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Existence result of the global attractor for a triply nonlinear thermistor problem

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ABSTRACT. We study the existence and uniqueness of a bounded weak solution for a triply nonlinear thermistor problem in Sobolev spaces. Furthermore, we prove the existence of an absorbing set and, consequently, the universal attractor.

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1. Introduction

The thermistor was discovered by Michael Faraday in 1833, who noticed that the temperature increases when the silver sulfides resistance decreases. A lot of studies of the thermistor problem can be found in [1, 9, 10, 14, 16].

A thermistor is a circuit component that may be used as a current limiter or as a temperature sensor. It is, typically, a tiny cylinder, constructed of a ceramic substance whose electrical conductivity is highly dependent on temperature. The thermistor regulates the heat created by an electrical current traveling through a conductor device. Thermistor problems have received a lot of attention. We refer the reader to [3, 7, 9, 12, 16, 19] and references therein.

Thermistors are commonly used as temperature control devices in a wide variety of industrial equipment, ranging from space vehicles to air conditioning controllers. They are also often used in the medical field, for localized and general body temperature measurement, in meteorology, for weather forecasting, and in chemical industries as process temperature sensors. A detailed description of thermistors and their applications in electronics and other industries can be found in [22].

There are two types of thermistors: NTC and PTC, which have a positive and negative temperature coefficient, respectively. An NTC thermistor is a temperature sensor that measures temperature using the resistance qualities of ceramic and metal composites. NTC sensors provide a number of benefits in terms of temperature sensing, including small size, great long-term stability, and high accuracy and precision. The operation of a PTC electric surge device is as follows: when the circuit's current is suddenly increased, the device heats up, causing a dramatic decline in its electrical conductivity, effectively shutting off the circuit. In this paper, we consider the following general nonlocal thermistor problem:

$$\begin{cases} \dfrac{\partial \alpha(v)}{\partial s} - \Delta_m v = \kappa \dfrac{f(v)}{\left(\int_\Omega f(v)dx\right)^2}, & \text{in} \quad Q, \\ \alpha(v(x,0)) = \alpha(v_0), & \text{in} \quad \Omega, \\ v = 0, & \text{on} \quad \Gamma \times]0, M[. \end{cases} \tag{1.1}$$

Problem (1.1) models the diffusion of the temperature produced when an electric current flows crossing a material, where $f(v)$ is the electrical resistance of the conductor and $\dfrac{f(v)}{\left(\int_\Omega f(v)dx\right)^2}$ represents the non-local term of (1.1). Here, $Q = \Omega \times [0, M]$, where Ω is an open bounded subset of $\mathbb{R}^N$, $N \geq 1$, and M is a positive constant.

Problem (1.1) is a generalization of the problem appearing in the work of Kavallaris and Nadzieja [15]. For $\alpha(v) = v$ and $m = 2$, one gets the classical model of the thermistor problem appearing in the work of Lacey [16], which is a transformation of the following problem:

$$\begin{gathered} \frac{\partial v}{\partial s} = \nabla \cdot (\kappa(v) \nabla v) + \rho(v) |\nabla \psi|^2, \\ \nabla \cdot (\rho(v) \nabla \psi) = 0, \end{gathered} \tag{1.2}$$

where κ is the thermal conductivity, ψ is the electrical potential, and $\rho(v)$ represents the electrical conductivity, which is normally a positive function supposed to drop sharply by several orders of magnitude at some critical temperature, and remains essentially zero for larger temperatures. This feature is essential for the intended functioning of thermistors as thermoelectric switches.

In the case $\alpha(v) = v$ and $m = 2$, existence and uniqueness results of bounded weak solutions to problem (1.1) were established in [9]. Existence of an optimal control has been obtained by many authors with different assumptions on f and m. We refer, for instance, to [13]. On the other hand, numerical computations of (1.1) and (1.2) have been carried out by other authors, see for example [6, 23, 24, 28], in which the chosen parameters correspond to actual devices. Moreover, a study of (1.2) in the case $N = 1$ can be found in [18]. Here, we extend the existing literature of the nonlocal thermistor problem to a triply nonlinear case.

