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Relation-changing models meet paraconsistency

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This paper is dedicated to Luís S. Barbosa, on the occasion of his sixtieth birthday. The authors are grateful for the time we have been able to spend working together with Luís on so many interesting Logic constructions. His inspiring and generous personality was always revealed while sharing his knowledge and experience.

ABSTRACT

Switch graphs are graph-like structures characterized by embedding higher-level edges (edges that link to other edges) to describe reactive phenomena. When an edge of such structure is traversed, the accessibility relation of this graph can be changed by adding/removing edges. Relation-changing models have been used to represent phenomena in diverse fields (from Biology to Computer Science) and some modal languages were introduced recently. In this paper we introduce four-valued local information in switch graphs, and propose a paraconsistent logic to study these systems.

1. Introduction and motivation

Areces and Van Benthem introduced Swap logic [1] and Sabotage logic [30], respectively. These logics are characterized by the addition of a new unary operator to Modal logic [5]. In both cases, the new operator has a modal flavour. The evaluation of traditional modal formulas with the usual diamond and box operators – $\Diamond$ and $\Box$ respectively – resorts to the accessibility relation in the underlying frame, as a means to obtain information about reachability between states, but without changing it. For example, a diamond-formula $\Diamond\varphi$ holds in a certain model and state if for some reachable state from the one where the evaluation is being carried and in the same model, φ holds. In the case of the operators introduced in Swap and Sabotage logics, the evaluation of the new modal-like formulas changes the accessibility relation in the underlying frame of the Kripke model. In short, after moving to one of the reachable states from the current one, the accessibility relation is updated, thus the model where the remaining evaluations are performed is not the same as the one at the start of the evaluation. For that reason, these operators are called relation-changing operators. Gabbay calls *reactive frames* to this kind of frame where the set of edges is reconfigured. Furthermore, Gabbay and Marcelino proposed a bimodal logic to describe this kind of frame [19].

Gabbay and Marcelino have also proposed, in [18], an alternative approach to reactivity, by considering reactive behaviours as intrinsic to the structure of the model, thus introducing the concept of *switch graph*. Switch graphs are graph-based structures which comprise a set of higher-level edges (edges connecting two other edges) in addition to usual edges and vertices. These higher-level edges are used to embed the reactive behaviours within this class of model.

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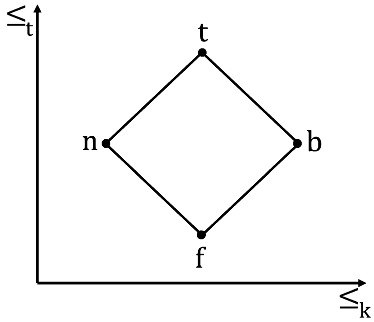

Fig. 1. Belnap's bilattice.

In [14], this class of structures was proposed as a tool to describe Reo connectors. The higher-level edges embedded in the structure of switch graphs can be used to describe the spacial structure of connectors like *lossy*. This was done by considering a hybrid multimodal language and a Kripke model capturing the admissible paths. Furthermore, this language was shown to be able to prove behavioural equivalences, namely by defining a notion of bisimulation for which, under some conditions, a Hennessy-Milner theorem was proven.

Moreover, switch graphs were shown to be useful to represent biological dynamics. In [13], they were applied to the study of biological regulatory networks. This field, which described the interaction of organic components (such as proteins and genes), makes use of models composed of a system with non-linear differential equations. The large size of these models often requires one to follow a simpler approach and consider graph-like discrete models. In [13] it was shown that switch graphs are a suitable class of discrete models to represent the dynamics of these systems and also that they preserve even more information about a system than regular graphs.

The study of switch graphs has been recently intensified. In [27], only one class of higher-level edges was considered and a fuzzy value representing the degree of existence was assigned to each higher-level edge. Also, a logic was introduced to reason within these structures. Later, in [6,7], two kinds of higher-level edges were considered (*activators* and *inhibitors*), both graded with fuzzy values. Some structural operations such as intersection and union were introduced.

The logics mentioned are classical in the sense that the evaluation of formulas takes only two values: either True or False. However, it may be the case that information that is found locally on each state is inconsistent or incomplete. That may occur due to several reasons: the use of different sources with opposite information, simple mistakes or loss of information. In order to be able to reason even in face of these situations, a paraconsistent and paracomplete approach is advised. In short, in the former the Principle of explosion is dropped, whereas in the latter it is the Principle of the excluded middle that is discarded. Therefore we can have inconsistencies without trivializing the whole system. In the 70s, Belnap introduced a 4-valued semantics [4]. The values are: (only) True (t), (only) False (f), Both True and False (b), and Neither True nor False (n). These four values may be arranged according to two partial orders; one deals with the quality of the information whereas the other deals with the quantity of information. The bilattice structure is represented in Fig. 1. Four-valued logics have been studied in many contexts, such as Computer Science and Artificial Intelligence and have been applied in areas such as symbolic model checking [8], semantics of logic programs [16] and inconsistency-tolerant systems.

There are some proposals of paraconsistent Modal and Hybrid logics (and extension of Modal logic with a new class of propositional variables whose behaviour allows them to name a specific state, plus an operator that allows jumps between named states when evaluating a formula), namely [25,26,17,12] for the modal ones, and [10,11,23] for the hybrid ones.

Following these works, we continue to enrich and analyze the family of reactive structures and take the first steps in the introduction of the paraconsistency paradigm in switch graph models.

This work introduces a modal-like logic where a frame is actually a switch graph and where propositional variables are four-valued. Our goal is to combine the formalism of switch graphs, as an excellent tool to describe reactive transition systems, with the imperfectness of information that surrounds us: sometimes with gluts, for example when different sources disagree on a topic, and sometimes with gaps, when no information at all is available. The result is a very versatile kind of model, whose potential will be explored further in this paper.

Outline. Section 2 is an overview of preliminary work on the topic of switch graphs. Section 3 introduces the syntax and semantics of a paraconsistent logic for switch graphs. For these paraconsistent models, we introduce, in Section 4, some inconsistency measures, with some accompanying examples. Finally, in Section 5 we conclude and present some lines to follow this research.

2. Preliminaries

In this section we introduce some preliminaries about switch graphs. We also briefly discuss the connection between switch graphs and the sabotage and swap operators in [1] and [30], respectively.

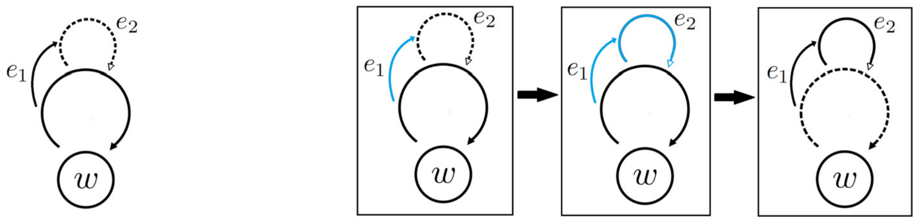

Fig. 2. Switch graph representing a 2-counter.

