Accepted version of:

SARRICO CS AND DYSON RG (2004). Restricting virtual weights in data envelopment analysis. *European Journal of Operational Research* 159: 1, 17-34. https://doi.org/10.1016/S0377-2217(03)00402-8

Restricting Virtual Weights in Data Envelopment Analysis

CLÁUDIA S. SARRICO[*]
Warwick Business School, University of Warwick, Coventry CV4 7AL, UK

ROBERT G. DYSON
Warwick Business School, University of Warwick, Coventry CV4 7AL, UK

* *Corresponding author.*
Dr C. S. Sarrico, Ciências Sociais, Jurídicas e Políticas, Universidade de Aveiro, 3810 – 193 AVEIRO, PORTUGAL. Tel: + 351 - 234 372 495. E-mail: C.S.Sarrico@csjp.ua.pt.

[*] Cláudia S. Sarrico acknowledges financial support from the Portuguese Foundation for Science and Technology.

Abstract

The consequences of the use of absolute weights restrictions (i.e. restricting the multipliers) on the efficiency score and targets of a DEA model have been explored elsewhere, the same is not true for the use of restrictions on the virtuals (i.e. the product of the input/ output factor by its multiplier). In this paper, a reflection on the uses of virtual weights restrictions is presented. The reasons for using virtual weights restrictions instead of absolute weights restrictions, in particular cases, are explained. Following a critique of Wong and Beasley's (1990) first proposed method for constraining the virtuals in DEA, a new classification scheme for virtual weights restrictions is presented, which brings the concept of assurance regions into virtual weights restrictions. It is shown that the use of simple virtual restrictions and virtual assurance regions are preferable to the use of the more generally advocated WB's proportional virtual weights restrictions. In recognition of levels of decision making at the unit, and external to the unit, the use of the terms unit of assessment (UOA) and controller is proposed. It is concluded that the use of *virtual assurance regions* applying to the target UOA can be a natural representation of preference structures and translate established patterns between the input-output divide. Also, the meaning of the efficiency score and targets in this approach most approximate traditional DEA. Alternatives to using virtual weights restrictions are considered, namely using absolute weights restrictions with a virtual meaning. Finally, an empirical example is offered.

Keywords: data envelopment analysis; virtuals, weights restrictions, assurance regions.

1 Introduction

The application of DEA to concrete situations has motivated the use of weights restrictions to curb the complete freedom of variation of weights allowed by the original DEA model. The problem of allowing total flexibility of the weights is that the values of the weights obtained by solving the unrestricted DEA program are often in contradiction to prior views or additional available information.

Thompson et al. (1986) were the first authors to propose the use of weights restrictions to increase discrimination of the results of a DEA problem to support the siting of a laboratory, where only six alternatives were under consideration. Their technique included the imposition of acceptable bounds on ratios of multipliers (weights), to solve a choice problem. Dyson and Thanassoulis (1988) were concerned about the omission of particular inputs or outputs from the efficiency score, through the assignment of zero weights. They suggested imposing meaningful bounds directly on individual multipliers based on average input levels per unit of output. Charnes et al. (1990), in another approach to the problem, suggested transforming input-output data to simulate weights restrictions, where DMUs are assessed on the basis of the input-output levels of pre-selected DMUs which were a priori recognised by experts as being efficient.

One of the problems with directly restricting the multipliers, i.e. *absolute* weights restrictions, is that they are dependent on the units of measurement of the inputs and outputs. Virtual input/ output, the product of the input/ output level and optimal weight for that input/ output, however, is dimensionless. The virtual inputs and outputs of a DMU reveal the relative contribution of each input and output to its efficiency rating. The higher the level of virtual input/ output, the more important that input/ output is in the efficiency rating of the DMU concerned. Therefore use of virtual inputs and outputs can help to identify strong and weak areas of performance. Additionally, it is hard to give a meaning to absolute weights restrictions. Virtual weights restrictions are, in most occasions, more intuitive for the decision-maker. In order to avoid the problems of *absolute* weights restrictions, Wong and Beasley (1990) proposed the use of *virtual* weights restrictions, and in particular, the use of *proportional* virtual weights restrictions, which were intended to make it easier for the decision maker to quantify value judgements in terms of percentage values. Roll and Golany (1993), to avoid the dependence on the units of measurement of the input and output factors in absolute weights restrictions, proposed instead the normalisation of the input-output data. One of the disadvantages of this method is that once results are obtained they must be transformed back to the original form in order to interpret the results. Also, absolute weights restriction can be a problem for the analyst when dealing with managers who do not necessarily understand DEA. In which case, it is easier to elicit from management virtual weights restrictions in terms of the proportional importance of the factors. The Roll and Golany approach overcomes some of the problems with absolute weights restrictions, but does not allow direct comparisons of the relative contributions of inputs and outputs to the efficiency rating.

A comprehensive review of the evolution, development and research directions on the use of weights restrictions can be found in Allen et al. (1997a). In this review the consequences for the interpretation of the results from DEA models with weights restrictions has been analysed for *absolute* weights restrictions. The analysis of the pros and cons of the use of *virtual* weights restrictions and how it compares with the use of absolute weights restrictions are proposed as a further direction of research. This paper proposes to contribute to that analysis.

The intention of incorporating value judgements might be, as seen above, to incorporate prior views or information regarding the assessment of efficient DMUs. On the other hand, there might be two levels of decision-making, the DMU (for instance, a department or university), and the corporate top management or external evaluator (for instance the State, or the applicant in relation to a university or department in Sarrico et al. (1997)). The DMU might use its value judgements if it wants to use the assessment for benchmarking itself against other DMUs (see, for instance, Sarrico and Dyson (2000)).

However, if an external agent does the evaluation, the expressions *DMU* and *decision maker* might be misleading, as the decision maker is, in fact, at a different level. In this case, the DEA assessment becomes a game between what can be called the *unit of assessment* (UOA) trying to show itself in the best possible light, and a higher level decision maker, i.e. a *controller* imposing its preference structure. Sarrico et al. (1997) and Sarrico and Dyson (2000) have used DEA in the assessment of UK universities' performance, where the university or department is the UOA with the applicant or the State, respectively, being the controller. A similar situation occurs in the regulated industries where the regulator is the controller. Virtual weights restrictions are particularly appealing in these circumstances when 'outside' judgements need to be translated into a DEA model weights restrictions. The higher the level of a virtual input or output, the more important that input or output for the efficiency rating of the DMU concerned. Absolute weights, however, do not normally have an obvious meaning to the controller.

Allen et al. (1997a) point out that the substantial changes to the UOA's current mix of input and output levels indicated by the imposition of weights restrictions might be beneficial. It might lead to the conclusion that the current mix is inadequate given the controller's preferences. The same goes for the deterioration of current levels of some inputs and outputs.

