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A NOTE ON MUCKENHOUPT WEIGHTS WITH NONSTANDARD GROWTH

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Abstract. We provide quantitative results on the inclusion of the family of variable power weights $x \mapsto |x-\xi|^{-\gamma(x)}$ in the Muckenhoupt A_1 class, under appropriate conditions on γ. We also present an application related to variable exponent Muckenhoupt classes $\mathscr{A}_{p(\cdot)}$.

1. Introduction

Qualitative results regarding the belongingness of the weight $x \mapsto |x|^{-\gamma(x)}$ to the Muckenhoupt A_1 class were obtained in [1], whenever γ satisfies (2.1) and (2.2) below. In this small note, we give quantitative results on the more general function $w_\xi^\gamma(x) := |x-\xi|^{-\gamma(x)}$, for fixed $\xi \in \mathbb{R}^n$, by making explicit the dependency of the constants on ξ and the exponent function $\gamma(x)$. Apart some few known cases, e.g. $|x|^{-\gamma}$ with constant $\gamma \in [0,n)$, it is hard to estimate the A_1 constant of a given weight due to the difficulty of computing the maximal function in general. So we find of special interest the study of Muckenhoupt weights of nonstandard growth. Additionally, we show that $w_\xi^\gamma \in \mathscr{A}_{p(\cdot)}$ for appropriate conditions on $\gamma(x)p(x)$.

2. Muckenhoupt Weights with Nonstandard Growth

Recall that a weight w is said to belong to the Muckenhoupt class A_1 if there exists a constant $C>0$ such that $Mw(x) \le C\, w(x)$ a.e., where M denotes the well-known maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{r^n} \int\limits_{B(x,r)} |f(y)|\, dy.$$

The A_1 constant of w, denoted by $[w]_{A_1}$, is the smallest value of C for which the inequality above holds. We refer to [7] for more details on classical Muckenhoupt weights.

We are interested in the family of weights

$$w_\xi^\gamma(x) := |\xi - x|^{-\gamma(x)},$$

indexed by $\xi \in \mathbb{R}^n$, where $\gamma : \mathbb{R}^n \to [0,\infty)$ satisfies regularity conditions of log-Hölder type.

Recall that a function $g : \mathbb{R}^n \to \mathbb{R}$ is said to be *locally* log-*Hölder continuous* if there exists $c_{\log}(g) > 0$ such that

$$|g(x)-g(y)| \le \frac{c_{\log}(g)}{\log(e+1/|x-y|)}, \quad \text{for all } x,y \in \mathbb{R}^n, \tag{2.1}$$

and satisfy the log-*Hölder continuity condition at infinity*, also known as the *decay condition*, if there exist $g_\infty \in [1,\infty)$ and $c^*_{\log}(g) > 0$ such that

$$|g(x)-g_\infty| \le \frac{c^*_{\log}(g)}{\log(e+|x|)}, \quad \text{for all } x \in \mathbb{R}^n. \tag{2.2}$$

Below we use the notation $\gamma_E^+ := \operatorname{ess\,sup}_{x\in E} \gamma(x)$ and $\gamma_E^- := \operatorname{ess\,inf}_{x\in E} \gamma(x)$. When $E = \mathbb{R}^n$, we simply write γ^+ and γ^-, respectively.

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Proposition 1. *Let γ be a function satisfying* (2.1) *and* (2.2), *and $0 \leq \gamma^- \leq \gamma^+ < n$. Then $w_\xi^\gamma \in A_1$ with*

$$[w_\xi^\gamma]_{A_1} \leq \frac{c(n)}{n-\gamma^+} \max\left\{e^{c_{\log}(\gamma)}, e^{c^*_{\log}(\gamma)}(e+|\xi|)^{2c^*_{\log}(\gamma)}\right\}, \tag{2.3}$$

where $c(n) > 0$ depends only on the dimension $n \in \mathbb{N}$.

Proof. Let $y \neq \xi \in \mathbb{R}^n$ and $r > 0$ be fixed. We split the proof in three cases.