Definition 1. A *switch graph* is a triple $\mathcal{G} = (\mathrm{W}, \mathrm{S}, \mathrm{I})$ where W is a non-empty set of states/nodes, S is a set of (generalized) edges and $\mathrm{I} : \mathrm{S} \to \{0, 1\}$ is an (initial) instantiation such that:

- $\mathrm{S}_0 = \mathrm{W} \times \mathrm{W}$;
- $\mathrm{S}_n = \mathrm{S}_0 \times \mathrm{S}_{n-1} \times \{\circ, \bullet\}$;
- $\mathrm{S} = \bigcup_{i \geq 0} \mathrm{S}_i$.

Also, an edge v is called an n-level edge if $v \in \mathrm{S}_n$. Furthermore, it is considered a higher-level edge if $n > 0$, or a regular edge if $n = 0$. We use letters $s, t, \ldots$ to represent arbitrary-level edges, and $e, f, \ldots$ to represent regular edges.

Note that an $n+1$-level edge starts at a 0-level edge and points to an n-level edge ($n \geq 0$).

The difference between a switch graph and a regular graph is the presence of higher-level edges, which are used to describe the reactive dynamics of the model.

Higher-level edges cannot be traversed; their sole purpose is to indicate how the graph changes when a regular edge is traversed. There are two sorts of higher-level edges, according to the third element on each tuple, which can be either $\circ$ or $\bullet$. This element defines the role of the edge, which can act as either an “inhibitor” (for $\circ$) or as an “activator” (for $\bullet$). According to this, the edge “temporarily removes” or “restores” an edge, respectively.

The instantiation function is used to indicate if an edge is (temporarily) present or (temporarily) absent in a graph, by assigning each edge either 0 or 1, respectively. A regular edge can only be traversed if it is instantiated with 1. An absent edge can never be traversed (if it is a regular edge), and can never activate nor inhibit other edges (if it is a higher-level edge). The dynamics of the graph is described by the instantiation function, which is updated whenever an edge is crossed.

We formally define the update of a switch graph as follows:

Definition 2. Let $\mathcal{G} = (\mathrm{W}, \mathrm{S}, \mathrm{I})$ be a switch graph. After traversing a regular edge $e \in \mathrm{S}$ such that $\mathrm{I}(e) = 1$, the updated switch graph $\mathcal{G}^e$ is defined as the tuple $(\mathrm{W}, \mathrm{S}, \mathrm{I}^e)$, where for each $s \in \mathrm{S}$, I^e is defined as follows:

$$\mathrm{I}^e(s) = \begin{cases} 1, & \text{if } (e, s, \bullet) \in \mathrm{S} \text{ and } \mathrm{I}(e, s, \bullet) = 1; \\ 0, & \text{if } (e, s, \circ) \in \mathrm{S} \text{ and } \mathrm{I}(e, s, \circ) = 1; \\ \mathrm{I}(s), & \text{otherwise.} \end{cases}$$

The notation $\mathcal{G}^e$ is only used when the updated switch graph is defined, *i.e.*, when indeed e is a regular edge present in $\mathcal{G}$.

The following example illustrates a regular application of this class of structures, which is the possibility of defining counters. This kind of dynamics can be useful to explore systems with limited resources, for instance.

Example 1 *([13]).* Consider the following switch graph: $\mathcal{G} = (\mathrm{W}, \mathrm{S}, \mathrm{I})$, where $\mathrm{W} = \{w\}$, S is defined as described in Definition 1 and the instantiation function I is such that $\mathrm{I}(w, w) = 1$, $\mathrm{I}((w, w), ((w, w), (w, w), \circ), \bullet) = 1$, and $\mathrm{I}(s) = 0$ for every other edge $s \in \mathrm{S}$. This corresponds to a 2-counter, *i.e.*, a graph where the only 0-level edge can be traversed exactly twice before being removed forever.

The switch graph $\mathcal{G}$ is represented in Fig. 2. The set of three frames on the right represents the update of the switch graph for each time the edge (w, w) is traversed. Traversing the (regular) edge (w, w) for the first time makes the higher-level edge e_1 activate the higher-level edge e_2 (which updates the switch graph from the first to the second configuration). Traversing the same edge a second time makes the higher-level edge e_2 inhibit (w, w) (which updates the switch graph from the second to the third configuration). From this point on, there are no regular edges that can be traversed.

Due to its simple and compact representation, switch graphs are a user-friendly model to represent relation-changing models. Moreover, although simple, this class of relation-changing structures is very versatile and is able to represent, for example, the behaviour of the sabotage ([30]) and swap ([1]) operators. In Fig. 3, we represent two simple examples of

Fig. 3. Switch graphs incorporate the dynamics of sabotage (left) and swap (right) operators.

switch graphs whose dynamics mimic the sabotage and the swap of an edge, respectively on the left and on the right. The sabotage operator inhibits the edge after it is traversed. The swap operator inhibits the edge after it is traversed, and furthermore activates the edge in the opposite direction.

3. A 4-valued logic for switch graphs

Several logics have been proposed to describe reactive dynamics: Areces and Van Benthem proposed logics whose reactivity was specified within the language ([2]) and Marcelino and Gabbay specified logics for models which embed reactivity in their structure ([19,18]); a hybrid multimodal logic was also proposed in the same context in [14].

The goal in this section is the introduction of a modal-like logic where a frame is a switch graph and the valuation is 4-valued. The use of four values is very natural; for example in a discussive approach, *i.e.*, one in which our knowledge of which propositional variables are true or false results from the collection of information from various sources. It is therefore possible that no information is found, or contradictions arise. In order to consider four values, we resort to a commonly used technique, where the valuation of propositional variables is decoupled into a positive and a negative valuation. The positive valuation takes care of evaluating propositional variables, whereas the negative one evaluates the negation of propositional variables. At the moment, gluts and gaps are not allowed anywhere else other than the propositional variables.

In this work we follow the lines used in [15,27,6] in combination with [11]. In the former, switch graphs are enriched with weights on edges and special focus is given to the scenario where these weights represent fuzzy measures. For these models, a fuzzy logic was defined and several concepts, such as bisimulation, were studied. In the latter, paraconsistency is present both at the level of propositional variables and the accessibility relation in Hybrid logic. We restrict paraconsistency to propositional variables only, and to the modal fragment. Transitions between states remain classical in the setting we introduce.