Although there are often references in the literature to WB's first proposed methods of restricting the virtuals in DEA, neither WB's original paper nor subsequent literature explore the consequences of their use in the interpretation of the efficiency score and targets thus obtained. Section 2 proposes to do that. In section 3 it is shown how WB's restrictions could be translated into equivalent absolute weights restrictions, and how easily some of their methods lead to infeasible problems. In section **Error! Reference source not found.**, the authors support the use of virtual weights restrictions, albeit of a different kind of WB's, for particular cases. They then propose a new classification of virtual weights restrictions, which introduce the concept of virtual assurance regions. In section 5 the consequences of using the authors proposed virtual weights restrictions is explored, and their advantage over WB's ascertained. Finally, in section 6 an empirical example, which illustrates the use of a range of types of weights restrictions including virtual assurance regions, is presented.

2 The Use of *Proportional* Virtual Weights Restrictions

It is noted that the ideas in this paper are developed with reference to the original DEA formulation by Charnes, Cooper and Rhodes (1978) below, which assumes constant returns to scale and that all input and output levels for all DMUs are strictly positive. Consideration of the use of virtual weights restrictions in relation to variable returns to scale formulations is left for further research.

The CCR model measures the efficiency of target unit j_o relative to a set of peer units:

CCR Model

$$e_0 = \max \frac{\sum_{r=1}^{s} u_r y_{rj_0}}{\sum_{i=1}^{m} v_i x_{ij_0}}$$

$$s.t. \quad \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \le 1, \quad j = 1, \dots, n$$

$$u_r, v_i \qquad \ge \varepsilon, \;\; \forall r \; and \; i$$

where

y_{rj}	=	amount of output *r* from unit *j*,
x_{ij}	=	amount of input *i* to unit *j*,
u_r	=	the weight given to output *r*,
v_i	=	the weight given to input *i*,
n	=	the number of units,
s	=	the number of outputs,
m	=	the number of inputs,
ε	=	a positive non-Archimedean infinitesimal.

It is assumed that there are *n* DMUs to be evaluated. Each DMU consumes varying amounts of *m* different inputs to produce *s* different outputs. The CCR model translates into the following: unit j_o is said to be efficient (e_o=1) if no other unit or combination of units can produce more than unit j_o on at least one output without producing less in some other output or requiring more of at least one input.

The method first developed by WB to restrict flexibility of the virtuals in DEA is based upon the use of proportions. Conceptually the proportional virtual output *r* of UOA *j* represents the importance attached to the output measure (a similar reasoning can be applied to an input factor). Thus the controller can set limits on this proportion to reflect value judgements, as follows:

$$a_r \leq \frac{y_{rj} u_r}{\sum_{r=1}^{s} y_{rj_0} u_r} \leq b_r \text{ for an output factor, and}$$

$$c_i \leq \frac{x_{ij} v_i}{\sum_{i=1}^{m} x_{ij} v_i} \leq d_i \text{ for an input factor.}$$

Note that these kinds of restrictions are UOA specific, which raises questions for their implementation, namely to which UOAs should the restrictions apply, and what effect they have on other UOAs. WB suggest three different alternatives. However, they did not explore the consequences of the imposition of their restrictions for the interpretation of the efficiency score and target setting, which we propose to do next.

An output-oriented model (i.e. where $\sum_{r=1}^{s} y_{rj_0} u_r = 1$) is considered to explore the consequences of the use of the different approaches. The use of an input-oriented model would lead to similar conclusions.

2.1 *Proportional virtual weights restrictions apply* only *to the target UOA* j*₀*.

When proportional weights restrictions are applied to the target UOA j_0, each UOA is assessed with two additional constraints for each factor (output or input) being restricted:

$$a_r \leq \frac{y_{rj_0} u_r}{\sum_{r=1}^{s} y_{rj_0} u_r} \leq b_r, \forall r \text{ and } c_i \leq \frac{x_{ij_0} v_i}{\sum_{i=1}^{m} x_{ij_0} v_i} \leq d_i, \forall i$$

Note that, so far as the target unit is concerned, the above constraints reduce to:

$$a_r \leq y_{rj_0} u_r \leq b_r, \forall r \text{ and } c_i \leq \frac{x_{ij_0} v_i}{\sum_{i=1}^{m} x_{ij_0} v_i} \leq d_i, \forall i$$

Adding the weights restrictions to the multipliers formulation, it becomes:

$$
\begin{array}{llll}
\frac{1}{e_0} = \min \sum_{i=1}^{m} x_{ij_0} v_i & & & \\
s.t. & & & \\
\sum_{r=1}^{s} y_{rj_0} u_r & = & 1 & \\
\sum_{i=1}^{m} x_{ij} v_i - \sum_{r=1}^{s} y_{rj} u_r & \geq & 0, & \forall j \\
y_{rj_0} u_r & \geq & a_r, & \forall r \\
-y_{rj_0} u_r & \geq & -b_r, & \forall r \\
x_{ij_0} v_i - c_i \sum_{i=1}^{m} x_{ij_0} v_i & \geq & 0, & \forall i \\
-x_{ij_0} v_i + d_i \sum_{i=1}^{m} x_{ij_0} v_i & \geq & 0, & \forall i \\
v_i & \geq & \varepsilon, & \forall i \\
u_r & \geq & \varepsilon, & \forall r
\end{array}
$$

Since these kind of restrictions are UOA specific, as they are dependent on the target unit's input/ output levels, the target unit will thus be 'imposing' different, and possibly unreasonable, restrictions on the virtuals of the other units when assessing the target unit.

The consequences for the measure of efficiency and target setting can be better appreciated from the envelopment formulation:

$$
\begin{array}{llllllll}
\frac{1}{e_0} = \max \theta_0 + \sum_{r=1}^{s} a_r \alpha_r - \sum_{r=1}^{s} b_r \beta_r + \varepsilon \sum_{i=1}^{m} s_i^- + \varepsilon \sum_{r=1}^{s} s_r^+ \\
s.t. \\
 & \sum_{j=1}^{n} x_{ij} \lambda_j & + x_{ij_0} \gamma_i - x_{ij_0} \sum_{i=1}^{m} c_i \gamma_i \\
 & & - x_{ij_0} \delta_i + x_{ij_0} \sum_{i=1}^{m} d_i \delta_i & + s_i^- & & = & x_{ij_0}, & \forall i \\
y_{rj_0} \theta_0 & - \sum_{j=1}^{n} y_{rj} \lambda_j & + y_{rj_0} \alpha_r - y_{rj_0} \beta_r & & + s_r^+ & = & 0, & \forall r \\
\theta_0 & & & & & & \text{free} \\
 & \lambda_j & & & & \geq & 0, & \forall j \\
 & & \gamma_i, \delta_i, & s_i^- & & \geq & 0, & \forall i \\
 & & \alpha_r, \beta_r, & & s_r^+ & \geq & 0, & \forall r
\end{array}
$$

The targets for UOA j_0 will thus be:

$$
\sum_{j=1}^{n} x_{ij} \lambda_j^* = x_{ij_0} \left(1 - \gamma_i^* + \sum_{i=1}^{m} c_i \gamma_i^* + \delta_i^* - \sum_{i=1}^{m} d_i \delta_i^* \right) - s_i^{-*}, \forall i
$$

$$
\sum_{j=1}^{n} y_{rj} \lambda_j^* = y_{rj_0} \left(\theta_0^* + \alpha_r^* - \beta_r^* \right) + s_r^{+*}, \forall r
$$

Two new variables α_r and β_r appear in the objective function of the linear program, which might not be negligible in comparison with θ_0. The efficiency measure e_0^* will no longer necessarily approximate the inverted radial expansion factor $1/\theta_0^*$, as it would in an unrestricted model. Moreover this measure will depend on the limits imposed for the proportional virtual outputs in an output-oriented model (and on the limits imposed for the proportional virtual inputs in an input-oriented model).

