,\ \Xi_{1v},\ \Theta_2$ and Ξ_{2v} $(v = 1, \ldots, m)$ in this case defined by the rule

$$\Theta_1(z) = \begin{cases} P_+\left(\frac{\varpi_+^{-1}(\cdot)\frac{1}{2}\sum_{j=1}^{2}(-1)^{j+1}f(\theta_{j+1}(\cdot))a(\theta_j(\cdot))(v(\cdot))^{j-1}}{[ab]_2^*(\cdot)+a_\theta(\cdot)}\right)(z), \\ P_+\left(\frac{\varpi_+^{-1}(\cdot)\frac{1}{2}\sum_{j=1}^{2}f(\theta_{j+1}(\cdot))a(\theta_j(\cdot))(v(\cdot))^{j-1}}{[ab]_1^*(\cdot)+a_\theta(\cdot)}\right)(z), \end{cases} \tag{6.118}$$

$$\Xi_{1v}(z) = \begin{cases} P_+\left(\frac{\varpi_+^{-1}(\cdot)\frac{1}{2}\sum_{j=1}^{2}(-1)^{j+1}a_v(\theta_{j+1}(\cdot))a(\theta_j(\cdot))(v(\cdot))^{j-1}}{[ab]_2^*(\cdot)+a_\theta(\cdot)}\right)(z), \\ P_+\left(\frac{\varpi_+^{-1}(\cdot)\frac{1}{2}\sum_{j=1}^{2}a_v(\theta_{j+1}(\cdot))a(\theta_j(\cdot))(v(\cdot))^{j-1}}{[ab]_1^*(\cdot)+a_\theta(\cdot)}\right)(z), \end{cases} \tag{6.119}$$

$$\Theta_2(z) = \begin{cases} P_-\left(\frac{\varpi_+^{-1}(\cdot)\frac{1}{2}\sum_{j=1}^{2}(-1)^{j+1}f(\theta_{j+1}(\cdot))a(\theta_j(\cdot))(v(\cdot))^{j-1}}{[ab]_2^*(\cdot)+a_\theta(\cdot)}\right)(z), \\ \\ P_-\left(\frac{\varpi_+^{-1}(\cdot)\frac{1}{2}\sum_{j=1}^{2}(f(\theta_{j+1}(\cdot))a(\theta_j(\cdot))(v(\cdot))^{j-1}}{[ab]_1^*(\cdot)+a_\theta(\cdot)}\right)(z), \end{cases} \tag{6.120}$$

$$\Xi_{2v}(z) = \begin{cases} P_-\left(\frac{\varpi_+^{-1}(\cdot)\frac{1}{2}\sum_{j=1}^{2}(-1)^{j+1}a_v(\theta_{j+1}(\cdot))a(\theta_j(\cdot))(v(\cdot))^{j-1}}{[ab]_2^*(\cdot)+a_\theta(\cdot)}\right)(z), \\ \\ P_-\left(\frac{\varpi_+^{-1}(\cdot)\frac{1}{2}\sum_{j=1}^{2}a_v(\theta_{j+1}(\cdot))a(\theta_j(\cdot))(v(\cdot))^{j-1}}{[ab]_1^*(\cdot)+a_\theta(\cdot)}\right)(z), \end{cases} \tag{6.121}$$

if $S_{\mathbb{T}}W = WS_{\mathbb{T}}$ or $S_{\mathbb{T}}W = -WS_{\mathbb{T}}$ correspondingly in each case. Substituting (6.116) and (6.117) into (6.115) we obtain

$$\begin{aligned}(S_{\mathbb{T}}\varphi_k)(t) =& \varpi_+(t)\Theta_1(t) + \varpi_-^{-1}t^{-\aleph}\Theta_2(t) - \sum_{v=1}^{m}\lambda_v(\varpi_+(t)\Xi_{1v}(t) \\ &+ \varpi_-^{-1}(t)\Xi_{2v}(t)t^{-\aleph}) + (\varpi_+(t) + \varpi_-^{-1}(t)t^{-\aleph})\sum_{j=1}^{\aleph}p_j t^{j-1}.\end{aligned}$$

Then, we can rewrite (6.114) in the form

$$\begin{aligned}\varphi(t) =& \frac{f(t) - b(t)\mathcal{P}_1[\varpi_+(t)\Theta_1(t) + \varpi_-^{-1}(t)t^{-\aleph}\Theta_2(t)]}{a(t)} \\ &- \sum_{j=1}^{m}\lambda_j\frac{a_j(t) - b(t)\mathcal{P}_1[\Xi_{1j}(t)\varpi_+(t) + \Xi_{2j}(t)\varpi_-^{-1}(t)t^{-\aleph}]}{a(t)} \\ &- \sum_{j=1}^{\aleph}p_j\frac{b(t)\mathcal{P}_1[(\varpi_+(t) + \varpi_-^{-1}(t)t^{-\aleph})t^{j-1}]}{a(t)}\end{aligned} \tag{6.122}$$

with $\varpi_\pm^{\pm 1}, \Theta_1, \Xi_{1j}, \Theta_2, \Xi_{2j}$ $(j = 1, \ldots, m)$ determined by (6.104), (6.118), (6.119), (6.120) and (6.121) respectively, (where we recall that the form of these functions depend on the commutative nature of the weighted Carleman shift operator W and $p_1, \ldots, p_\aleph$ are arbitrary). The function φ is a solution of the equation (6.2) if it satisfies the condition (6.92) that is:

$$\mathcal{M}_{b_j}(\varphi) = \lambda_j \qquad j = 1, \ldots, m.$$

Substituting (6.122) into the last condition we obtain

$$\lambda_\iota = d_\iota - \sum_{j=1}^{m}\lambda_j e_{\iota j} - \sum_{j=1}^{\aleph}p_j g_{\iota j}, \quad \iota = 1, \ldots, m \tag{6.123}$$

where d_ι, $e_{\iota j}$ and $g_{\iota j}$ are given by

$$\begin{aligned} d_\iota(t) :=& \mathcal{M}_{b_\iota}\left(\frac{f(t) - b(t)\mathcal{P}_1[\varpi_+(t)\Theta_1(t) + \varpi_-^{-1}(t)t^{-\aleph}\Theta_2(t)]}{a(t)}\right), \\ e_{\iota j}(t) :=& \mathcal{M}_{b_\iota}\left(\frac{a_j(t) - b(t)\mathcal{P}_1[\Xi_{1j}(t)\varpi_+(t) + \Xi_{2j}(t)\varpi_-^{-1}(t)t^{-\aleph}]}{a(t)}\right), \\ g_{\iota j}(t) :=& \mathcal{M}_{b_\iota}\left(\frac{b(t)\mathcal{P}_1\left(t^{j-1}(\varpi_+(t) + \varpi_-^{-1}(t)t^{-\aleph})\right)}{a(t)}\right). \end{aligned} \tag{6.124}$$

So, we can rewrite equation (6.123) in the form of the following matricial identity:

$$(I_{m\times m} + \mathcal{E})\lambda = \mathcal{D} - \mathcal{G}P \tag{6.125}$$

with $\mathcal{E}$, $\mathcal{D}$, $\mathcal{G}$, P and λ as in (6.75) but with the entries on (6.124). Thus, we can formulate that the function determined by (6.122) is a solution of (6.7) if and only if $(\lambda_1, \ldots, \lambda_m)$ satisfies the condition (6.125).