The resulting logic is helpful to model and reason within various scenarios. For example, consider the case of an agent that must be submitted to a serology test in order to determine the presence or absence of a determined substance in her blood. There are limited resources available, so the agent can only be tested at most twice. We consider that possible results of the test are “positive”, “negative” and “inconclusive”. At first, the agent has no information whether the substance is present (p) or not ($\neg p$); the agent is in a paracomplete state. If the first test comes back positive, there is no need to perform a second one. In case the first test is inconclusive, the agent performs a second test. Likewise, if the first test is negative, the agent performs a second test in order to strengthen the result. In case the second test comes back positive, the agent finds herself in a paraconsistent state.

We start by defining our syntax, which is nothing more than what is standard in Modal logic:

Definition 3. Let Prop be a set of propositional variables. The set Form(Prop) of formulas over Prop, which for ease of notation we denote simply as Form, is obtained recursively as:

- $p \in \mathrm{Form}$, for $p \in \mathrm{Prop}$;
- $\bot \in \mathrm{Form}$;
- $\neg\varphi \in \mathrm{Form}$ whenever $\varphi \in \mathrm{Form}$;
- $\varphi \vee \psi \in \mathrm{Form}$ whenever $\varphi, \psi \in \mathrm{Form}$;
- $\varphi \to \psi \in \mathrm{Form}$ whenever $\varphi, \psi \in \mathrm{Form}$;
- $\Diamond\varphi \in \mathrm{Form}$ whenever $\varphi \in \mathrm{Form}$.

The operators $\wedge$ and $\Box$ are introduced as usual abbreviations.

In our models, we consider that the frame is a switch graph and that there are two valuations, as mentioned before. Formally:

Definition 4. A *paraconsistent switch graph model*, for short *model*, $\mathcal{M}$ is a tuple $(\mathrm{W}, \mathrm{S}, \mathrm{I}, \mathrm{V}^+, \mathrm{V}^-)$ where $(\mathrm{W}, \mathrm{S}, \mathrm{I})$ is a switch graph and $\mathrm{V}^+, \mathrm{V}^- : \mathrm{Prop} \to 2^{\mathrm{W}}$ are two valuations. Intuitively V^+ assigns to each propositional variable the set of states where they hold, whereas V^- assigns to each propositional variable the set of states where their negation holds.

Observe that the 4 values of a propositional variable are determined by its positive and negative valuations. Namely, for a propositional variable $p \in \mathrm{Prop}$:

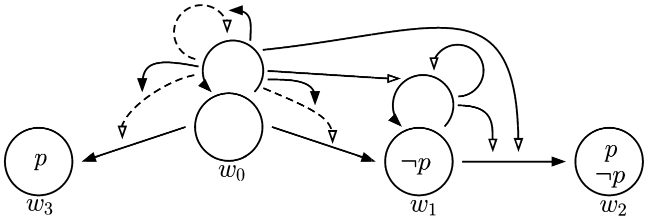

Fig. 4. A paraconsistent switch graph model.

(i)	$\mathcal{M}, w \vDash \varphi \Leftrightarrow w \in \mathrm{V}^+(\varphi)$, whenever $\varphi \in \mathrm{Prop}$;
(ii)	$\mathcal{M}, w \vDash \neg\varphi \Leftrightarrow w \in \mathrm{V}^-(\varphi)$, whenever $\varphi \in \mathrm{Prop}$;
(iii)	$\mathcal{M}, w \vDash \perp$ never;
(iv)	$\mathcal{M}, w \vDash \neg\perp$ always;
(v)	$\mathcal{M}, w \vDash \varphi \vee \psi \Leftrightarrow \mathcal{M}, w \vDash \varphi$ or $\mathcal{M}, w \vDash \psi$;
(vi)	$\mathcal{M}, w \vDash \neg(\varphi \vee \psi) \Leftrightarrow \mathcal{M}, w \vDash \neg\varphi$ and $\mathcal{M}, w \vDash \neg\psi$;
(vii)	$\mathcal{M}, w \vDash \varphi \to \psi \Leftrightarrow \mathcal{M}, w \vDash \varphi$ implies $\mathcal{M}, w \vDash \psi$;
(viii)	$\mathcal{M}, w \vDash \neg(\varphi \to \psi) \Leftrightarrow \mathcal{M}, w \nvDash \neg\varphi$ and $\mathcal{M}, w \vDash \neg\psi$;
(ix)	$\mathcal{M}, w \vDash \Diamond\varphi \Leftrightarrow \mathcal{M}^{(w,w')}, w' \vDash \varphi$, for some $w' \in \mathrm{W}$: $(w, w') \in \mathrm{S}$ and $\mathrm{I}(w, w') = 1$;
(x)	$\mathcal{M}, w \vDash \neg\Diamond\varphi \Leftrightarrow \mathcal{M}^{(w,w')}, w' \vDash \neg\varphi$, for all $w' \in \mathrm{W}$: $(w, w') \in \mathrm{S}$ and $\mathrm{I}(w, w') = 1$;
(xi)	$\mathcal{M}, w \vDash \neg\neg\varphi \Leftrightarrow \mathcal{M}, w \vDash \varphi$;

where $\mathcal{M}^e = (\mathrm{W}, \mathrm{S}, \mathrm{I}^e, \mathrm{V}^+, \mathrm{V}^-)$ is called the e-update of $\mathcal{M}$.

Fig. 5. Definition of the satisfaction relation $\mathcal{M}, w \vDash \varphi$.

- p is only true at state w iff $w \in \mathrm{V}^+(p)$ and $w \notin \mathrm{V}^-(p)$;
- p is only false at state w iff $w \notin \mathrm{V}^+(p)$ and $w \in \mathrm{V}^-(p)$;
- p is both true and false at state w iff $w \in \mathrm{V}^+(p)$ and $w \in \mathrm{V}^-(p)$;
- p is neither true nor false at state w iff $w \notin \mathrm{V}^+(p)$ and $w \notin \mathrm{V}^-(p)$.

The following example corresponds to the formalization of the motivating example in the prelude of the section:

Example 2. Consider the paraconsistent switch graph model represented in Fig. 4 intended to model a situation where an agent is submitted to a serology test in order to determine the presence or absence of a substance. The test can be performed at most twice. The result of the test may be inconclusive, positive or negative. If the first result is positive, then it is not necessary to perform a second test. Otherwise, a second test is executed.

State w_0 represents the initial state where the agent has no information whether p or $\neg p$ holds. Transitions between states represent a test. The model allows at most two transitions from w_0, which corresponds to the execution of at most two tests. Observe that if a first test is inconclusive, meaning that the transition from w_0 to w_0 is performed, then the underlying switch graph is updated so that the inhibitors whose starting point is the edge (w_0, w_0) and that were initially inactive, become active. Furthermore, the transitions from world w_1 are both removed, which means that if the result of the second test is negative, no other test will be performed, and the last state visited is w_1. If a second test is again inconclusive, then all edges from w_0 are removed and no transitions from w_0 are allowed anymore.