As to the targets, two different situations arise for inputs and outputs. For both input and output factors, the targets obtained can mean either an improvement or deterioration in the current level. However, for an input factor, its target will also depend on the limits imposed (c_i and d_i) for all the proportional virtual inputs. Because of this characteristic, it would seem more appropriate to impose this kind of weights restriction only on the virtual outputs for an output-oriented model, and only on the virtual inputs for a virtual input model. This procedure would be more in line with ‘classical’ DEA, where in the input-oriented models, one focuses on maximal movement towards the frontier through proportional reduction of inputs, whereas in the output-oriented models, one focuses on maximal movement via proportional augmentation of outputs, but not both simultaneously. The choice of model will depend on which factors under consideration are more easily controlled by the UOA.

In conclusion, in this alternative the target UOA might impose unreasonable restrictions on the virtuals of the other UOAs. Also, in an output-oriented model, the restrictions applied to the target unit for the outputs affect the efficiency score and output targets, whereas the restrictions applied to the inputs will affect the input targets in an unconventional way.

2.2 Proportional virtual weights restrictions apply to all *UOAs* j

As seen in the previous section, applying proportional virtual weights restrictions only to the target UOA might impose unreasonable virtuals on the other UOAs. WB’s second approach, however, proposes that all virtual weights restrictions should apply to all UOAs:

$$a_r \le \frac{y_{rj} u_r}{\sum_{r=1}^{s} y_{rj} u_r} \le b_r, \forall r, j \quad \text{and} \quad c_i \le \frac{x_{ij} v_i}{\sum_{i=1}^{m} x_{ij} v_i} \le d_i, \forall i, j$$

For each input or output factor being constrained $2n$ inequalities are added to each linear program.

Adding the weights restrictions to the multipliers formulation, it becomes:

$$
\begin{array}{lllll}
\frac{1}{e_0} = \min \sum_{i=1}^{m} x_{ij_0} v_i & & & & \\
s.t. & & & & \\
 & \sum_{r=1}^{s} y_{rj_0} u_r & = & 1 & \\
\sum_{i=1}^{m} x_{ij} v_i & -\sum_{r=1}^{s} y_{rj} u_r & \geq & 0, & \forall j \\
 & y_{rj_0} u_r & \geq & a_r, & \forall r \\
 & -y_{rj_0} u_r & \geq & -b_r, & \forall r \\
 & y_{rj} u_r - a_r \sum_{r=1}^{s} y_{rj} u_r & \geq & 0, & \forall r, j \neq j_0 \\
 & -y_{rj} u_r + b_r \sum_{r=1}^{s} y_{rj} u_r & \geq & 0, & \forall r, j \neq j_0 \\
x_{ij} v_i - c_i \sum_{i=1}^{m} x_{ij} v_i & & \geq & 0, & \forall i, j \\
-x_{ij} v_i + d_i \sum_{i=1}^{m} x_{ij} v_i & & \geq & 0, & \forall i, j \\
v_i & & \geq & \varepsilon, & \forall i \\
 & u_r & \geq & \varepsilon, & \forall r
\end{array}
$$

The envelopment formulation becomes:

$$
\begin{aligned}
&\frac{1}{e_0} = \max \theta_0 + \sum_{r=1}^{s} a_r \alpha_{r_0} - \sum_{r=1}^{s} b_r \beta_{r_0} + \varepsilon \sum_{i=1}^{m} s_i^- + \varepsilon \sum_{r=1}^{s} s_r^+ \\
&s.t. \\
&\sum_{j=1}^{n} x_{ij}\lambda_j + \sum_{j=1}^{n}\left[x_{ij}\left(\gamma_{ij} - \sum_{i=1}^{m} c_i \gamma_{ij} - \delta_{ij} + \sum_{i=1}^{m} d_i \delta_{ij} \right)\right] + s_i^- = x_{ij_0}, \quad \forall i \\
&y_{rj_0}\theta_0 - \sum_{j=1}^{n} y_{rj}\lambda_j + y_{rj_0}\alpha_{r_0} - y_{rj_0}\beta_{r_0} \\
&\qquad + \sum_{\substack{j=1 \\ j \neq j_0}}^{n}\left[y_{rj}\left(\alpha_{rj} - \sum_{r=1}^{s} a_r \alpha_{rj} - \beta_{rj} + \sum_{r=1}^{s} b_r \beta_{rj} \right)\right] + s_r^+ = 0, \quad \forall r \\
&\theta_0 \quad \text{free} \\
&\lambda_j \geq 0, \quad \forall j \\
&\gamma_{ij}, \delta_{ij} \geq 0, \quad \forall i, j \\
&s_i^- \geq 0, \quad \forall i \\
&\alpha_{rj}, \beta_{rj} \geq 0, \quad \forall r, j \\
&s_r^+ \geq 0, \quad \forall r
\end{aligned}
$$

The targets:

$$
\sum_{j=1}^{n} x_{ij}\lambda_j^* = x_{ij_0} - \sum_{j=1}^{n}\left[x_{ij}\left(\gamma_{ij}^* - \sum_{i=1}^{m} c_i \gamma_{ij}^* - \delta_{ij}^* + \sum_{i=1}^{m} d_i \delta_{ij}^* \right)\right] - s_i^{-*}, \forall i
$$

$$
\sum_{j=1}^{n} y_{rj}\lambda_j^* = y_{rj_0}\left(\theta_0^* + \alpha_{r_0}^* - \beta_{r_0}^* \right) + \sum_{\substack{j=1 \\ j \neq j_0}}^{n}\left[y_{rj}\left(\alpha_{rj}^* - \sum_{r=1}^{s} a_r \alpha_{rj}^* - \beta_{rj}^* + \sum_{r=1}^{s} b_r \beta_{rj}^* \right)\right] + s_r^{+*}, \forall r
$$

As with the previous approach the efficiency score e_0^* will no longer necessarily approximate the inverted radial expansion factor $1/\theta_0^*$. Moreover, the interpretation of the targets becomes increasingly difficult. The targets for a factor (either input or output) become dependent, not only on the value of that factor for the target unit, but also on the value of that factor for all the other units. Additionally, they become dependent on the limits imposed for the virtuals (a_r and b_r for outputs and c_i and d_i for inputs) of all other factors.

WB's argument that this alternative is computationally expensive is probably the least of its caveats as software and hardware become increasingly more powerful. This approach applies each unit's restrictions to all the other units, and is thus applying the 'worst case' to each unit. As a consequence, and more worrying is the propensity of this approach to lead to infeasible linear programs, as will be shown in section 3 of this paper.

As an alternative, Roll and Golany (1993) propose the "common set of weights" approach, where all units are treated the same. However, this approach requires additional (often subjective) information to be introduced into the analysis.