Case $\aleph < 0$ In this case from Theorems 6.12 and 6.13 we have that equation (6.93) has solutions if the condition (6.112) is satisfied. Since $p_{\aleph-1} \equiv 0$, then the solutions of (6.93) are given by

$$\begin{aligned} \varphi(t) =& \frac{f(t) - b(t)\mathcal{P}_1[\varpi_+(t)\Theta_1(t) + \varpi_-^{-1}(t)t^{-\aleph}\Theta_2(t)]}{a(t)} \\ &- \sum_{j=1}^{m}\lambda_j \frac{a_j(t) - b(t)\mathcal{P}_1[\Xi_{1j}(t)\varpi_+(t) + \Xi_{2j}(t)\varpi_-^{-1}(t)t^{-\aleph}]}{a(t)}. \end{aligned} \tag{6.126}$$

The equation (6.126) is a solution of equation (6.91) if and only if $(\lambda_1, \ldots, \lambda_m)$ satisfy the following matricial condition

$$(I_{m\times m} + \mathcal{E})\lambda = \mathcal{D}. \tag{6.127}$$

On the other hand, since condition (6.112) is necessary for the solvability of the problem (6.103), then we rewrite it (using (6.95)) as

$$d'_\eta = \sum_{v=1}^{m} e'_{\eta v}\lambda_v \tag{6.128}$$

with d'_η and $e'_{\eta v}$ given by

$$d'_\eta = \begin{cases} \int_{\mathbb{T}} \frac{\varpi_+^{-1}(\tau)\frac{1}{2}\sum_{j=1}^{2}(-1)^{j+1}f(\theta_{j+1}(t))a(\theta_j(\tau))(v(\tau))^{j-1}}{[ab]_2^*(\tau)+a_\theta(\tau)}\tau^\eta d\tau, & \text{if } S_{\mathbb{T}}W = WS_{\mathbb{T}} \\ \int_{\mathbb{T}} \frac{\varpi_+^{-1}(\tau)\frac{1}{2}\sum_{j=1}^{2}f(\theta_{j+1}(t))a(\theta_j(\tau))(v(\tau))^{j-1}}{[ab]_1^*(\tau)+a_\theta(\tau)}\tau^\eta d\tau, & \text{if } S_{\mathbb{T}}W = -WS_{\mathbb{T}} \end{cases} \tag{6.129}$$

and

$$e'_{\eta v} = \begin{cases} \int_{\mathbb{T}} \frac{\varpi_+^{-1}(\tau)\frac{1}{2}\sum_{j=1}^{2}(-1)^{j+1}a_v(\theta_{j+1}(t))a(\theta_j(\tau))(v(\tau))^{j-1}}{[ab]_2^*(\tau)+a_\theta(\tau)}\tau^\eta d\tau, & S_{\mathbb{T}}W = WS_{\mathbb{T}} \\ \int_{\mathbb{T}} \frac{\varpi_+^{-1}(\tau)\frac{1}{2}\sum_{j=1}^{2}a_v(\theta_{j+1}(t))a(\theta_j(\tau))(v(\tau))^{j-1}}{[ab]_1^*(\tau)+a_\theta(\tau)}\tau^\eta d\tau, & \text{if } S_{\mathbb{T}}W = -WS_{\mathbb{T}}. \end{cases} \tag{6.130}$$

As in (6.68), equality (6.128) can be written in the following matricial form

$$\mathcal{D}' = \mathcal{E}'\lambda. \tag{6.131}$$

The matrices $\mathcal{D}'$ and $\mathcal{E}'$ are defined as in (6.82) with entries given in (6.129) and (6.130). Combining (6.127) and (6.131) we conclude that φ determined by (6.126) is a solution of (6.91) if and only $(\lambda_1, \dots \lambda_m)$ satisfies the following matricial condition

$$\begin{pmatrix} I_{m\times m} + \mathcal{E} \\ \mathcal{E}' \end{pmatrix}_{(m-\aleph)\times m} \lambda = \begin{pmatrix} \mathcal{D} \\ \mathcal{D}' \end{pmatrix}_{(m-\aleph)\times 1}. \tag{6.132}$$

Theorem 6.14. *Suppose that the functions* $[ab]_k^*(t) \pm a_\theta(t)$ $(k = 1, 2)$ *do not vanish on* $\mathbb{T}$, *and that the functions* $\varpi = \frac{[ab]_k^* - a_\theta}{[ab]_k^* + a_\theta}$ $(k = 1, 2)$ *admit a factorization in* $L^p(\mathbb{T})$, *say* $\varpi(t) = \varpi_+(t)t^\aleph\varpi_-(t)$.

(1.) *If* $\aleph \geq 0$, *consider*

$$r = \operatorname{rank}((I_{m\times m} + \mathcal{E}) \quad \mathcal{G})_{m\times(m+\aleph)},$$

where $\mathcal{E}$ *and* $\mathcal{G}$ *are defined as in* (6.75) *but with entries given by* (6.124). *Then, the equation* (6.93) *is solvable if and only if the matrix* $\mathcal{D}$, *determined as in* (6.75) *and having entries defined by* (6.124), *satisfies the condition*

$$\operatorname{rank}((I_{m\times m} + \mathcal{E}) \quad \mathcal{G} \quad \mathcal{D})_{m\times(m+\aleph+1)} = r.$$

If this is the case, the solutions of the equation (6.93) *are given by the formula* (6.122), *where* $(\lambda_1, \dots, \lambda_m, p_1, \dots, p_\aleph)$ *satisfies* (6.125). *Moreover, we can choose* $m + \aleph - r$ *coefficients in* $\{\lambda_1, \dots, \lambda_m, p_1, \dots, p_\aleph\}$ *which are arbitrary so that* φ *is uniquely determined by these coefficients. In particular, if* $r = m$ *then the equation* (6.93) *is solvable for any function* f.

(2.) *If* $\aleph < 0$, *let*

$$r = \operatorname{rank}\begin{pmatrix} I_{m\times m} + \mathcal{E} \\ \mathcal{E}' \end{pmatrix}_{(m-\aleph)\times m},$$

with $\mathcal{E}$ and $\mathcal{E}'$ as in (6.75) *and* (6.82) *whose entries are given in* (6.124) *and* (6.129)*, respectively. The equation* (6.93) *is solvable if and only if the function f determines $\mathcal{D}$ and $\mathcal{D}'$ as in the formulas* (6.75) *and* (6.82) (*with entries on* (6.124) *and* (6.129)*, respectively*) *which satisfy the following matricial condition*

$$\operatorname{rank}\begin{pmatrix} I_{m\times m}+\mathcal{E} & \mathcal{D} \\ \mathcal{E}' & \mathcal{D}' \end{pmatrix}_{(m-\aleph)\times(m+1)} = r. \tag{6.133}$$

If this is the case, the solutions of the equation (6.93) *are given by the formula* (6.126)*, where* $(\lambda_1\ldots,\lambda_m)$ *satisfies* (6.132)*. In particular, if $r=m$ and the condition* (6.133) *is satisfied, then the equation* (6.93) *has an unique solution.*

Proof. The proof runs analogously to the proof of Theorem 6.8. □

6.6 The solvability of equation (6.3)

This last part of the chapter will be devoted to the existence and uniqueness of the eventual solutions of equation (6.3). We are going to adapt the results of the previous sections to the present case.