Note that the valuation function does not change with the different instantiations of the underlying switch graph.

Satisfaction of formulas in a given model and state is defined as follows:

Definition 5. A satisfaction relation $\vDash$ between a model $\mathcal{M}$, a state w and a formula φ is defined by structural induction on φ in Fig. 5.

We say that a formula φ is *globally satisfied* if $\mathcal{M}, w \vDash \varphi$ for all $w \in \mathrm{W}$; we denote it as $\mathcal{M} \vDash \varphi$. A formula is *valid*, denoted $\vDash \varphi$, if it is globally satisfied in all models.

The choice of semantics for $\to$ is distinct from those of weak and strong implications. The case (vii) is easily explained: $\varphi \to \psi$ is interpreted as regular implication. For the case with negation on the left, (viii), the reasoning is as follows: observe that $\mathcal{M}, w \vDash \varphi \to \psi \Leftrightarrow \mathcal{M}, w \nvDash \varphi$ or $\mathcal{M}, w \vDash \psi$. Rather than having that $\varphi \to \psi$ is equivalent to $\neg\varphi \vee \psi$, we actually have that the former is equivalent to $\sim\varphi \vee \psi$ where $\sim$ should be interpreted as a classical negation such that $\mathcal{M}, w \vDash \sim \varphi \Leftrightarrow \mathcal{M}, w \nvDash \varphi$. We then want to force $\neg(\varphi \to \psi)$ to be equivalent to $\neg(\sim\varphi \vee \psi)$. Note that $\neg\sim\varphi$ is not equivalent to φ, however, it is equivalent to $\sim\neg\varphi$. From this we obtain the interpretation used. (For further discussion about this topic, consult [9].)

The following example illustrates the satisfaction of some formulas:

Example 3. Consider the paraconsistent switch graph model represented in Fig. 4. The following hold:

$\mathcal{M}, w_0 \models \Diamond\Diamond p$: there are two witnesses for this statement (1.) a transition from w_0 to w_0 followed by a transition from w_0 to w_3, (2.) a transition from w_0 to w_1 followed by a transition from w_1 to w_2;

$\mathcal{M}, w_0 \not\models \Diamond\Diamond\Diamond\neg\bot$: it is not possible to perform three consecutive transitions from w_0;

$\mathcal{M}, w_0 \not\models q \vee \neg q$ is a byproduct of paracompleteness (at w_0);

$\mathcal{M}, w_1 \models \Diamond(p \wedge \neg p)$ is a byproduct of paraconsistency (at w_1).

Proposition 1. *The Necessitation rule ($\models \varphi$ implies $\models \Box\varphi$) and the K-axiom ($\models (\Box\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$) both hold.*

Proof. In order to prove that the Necessitation rule holds, note that $\Box\varphi \equiv \neg\Diamond\neg\varphi$.

Let φ be a valid formula, *i.e.* $\models \varphi$. Then, for any model $\mathcal{M} = (\mathrm{W}, \mathrm{S}, \mathrm{I}, \mathrm{V}^+, \mathrm{V}^-)$ and state $w \in \mathrm{W}$, $\mathcal{M}, w \models \Box\varphi \Leftrightarrow \mathcal{M}^{(w,w')}, w' \models \neg\neg\varphi$, for all $w' \in \mathrm{W}$ such that $\mathrm{I}(w, w') = 1 \Leftrightarrow \mathcal{M}^{(w,w')}, w' \models \varphi$, for all $w' \in \mathrm{W}$ such that $\mathrm{I}(w, w') = 1$. This holds because $\mathcal{M}^{(w,w')}$ is itself a model and φ is valid.

In order to prove that the K-axiom holds, note that $K \equiv \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \equiv \neg\Diamond\neg(\varphi \to \psi) \to (\neg\Diamond\neg\varphi \to \neg\Diamond\neg\psi)$.

In order to prove the validity of K, which has shape $A \to (B \to C)$, we take an arbitrary model and state such that A and B both hold, and conclude that C must hold in the same model and state.

Take a model $\mathcal{M} = (\mathrm{W}, \mathrm{S}, \mathrm{I}, \mathrm{V}^+, \mathrm{V}^-)$ and state $w \in \mathrm{W}$, such that $\mathcal{M}, w \models \neg\Diamond\neg(\varphi \to \psi) \Leftrightarrow \mathcal{M}^{(w,w')}, w' \models \varphi \to \psi$, for all $w' \in \mathrm{W}$ such that $\mathrm{I}(w, w') = 1$. (A)

Consider also that $\mathcal{M}, w \models \neg\Diamond\neg\varphi \Leftrightarrow \mathcal{M}^{(w,w')}, w' \models \varphi$, for all $w' \in \mathrm{W}$ such that $\mathrm{I}(w, w') = 1$. (B)

Because we assumed (A), (B) implies that $\mathcal{M}^{(w,w')}, w' \models \psi$, for all $w' \in \mathrm{W}$ such that $\mathrm{I}(w, w') = 1$, *i.e.*, $\mathcal{M}, w \models \neg\Diamond\neg\psi$.

Therefore, K is valid. $\Box$

For switch graph models, there is an adapted notion of bisimulation, in order to accommodate the changes in the accessibility relation. Note that the usual *Zig* and *Zag* conditions which consider a single step are insufficient to grasp the changes that occur after a transition is made. Because of this we introduce a notion of bisimulation based on walks.

Definition 6. A *walk* K in a model $\mathcal{M} = (\mathrm{W}, \mathrm{S}, \mathrm{I}, \mathrm{V}^+, \mathrm{V}^-)$ is a sequence of states $[w_0, \ldots, w_n]$ (with possible repetitions), for $n \in \mathbb{N}_0$ such that the models $\mathcal{M}_0 = \mathcal{M}$, $\mathcal{M}_{i+1} = \mathcal{M}_i^{(w_i, w_{i+1})}$, for $i \in \{0, \ldots, n-1\}$ are defined, *i.e.*, for $\mathrm{I}_0 = \mathrm{I}$, then $\mathrm{I}_0(w_0, w_1) = 1$ and for $\mathrm{I}_{i+1} = \mathrm{I}_i^{(w_i, w_{i+1})}$ then $\mathrm{I}_{i+1}(w_i, w_{i+1}) = 1$, for $i \in \{0, \ldots, n-1\}$.

Thus a walk is a sequence of states such that, after traversing the first edge (*i.e.*, the edge constituted by the first two states in the sequence) in the walk, the sequence of states that results from removing the first one is also a walk in the updated model.