2.3 Proportional virtual weights restrictions apply to target and *to an 'average' artificial UOA* j_a

In order to keep to the spirit of their second approach in section 2.2, but to avoid its computational problems, WB favour a third approach involving an artificial 'average' UOA. The logic behind it is that the proportional weights restriction should not only apply to the target UOA, but to the average UOA in the set. The average artificial UOA j_a is defined as: $\left(x_{ij_a}, y_{rj_a}\right) = \left(\sum_{j=1}^{n} \frac{x_{ij}}{n}, \sum_{j=1}^{n} \frac{y_{rj}}{n}\right)$, and the following restrictions are added to each linear program in the initial approach in section 2.1:

$$a_r \leq \frac{\left(\sum_{j=1}^{n} \frac{y_{rj}}{n}\right) u_r}{\sum_{r=1}^{s} \left(\sum_{j=1}^{n} \frac{y_{rj}}{n}\right) u_r} \leq b_r, \forall r \quad \text{and} \quad c_i \leq \frac{\left(\sum_{j=1}^{n} \frac{x_{ij}}{n}\right) v_i}{\sum_{i=1}^{m} \left(\sum_{j=1}^{n} \frac{x_{ij}}{n}\right) v_i} \leq d_i, \forall i$$

Thus, when this approach is implemented, each UOA is assessed with four additional constraints for each factor being restricted:

$$
\begin{array}{lllll}
\frac{1}{e_0} = \min \sum_{i=1}^{m} x_{ij_0} v_i & & & & \\
s.t. & & & & \\
 & \sum_{r=1}^{s} y_{rj_0} u_r & = & 1 & \\
\sum_{i=1}^{m} x_{ij} v_i & -\sum_{r=1}^{s} y_{rj} u_r & \geq & 0, & \forall j \\
 & y_{rj_0} u_r & \geq & a_r, & \forall r \\
 & -y_{rj_0} u_r & \geq & -b_r, & \forall r \\
x_{ij_0} v_i - c_i \sum_{i=1}^{m} x_{ij_0} v_i & & \geq & 0, & \forall i \\
-x_{ij_0} v_i + d_i \sum_{i=1}^{m} x_{ij_0} v_i & & \geq & 0, & \forall i \\
 & \left(\sum_{j=1}^{n} \frac{y_{rj}}{n} \right) u_r - a_r \sum_{r=1}^{s} \left(\sum_{j=1}^{n} \frac{y_{rj}}{n} \right) u_r & \geq & 0, & \forall r \\
 & -\left(\sum_{j=1}^{n} \frac{y_{rj}}{n} \right) u_r + b_r \sum_{r=1}^{s} \left(\sum_{j=1}^{n} \frac{y_{rj}}{n} \right) u_r & \geq & 0, & \forall r \\
\left(\sum_{j=1}^{n} \frac{x_{ij}}{n} \right) v_i - c_i \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \frac{x_{ij}}{n} \right) v_i & & \geq & 0, & \forall i \\
-\left(\sum_{j=1}^{n} \frac{x_{ij}}{n} \right) v_i + d_i \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \frac{x_{ij}}{n} \right) v_i & & \geq & 0, & \forall i \\
v_i & & \geq & \varepsilon, & \forall i \\
 & u_r & \geq & \varepsilon, & \forall r
\end{array}
$$

This averaging construct will penalise units with small or large, input or output values, as it imposes a ‘majority rule’, which is clearly against the spirit of traditional DEA.

The envelopment formulation becomes:

$$
\begin{aligned}
&\frac{1}{e_0} = \max \theta_0 + \sum_{r=1}^{s} a_r \alpha_r - \sum_{r=1}^{s} b_r \beta_r + \varepsilon \sum_{i=1}^{m} s_i^- + \varepsilon \sum_{r=1}^{s} s_r^+ \\
&s.t. \\
&\quad \sum_{j=1}^{n} x_{ij} \lambda_j + x_{ij_0} \left(\gamma_i - \sum_{i=1}^{m} c_i \gamma_i - \delta_i + \sum_{i=1}^{m} d_i \delta_i \right) + \\
&\qquad + \left(\sum_{j=1}^{n} \frac{x_{ij}}{n} \right) \left(\gamma_{i_a} - \sum_{i=1}^{m} c_i \gamma_{i_a} - \delta_{i_a} + \sum_{i=1}^{m} d_i \delta_{i_a} \right) + s_i^- \qquad = x_{ij_0}, \forall i \\
&y_{rj_0} \theta_0 - \sum_{j=1}^{n} y_{rj} \lambda_j + y_{rj_0} \left(\alpha_r - \beta_r \right) + \\
&\qquad + \left(\sum_{j=1}^{n} \frac{y_{rj}}{n} \right) \left(\alpha_{r_a} - \sum_{r=1}^{s} a_r \alpha_{r_a} - \beta_{r_a} + \sum_{r=1}^{s} b_r \beta_{r_a} \right) \quad + s_r^+ \qquad = 0, \forall r \\
&\theta_0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{free} \\
&\quad \lambda_j \qquad\qquad\qquad\qquad\qquad\qquad\qquad \geq 0, \forall j \\
&\qquad \gamma_i, \delta_i, \gamma_{i_a}, \delta_{i_a}, \qquad s_i^- \qquad \geq 0, \forall i \\
&\qquad \alpha_r, \beta_r, \alpha_{r_a}, \beta_{r_a}, \qquad s_r^+ \qquad \geq 0, \forall r
\end{aligned}
$$

The obtained targets are then:

$$
\begin{aligned}
&\sum_{j=1}^{n} x_{ij} \lambda_j^* = x_{ij_0} \left(1 - \gamma_i^* + \sum_{i=1}^{m} c_i \gamma_i^* + \delta_i^* - \sum_{i=1}^{m} d_i \delta_i^* \right) - \\
&\qquad - x_{ij_a} \left(\gamma_{i_a}^* - \sum_{i=1}^{m} c_i \gamma_{i_a}^* - \delta_{i_a}^* + \sum_{i=1}^{m} d_i \delta_{i_a}^* \right) - s_i^{-*}, \forall i \\
&\sum_{j=1}^{n} y_{rj} \lambda_j^* = y_{rj_0} \left(\theta_0^* + \alpha_r^* - \beta_r^* \right) + \\
&\qquad + y_{rj_a} \left(\alpha_{r_a}^* - \sum_{r=1}^{s} a_r \alpha_{r_a}^* - \beta_{r_a}^* + \sum_{r=1}^{s} b_r \beta_{r_a}^* \right) + s_r^{+*}, \forall r
\end{aligned}
$$

Being a compromise between the first (2.1) and second (2.2) approaches, the third approach has characteristics of both of them. As in the previous approaches, the efficiency measure e_0^* will no longer necessarily approximate the inverted radial expansion factor $1/\theta_0^*$. And, the targets are increasingly difficult to interpret: not only are they dependent on the current levels of input and output of the target UOA, they are also dependent on the average value of all the units. As with the previous approach, this alternative can also easily lead to infeasible linear programs, as will be shown in section 3 of this paper.