6.6.1 Main result

We start by introducing the following facts and notation: Let

$$\varphi_k(t) := \frac{1}{2}(\varphi(t)+(-1)^k v(t)\varphi(\alpha(t))), \quad k=1,2$$

and consider

$$\Phi_k(z) := \frac{1}{2\pi i}\int_\Gamma \frac{\varphi_k(\tau)}{\tau - z}d\tau, \qquad z\in\mathbb{C}\setminus\Gamma,$$

which is known to be an analytic function in $D^\pm$, with $\Phi_k(\infty)=0$, and admitting the non-tangential limits $\Phi_k^\pm(t) := \lim_{D^\pm\ni z\to t}\Phi_k(z)$ almost everywhere on Γ. Moreover, from the Sokhotskii-Plemelj formulas, we know that

$$\begin{aligned}\varphi_k(t) =& \Phi_k^+(t)-\Phi_k^-(t)\\ (S_\Gamma\varphi_k)(t) =& \Phi_k^+(t)+\Phi_k^-(t).\end{aligned}$$

In addition, let us introduce the following functions

$$G(t) = \begin{cases} \frac{f_\alpha(t)-[fg]_2^*(t)}{f_\alpha(t)+[fg]_2^*(t)}, & \text{if } S_\Gamma W = WS_\Gamma \\ \frac{f_\alpha(t)-[fg]_1^*(t)}{f_\alpha(t)+[fg]_1^*(t)}, & \text{if } S_\Gamma W = -WS_\Gamma, \end{cases} \tag{6.134}$$

and

$$H(t) = \begin{cases} \frac{[fh]_2(t)}{f_\alpha(t)+[fg]_2^*(t)}, & \text{if } S_\Gamma W = W S_\Gamma \\ \frac{[fh]_1(t)}{f_\alpha(t)+[fg]_1^*(t)}, & \text{if } S_\Gamma W = -W S_\Gamma \end{cases} \tag{6.135}$$

where, for $k = 1, 2$,

$$f_\alpha(t) := f(t) f(\alpha(t)), \tag{6.136}$$

$$[fg]_k^*(t) := f(\alpha(t)) g(t) + (-1)^k f(t) g(\alpha(t)), \tag{6.137}$$

$$[fh]_k(t) := \frac{1}{2}(f(\alpha(t)) h(t) + (-1)^k v(t) f(t) h(\alpha(t))). \tag{6.138}$$

We assume that $G(t)$ is a continuous function and that both $f(t)$ and $G(t)$ are non-vanishing on Γ. Putting

$$n := \frac{1}{2\pi} \int_\Gamma d(\arg G(t))$$

and fixing a branch of $\ln(t^{-n} G(t))$, we introduce the functions

$$\chi^+(z) = e^{\Upsilon^+(z)}, \qquad \chi^-(z) = z^{-n} e^{\Upsilon^-(z)}, \tag{6.139}$$

considering

$$\Upsilon(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\ln(\tau^{-n} G(\tau))}{\tau - z} d\tau,$$

and having $\Upsilon^+(z) = \Upsilon(z)$ if $z \in D^+$ and $\Upsilon^-(z) = \Upsilon(z)$ if $z \in D^-$.

We are now in condition to state the main theorem of this section.

Theorem 6.15. *Equation* (6.3) *has solutions and they are given by*

$$\varphi(t) = \frac{h(t) - 2g(t)(S_\Gamma \varphi_k)(t)}{f(t)}, \quad k = 1, 2,$$

where $k = 1$ in case of $S_\Gamma W = W S_\Gamma$, and $k = 2$ if $S_\Gamma W = -W S_\Gamma$. In addition, for computing $(S_\Gamma \varphi_k)(t) = \Phi_k^+(t) + \Phi_k^-(t)$, we have the following different situations:

$(n \geq 0)$**:** *In this case we have*

$$\Phi_k^\pm(z) = \chi^\pm(z) \Psi^\pm(z) + \chi^\pm(z) P_{n-1}(z) \tag{6.140}$$

where

$$\Psi(z) = \frac{1}{2\pi i} \int_\Gamma \frac{H(\tau)}{\chi^+(\tau)} \frac{d\tau}{\tau - z}$$

and $P_{n-1}(z) \equiv 0$ if $n = 0$ and $P_{n-1}(z)$ is a polynomial of degree no greater than $n-1$ with arbitrary complex coefficients $c_0, c_1, \ldots, c_{n-1}$ for $n > 0$. If $n = 0$, then the equation (6.3) *has a unique solution.*

$(n<0)$**:** *For this case, we assume:*

$$\int_\Gamma \frac{H(t)t^{\eta-1}}{\chi^+(t)}dt = 0, \quad \eta = 1,2,\ldots,-n.$$

Then, we have that $P_{n-1}(z) \equiv 0$ *in equality* (6.140).

The proof of this result will be presented in the last part of the section, after obtaining some auxiliary results in the next subsection.

6.6.2 On the eventual solutions of equation (6.3)

Notice that to equation (6.3) is associated the singular integral operator

$$fI_\Gamma + gS_\Gamma + gWS_\Gamma : L^p(\Gamma) \longrightarrow L^p(\Gamma) \tag{6.141}$$

where W is the weighted Carleman shift operator induced by the complex-valued function v and the shift function α, i.e.,

$$(W\phi)(t) = v(t)\phi(t).$$

Let us consider, in this case, the following complementary projections

$$P_1 := \frac{1}{2}(I_\Gamma - W) \quad \text{and} \quad P_2 := \frac{1}{2}(I_\Gamma + W) \tag{6.142}$$

which, as before, are rewriten as

$$P_k = \frac{1}{2}\sum_{j=1}^{2}(-1)^{k(1-j)}W^{j+1}, \quad k = 1,2. \tag{6.143}$$

Notice that for these complementary projections, Proposition 6.9 holds. Also note that with the projection operators P_2 we can rewrite equation (6.3) as follows:

$$f(t)\varphi(t) + 2g(t)(P_2S_\Gamma\varphi)(t) = h(t). \tag{6.144}$$

Observe that with these projections, the functions in (6.138) can be rewritten as

$$[fh]_k(t) = P_k[f(\alpha(t))h(t)].$$

Proposition 6.16. *Let* $\varphi \in L^p(\Gamma)$*. Then* φ *is a solution of* (6.144) *if and only if* $\{\varphi_k := P_k\varphi, \quad k = 1,2\}$ *is a solution of the following system*

$$\begin{array}{l} f_\alpha(t)\varphi_k(t) + [fg]_k^*(t)(S_\Gamma\varphi_2)(t) = [fh]_k(t), \quad \text{if } S_\Gamma W = WS_\Gamma \\ \text{or} \\ f_\alpha(t)\varphi_k(t) + [fg]_k^*(t)(S_\Gamma\varphi_1)(t) = [fh]_k(t), \quad \text{if } S_\Gamma W = -WS_\Gamma \end{array} \tag{6.145}$$

where for $k = 1,2$ *the elements* $f_\alpha(t)$, $[fg]_k^*(t)$ *and* $[fh]_k(t)$ *are defined in* (6.136), (6.137) *and* (6.138), *respectively.*

Proof. Suppose that $\varphi \in L^p(\Gamma)$ is a solution of (6.144). Multiplying by $f(\alpha(t))$ we have

$$f(\alpha(t))f(t)\varphi(t) + 2f(\alpha(t))g(t)(P_2S_\Gamma\varphi)(t) = f(\alpha(t))h(t).$$