Given a walk $\mathrm{K} = [w_0, \ldots, w_n]$, any walk $\mathrm{K}' = [w_0, \ldots, w_i]$, $0 \le i \le n$ is called a sub-walk of K, and is denoted by $\mathrm{K}' \subseteq \mathrm{K}$. The element w_0 is called the initial state, and w_n is the last state visited in the walk K. The length of a walk is the number of states in the walk.

It is also useful to formally define terminal walks:

Definition 7. Given a model $\mathcal{M} = (\mathrm{W}, \mathrm{S}, \mathrm{I}, \mathrm{V}^+, \mathrm{V}^-)$, we say that $\mathrm{K} = [w_0, \ldots, w_n]$ is a *terminal walk* if there is no $z \in \mathrm{W}$ such that $[w_0, \ldots, w_n, z]$ is a walk in $\mathcal{M}$.

It is helpful to use the following terminology and notation:

Definition 8. We say that a pointed model $(\mathcal{M}', w')$ is an *update* of $(\mathcal{M}, w)$ if $\mathrm{I}(w, w') = 1$ and $\mathcal{M}' = \mathcal{M}^{(w,w')}$; we denote it as $(\mathcal{M}, w) \xrightarrow{\mathrm{upd}} (\mathcal{M}', w')$.

Let $\mathbf{r}(\mathcal{M}, w)$ denote the set of all updates of $(\mathcal{M}, w)$; formally: $\mathbf{r}(\mathcal{M}, w) = \{(\mathcal{M}', w') \mid (\mathcal{M}, w) \xrightarrow{\mathrm{upd}} (\mathcal{M}', w')\}$. Furthermore, let $\mathbf{r}^*(\mathcal{M}, w)$ denote the set of all pointed models which are reachable from $(\mathcal{M}, w)$ by the reflexive and transitive closure of $\xrightarrow{\mathrm{upd}}$.

Then the definition of bisimulation, which is based in the definition of bisimulation found in [3], comes as follows:

Definition 9. Let $\mathcal{M}_1 = (\mathrm{W}_1, \mathrm{S}_1, \mathrm{I}_1, \mathrm{V}_1^+, \mathrm{V}_1^-)$ and $\mathcal{M}_2 = (\mathrm{W}_2, \mathrm{S}_2, \mathrm{I}_2, \mathrm{V}_2^+, \mathrm{V}_2^-)$ be two paraconsistent switch graph models.

A *(walk) bisimulation* between pointed models $(\mathcal{M}_1, w_1)$ and $(\mathcal{M}_2, w_2)$ is a non-empty binary relation $\mathrm{B} \subseteq \mathbf{r}^*(\mathcal{M}_1, w_1) \times \mathbf{r}^*(\mathcal{M}_2, w_2)$ such that the following conditions hold:

(Prop) if $((\mathcal{M}_1, w_1), (\mathcal{M}_2, w_2)) \in \mathrm{B}$, then $w_1 \in \mathrm{V}_1^+(p)$ iff $w_2 \in \mathrm{V}_2^+(p)$, for all $p \in \mathrm{Prop}$ and analogously with V_i^-;

(w-Zig) if $((\mathcal{M}_1, w_1), (\mathcal{M}_2, w_2)) \in \mathrm{B}$, then for every (1-step) walk in $\mathcal{M}_1$, $[w_1, w_1']$, there exists a (1-step) walk in $\mathcal{M}_2$, $[w_2, w_2']$ such that $((\mathcal{M}_1^{(w_1,w_1')}, w_1'), (\mathcal{M}_2^{(w_2,w_2')}, w_2')) \in \mathrm{B}$;

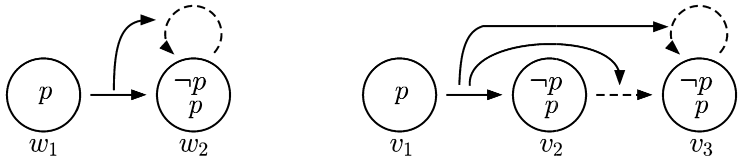

Fig. 6. Two bisimilar paraconsistent switch graph models.

(w-Zag) if $((\mathcal{M}_1, w_1), (\mathcal{M}_2, w_2)) \in \mathrm{B}$, then for every (1-step) walk in $\mathcal{M}_2$, $[w_2, w_2']$, there exists a (1-step) walk in $\mathcal{M}_1$, $[w_1, w_1']$ such that $((\mathcal{M}_1^{(w_1,w_1')}, w_1'), (\mathcal{M}_2^{(w_2,w_2')}, w_2')) \in \mathrm{B}$.

Two pointed models $(\mathcal{M}_1, w_1)$ and $(\mathcal{M}_2, w_2)$ are *bisimilar* if there is a bisimulation B such that $((\mathcal{M}_1, w_1), (\mathcal{M}_2, w_2)) \in \mathrm{B}$.

The following example illustrates two bisimilar models:

Example 4. Consider the two paraconsistent switch graph models represented in Fig. 6; call them $\mathcal{M}$ and $\mathcal{M}'$ respectively. The relation $\mathrm{B} = \{((\mathcal{M}, w_1), (\mathcal{M}', v_1)), ((\mathcal{M}^{(w_1,w_2)}, w_2), (\mathcal{M}'^{(v_1,v_2)}, v_2)), ((\mathcal{M}^{(w_1,w_2)}, w_2), (\mathcal{M}'^{(v_1,v_2)}, v_3))\}$ is a bisimulation.

Note that traversing (w_1, w_2) and (v_1, v_2) updates the respective underlying models. However, traversing (w_2, w_2), (v_2, v_3) or (v_3, v_3) in the (respective, and already) updated models does not make any changes to them.

We now conclude this section by proving that bisimulation preserves satisfaction of formulas.

Theorem 1. *Let* $\mathcal{M}_1 = (\mathrm{W}_1, \mathrm{S}_1, \mathrm{I}_1, \mathrm{V}_1^+, \mathrm{V}_1^-)$ *and* $\mathcal{M}_2 = (\mathrm{W}_2, \mathrm{S}_2, \mathrm{I}_2, \mathrm{V}_2^+, \mathrm{V}_2^-)$ *be two paraconsistent switch graph models. If* $(\mathcal{M}_1, w_1)$ *and* $\mathcal{M}_2, w_2))$ *are bisimilar, then* $\mathcal{M}_1, w_1 \models \varphi \Leftrightarrow \mathcal{M}_2, w_2 \models \varphi$, *for any* $\varphi \in \mathrm{Form}(\mathrm{Prop})$.

Proof. We prove the equivalence by induction over the structure of formulas.

- Cases $\varphi = \bot, \neg\bot, p, \neg p$, with $p \in \mathrm{Prop}$: follows directly from the definition.