3 Using Absolute Weights Restriction with a 'Virtual' Meaning

A problem with virtual weights restrictions is that they are UOA specific. Allen et al (1997b) (see also Dyson et al (2001)) have suggested that virtual weights restriction, as proposed by WB's second alternative (2.2), could be converted into absolute weights restrictions, in the following manner:

When considering a lower bound on output *r*, of a_r, such as $a_r \le y_{rj} u_r, \forall j$, clearly the virtual restriction corresponding to the UOA with the lowest output level can be binding. Similarly, if a virtual upper bound restriction were imposed on an output *r*, of b_r, such as $y_{rj} u_r \le b_r, \forall j$, the binding virtual restriction would be the one corresponding to the UOA with the largest input or output level. Allen et al (1997b) conclude that a more economical approach to WB's would be to add only the required binding absolute restrictions. The idea would be very useful in transforming simple virtual weights restrictions, when only lower or upper bounds have been determined for each factor, but not both simultaneously, as in WB's second (2.2) and third (2.3) approaches. In those cases Allen et al's suggestion of transforming proportional virtual weights restrictions into absolute ones often leads to infeasibility, as it is shown next.

Consider the example of a factor, whose importance should, according to the controller, be between 5% and 15%. If the maximum value for this factor is 100, and the minimum 1 (as is the case in some of the factors in Sarrico et al. (1997), for example), the following expression is derived:

$$5\% \le y_{rj}u_r \le 15\%, \forall j$$

Using Allen et al's approach, it would lead to the following:

$$\frac{0.05}{\min(y_{rj})} \le u_r \le \frac{0.15}{\max(y_{rj})} \Leftrightarrow 0.05 \le u_r \le 0.0015\,!$$

In conclusion, the alternative will easily lead to infeasible results for some 'intuitive' set of bound on the virtuals. In fact, for the use of *proportional* virtual weights restrictions to be feasible for all units in each linear program in section 2.2, the bounds have to be carefully chosen, such that:

$$\frac{a_r}{\min(y_{rj})} \le u_r \le \frac{b_r}{\max(y_{rj})}$$

is feasible for all the units. This will mean:

* setting up the lower limit a_r, and then calculate the upper limit b_r as follows:

$$a_r \le y_{rj}u_r \le \frac{\max(y_{rj})}{\min(y_{rj})} \cdot a_r \le b_r, \forall r, j$$

* or, setting up the upper limit b_r, and then a_r is calculated as follows:

$$a_r \le \frac{\min(y_{rj})}{\max(y_{rj})} \cdot b_r \le y_{rj}u_r \le b_r, \forall r, j\ .$$

Going back to our previous example, if a_r were set at the 5% level, then b_r would have to be at least 500%, for the restriction to be feasible for all UOAs in section 2.2. If b_r were set at the 15% level, then a_r could not exceed 0.15%. These results are clearly no longer intuitive, as promoted by WB.

A similar reasoning for approach in section 2.3, where the restrictions apply to both target and average unit simultaneously, can be made:

$$\begin{cases} \dfrac{a_r}{y_{rj_0}} \le u_r \le \dfrac{b_r}{y_{rj_0}} \\ \dfrac{a_r}{y_{rj_a}} \le u_r \le \dfrac{b_r}{y_{rj_a}} \end{cases} \Leftrightarrow \begin{cases} y_{rj_0} \le y_{rj_a} \\ \dfrac{a_r}{y_{rj_0}} \le u_r \le \dfrac{b_r}{y_{rj_a}} \end{cases} \vee \begin{cases} y_{rj_0} \ge y_{rj_a} \\ \dfrac{a_r}{y_{rj_a}} \le u_r \le \dfrac{b_r}{y_{rj_0}} \end{cases}$$

for each linear program.

Two situations arise; either the level of the output for the target unit is less than the average, or greater than the average. However, for all linear programs to be feasible, in either case, the same conclusion as for section 2.2 is reached:

$$a_r \le \frac{\min(y_{rj})}{\max(y_{rj})} \cdot b_r \, .$$

It is thus demonstrated that if both upper and lower bounds are considered in WB's second (2.2) and third (2.3) approaches, the programs will often be infeasible.

WB were optimistic that obtaining the limits, a_r and b_r corresponding to value judgements, was 'not too difficult', but the multicriteria decision making (MCDM) perspective on the use of DEA disagrees (see, for instance, Belton and Vickers, 1993; Stewart, 1996; Belton and Stewart, 1997). Recently, Belton and Stewart on a discussion on the interactions between MCDM and DEA, point out that the DEA field could learn from the extensive experience of MCDM in eliciting and working with value judgements. They consider that it is difficult to set meaningful bounds on the weights, especially in the case of multiple input - multiple output problems, except in terms of ratios. Moreover 'intuitive' limits expressed as a percentage, as seen, can easily lead to infeasibility in WB's approaches described above. Interdisciplinary research between the MCDM and DEA fields looks like fertile ground for further research.

The realisation that using explicit boundaries for weights is, in general, a difficult task, has led the authors of this paper to advocate the use of virtual assurance regions, similar to the use of assurance regions with absolute weights, rather than the simple direct restrictions on the virtuals. In the next section, a new classification of virtual weights restrictions is proposed and their advantages in relation to WB's original proposals are shown.

4 Virtual Assurance Regions

4.1 Why use virtual weights restrictions?

As described above there are problems with using WB's approaches, in that the choice of bounds by the decision maker/ controller can easily render the problems infeasible, and in addition, the efficiency score and targets are not always readily interpreted. Despite the portrayed problems, the authors still think that the use of virtual weights restrictions is valuable when dealing with an external controller. When the different factors in the assessment do not have a common unit, such as a monetary unit, it is often easier to translate the external decision-maker's preference structure into restrictions on the virtuals, rather than on absolute weights.

However, although inspired by WB's virtual weights restrictions, the virtual weights restrictions advocated by the authors of this paper are different. It will be shown that the restrictions proposed present advantages relative to the original proportional virtual weights restrictions proposed by WB. Especially, the introduction of the concept of virtual assurance regions will avoid the problem of the decision maker having to choose explicit bounds on the virtuals, when no objective information exists to allow him to make such a decision.

4.2 A New Classification

Thompson et al. (1990) proposed a classification of absolute weights restrictions into simple absolute weights restrictions and assurance regions. They named assurance regions of type I (ARI) the absolute weights restrictions that incorporate into the analysis the relative ordering or values of the inputs or outputs; and assurance regions of type II (ARII), the absolute weights restrictions that translate relationships across the input and output divide. In fact, information about a numerical range to translate the importance of the input or output factor as in simple direct restrictions on the multipliers, in general, might be difficult to obtain, and an ordering of preferences, as in assurance regions, might be more suitable.

For this reason, in this section a similar classification for virtual weights restrictions is proposed. That is, to have virtual weights restrictions equivalent to simple absolute weights restrictions, as well as equivalent to assurance regions of type I (ARI) and of type II (ARII). Thus the ordering of preferences by applicants to universities in Sarrico et al. (1997) are translated by *virtual* weights restrictions of the type I. Indeed, most applicants would not be able to specify explicit weights. The linking of teaching and research outputs produced by a cost group in each university to the inputs available to that cost group, when assessing the performance of universities from the perspective of the State in Sarrico (1998), is an example of *virtual* weights restrictions of type II, a small example of which will be shown in section 6 of this paper.