Applying the projections P_k $(k = 1, 2)$ to both sides of the above equation, we get

$$P_k[f(\alpha(t))f(t)\varphi](t) + 2P_k[f(\alpha(t))g(t)(P_2S_\Gamma\varphi)](t) = P_k[f(\alpha(t))h(t)]. \quad (6.146)$$

By using (6.143) and the fact that $WP_2 = P_2$, we can verify that

$$\begin{aligned} P_k[f(\alpha(t))f(t)\varphi](t) =& f(\alpha(t))f(t)(P_k\varphi)(t) \\ P_k[f(\alpha(t))g(t)(P_2S_\Gamma\varphi)](t) =& [fg]_k^*(t)(P_2S_\Gamma\varphi)(t). \end{aligned}$$

Therefore, we can rewrite (6.146) as

$$f(\alpha(t))f(t)(P_k\varphi)(t) + 2[fg]_k^*(t)(P_2S_\Gamma\varphi)(t) = P_k[f(\alpha(t))h(t)].$$

Now, by Proposition 6.9 (apply to this case) we have that $P_2S_\Gamma = S_\Gamma P_2$ for the $S_\Gamma W = WS_\Gamma$ case and $P_2S_\Gamma = S_\Gamma P_1$ for the case of $S_\Gamma W = -WS_\Gamma$. Thus $(P_1\varphi, P_2\varphi)$ is a solution of (6.145).

Conversely, suppose that there exists φ such that $(P_1\varphi, P_2\varphi)$ is a solution of (6.145). Summing k from 1 to 2, we directly obtain that

$$\sum_{k=1}^{2} [f_\alpha(t)(\varphi_k)(t) + 2[fg]_k^*(t)(P_2S_\Gamma\varphi_i)(t)] = \sum_{k=1}^{2} P_k[f(\alpha(t))h(t)], i = 1, 2,$$

is equivalent to $f(\alpha(t))f(t)\varphi(t) + 2f(\alpha(t))g(t)(P_2S_\Gamma\varphi_i)(t) = f(\alpha(t))h(t)$, and this implies that $f(t)\varphi(t) + 2g(t)(P_2S_\Gamma\varphi)(t) = h(t)$. $\square$

Proposition 6.17. *If* (ϕ_1, ϕ_2) *is a solution of the system* (6.145)*, then* $(P_1\phi_1, P_2\phi_2)$ *is also a solution of* (6.145).

Proof. Let (ϕ_1, ϕ_2) be a solution of the system (6.145). Applying the projections P_k to both sides of (6.145), we have

$$P_k\left(f_\alpha(t)\phi_k(t) + [fg]_k^*(t)(S_\Gamma\phi_i)(t)\right) = P_k([fh]_k(t)), \quad k, i = 1, 2. \quad (6.147)$$

Notice that $P_k[f_\alpha(t)\phi_k](t) = f_\alpha(t)P_k\phi_k(t)$ and

$$\begin{aligned} P_k([(fg)]_k^*(t)(S_\Gamma\phi_i))(t) =& \frac{1}{2}\left\{[fg]_k^*(t)(S_\Gamma\phi_i)(t) + (-1)^k[fg]_k^*(\alpha(t))W(S_\Gamma\phi_i)(t)\right\} \\ =& [fg]_k^*(t)\frac{1}{2}\{(S_\Gamma\phi_i)(t) + W(S_\Gamma\phi_i)(t)\}. \end{aligned} \quad (6.148)$$

Equality (6.148) holds because $[fg]_k^*(t) = (-1)^k [fg]_k^*(\alpha(t))$. Then the right-hand side of equality (6.148) can be rewritten as $[fg]_k^*(t) P_2(S_\Gamma \phi_i)(t)$. From (6.145), the value of the index i depends on the commuting property of the shift operator with S_Γ. Therefore,

$$P_k[[fg]_k^*(t)(S_\Gamma \phi_i)](t) = [fg]_k^*(t)(S_\Gamma P_i \phi_i)(t).$$

Finally, note that $P_k([fh]_k)(t) = [fh]_k(t)$. Therefore, $(P_1\phi_1, P_2\phi_2)$ is a solution of (6.145). $\square$

Theorem 6.18. *The equation* (6.144) *has solutions in* $L^p(\Gamma)$ *if and only if the following equation*

$$\begin{aligned} &f_\alpha(t)\varphi_2(t) + [fg]_2^*(t)(S_\Gamma \varphi_2)(t) = [fh]_2(t), \quad \textit{in the } S_\Gamma W = W S_\Gamma \textit{ case} \\ &\textit{or} \\ &f_\alpha(t)\varphi_1(t) + [fg]_1^*(t)(S_\Gamma \varphi_1)(t) = [fh]_1(t), \quad \textit{in the } S_\Gamma W = -W S_\Gamma \textit{ case} \end{aligned} \tag{6.149}$$

has solutions. Moreover, if $\varphi_k(t)$ $(k = 1, 2)$ *is a solution of* (6.149), *then equation* (6.144) *has a solution which is given by the formula*

$$\varphi(t) = \begin{cases} \frac{h(t) - 2g(t)(S_\Gamma \varphi_2)(t)}{f(t)}, & \textit{if } S_\Gamma W = W S_\Gamma \\ \frac{h(t) - 2g(t)(S_\Gamma \varphi_1)(t)}{f(t)}, & \textit{if } S_\Gamma W = -W S_\Gamma. \end{cases} \tag{6.150}$$

Proof. Suppose that $\varphi \in L^p(\Gamma)$ is a solution of equation (6.144). By Proposition 6.16 we know that $(P_1\varphi, P_2\varphi)$ is a solution of system (6.145). Hence, for the $S_\Gamma W = W S_\Gamma$ case, $P_2\varphi$ is a solution of (6.149) and $P_1\varphi$ is the corresponding solution for the $S_\Gamma W = -W S_\Gamma$ case.

Conversely, suppose that φ_2 is a solution of (6.149). Without loss of generality, we assume now that $S_\Gamma W = W S_\Gamma$ (since the situation of $S_\Gamma W = -W S_\Gamma$ is dealt with similarly). In this case, the system (6.145) has a solution (φ_1, φ_2) determined by

$$\varphi_1(t) = \frac{[fh]_1(t) - [fg]_1^*(t)(S_\Gamma \varphi_2)(t)}{f_\alpha(t)}. \tag{6.151}$$

By Proposition 6.17 we have that $(P_1\varphi_1, P_2\varphi_2)$ is also a solution of (6.145). Set $\varphi = P_1\varphi_1 + P_2\varphi_2$. It is clear that $P_k\varphi = P_k\varphi_k$. This means that $(P_1\varphi, P_2\varphi)$ is a solution of (6.146). From Proposition 6.16 it follows that φ is a solution of (6.145). Moreover, from (6.151), we obtain

$$\varphi(t) = \sum_{k=1}^{2} P_k \left[\frac{[fh]_k(t) - [fg]_k^*(S_\Gamma \varphi_2)(t)}{f_\alpha(t)} \right]. \tag{6.152}$$

As before, we can see that

$$\sum_{k=1}^{2} P_k[fh]_k(t) = f(\alpha(t))h(t),$$
$$\sum_{k=1}^{2} P_k([fg]_k^*(t)(S_\Gamma\varphi_2)(t)) = 2f(\alpha(t))g(t)(S_\Gamma\varphi_2)(t).$$