Case 1. $r < |\xi - y|/2$. We have $\frac{|\xi-y|}{2} \leq |\xi - z| \leq \frac{3|\xi-y|}{2}$, when $z \in B(y,r)$. This yields the inequality

$$|z-\xi|^{-\gamma(z)} \leq 2^n |y-\xi|^{-\gamma(y)} |y-\xi|^{\gamma(y)-\gamma(z)}. \tag{2.4}$$

We now prove the estimate $|y-\xi|^{\gamma(x)-\gamma(z)} \leq c(n)\max\{e^{c_{\log}(\gamma)}, e^{c^*_{\log}(\gamma)}(e+|\xi|)^{2c^*_{\log}(\gamma)}\}$, splitting it in three cases:

Case 1a. $|y-\xi| \geq 2e$. It suffices to estimate the case $\gamma(y)-\gamma(z) > 0$. We have

$$\begin{aligned}
&|y-\xi|^{|\gamma(y)-\gamma(z)|} \\
&\quad\leq \exp\left(|\gamma(y)-\gamma_\infty|\log(e+|y|)\frac{\log|y-\xi|}{\log(e+|y|)} + |\gamma(z)-\gamma_\infty|\log(e+|z|)\frac{\log|y-\xi|}{\log(e+|z|)}\right) \\
&\quad\leq \exp\left\{c^*_{\log}(\gamma)\log(e+|\xi|) + c^*_{\log}(\gamma)[1+\log(e+|\xi|)]\right\} \\
&\quad= e^{c^*_{\log}(\gamma)}(e+|\xi|)^{2c^*_{\log}(\gamma)},
\end{aligned} \tag{2.5}$$

where the second inequality follows from the fact that, for $\eta > 0$, $\mathbb{R}_+ \ni t \mapsto \log(t+\eta)/\log(t)$ is decreasing and the inequality $\log|y-\xi| \leq \log(2(|z|+|\xi|))$.

Case 1b. $1 \leq |y-\xi| \leq 2e$. It is immediate that $|y-\xi|^{\gamma(y)-\gamma(z)} \leq (2e)^n$.

Case 1c. $|y-\xi| \leq 1$. It is enough to study the case $\gamma(y)-\gamma(z) < 0$. Since $|y-\xi| > 2|y-z|$ we get

$$|y-\xi|^{-|\gamma(y)-\gamma(z)|} \leq \exp\left(\log(2|z-y|)^{-|\gamma(y)-\gamma(z)|}\right) \leq e^{c_{\log}(\gamma)}. \tag{2.6}$$

Taking into account all the previous estimates, we obtain

$$\int_{B(y,r)} |z-\xi|^{-\gamma(z)}dz \leq c(n)\, r^n\, |y-\xi|^{-\gamma(y)} \max\left\{e^{c_{\log}(\gamma)}, e^{c^*_{\log}(\gamma)}(e+|\xi|)^{2c^*_{\log}(\gamma)}\right\}.$$

Case 2. $|y-\xi|/2 \leq r < 2|y-\xi|$. We decompose the integral $\int_{B(y,r)} |z-\xi|^{-\gamma(z)}dz$ as

$$\int_{\substack{B(y,r)\\ |y-\xi|<|z-\xi|<3|y-\xi|}} |z-\xi|^{-\gamma(z)}dz + \int_{\substack{B(y,r)\\ 1<|z-\xi|<|y-\xi|}} |z-\xi|^{-\gamma(z)}dz + \int_{\substack{B(y,r)\\ |z-\xi|<|y-\xi|\wedge|z-\xi|<1}} |z-\xi|^{-\gamma(z)}dz =: I_1+I_2+I_3.$$

Estimation of I_1. The proof of

$$I_1 \leq c(n)\, r^n |y-\xi|^{-\gamma(y)} \max\{e^{c_{\log}(\gamma)}, e^{c^*_{\log}(\gamma)}(e+|\xi|)^{2c^*_{\log}(\gamma)}\}$$

proceeds along the same lines as the proof of Case 1, since (2.4) is valid up to a multiplicative constant depending only on n.