All proposed virtual weights restrictions can be described by the general set of $w=1...t$ weights restrictions, applying to the target unit:

$$\sum_{i=1}^{m} a_{iw} x_{ij_0} v_i + \sum_{r=1}^{s} b_{rw} y_{rj_0} u_r \geq k_w, \qquad \forall w.$$

This expression encapsulates the three different kinds of virtual weights restrictions, of the new classification presented below.

Simple virtual weights restrictions

Simple virtual weights restrictions involve constraining the virtual of a single factor. This approach is equivalent to using *proportional* virtual weights restrictions applied to output virtuals in an output-oriented model, as the denominator is 1, (i.e. $\sum_{r=1}^{s} y_{rj_0} u_r = 1$). If applied also to input virtuals in an output-oriented model, it will only be equivalent for the units that are efficient (i.e. $\sum_{i=1}^{m} x_{ij_0} v_i = 1$), and therefore define the frontier. They are of the form:

$$\begin{aligned} & a_{iw} x_{ij_0} \geq k_w, i = i' \\ & a_{iw} = 0, \forall i \neq i' \\ & b_{rw} = 0, \forall r \end{aligned}$$

for restricting the virtual input *i'*; and

$$\begin{aligned} & b_{rw} y_{rj_0} \geq k_w, r = r' \\ & b_{rw} = 0, \forall r \neq r' \\ & a_{iw} = 0, \forall i \end{aligned}$$

for restricting virtual output *r'*.

These restrictions are useful when the decision maker is able to specify particular bounds, or wants to assure that a certain factor attains a threshold value, for instance.

Virtual assurance regions of type I

Virtual ARI restrictions are equivalent to absolute weights restrictions ARI. They link virtual inputs or outputs to translate an ordering of preference. They are:

$$\sum_{i=1}^{m} a_{iw} x_{ij_0} v_i \geq 0$$
$$b_{rw} = 0, \forall r$$

to link virtual inputs, and

$$\sum_{r=1}^{s} b_{rw} y_{rj_0} u_r \geq 0$$
$$a_{iw} = 0, \forall i$$

to link virtual outputs.

These restrictions are useful when the decision maker cannot assign particular bounds to the factors, but is able to decide that a factor is more important than another, twice as important, etc.

Virtual assurance regions of type II

Finally, virtual ARII restrictions are equivalent to absolute weights restrictions ARII. They link the input-output divide. They can be translated by:

$$\sum_{i=1}^{m} a_{iw} x_{ij_0} v_i + \sum_{r=1}^{s} b_{rw} y_{rj_0} u_r \geq 0,$$

where at least one $a_{iw} \neq 0$ and one $b_{rw} \neq 0$.

These restrictions are useful when there is a known relationship between an input and an output. For instance, it is known that to produce a certain output, one needs to have a certain level of a certain input.

In the next section the consequences of the use of this section virtual weights restrictions for the interpretation of the efficiency score and targets is explored, and reasons given why they are preferred to WB's.

5 Virtual Assurance Regions with Factor Linkages

5.1 Virtual weights restrictions apply to the target *UOA* j_0

When combinations of different types of virtual weights restrictions are used in a model, the multipliers formulation becomes:

$$
\begin{aligned}
&\frac{1}{e_0} = \min \sum_{i=1}^{m} x_{ij_0} v_i \\
&s.t. \\
&\qquad\qquad \sum_{r=1}^{s} y_{rj_0} u_r \quad = \quad 1 \\
&\sum_{i=1}^{m} x_{ij} v_i \quad - \sum_{r=1}^{s} y_{rj} u_r \quad \geq \quad 0, \quad \forall j \\
&\sum_{i=1}^{m} a_{iw} x_{ij_0} v_i \quad + \sum_{r=1}^{s} b_{rw} y_{rj_0} u_r \quad \geq \quad k_w, \quad \forall w \\
&v_i \quad \geq \quad \varepsilon, \quad \forall i \\
&\qquad u_r \quad \geq \quad \varepsilon, \quad \forall r
\end{aligned}
$$

The effect on the envelopment formulation is as below:

$$
\begin{aligned}
&\frac{1}{e_0} = \max \theta_0 + \sum_{w=1}^{t} k_w \rho_{w_0} + \varepsilon \sum_{i=1}^{m} s_i^- + \varepsilon \sum_{r=1}^{s} s_r^+ \\
&s.t. \\
&\qquad \sum_{j=1}^{n} x_{ij} \lambda_j \quad + \sum_{w=1}^{t} a_{iw} x_{ij_0} \rho_{w_0} + s_i^- \quad = \quad x_{ij_0}, \quad \forall i \\
&y_{rj_0} \theta_0 \quad - \sum_{j=1}^{n} y_{rj} \lambda_j \quad + \sum_{w=1}^{t} b_{rw} y_{rj_0} \rho_{w_0} \quad + s_r^+ \quad = \quad 0, \quad \forall r \\
&\theta_0 \quad \text{free} \\
&\qquad \lambda_j \quad \geq \quad 0, \quad \forall j \\
&\qquad\qquad \rho_{w_0} \quad \geq \quad 0, \quad \forall w \\
&\qquad\qquad\qquad s_i^- \quad \geq \quad 0, \quad \forall i \\
&\qquad\qquad\qquad\qquad s_r^+ \quad \geq \quad 0, \quad \forall r
\end{aligned}
$$

And the targets:

$$\sum_{j=1}^{n} x_{ij}\lambda_j^* = x_{ij_0}\left(1-\sum_{w=1}^{t} a_{iw}\rho_{w_0}^*\right) - s_i^{-*}, \forall i$$
$$\sum_{j=1}^{n} y_{rj}\lambda_j^* = y_{rj_0}\left(\theta_0^* + \sum_{w=1}^{t} b_{rw}\rho_{w_0}^*\right) + s_r^{+*}, \forall r$$

If the model includes *simple virtual weights restrictions*, where either a minimum and/ or maximum virtual is imposed for some or all factors, then the efficiency score e_0^* will no longer necessarily approximate the inverted radial expansion factor $1/\theta_0^*$, as the objective function contains a term with the new variable ρ_{w_0} to be maximised. On the other hand, if the model contains only *virtual weights restrictions of the type ARI and ARII*, the efficiency score will converge to be $1/\theta_0^*$, as the term $\sum_{w=1}^{t} k_w \rho_{w_0}$ will not exist. As for the targets, their interpretation is easier than in the models with *proportional* virtual weights restrictions. Either an improvement or deterioration of current levels of the factors is possible, but in any case they can still be interpreted as a contraction or expansion of the current levels of the factors of the target unit.

The same problems discussed, when *proportional* virtual weights restrictions apply only to the target unit, occur. Restrictions applying to all units can be envisaged, as in the next section.