Thus, substituting these in (6.152), we have

$$\varphi(t) = \frac{h(t) - 2g(t)(S_\Gamma\varphi_2)(t)}{f(t)}$$

□

6.6.3 Proof of the main result

From Theorem 6.18 we know that equation (6.3) has solutions if and only if equation (6.149) has solutions. Furthermore, the solutions of (6.3) are given by (6.150). Thus, we will compute the solutions of equation (6.149). For such a goal, we will use the corresponding Riemann boundary value problem associated to equation (6.149). Namely, by means of the Sokhotskii-Plemelj formulas, the equation (6.149) is reduced to the following boundary problem: Find a sectionally analytic function $\Phi(z)$ ($\Phi(z) = \Phi^+(z)$ for $z \in D^+$, $\Phi(z) = \Phi^-(z)$ for $z \in D^-$) vanishing at infinity and satisfying the condition

$$\Phi^+(t) = G(t)\Phi^-(t) + H(t) \tag{6.153}$$

where the functions $G(t)$ and $H(t)$ are defined in (6.134) and (6.135), respectively. We are now able to use the results in [64], under the assumptions imposed on $G(t)$ in Subsection 6.6.1. Thus, the solutions of the problem (6.153) read as follows:

(1) *Case* $n \geq 0$. In this case the solutions are given by

$$\Phi^\pm(z) = \chi^\pm(z)\Psi^\pm(z) + \chi^\pm(z)P_{n-1}(z) \tag{6.154}$$

with

$$\Psi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{H(\tau)}{\chi^+(\tau)}\frac{d\tau}{\tau - z}$$

and $P_{n-1}(z) \equiv 0$ if $n = 0$ and $P_{n-1}(z)$ is a polynomial of degree no greater than $n-1$ with arbitrary complex coefficients $c_0, c_1, \ldots, c_{n-1}$, for $n > 0$. The second item in the right-hand side of formula (6.154) is the general solution of the homogeneous ($H(t) \equiv 0$) Riemann problem (6.153) and the first item is a particular solution of the corresponding non-homogeneous problem (6.153). If $n = 0$, then the problem (6.153) has a unique solution.

(2) *Case* $n < 0$. For this case $P_{n-1}(z) \equiv 0$ and

$$\int_\Gamma \frac{H(t)t^{\eta-1}}{\chi^+(t)}dt = 0, \quad \eta = 1, 2, \ldots, -n$$

is a necessary condition to the solvability of equation (6.153).

This completes the proof of Theorem 6.15. $\square$

Appendix A

Singular Integral Operators on Topological Groups

The theory of analytic functions on the unit disc can be extended in several ways. For instance, the unit disc can be replace by other plane domains, or by domains in spaces of several complex variables. For all these kinds of extensions, generalizations of the Cauchy integral operator are also given and considered. For example, defining these operators on different type of curves [67], in octonionic spaces [62], as well as defining the so-called Calderón-Zygmund type operators on surfaces [72, 82], singular convolution operators on the Heisenberg group [65] and right convolution operators on homogeneous groups [77], are some of the most popular generalizations related to the Cauchy integral operator. All these generalizations were motivated by the large diversity of applications suitable to be considered in those corresponding frameworks.

On the other hand, the classic generalization of trigonometric series on the unit circle,

$$\varphi(t) = \sum a_n e^{int} ,$$

in the abstract harmonic analysis approach, is given by replacing the unit circle $\mathbb{T}$ by any abelian locally compact group G, while the set of indices n is taken as the dual group Γ of G. In this way, algebras of functions on G can the defined such that the unit disc becomes the space of the maximal ideal of the algebra and the group G becomes de boundary of the disc. For more information on this see, for instance, the pioneering works [1, 66]. Using these ideas, we will define a generalization of the Cauchy integral operator over a connected, compact, abelian group, such that we will therefore study the conditions that guarantee the existence of a (one-sided) inverse of the corresponding singular integral operator.

A.1 Some notions of harmonic analysis on groups

As it was announced in the Introduction, we are going to define the Cauchy integral operator over a topological group.

A.1.1 Ordered groups

In the sequel let G be a compact, abelian and connected group. Let P be a closed subset of G satisfying the semigroup condition $P + P \subset P$, and the two additional properties

$$P \cap (-P) = \{0\}, \qquad P \cup (-P) = G. \tag{A.1}$$

Under these conditions, P induces an order in G. Defining $x \geq y$ if $x-y \in P$, the axioms for a linear order are satisfied: if $x - y \geq 0$ and $y - z \geq 0$, then $x - z \geq 0$. Since P is a semigroup, and from conditions (A.1) we have that each pair x, y satisfies one and only one of the relations $x > y$, $x = y$, $y > x$. Also, if $x > y$, then $x+z > y+z$ (for any z). The choice of a semigroup P with the above properties (i.e., the choice of an order on G which is compatible with the group operations) makes G to be an *ordered group*. Notice that a given group G may have many different orders (for additional information about ordered groups see [86, Chapter 8]). The additive dual group of G equipped with the discrete topology will be denoted by Γ and, as usual, by the Pontryagin duality it is customary to write $\gamma(x)$ in the place of (γ, x) for every $\gamma \in \Gamma$ and $x \in G$. Since G is compact and connected, Γ can be ordered. In applications, often Γ is an additive subgroup of $\mathbb{R}^k$ so that G it is his Bohr compactification, or $\Gamma = \mathbb{Z}^d$ so that $G = \mathbb{T}^d$ is the d-torus.

With respect to any fixed order, let $\Omega(G)$ be the set of all *trigonometric polynomials* p on G of the form

$$p(x) = \sum_{\gamma} a_\gamma (x, \gamma), \quad x \in G\,.$$

The *conjugate function* of p is the trigonometric polynomial

$$v(x) = -i \sum_{\gamma > 0} a_\gamma (x, \gamma) + i \sum_{\gamma < 0} a_\gamma (x, \gamma)\,.$$

With this function, the *Cauchy representation* $\mathbb{S}_G p$ of p can be defined as

$$\mathbb{S}_G p(x) := -iv(x),$$

and the *analytic contraction* of p is given by

$$p_+(x) := \sum_{\gamma \geq 0} a_\gamma (x, \gamma).$$

Thus, the equation

$$\Phi p = p_+$$

defines a linear operator on the space of all trigonometrical polynomials on G.

Let $\ell^1(\Gamma)$ stand for the complex Banach space of all complex-valued Γ-indexed sequences $x = \{x_\gamma\}_{\gamma\in\Gamma}$ such that

$$\|x\|_1 = \sum_{\gamma\in\Gamma} |x_\gamma| < \infty.$$

It is clear that for each x at most countable many x_γ's are different from zero. $\ell^1(\Gamma)$ is a commutative Banach algebra with respect to the convolution product $(x * y)_\gamma = \sum_{k\in\Gamma} x_k y_{\gamma-k}$, where $x = \{x_\gamma\}_{\gamma\in\Gamma}$, $y = \{y_\gamma\}_{\gamma\in\Gamma}$ and the sequence e having 1 in its 0th position and zeroes elsewhere is the unit element. Further, introducing the additive semigroups $\Gamma_+ = \{\gamma \in \Gamma \ : \ \gamma \geq 0\}$ and $\Gamma_- = \{\gamma \in \Gamma \ : \ \gamma \leq 0\}$, we have that $\ell^1(\Gamma_+)$ and $\ell^1(\Gamma_-)$ are closed subalgebras of $\ell^1(\Gamma)$ containing e.