Estimation of I_2. In this case, we have

$$|z-\xi|^{-\gamma(z)} = |z-\xi|^{-\gamma(y)}|z-\xi|^{\gamma(y)-\gamma(z)} \leq (e+|\xi|)^{2c^*_{\log}(\gamma)}|z-\xi|^{-\gamma(y)}, \tag{2.7}$$

where the proof follows very closely the proof of (2.5), except for the replacement of $|y-\xi|$ by $|z-\xi|$ and the usage of the inequality $|z-\xi| < |y-\xi|$. We have

$$\int\limits_{\substack{B(y,r)\\ 1<|z-\xi|<|y-\xi|}} |z-\xi|^{-\gamma(y)}dz \le \int\limits_{|z-\xi|<|y-\xi|} |z-\xi|^{-\gamma(y)}dz$$

$$= c(n)\int\limits_0^{|y-\xi|} t^{n-\gamma(y)-1}dt \le \frac{c(n)}{n-\gamma^+} r^n |y-\xi|^{-\gamma(y)}, \tag{2.8}$$

which, together with (2.7), yields

$$I_2 \le \frac{c(n)}{n-\gamma^+} r^n (e+|\xi|)^{2c^*_{\log}(\gamma)} |y-\xi|^{-\gamma(y)}.$$

Estimation of I_3. Letting $B := B(y,r)$ and using polar coordinates like in (2.8), we derive the estimate

$$I_3 \le \int\limits_{|z-\xi|<|y-\xi|} |z-\xi|^{-\gamma_B^+}dz \le \frac{c(n)}{n-\gamma^+}\, r^n\, e^{c_{\log}(\gamma)} |y-\xi|^{-\gamma_B^+}.$$

If $|y-\xi| \ge 1$, then $|y-\xi|^{-\gamma_B^+} \le |y-\xi|^{-\gamma(y)}$. In the remaining case, we have

$$|y-\xi|^{-\gamma_B^+} \le c\cdot e^{c_{\log}(\gamma)} |y-\xi|^{-\gamma(y)},$$

which follows from the inequality $|y-u| \le 4|y-\xi|$ and

$$|y-\xi|^{\gamma(y)-\gamma_B^+} = |y-\xi|^{-|\gamma(y)-\gamma(u)|} = \exp\left(c_{\log}(\gamma)\frac{\log(1/|y-\xi|)}{\log(e+1/|y-u|)}\right),$$

for some $u \in \overline{B}$, due to the continuity of γ.

Case 3. $r \ge 2|y-\xi|$. Defining $\Lambda := B(y, 2|y-\xi|)$ and $\Xi := B(y,r) \setminus B(y,2|y-\xi|)$, set

$$\int\limits_{B(y,r)} |z-\xi|^{-\gamma(z)}dz = \left(\int\limits_{\substack{\Lambda\\ |z-\xi|\le 1}} + \int\limits_{\substack{\Lambda\\ |z-\xi|\ge 1}} + \int\limits_{\substack{\Xi\\ |y-\xi|\ge 1}} + \int\limits_{\substack{\Xi\\ |z-\xi|\le 1}} + \int\limits_{\substack{\Xi\\ |y-\xi|\le 1 \wedge |z-\xi|\ge 1}}\right)|z-\xi|^{-\gamma(z)}dz$$

$$=: J_{1,1} + J_{1,2} + J_{2,1} + J_{2,2} + J_{2,3}.$$

Estimation of $J_{1,1}$. The inequality $|z-\xi| \le 3|y-\xi|$, together with similar arguments to those used in the estimation of I_3 above, yields the bound

$$J_{1,1} \le c(n)\, e^{c_{\log}(\gamma)}\, r^n |y-\xi|^{-\gamma(y)}.$$

Estimation of $J_{1,2}$. The integrand $|z-\xi|^{-\gamma(z)}$ in the integral $J_{1,2}$ satisfies the inequality

$$|z-\xi|^{-\gamma(z)} \le c(n)(e+|\xi|)^{2c^*_{\log}(\gamma)}|z-\xi|^{-\gamma(y)},$$

which can be obtained following the same lines as in (2.5). From this inequality and passing to polar coordinates, as done in (2.8), it follows the inequality

$$J_{1,2} \le \frac{c(n)}{n-\gamma^+}(e+|\xi|)^{2c^*_{\log}(\gamma)} r^n |y-\xi|^{-\gamma(y)}.$$