(I.H.) Consider that the result holds for every proper subformula of φ and their negations.

- Case $\varphi = \phi \vee \psi$: let $\mathcal{M}_1, w_1 \models \phi \vee \psi$. By definition, $\mathcal{M}_1, w_1 \models \phi$ or $\mathcal{M}_1, w_1 \models \psi$. By Induction Hypothesis, $\mathcal{M}_2, w_2 \models \phi$ or $\mathcal{M}_2, w_2 \models \psi$ and equivalently, $\mathcal{M}_2, w_2 \models \phi \vee \psi$. The reciprocal is proved analogously.
- Cases $\varphi = \neg(\phi \vee \psi), \phi \to \psi, \neg(\phi \to \psi), \neg\neg\phi$: the proof is analogous to the previous case.
- Case $\varphi = \Diamond\phi$: let $\mathcal{M}_1, w_1 \models \Diamond\phi$. By definition, $\mathcal{M}_1^{(w_1,w_1')}, w_1' \models \phi$, for some $w_1' \in \mathrm{W}_1$ such that $(w_1, w_1') \in \mathrm{S}_1$ and $\mathrm{I}_1(w_1, w_1') = 1$. Thus $[w_1, w_1']$ is a (1-step) walk in $\mathcal{M}_1$. By (w-Zig) in the definition of bisimulation, $\exists w_2' \in \mathrm{W}_2$ such that $[w_2, w_2']$ is a (1-step) walk in $\mathcal{M}_2$, (*i.e.*, such that $(w_2, w_2') \in \mathrm{S}_2$, $\mathrm{I}_2(w_2, w_2') = 1$) and $((\mathcal{M}_1^{(w_1,w_1')}, w_1'), (\mathcal{M}_2^{(w_2,w_2')}, w_2')) \in \mathrm{B}$. Thus, by the Induction Hypothesis, we get $\mathcal{M}_2^{(w_2,w_2')}, w_2' \models \phi$, for some $w_2' \in \mathrm{W}_2$ such that $(w_2, w_2') \in \mathrm{S}_2$ and $\mathrm{I}_2(w_2, w_2') = 1$. Equivalently, $\mathcal{M}_2, w_2 \models \Diamond\phi$. The reciprocal is proved analogously resorting to (w-Zag).
- Case $\varphi = \neg\Diamond\phi$: let $\mathcal{M}_1, w_1 \models \neg\Diamond\phi$. By definition, $\mathcal{M}_1^{(w_1,w_1')}, w_1' \models \neg\phi$, for all $w_1' \in \mathrm{W}_1$ such that $(w_1, w_1') \in \mathrm{S}_1$ and $\mathrm{I}_1(w_1, w_1') = 1$. Then, for each (1-step) walk in $\mathcal{M}_1$, $[w_1, w_1']$, by (w-Zig) in the definition of bisimulation, $\exists w_2' \in \mathrm{W}_2$ such that $[w_2, w_2']$ is a (1-step) walk in $\mathcal{M}_2$ (*i.e.*, such that $(w_2, w_2') \in \mathrm{S}_2$, $\mathrm{I}_2(w_2, w_2') = 1$) and for every (1-step) walk from w_2 in $\mathcal{M}_2$ as such, $((\mathcal{M}_1^{(w_1,w_1')}, w_1'), (\mathcal{M}_2^{(w_2,w_2')}, w_2')) \in \mathrm{B}$. Thus, by the Induction Hypothesis, we get $\mathcal{M}_2^{(w_2,w_2')}, w_2' \models \neg\phi$, for all $w_2' \in \mathrm{W}_2$ such that $(w_2, w_2') \in \mathrm{S}_2$ and $\mathrm{I}_2(w_2, w_2') = 1$. Equivalently, $\mathcal{M}_2, w_2 \models \neg\Diamond\phi$. The reciprocal is proved analogously resorting to (w-Zag). $\square$

4. Inconsistency measures

The idea of measuring the amount of inconsistent information in paraconsistent structures has been addressed for example in [20–22], where different measures have been proposed by a range of different authors.

An inconsistency measure [10] is a function that assigns a non-negative real value to a model where contradictory information can be found. Each inconsistency measure is a strategy for analysing inconsistent information by showing how many conflicts each model contains. Some measures are more fine-grained than others, but in general what they do is to allow us to compare between sets of information.

The measure of inconsistency of a model is crucial in a diverse range of applications in artificial intelligence to compare knowledge bases. As supported in [24], it may be a useful tool in analysing various information types, such as news reports, software specifications, integrity constraints and e-commerce protocols. Furthermore, this topic has also been explored in terms of complexity and expressivity in [28,29].

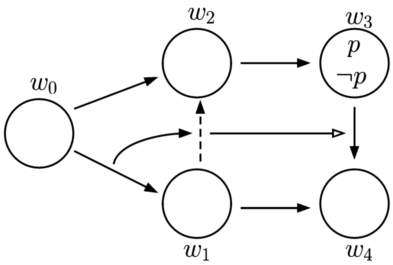

Fig. 7. A paraconsistent switch graph model.

Inconsistency measures can be classified in various ways and may satisfy certain properties. One distinction is between absolute measures, that measure the total amount of contradictions, and relative measures, that use a ratio to determine how much of a structure is inconsistent.

In the particular case of switch graph models, the reactivity that is adjacent to modal formulas makes it natural to think of measures of inconsistency not only in a static approach, where we would check the amount of inconsistencies in each individual state and combine them after investigating all states, but it is rather helpful to define measures of inconsistency in paths that can be traversed in those models. There are cases when traversing an edge makes it so that other edges that would lead to local inconsistencies are inhibited, whereas in other cases, traversing an edge activates an edge connected to a state where inconsistent information is present.

Before going any further, we must formally define what is a path in a model:

Definition 10. An (admissible) *path* P in a model $\mathcal{M} = (\mathrm{W}, \mathrm{S}, \mathrm{I}, \mathrm{V}^+, \mathrm{V}^-)$ is a walk without repetitions.

A simple consequence of the definition of admissible path is the following:

Proposition 2. *Given a model $\mathcal{M} = (\mathrm{W}, \mathrm{S}, \mathrm{I}, \mathrm{V}^+, \mathrm{V}^-)$ and a path $\mathrm{P} = [w_0, \dots, w_n]$, the sequences $[w_0, \dots, w_i]$ are also admissible paths, for $0 \leq i \leq n$.*

The notions of initial state and last state visited in a path, sub-path, and terminal path are analogous to the definitions for walks.