For a function $a = \{a_\gamma\}_{\gamma\in\Gamma} \in \ell^1(\Gamma)$, by the *symbol* of a we mean the complex-valued continuous function $\hat{a}$ on G defined by

$$\hat{a}(x) = \sum_{\gamma\in\Gamma} a_\gamma (x, \gamma), \qquad x \in G.$$

The set $\sigma(\hat{a}) = \{\gamma \in \Gamma \ : \ a_\gamma \neq 0\}$ will be called the *Fourier spectrum* of $\hat{a}$. We will use the shorthand notation e_γ for the function

$$e_\gamma(x) = (x, \gamma), \qquad x \in G\,;$$

thus, $e_{\alpha+\beta} = e_\alpha e_\beta$. The set of all symbols of elements $a = \{a_\gamma\}_{\gamma\in\Gamma} \in \ell^1(\Gamma)$ forms an algebra of continuous functions on G. The algebra $\mathbb{W}(G)$ (so-called the *Wiener algebra*) with pointwise multiplication and addition is isomorphic to $\ell^1(\Gamma)$ by letting $\Lambda : a \mapsto \hat{a}$ to be an isometry. In fact this is possible since Λ is injective ([86, Sec. 1.3.6]). Standard Gelfand theory implies that the algebra $\mathbb{W}(G)$ is inverse closed in the algebra of all continuous functions on G. We denote by $\mathbb{W}(G)_+$ (resp., $\mathbb{W}(G)_-$) the algebra of symbols of elements in $\ell^1(\Gamma_+)$ (resp., $\ell^1(\Gamma_-)$).

A.1.2 Cauchy integral operators on G

In this paper we will consider the *Lebesgue space* $L^p(G)$, $1 \leq p < \infty$, of all functions on the group G for which the norm

$$\|f\|_p^p := \int_G |f(x)|^p dx$$

is finite. Here dx is the uniquely determined Haar measure on G. By $L^\infty(G)$ we mean the space of all bounded Borel functions on G, normed by

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x\in G} |f(x)|.$$

Any function $f \in L^1(G)$ is of *analytic type* if $\hat{f}(\gamma) = 0$ for $\gamma < 0$, where

$$\hat{f}(\gamma) = \int_G f(x)e_\gamma(x)dx, \quad \gamma \in \Gamma\,,$$

is the *Fourier transform* of f. The set of all functions $\hat{f}$ obtained in this way will be denoted throughout by $\mathbb{A}(\Gamma)$ and the set of all functions of analytic type which belong to $L^p(G)$ will be denoted by $H^p_+(G)$ $(1 \leq p \leq \infty)$. Notice that this class does not only depends on G, but depends also on the particular order which is imposed on Γ.

In this paper, by using the Fourier coefficients of functions on $L^p(G)$, for $p \in (1,\infty)$, we will give a generalization of the Cauchy integral operator. In general, if $u \in L^1(G)$, and if $\hat{u}\chi \in \mathbb{A}(\Gamma)$, where $\chi(\gamma) = 1$ for $\gamma \geq 0$, $\chi(\gamma) = 0$ for $\gamma < 0$, then the function F defined by the equation $\hat{F} = \hat{u}\chi$ will be called the analytic contraction of u and we will write $F = \Phi u$. Theorem 8.7.2 in [86] gives us, in particular, that the maps $p \mapsto -iv$ and $p \mapsto p_+$ are bounded on $L^p(G)$ for $1 < p < \infty$. I.e., the *Cauchy integral operator* over G, defined for a function $\phi(x) = \sum_\gamma c_\gamma e_\gamma(x)$ on $L^p(G)$, by the formula (cf., (1.9))

$$(S_G\phi)(x) := \sum_{\gamma\geq 0} c_\gamma e_\gamma(x) - \sum_{\gamma<0} c_\gamma e_\gamma(x), \tag{A.2}$$

is in fact well-defined and bounded, as well as, the operator

$$(P_G\phi)(x) := \sum_{\gamma\geq 0} c_\gamma e_\gamma(x). \tag{A.3}$$

Notice that S_G is an involution ($S_G^2 = I_G$) whereas, P_G is a projection ($P_G^2 = P_G$). In case that $G = \mathbb{T}$ and $\Gamma = \mathbb{Z}$, we have that S_G is the Cauchy integral operator along $\mathbb{T}$ and P_G is the Riesz projection $P_+ = \frac{1}{2}(I_\mathbb{T} + S_\mathbb{T})$.

In this way, we have, as in the classic case, that $H^p_+(G) \equiv P_G L^p(G)$, $p \in (1,\infty)$. So, defining $Q_G := I_G - P_G$ we write $\dot{H}^p_-(G) \equiv Q_G L^p(G)$.

A.2 On the invertibility of SIO's on G with trigonometrical polynomial coefficients

The fact that the Cauchy integral operator S_G is bounded on $L^p(G)$, for $p \in (1,\infty)$, implies that for essentially bounded functions a, b on G ($a, b \in$

$L^\infty(G)$), the following operator

$$\mathcal{A} := aP_G + bQ_G$$

where P_G and Q_G are the Riesz projections $P_G = \frac{1}{2}(I_G + S_G)$, $Q_G = \frac{1}{2}(I_G - S_G)$ is also bounded on $L^p(G)$. The operator $\mathcal{A}$ just now defined will be called the *singular integral operator* over G.

The following subsets of $\Omega(G) \subset L^\infty(G)$ will be useful in the sequel: $\Omega_+(G)$ the set of the trigonometric polynomials of analytic type. I.e., the polynomial functions of the form

$$p(x) = \sum_{\gamma \geq 0} c_\gamma e_\gamma(x).$$

By $\Omega_-(G)$ we denote the set of all polynomial functions without the coefficients c_γ with $\gamma \geq 0$:

$$q(x) = \sum_{\gamma < 0} d_\gamma e_\gamma(x).$$

Proposition A.1. *The operator $\mathfrak{C}_p := pP_G + Q_G$, where $p \in \Omega_+(G)$ and with $p(x) \neq 0$, for $x \in G$, is left invertible with left-inverse given by*

$$\mathfrak{C}_p^{(-1)} = p^{-1}P_G + Q_G. \tag{A.4}$$

In the case of $p \in \mathcal{G}\Omega_+(G)$, then $\mathfrak{C}_p$ is invertible with inverse given by equality (A.4).

Proof. Since $p(x) \neq 0$, $x \in G$, then p^{-1} exists. Now, we will perform the computation of $\mathfrak{C}_p^{(-1)}\mathfrak{C}_p$:

$$\begin{aligned} \mathfrak{C}_p^{(-1)}\mathfrak{C}_p &= (p^{-1}P_G + Q_G)(pP_G + Q_G) \\ &= p^{-1}P_G p P_G + Q_G. \end{aligned}$$

Because $p \in \Omega_+(G)$, we have that $P_G p P_G = pP_G$. Therefore we get

$$\mathfrak{C}_p^{(-1)}\mathfrak{C}_p = p^{-1}P_G p P_G + Q_G = P_G + Q_G = I_G.$$

Similarly, $p \in \mathcal{G}\Omega_+(G)$ implies that $P_G p^{-1} P_G = p^{-1}P_G$. Thus

$$\begin{aligned} \mathfrak{C}_p\mathfrak{C}_p^{(-1)} &= (pP_G + Q_G)(p^{-1}P_G + Q_G) \\ &= pP_G p^{-1} P_G + Q_G = P_G + Q_G \\ &= I_G. \end{aligned}$$

□

The following proposition can be proved analogously.