Estimation of $J_{2,1}$. Since $|y-\xi| \le |z-\xi|$, when $z \in \Xi$, we have $|z-\xi|^{-\gamma(z)} \le |y-\xi|^{-\gamma(y)}|y-\xi|^{\gamma(y)-\gamma(z)}$. Following the scheme of proof of (2.5), we obtain $|y-\xi|^{\gamma(y)-\gamma(z)} \le (e+|\xi|)^{2c^*_{\log}(\gamma)}$, from which we get

$$I_{2,1} \le c(n)(e+|\xi|)^{2c^*_{\log}(\gamma)} r^n |y-\xi|^{-\gamma(y)}.$$

Estimation of $J_{2,2}$. We have $|z-\xi|^{-\gamma(z)} \le c(n)e^{c_{\log}(\gamma)}|y-\xi|^{\gamma(y)}$. This is a consequence of the inequality $|z-\xi| \le 1$ and following, mutatis mutandis, the arguments in (2.6). The above mentioned estimates yield

$$J_{2,2} \le c(n)\, e^{c_{\log}(\gamma)} r^n |y-\xi|^{-\gamma(y)}.$$

Estimation of $J_{2,3}$. The estimate $J_{2,3} \leq c(n)\, r^n |y-\xi|^{-\gamma(y)}$ is immediate, taking into account that $|z-\xi| \leq 1$, $|y-\xi| \leq 1$, and $2|y-\xi| \leq |z-\xi|$.

Taking all previous estimates into account, for $\xi, y \in \mathbb{R}^n$, $y \neq \xi$, and $r > 0$, we have

$$\int\limits_{B(y,r)} |z-\xi|^{-\gamma(z)} dz \leq \frac{c(n)}{n-\gamma^+} r^n \max\left\{ e^{c_{\log}(\gamma)}, e^{c^*_{\log}(\gamma)} (e+|\xi|)^{2c^*_{\log}(\gamma)} \right\} |y-\xi|^{-\gamma(y)},$$

which yields (2.3). □

Basing ourselves on Proposition 1, we obtain results regarding $\mathscr{A}_{p(\cdot)}$-weights in Corollary 2. Before we proceed, it is necessary to first recall some definitions.

The class $\mathscr{P}$ is defined as the set of all bounded measurable functions $p : \mathbb{R}^n \to [1, \infty)$. The variable exponent Lebesgue space, denoted by $L^{p(\cdot)}(\mathbb{R}^n)$, with $p \in \mathscr{P}$, is the space of all measurable functions f such that

$$\|f\|_{p(\cdot)} := \inf \left\{ \lambda > 0 : \int\limits_{\mathbb{R}^n} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \leqslant 1 \right\} < \infty, \tag{2.9}$$

for more information on variable exponent Lebesgue spaces, see [3, 5]. For a function $p \in \mathscr{P}$, we say that a weight w belongs to the variable exponent class $\mathscr{A}_{p(\cdot)}$ if

$$[w]_{\mathscr{A}_{p(\cdot)}} := \sup_B |B|^{-1} \|w \mathbf{1}_B\|_{p(\cdot)} \|w^{-1} \mathbf{1}_B\|_{p'(\cdot)} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$. For more information on these classes, see [2,4,6].

Corollary. *Let* $\gamma(x)p(x)$ *satisfy* (2.1) *and* (2.2)*, and* $0 \leqslant (\gamma p)^- \leqslant (\gamma p)^+ < n$*. Then* $w_\xi^\gamma \in \mathscr{A}_{p(\cdot)}$.

Proof. We know, see [1], that $w \in A_1 \Rightarrow w^{\frac{1}{p(\cdot)}} \in \mathscr{A}_{p(\cdot)}$. The result now follows from Proposition 1. □

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