5.2 *Virtual weights restrictions apply to* all *UOA* j

The multipliers formulation with the virtual weights restrictions applying to all UOAs are as below:

$$
\begin{aligned}
&\frac{1}{e_0} = \min \sum_{i=1}^{m} x_{ij_0} v_i \\
&s.t. \\
&\qquad \sum_{r=1}^{s} y_{rj_0} u_r = 1 \\
&\sum_{i=1}^{m} x_{ij} v_i - \sum_{r=1}^{s} y_{rj} u_r \geq 0, \quad \forall j \\
&\sum_{i=1}^{m} a_{iw} x_{ij} v_i + \sum_{r=1}^{s} b_{rw} y_{rj} u_r \geq k_w, \quad \forall w, j \\
&v_i \geq \varepsilon, \quad \forall i \\
&\qquad u_r \geq \varepsilon, \quad \forall r
\end{aligned}
$$

The effect on the envelopment formulation would be as below:

$$
\begin{aligned}
&\frac{1}{e_0} = \max \theta_0 + \sum_{j=1}^{n} \sum_{w=1}^{t} k_w \rho_{wj} + \varepsilon \sum_{i=1}^{m} s_i^- + \varepsilon \sum_{r=1}^{s} s_r^+ \\
&s.t. \\
&\qquad \sum_{j=1}^{n} x_{ij} \lambda_j + \sum_{j=1}^{n} \sum_{w=1}^{t} a_{iw} x_{ij} \rho_{wj} + s_i^- = x_{ij_0}, \quad \forall i \\
&y_{rj_0} \theta_0 - \sum_{j=1}^{n} y_{rj} \lambda_j + \sum_{j=1}^{n} \sum_{w=1}^{t} b_{rw} y_{rj} \rho_{wj} + s_r^+ = 0, \quad \forall r \\
&\theta_0 \quad \text{free} \\
&\lambda_j \geq 0, \quad \forall j \\
&\rho_{wj} \geq 0, \quad \forall w, j \\
&s_i^- \geq 0, \quad \forall i \\
&s_r^+ \geq 0, \quad \forall r
\end{aligned}
$$

And the targets:

$$\sum_{j=1}^{n} x_{ij}\lambda_j^* = x_{ij_0} - \sum_{j=1}^{n}\sum_{w=1}^{t} a_{iw} x_{ij}\rho_{wj}^* - s_i^{-*}, \forall i$$
$$\sum_{j=1}^{n} y_{rj}\lambda_j^* = y_{rj_0}\theta_0^* + \sum_{j=1}^{n}\sum_{w=1}^{t} b_{rw} y_{rj}\rho_{wj}^* + s_r^{+*}, \forall r$$

As in the previous section, if the model includes *simple virtual weights restrictions*, where either a minimum and/ or maximum virtual is imposed for some or all factors, then the efficiency score e_0^* will no longer necessarily approximate the inverted radial expansion factor $1/\theta_0^*$. On the other hand, if the model contains only *virtual weights restrictions of the type ARI and ARII,* the efficiency score will converge to be $1/\theta_0^*$.

However, the interpretation of the targets as a contraction or expansion of the current levels of inputs or outputs, depending on the controller's preferences translated by the weights restrictions imposed, no longer applies. The expression of the targets for the UOA under analysis has a new term, which not only depends on the current levels of the factor for the target unit but also for all the other units. However, it is still an improvement from the targets obtained from the use of *proportional* virtual weights restrictions applying to all units, in that the target for the factor under analysis does not depend on the virtual limits imposed on all the other factors.

In conclusion, the use of the virtual weights restrictions applying to the target UOA j_0 only, as in section 5.1, seems to be the best approach. This approach was widely used in the context of performance measurement in UK universities in Sarrico (1998), an example of which will be shown in the next section. It allows for the natural representation of preference structures; linkages between inputs and outputs translating established patterns; and finally, the meaning of the efficiency score and targets are most easily interpreted.

6 An application to the UK university sector for the academic year 1995/96

6.1 Units of assessment

Assume that the efficiency of universities is to be compared. Handling subject mix in the context of university performance measurement has long been a problem. A procedure is required which ensures that universities with low cost subjects do not have an unfair advantage in the assessment of performance. In the past, universities have been divided into different comparable sets, and assessed separately. In this empirical application a solution for taking into account subject mix in universities is arrived at by linking inputs and outputs across the input-output divide, via virtual assurance regions. This represents a novel application of the method devised by Thanassoulis et al. (1995) in relation to absolute weights restrictions, which prevents units from taking undue advantage of weight flexibility contrary to the known links between certain inputs and outputs.

Consider 89 UK universities, which can develop teaching and research activities in four different cost bands considered by the funding councils. The cost bands are as follows: Group A: Clinical, Group B: Science, Engineering and Technology, Group C: Other high cost subjects with a studio, laboratory or fieldwork element, and Group D: All other subjects.

6.2 Experimental design

To disentangle the subject-mix effect from the measurement of institutional efficiency, the four cost band groups are considered. The inputs, outputs and weights restrictions used in the CCR model are as in Table 1. The variables used in the model are defined as follows:

EXP_A: It is the sum of *Academic Departments* expenditure for academic cost centres (ACCs) belonging to Group A in £ thousands, from Higher Education Statistics Agency (HESA).

EXP_B, EXP_C, EXP_D: As in EXP_A, but for ACCs belonging to groups B, C, and D, respectively.

CNTRL_EXP: It is the sum of *Academic Services*, *Administration and Central Services*, *Premises*, *Residences and Catering Operations*, and *Research Grants and Contracts* expenditures, in £ thousands, from HESA statistics.

TEACH_A: It is the total number of full-time-equivalent (FTE) students in the subject areas (from HESA statistics) belonging to Group A, as the measure of volume, multiplied by the average teaching rating (1-24) for the university, as a measure of quality.

TEACH_B, TEACH_C, TEACH_D: As in TEACH_A, but for subject areas belonging to groups B, C, and D, respectively.

RES_A: It is the total number of FTE academic staff in UOAs (from the Research Assessment Exercise 1996 database) belonging to Group A, as the measure of volume, multiplied by the average research rating (1 to 7) for the university, as a measure of quality.

RES_B, RES_C, RES_D: As in RES_A, but for UOA belonging to groups B, C, and D.

Table 1: University efficiency taking account of subject mix

Variable Set			
Inputs	*Outputs*	*Weights Restrictions*	*Type*
EXP_A	TEACH_A	*V*TEACH_A + *V*RES_A > *V*EXP_A	Virtual II
EXP_B	TEACH_B	*V*TEACH_B + *V*RES_B > *V*EXP_B	Virtual II
EXP_C	TEACH_C	*V*TEACH_C + *V*RES_C > *V*EXP_C	Virtual II
EXP_D	TEACH_D	*V*TEACH_D + *V*RES_D > *V*EXP_D	Virtual II
CNTRL_EXP	RES_A	*W*EXP_A > 4.5 *W*EXP_D	Absolute I
	RES_B	*W*EXP_B > 2.0 *W*EXP_D	Absolute I
	RES_C	*W*EXP_C > 1.5 *W*EXP_D	Absolute I
	RES_D	*V*EXP_D > 0.05	Virtual Simple
		*V*CNTRL_EXP > 0.40	Virtual Simple

Note: The prefix *V* indicates the virtual associated with the factor, and the prefix *W* indicates the multiplier associated with the factor.