The following example will accompany us throughout this section:

Example 5. Consider the switch graph represented in Fig. 7. Admissible paths include: $\mathrm{P}_1 = [w_0, w_2, w_3, w_4]$, $\mathrm{P}_2 = [w_0, w_1, w_2, w_3]$, $\mathrm{P}_3 = [w_0, w_2, w_3]$ and $\mathrm{P}_4 = [w_0, w_1, w_4]$, among others. $\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_4$ are all terminal paths; P_3 is not terminal.

In order to simplify notation later, we define a local inconsistency measure for counting the amount of propositional variables p such that both p and $\neg p$ simultaneously hold at a certain state.

Definition 11 *(Local inconsistency measure).* For a model $\mathcal{M} = (\mathrm{W}, \mathrm{S}, \mathrm{I}, \mathrm{V}^+, \mathrm{V}^-)$ and a state $w \in W$, a local inconsistency measure $\mathrm{MInc}(\mathcal{M}, w)$ is defined as follows:

$$\mathrm{MInc}(\mathcal{M}, w) = |\{p : w \in \mathrm{V}^+(p) \cap \mathrm{V}^-(p)\}|.$$

We can now define our first inconsistency measure for a path in a model.

Definition 12 *(Path inconsistency measure).* For a model $\mathcal{M} = (\mathrm{W}, \mathrm{S}, \mathrm{I}, \mathrm{V}^+, \mathrm{V}^-)$ and an admissible path $\mathrm{P} = [w_0, \dots, w_n]$ in $\mathcal{M}$, an inconsistency measure $\mathrm{MInc}(\mathcal{M}, \mathrm{P})$ is defined as follows:

$$\mathrm{MInc}(\mathcal{M}, \mathrm{P}) = \sum_{i=0}^{n} \mathrm{MInc}(\mathcal{M}, w_i).$$

Lemma 1 *(Monotonicity). If P is a path in a model $\mathcal{M}$ and $\mathrm{P}' \subseteq \mathrm{P}$, then* $\mathrm{MInc}(\mathcal{M}, \mathrm{P}') \leq \mathrm{MInc}(\mathcal{M}, \mathrm{P})$.

Example 6. Consider the switch graph and paths in Example 5. We can calculate the inconsistency measure $\mathrm{MInc}(\mathcal{M}, \mathrm{P}_i)$ for each $i \in \{1, 2, 3, 4\}$, which gives us:

$$\mathrm{MInc}(\mathcal{M}, \mathrm{P}_j) = 1, \text{ for } j \in \{1, 2, 3\} \qquad \mathrm{MInc}(\mathcal{M}, \mathrm{P}_4) = 0.$$

However, there is no relation between the number of elements in a path and the inconsistency measure. Sometimes it may be the case that longer paths are less inconsistent than shorter ones.

Example 7. Consider once again the switch graph represented in Fig. 7. The path $\mathrm{P}_5 = [w_2, w_3]$ has length 2 and $\mathrm{MInc}(\mathcal{M}, \mathrm{P}_5) = 1$, whereas $\mathrm{P}_4 = [w_0, w_1, w_4]$ has length 3 and $\mathrm{MInc}(\mathcal{M}, \mathrm{P}_4) = 0$.

Definition 13. The best (absolute) path between two states w, w' in a model $\mathcal{M}$ is a path that encounters the least number of inconsistencies, *i.e.*, is a path $\mathrm{P} = [w, \dots, w']$ such that $\mathrm{MInc}(\mathcal{M}, \mathrm{P}) \leq \mathrm{MInc}(\mathcal{M}, \mathrm{P}')$ for any path P' between w and w'.

Example 8. For the switch graph represented in Fig. 7, the best absolute path between w_0 and w_4 is clearly $\mathrm{P}_4 = [w_0, w_1, w_4]$.

Note that the best absolute path may not be unique.

In order to distinguish between paths with the same (global) inconsistency measure, we may be interested in considering the amount of states visited in each path. For that, we introduce a relative inconsistency measure as follows:

Definition 14 *(Path relative inconsistency measure).* For a model $\mathcal{M} = (\mathrm{W}, \mathrm{S}, \mathrm{I}, \mathrm{V}^+, \mathrm{V}^-)$ and an admissible path $\mathrm{P} = [w_0, \dots, w_n]$ in $\mathcal{M}$, a relative inconsistency measure $\mathrm{RelInc}(\mathcal{M}, \mathrm{P})$ is defined as follows:

$$\mathrm{RelInc}(\mathcal{M}, \mathrm{P}) = \frac{\mathrm{MInc}(\mathcal{M}, \mathrm{P})}{(n+1)}.$$

This measure is neither monotonic nor anti-monotonic. Consider the following example:

Example 9. Consider once again the switch graph represented in Fig. 7 and the paths introduced in Example 5. Take $\mathrm{P}_6 = [w_0, w_2]$. Then, $\mathrm{P}_6 \subseteq \mathrm{P}_3 \subseteq \mathrm{P}_1$, and $\mathrm{RelInc}(\mathcal{M}, \mathrm{P}_6) = 0 \leq \mathrm{RelInc}(\mathcal{M}, \mathrm{P}_3) = \frac{1}{3}$. But on the other hand, $\mathrm{RelInc}(\mathcal{M}, \mathrm{P}_3) = \frac{1}{3} \geq \mathrm{RelInc}(\mathcal{M}, \mathrm{P}_1) = \frac{1}{4}$.

Note that for a relative inconsistency measure to be greater than 1, it must be the case that on average each state contains more than one inconsistency.

Definition 15. The best (relative) path between two states w, w' in a model $\mathcal{M}$ is a path with the lowest ratio of inconsistencies per state, *i.e.*, is a path $\mathrm{P} = [w, \dots, w']$ such that $\mathrm{RelInc}(\mathcal{M}, \mathrm{P}) \leq \mathrm{RelInc}(\mathcal{M}, \mathrm{P}')$ for any path P' between w and w'.

Example 10. For the switch graph represented in Fig. 7, the best relative path between w_0 and w_3 is $\mathrm{P}_2 = [w_0, w_1, w_2, w_3]$, which is such that $\mathrm{RelInc}(\mathcal{M}, \mathrm{P}_2) = \frac{1}{4}$.

We may also be interested in checking, for a certain state, the amount of inconsistent information that is possible to reach from it. The idea is to explore all the states that can be reached from an initial state. We formally introduce the notion of reachable set as follows:

Definition 16. Given a model $\mathcal{M} = (\mathrm{W}, \mathrm{S}, \mathrm{I}, \mathrm{V}^+, \mathrm{V}^-)$, and a state w, the reachable set $\mathrm{Reach}(w)$ is the set of all w' such that $w' \in \mathrm{P}$ for some admissible path P that has initial state w.