Proposition A.2. *Let us consider the operator* $\mathfrak{D}_p := P_G + pQ_G$*, for an invertible* $p \in \Omega_-(G)$*.*

(i) $\mathfrak{D}_p$ *is a left invertible operator with left inverse*

$$\mathfrak{D}_p^{(-1)} = P_G + p^{-1}Q_G. \tag{A.5}$$

(ii) *If* $p \in \mathcal{G}\Omega_-(G)$*, then* $\mathfrak{D}_p$ *is invertible with inverse given by* (A.5)*.*

A.3 Factorization of symbols in the Wiener algebra and invertibility of $\mathcal{A}$

In this part we will consider the coefficients a and b of the singular integral operator $\mathcal{A}$ on the Wiener algebra $\mathbb{W}(G)$.

For this kind of symbols the notion of (left) *factorization* is introduced in order to show how such a factorization has an influence in the invertibility of the operator A.

Definition A.1 (cf. [83]). A (left) factorization of $a \in \mathbb{W}(G)$ with $a(x) \neq 0$, for every $x \in G$, is a representation of the form

$$a(x) = a_+(x)e_\gamma(x)a_-(x), \qquad x \in G \tag{A.6}$$

where $a_+ \in \mathcal{G}(\mathbb{W}(G)_+)$, $a_- \in \mathcal{G}(\mathbb{W}(G)_-)$ and $\gamma \in \Gamma$ is the abstract *winding number* of a which is uniquely determined.

If $\gamma = 0$, then the factorization is called *canonical*. Moreover, for any two canonical factorizations of a, say $a = a_+a_-$ and $a = b_+b_-$, there exists a nonzero complex number c such that $a_+ = cb_+$ and $a_- = cb_-$.

To emphasize that all the factorization factors have their Fourier spectrum in Γ, we will say that factorization (A.6) is a Γ-factorization. If a factorization for a exists, the function a is called *factorable*. Clearly, invertibility of a in $\mathbb{W}(G)$ is a necessary condition for its factorability. A *right* factorization differs only from the left factorization in the circumstance that the factors $a_\pm$ interchange their positions.

For $\Gamma = \mathbb{Z}$ and G the unit circle $\mathbb{T}$, Definition A.1 yields the classical Wiener-Hopf factorization and in the case $\Gamma = \mathbb{R}$, the group G becomes the Bohr compactification of $\mathbb{R}$ and $\mathbb{W}(G)$ turns into the Wiener algebra APW of Bohr almost periodic functions. A full characterization and properties (as the uniqueness of factorization indices, hereditary properties and connectedness) of the Γ-factorization can be found in [37, 83] as in references therein.

Proposition A.3. *Let $\gamma \in \Gamma$. The operator $\mathfrak{C}_\gamma = e_\gamma P_G + Q_G$ acting between $L^p(G)$, spaces, for $p \in (0, \infty)$, is left (resp., right) invertible if $\gamma > 0$ (resp., $\gamma < 0$). In both cases the inverse takes the form*

$$\mathfrak{C}_\gamma^{(-1)} = e_{-\gamma} P_G + Q_G.$$

Proof. Let $\gamma > 0$, then $P_G e_\gamma P_G = e_\gamma P_G$. Thus

$$(e_{-\gamma} P_G + Q_G)(e_\gamma P_G + Q_G) = I \tag{A.7}$$

and, for $\gamma < 0$ we get $P_G e_{-\gamma} P_G = e_{-\gamma} P_G$, so

$$(e_\gamma P_G + Q_G)(e_{-\gamma} P_G + Q_G) = I. \tag{A.8}$$

Notice that in both (A.7) and (A.8) the left-hand side factors do not commute.

□

Theorem A.4. *Let a, b be arbitrary non-vanishing functions on $\mathbb{W}(G)$ and let*

$$c(x) = c_+(x) e_\gamma(x) c_-(x), \quad x \in G,$$

be the Γ-factorization of c, where $c = ab^{-1}$. Then the operator $\mathcal{A} = aP_G + bQ_G$ on the space $L^p(G)$ for $p \in (1, \infty)$:

(i) is invertible if $\gamma = 0$. In such a case, the inverse takes the form

$$\mathcal{A}^{-1} = (c_+ P_G + c_- Q_G) c_-^{-1} b^{-1};$$

(ii) is left (resp., right) invertible if $\gamma > 0$ (resp., $\gamma < 0$). The form of the one-sided inverse is given in both cases by

$$\mathcal{A}^{(-1)} = (e_{-\gamma} P_G + Q_G)(c_+^{-1} P_G + c_- Q_G) c_-^{-1} b^{-1}. \tag{A.9}$$

Proof. First let $\gamma = 0$. Choose a $r \in \Omega(G)$ which approximates the function $c = ab^{-1}$ sufficiently well, namely, let

$$\max_{x \in G} |d(x)| < 1/\|P_G\|, \tag{A.10}$$

where $d = cr^{-1} - 1$, and therefore $c = r(1 + d)$. Let $r = r_+ r_-$ the canonical factorization that approximates ab^{-1}. Notice that

$$\mathcal{A} = br_-(I + dP_G)(r_+ P_G + r_-^{-1} Q_G). \tag{A.11}$$

The three factors on the right-hand side of above equality are invertible (note that by virtue of (A.10) and $\|P_G\| \geq 1$, we have that $\|dP_G\| < 1$). Thus, the operator $\mathcal{A}$ is invertible. We shall prove that

$$\mathcal{A}^{-1} = (c_+^{-1} P_G + c_- Q_G) c_-^{-1} b^{-1}. \tag{A.12}$$

For such a goal first we will prove the following fact:

Claim 1. *Let the function a be factorable as in* (A.6). *Then the operator*

$$a_+^{-1} P_G a_-^{-1}$$

is bounded on the space $L^p(G)$, for $p \in (1, \infty)$.

Proof. Let $\mathfrak{B} = (a_+^{-1} P_G + a_- Q_G) a_-^{-1}$. We can verify that for $\varphi \in L^p(G)$

$$\mathfrak{B} \mathfrak{Z}_\gamma \varphi = \varphi$$

where the operator $\mathfrak{Z}_\gamma = a e_{-\gamma} P_G + Q_G$ is invertible and bounded. From this it follows that $\mathfrak{B} = \mathfrak{Z}_\gamma^{(-1)}$ is bounded on $L^p(G)$. Let $c = a_+ a_-$ and we will assume, without loss of generality, that $|c(x)| \leq m < 1$. The representation $\mathfrak{B} = I + (1 - c) a_+^{-1} P_G a_-^{-1}$ then immediately implies the boundedness of the operator $a_+^{-1} P_G a_-^{-1}$ on $L^p(G)$. □

Notice that by virtue of (A.6) the operators

$$a_+^{-1} P_G a_-^{-1}, \quad a_+ P_G a_-, \quad a_- P_G a_-^{-1}, \quad a_+^{-1} P_G a_+$$

are either simultaneously bounded on $L^p(G)$, $1 < p < \infty$, or not.