The weights restrictions reflect the following:

* The first four virtual assurance regions reflect the fact that the teaching and research output virtual weights combined for each group should be at least the same as the departmental expenditure virtual weight for that group. This reflects the fact that both departmental expenditure and a proportion of central expenditure contribute for the departmental output. Since the variables are expressed in different units, virtual weights restrictions are preferred. Assurance regions are preferred because no meaningful bounds on the variables could be ascertained.
* The absolute weight for Group A should be at least 4.5 times that for Group D, for Group C 2.0 times, and for Group B 1.5 times. This reflects empirical evidence collected by the funding councils. Since all expenditures refer to the same monetary unit, absolute weights restrictions are used.
* The two last virtual weights restrictions reflect the evidence in the data; that for all units under assessment, at least 5% of the total expenditure is academic departmental expenditure in Group D, and at least 40% of the total expenditure is in central activities. Simple virtual weights restrictions are used, since meaningful bounds on the weights can be ascertained, and to avoid that some of the factors are not considered by the UOA by assigning zero weights to some of the factors.

The model thus requires several categories of weights restrictions to capture the required preferences. Several categories of virtual weights restrictions were also used in the Kenilworth School case study in Sarrico et al (1997).

6.3 Results

The results for the model in Table 1, taking into account of the effect of a university's subject mix in the measurement of efficiency, are shown in Table 2. There does not seem to be much of a difference in the results between clinical and non-clinical, English and non-English, and London and Non-London universities. As for the difference in results between old and new universities, it seems that under CRS assumption new universities are more efficient than old universities. New universities were for their majority polytechnics, which were allowed to acquire university status following the

1992 Further and Higher Education Act. Oxford is efficient, whereas Cambridge is inefficient (88.9%). This incongruence in the results for Oxbridge, keeps the debate open as to the legitimacy of their different funding structure, from the other universities. The allocation of virtual weights between teaching and research activities is, also, quite different. Old universities, despite years of University Grants Committee (UGC) funding based on that teaching and research should be pursued in a ratio of approximately 2:1, put considerable more emphasis on research than teaching, whereas the weight put on teaching and research by the new universities is more balanced.

Table 2: Results

	Count	#Efficient Units	Average Efficiency	Average Sum of Teaching Virtuals	Average Sum of Research Virtuals
All	89	40	93.7	42.6	57.4
Clinical	21	9	95.5	34.4	65.6
Non Clinical	68	31	93.1	45.1	54.9
Non English	20	11	94.4	30.0	70.0
English	69	29	93.4	46.2	53.8
New	40	24	97.1	55.0	45.0
Old	49	16	90.9	32.5	67.6
London	11	2	91.2	53.3	46.8
Non London	78	38	94.0	41.1	59.0

In the past universities have been pooled in different groups of similar subject mix before their efficiency could be compared. The use of *virtual assurance regions of type II* linking the input-output divide for each subject group, allows efficiency comparisons to be made for all universities irrespective of their subject mix.

7 Concluding Remarks

Absolute weights restrictions, and assurance regions have been advocated as ways of restricting the values of weights (multipliers) in DEA. However they can also have limited scope. This paper has advocated the use of restrictions on virtual weights in DEA on the grounds that they often provide a natural representation of preferences. Restrictions on virtual weights were proposed first by Wong and Beasley. The paper has shown that their *proportional* weights restrictions can lead to problems of feasibility and in the interpretation of targets and efficiency scores. As an alternative, the paper proposes the use of (non-proportional) *virtual* weights restrictions, and categorises them as simple virtual weights restrictions, and virtual assurance regions of types I and II, following the categories established for *absolute* weights restrictions. It is concluded that these categories overcome many of the difficulties of *proportional* weights restrictions whist retaining the benefit of the natural representation of preference structures, which often need to be imposed on the basic DEA model. Additionally they can translate established patterns across the input-output divide, and the meaning of the efficiency score and targets in this approach are more easily interpreted. Simple virtual restrictions can be used when meaningful bounds can be established.

The paper concludes with an application that illustrates how a range of categories of weights restrictions may be required to capture the preference structures in the context of higher education. The overall conclusion is thus that different approaches to weights restrictions may be appropriate to capture particular preference information, and that virtual weights restrictions have a key role to play in that arena.

References

Allen, R., Athanassopoulos, A. D., Dyson, R. G., Thanassoulis, E., 1997a. Weights restrictions and value judgements in data envelopment analysis: evolution, development and future directions. Annals of Operations Research 73, 13-34.

Allen, R., Dyson, R. G., Santos, A. R., Shale, E., 1997b. Pitfalls and protocols in the application of DEA. Presented at the 39th Annual Conference of the Operational Research Society. September 1997, Bath, UK.

Belton, V., Vickers, S. P., 1993. Demystifying DEA - a visual interactive approach based on multiple criteria analysis. Journal of the Operational Research Society 44, 883-896.

Belton, V., Stewart, T. J., 1998. DEA and MCDM: competing or complementary approaches? In Meskens, N., Roubens, M. (Eds.), Advances in Decision Analysis, Kluwer, Boston.

Charnes, A., Cooper, W. W., Rhodes, E., 1978. Measuring the efficiency of decision making units. European Journal of Operational Research 2, 429-444.

Charnes, A., Cooper, W. W., Huang, Z. M., Sun, D. B., 1990. Polyhedral cone-ratio DEA models with an illustrative application to large commercial banks. Journal of Econometrics 46, 73-91.

Dyson, R. G., Thanassoulis, E., 1988. Reducing weight flexibility in data envelopment analysis. Journal of the Operational Research Society 39, 563-576.

Dyson, R. G., Allen, R., Camanho, A. S., Podinovski, V. V., Sarrico, C. S., Shale, E. A., 2001. Pitfalls and protocols in DEA. European Journal of Operational Research 132, 245-259.

Roll, Y., Golany, B., 1993. Alternate methods of treating factor weights in DEA. Omega 21, 99-109.

Sarrico, C. S., 1998. Performance Measurement in UK Universities: Bringing in the Stakeholders' Perspectives Using Data Envelopment Analysis. PhD Thesis, University of Warwick.

Sarrico, C. S., Dyson, R. G., 2000. Using DEA for planning UK universities - an institutional perspective. Journal of the Operational Research Society 51, 789-800.

Sarrico, C. S., Hogan, S. M., Dyson, R. G., Athanassopoulos, A. D., 1997. Data envelopment analysis and university selection. Journal of the Operational Research Society 48, 1163-1177.

Stewart, T. J., 1996. Relationships between data envelopment analysis and multicriteria decision analysis. Journal of the Operational Research Society 47, 654-665.

Thanassoulis, E., Boussofiane, A., Dyson, R. G., 1995. Exploring output quality targets in the provision of perinatal care in England using data envelopment analysis. European Journal of Operational Research 80, 588-607.

Thompson, R. G., Singleton, F. D., Thrall, R. M., Smith, B. A., 1986. Comparative site evaluations for locating a high-energy physics lab in Texas. Interfaces 16, 35-49.

Thompson, R. G., Langemeier, L. N., Lee, C.-T., Lee, E., Thrall, R. M., 1990. The role of multiplier bounds in efficiency analysis with application to Kansas farming. Journal of Econometrics 46, 93-108.

Wong, Y.-H. B., Beasley, J. E., 1990. Restricting weight flexibility in data envelopment analysis. Journal of the Operational Research Society 41, 829-835.