Note the following: in a reactive model, if there is an edge from w_1 to w_2, it is not the case that the reachable states from w_1 include the reachable states from w_2. The moment the first traverse, from w_1 to w_2, is made, it changes the underlying relation in the frame. So it may be the case that in the initial configuration w_3 was reachable from w_2 but the traverse from w_1 to w_2 inhibits it, making w_3 unreachable from w_2 in the second configuration. Thus, even though w_3 was part of the set of states reachable from w_2, it is not part of the set of states reachable from w_1. This is represented in the switch graph on the left in Fig. 6.

We can measure the inconsistencies in a reachable set in a standard way: given a model $\mathcal{M} = (\mathrm{W}, \mathrm{S}, \mathrm{I}, \mathrm{V}^+, \mathrm{V}^-)$ and a set of states $\mathrm{A} \subseteq \mathrm{W}$, $\mathrm{MInc}(\mathcal{M}, \mathrm{A}) = \sum_{w \in \mathrm{A}} \mathrm{MInc}(\mathcal{M}, w)$. Analogously, we can have a relative measure, $\mathrm{RelInc}(\mathcal{M}, \mathrm{A}) = \frac{\mathrm{MInc}(\mathcal{M}, \mathrm{A})}{|A|}$. The behaviour of these measures in what concerns monotonicity or anti-monotonicity is exactly the same as for the case with paths.

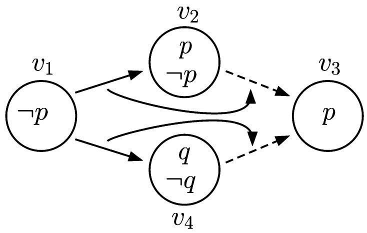

Fig. 8. A paraconsistent switch graph model.

Recall that the paraconsistent switch graph model is a way to represent a machine with varying configurations depending on the traverses that are made, and that each propositional variable represents a property verified at some states in the machine.

If we had to choose an initial state so that we would be able to reach the least number of inconsistencies, without knowing about the model, we would intuitively choose the state w for which $\mathrm{RelInc}(\mathcal{M}, \mathrm{Reach}(w))$ is lowest. Of course, for any inconsistency measure other than 0, that would not be a guarantee that we would start at the best possible point. But obviously, a state for which $\mathrm{RelInc}(\mathcal{M}, \mathrm{Reach}(w)) = 0$ is the most desirable.

Before concluding this section, observe that not all properties have the same impact on the working of a machine. There may be some inconsistencies that really do not affect much the work of a machine, while others do. In order to account for these scenarios, we can introduce a weighted inconsistency measure, which is such that for each propositional variable there is a value that represents its importance. A propositional variable with higher importance has a larger value, and contradictory information about it has more influence in the calculation of the measure of inconsistency.

First, consider a function that for each propositional variable assigns a positive real value. Take $\mathrm{wgt} : \mathrm{Prop} \to \mathbb{R}^+$. A weighted local inconsistency measure is given as follows:

Definition 17. For a model $\mathcal{M} = (\mathrm{W}, \mathrm{S}, \mathrm{I}, \mathrm{V}^+, \mathrm{V}^-)$, a state $w \in W$, and a weighting function $\mathrm{wgt} : \mathrm{Prop} \to \mathbb{R}^+$, a weighted local inconsistency measure $\mathrm{MInc}_{\mathrm{wgt}}(\mathcal{M}, w)$ is defined as follows:

$$\mathrm{MInc}_{\mathrm{wgt}}(\mathcal{M}, w) = \sum_{p : w \in \mathrm{V}^+(p) \cap \mathrm{V}^-(p)} \mathrm{wgt}(p).$$

For a global weighted measure in a path, the definition is straightforward: it is the sum of the weighted inconsistency measures in each state of the path. The standard definition applies for a relative version, it is the global amount divided by the number of states visited.

Example 11. Consider the paraconsistent switch graph model represented in Fig. 8 and the two only paths between v_1 and v_3: $\mathrm{P}_1 = [v_1, v_2, v_3]$ and $\mathrm{P}_2 = [v_1, v_4, v_3]$. Consider also $\mathrm{wgt}(p) = 0.2$ and $\mathrm{wgt}(q) = 0.8$. Then,

$$\mathrm{MInc}_{\mathrm{wgt}}(\mathcal{M}, \mathrm{P}_1) = 0.2 \qquad \mathrm{MInc}_{\mathrm{wgt}}(\mathcal{M}, \mathrm{P}_2) = 0.8.$$
$$\mathrm{RelInc}_{\mathrm{wgt}}(\mathcal{M}, \mathrm{P}_1) = \frac{0.2}{3} \qquad \mathrm{RelInc}_{\mathrm{wgt}}(\mathcal{M}, \mathrm{P}_2) = \frac{0.8}{3}.$$

Therefore, we can say that the path P_1 is *better*, as it is less inconsistent.

5. Conclusion and future work

In this paper we build on recent work on logics for switch graphs. We extend the usual Modal logic so that a frame is not a graph structure but rather a switch graph, and our models admit four-valued propositional variables which are, as usual, evaluated locally. In the proposed modal-like logic, the Necessity rule and the K-axiom still hold. A notion of walk bisimulation was introduced and it was proven that bisimilar states satisfy the same formulas. We would like to point out that this definition of bisimulation does not follow the line of the one introduced in [7,27]. We expect to explore the connection between these two definitions in future. Afterwards, we proposed several inconsistency measures, as a tool to assess the quality of a model. These measures evaluated paths and sets of reachable states, and some allowed weights for each propositional variable. Given the nature of switch graphs, with their relation-changing accessibility quirks, traversing some edges could lead to the activation of edges that would lead to inconsistent states, or, on the other hand, prevent us from reaching inconsistent states. We aimed at providing several examples throughout the paper to illustrate the concepts introduced.

In the future we intend to incorporate paraconsistency and paracompleteness on edges. This has been done previously for Modal and Hybrid logics (in [9,12], among others), but never when frames are switch graphs. The interconnection between 4-valued edges and higher-level edges shall constitute an interesting challenge.

Another topic on our to-do list is the addition of machinery from Hybrid logic, such as nominals and the satisfaction operator to modal-like logics where frames are switch graphs, thus making our language more expressive. This extension will inherit all properties of the base system. However, the classical notion of Robinson's diagram will be insufficient to capture the dynamic behaviour of a switch graph; indeed one such set of atoms will only be able to describe particular instances of the switch graph. In the future we aim at adapting Robinson's diagrams in a way that the resulting notion will be apt to fully capture the dynamic aspect of a given switch graph.

Last on the list is the study of inconsistency measures in bisimilar models. Bisimilar models contain walks of the same length such that each pair of elements in one walk and the other are in the bisimulation. However, we have been considering measures for paths, in order to avoid duplicating already known inconsistencies. Two models having walks of the same length does not imply them having paths of the same length. Thus some new measures shall be explored to take this into account.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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