From the previous fact and Claim 1 we get that the operator (A.12) is bounded on $L^p(G)$. Let $r \in \Omega(G)$, then

$$(c_+^{-1} P_G + c_- Q_G) c_-^{-1} b^{-1} \mathcal{A} r = (c_+^{-1} P_G + c_- Q_G)(c_+ r_+ + c_-^{-1} r_-) \tag{A.13}$$

where $r_+ = P_G r$, $r_- = Q_G r$. Since $c_+ r_+ \in H_+^p(G)$ and $c_-^{-1} r_- \in \dot{H}_-^p(G)$, the right-hand side of (A.13) is equal to r and, consequently, the operator defined by (A.12) is the left inverse of $\mathcal{A}$.

Now we will prove that this operator is also the right inverse of $\mathcal{A}$. In fact,

$$\begin{aligned} \mathcal{A}(c_+^{-1} P_G + c_- Q_G) c_-^{-1} b^{-1} r =& b(c_+ c_- P_G + Q_G)(c_+^{-1} P_G + c_- Q_G) c_-^{-1} b^{-1} r, \\ =& b(c_- P_G + c_- Q_G) c_-^{-1} b^{-1} r = r. \end{aligned}$$

Therefore, (A.12) is the inverse of $\mathcal{A}$, proving in this way proposition (i). On the other hand, if $\gamma > 0$, then the operator $\mathcal{A}$ can be represented in the form

$$\mathcal{A} = b(ab^{-1} e_{-\gamma} P_G + Q_G)(e_\gamma P_G + Q_G)$$

where, from Proposition A.3, $e_\gamma P_G + Q_G$ is a left invertible operator with left inverse $e_{-\gamma} P_G + Q_G$ and the invertibility of $ab^{-1} e_{-\gamma} P_G + Q_G$ was already proved. It follows then that (A.9) is a left inverse of $\mathcal{A}$.

Now let $\gamma < 0$. Then the operator

$$\mathfrak{B} = b(ab^{-1} e_{-\gamma} P_G + Q_G) = \mathcal{A}(e_{-\gamma} P_G + Q_G)$$

is invertible and the inverse $\mathfrak{B}^{-1}$ can be obtained using (A.12). This proves that the operator (A.9) is the right inverse of $\mathcal{A}$.

□

Conclusion

The fundamental theory of singular integral operators with shifts is not only interesting by pure theoretical reasons but also due to the possibility of increasing the range of applications of these operators. The operator equivalence relations have been proven to be a powerful tool by transferring some fundamental theory from a “simpler” operator to the operator under study.

Thus, in this thesis we studied the regularity properties of singular integral operators with the action of the reflection shift operator and also with the so-called flip shift operator defined on weighted Lebesgue spaces on the unit circle. For these operators, explicit equivalence relations were exhibited in Chapter 2 which allows us, in Chapter 3, to determinate its Fredholm property when its coefficients belong to the classes of continuous, piecewise continuous and semi almost periodic functions, as well as an invertibility criterion, including the form of the (lateral) inverse(s), for generalized factorable coefficients. We would like to point out that by the nature of the equivalence relation after extension performed to the case of the singular integral operator with flip, it was necessary to define this operator on a particular weighted Lebesgue space, remaining open the case of considering it in a weighted Lebesgue space with general Hunt–Muckenhoupt–Wheeden weights. Notice that due to the mentioned equivalence relations, the regularity properties for those operators with some other classes of essentially bounded coefficients are available as the corresponding properties exist in the literature for matrix Toeplitz, pure matrix singular integral and matrix Toeplitz plus Hankel operators.

On the other hand, in Chapter 4, with the results of Chapters 2 and 3, we were able to compute the kernel dimensions of the mentioned singular integral operators with shift and piecewise continuous coefficients, by using C^*-algebra theory as a tool in the framework of numerical analysis. Also, the rate of convergence of the methods used to compute such kernel dimensions was calculated for the case of smooth coefficients, as well as the Moore-Penrose inverse of the initial operators. This approach, which includes notions as:

approximation numbers, projection methods, stability, algebraization of the stability and collocation methods, among others, is becoming popular in the investigation of concrete discretization procedures also in the solution of integral equations, convolution equations (of Wiener-Hopf or Mellin type), or even pseudodifferential equations. In the final part of the chapter, with the help of some estimations and an operator equivalence relation after extension, we estimate upper bounds for the kernel dimensions of singular integral operators with generic preserving orientation weighted Carleman shift operators having continuous coefficients. These bounds complement the existing results for preserving orientation non-Carleman shift operators, remaining open the case of reverting orientation shift operators.

In general, the construction of explicit equivalence relations between operators is not an easy task, however, in counterpart, the profit of these relationships makes it worth trying to build. In the actual literature there are a vast quantity of singular integral operators with shifts, as the so-called fractional shift operator and the complex conjugation shift operator, which can be used in a wide range of applications whose fundamental theory is not yet complete, making them natural candidates for applying this technique. Also the interest on the fundamental theory of convolution type operators defined on Lebesgue spaces with variable exponent, or acting on curves with cusps is growing. Therefore, equivalence relations between operators can play a central role on the extension of those results to the corresponding case of singular integral operators with shift in those spaces and curves.

Nicer representations in the form of the identity plus a compact operator or of the identity plus an operator with small norm, are very convenient in order to use numerical analysis tools to compute numerically the solutions of operator equations. In Chapter 5, we gave conditions that assure a nice representation of singular integral operators with shift and piecewise continuous symbols. That representation was possible by using a symbol calculus from C^*-algebra generated by two idempotents and a flip. We would like to point out that, in a similar way, with a symbol calculus for the algebra generated by two idempotents, some conditions for the locally strong ellipticity of pure singular integral operators with piecewise continuous are already known, however the case of operators with shift cannot be obtained directly by using that technique. Thus, conditions for the locally strong ellipticity of singular integral operators with shift are still unknown.

In the thesis, it was also paid attention to singular integral equations with shifts on weighted Lebesgue spaces. More precisely, in Chapter 6 we studied the solvability conditions and the representations of the solutions of a class of singular integral equations with the reflection shift function on weighted Lebesgue spaces on the unit circle and also a class of equations with generic Carleman shift functions inducing preserving or reverting orientation weighted Carleman shift operators (satisfying a commutative relation with the Cauchy integral operator) on Lebesgue spaces over a Carleson curve. The strategy here was to use some projection methods and operator identities in order to transfer the solvability conditions from related systems of pure singular integral equations, to the original equations with shift. These systems were analyzed by means of a corresponding Riemann boundary value problem. Thus, the solutions of the initial equations were constructed from the solutions of the Riemann boundary value problems. A key assumption here was the commutative relations between the shift operator and the Cauchy integral operator. However, it is well-known that in general $S_{\mathbb{T}}W - WS_{\mathbb{T}}$ is a compact operator. Nevertheless, the procedure can be applied once it is known that compact operator. On the other hand, due to the fact that this strategy avoid the use of more complicated methods like the gluing technique or the reduction to a Haseman problem, seems plausibly to try to extend this procedure to other singular integral equations with shift.

We would like to stress that generalizations of singular integral operators have appeared in different circumstances, for instance, defined on surfaces and Euclidean groups like octonian spaces and on the Heisenberg group. Here, in Appendix A, an extension of the definition of Cauchy integral operators was introduced by defining these operators on Lebesgue spaces over abelian topological groups. This definition was motivated by the large diversity of applications suitable to be consider on this framework. We investigated invertibility conditions, as well as the form of the both-sided lateral inverses, for these operators with polynomial trigonometrical and essentially bounded factorable coefficients, leaving open the rest of the fundamental theory of those operators.

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