Let B be the area of Ω, I the current such that $\kappa = I^2/B^2$, and Δ_m be defined by

$$\Delta_m v = \operatorname{div}(| \nabla v |^{m-2} \nabla v) \ \ \forall m \geq 2.$$

We further specify the terms in (1.1). We assume:

- (H1) $v_0 \in L^\infty(\Omega)$;
- (H2) $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ is a Lipschitz continuous increasing function such that $\alpha(0) = 0$ and $\alpha'(s) \geq \lambda > 0$ for all $s \in \mathbb{R}$;
- (H3) f is a Lipshitz continuous function, with compact support, verifying

$$\sigma \leq f(s), \text{ for all } s \in \mathbb{R}, \text{ for a positive constant } \sigma.$$

The rest of the paper is organized as follows. In Section 2, we collect some basic concepts and a few known results that are useful to our development. Section 3 is devoted to the existence of a classical solution to the regularized problem of (1.1). In Section 4, existence of a bounded weak solution to the regularized problem is proved. Then, in Section 5, we provide sufficient conditions under which the solution is unique. Existence of an absorbing set, as well as the global attractor, are proved in Section 6. Finally, we present some concluding remarks in Section 7.

2. Preliminaries

In this section we collect a few known results that are useful to us.

Definition 1 (See [5]). Let α be a continuous increasing function with $\alpha(0) = 0$. For $s \in \mathbb{R}$ we define

$$\Psi(s) = \int_0^s \alpha(t)dt.$$

The Legendre transform Ψ^* of Ψ is defined by

$$\Psi^*(t) = \sup_{r\in\mathbb{R}}\{rt - \Psi(t)\}. \tag{2.1}$$

In particular, we get

$$\Psi^*(\alpha(t)) = t\alpha(t) - \Psi(t). \tag{2.2}$$

Remark 2. *If $v \in L^\infty(Q)$, then $\alpha(v) \in L^\infty(Q)$. It turns out, from equality (2.2), that $\Psi^*(\alpha(v))$ is also bounded.*

Lemma 3 (See [21])**.** *Assume that z is a non-negative, absolutely continuous function, satisfying the following inequality:*

$$z'(s) \leq hz(s) + g(s), \text{ for } s \geq s_0,$$

where h and g are two non-negative integrable functions on $[0, M]$. Then, for each $s \in [0, M]$,

$$z(s) \leq \exp\left(\int_0^s h(\tau)d\tau\right) \cdot \left[z(0) + \int_0^s g(\tau)d\tau\right].$$

Lemma 4 (Ghidaglia lemma [21])**.** *Let z be a positive and absolutely continuous function on $]0, \infty[$ such that the inequality*

$$z' + \delta z^q \leq \eta$$

holds, where $q > 1$, $\delta > 0$, $\eta \geq 0$. Then,

$$z(s) \leq \left(\frac{\eta}{\delta}\right)^{1/q} + (\delta(q-1)s)^{-1/(q-1)}$$

for all $s \geq 0$.

Lemma 5 (See [2])**.** *If $v \in L^m\left(0, M; W^{1,m}(\Omega)\right)$ with*

$$\frac{\partial \alpha(v)}{\partial s} \in L^{m'}\left(0, M; W^{-1,m'}(\Omega)\right),$$

then

$$\left\langle \frac{\partial \alpha(v)}{\partial s}, v \right\rangle_{W^{-1,m'}(\Omega), W^{1,m}(\Omega)} = \frac{d}{ds} \int_\Omega \Psi^*(\alpha(v)).$$

In order to study the existence of the global (universal) attractor, we introduce the following definitions.

Definition 6 (See [21])**.** Let us consider $\mathcal{B} \subset F$ and $\mathcal{U}$ an open bounded set such that $\mathcal{U} \subset \mathcal{B}$. Then $\mathcal{B}$ is an absorbing set in $\mathcal{U}$ if the orbit of each bounded set of $\mathcal{U}$ enters into $\mathcal{B}$ after a given period of time (which may depend on the set):

$$\forall \mathcal{B}_0 \subset \mathcal{U}, \quad \mathcal{B}_0 \text{ bounded},$$
$$\exists s_0(\mathcal{B}_0) \text{ such that } S(s)\mathcal{B}_0 \subset \mathcal{B}, \quad \forall s \geq s_0(\mathcal{B}_0).$$

Definition 7 (See [21])**.** The set $A \subset F$ is said to be an universal attractor for the semigroup $(S(s))_{s\geq 0}$, if the following conditions hold:

(1) $A \subset F$ is a nonempty invariant compact set,
(2) the set $A \subset F$ attracts any bounded set $\mathcal{B} \subset F$, that is,

$$dist\,(S(s)\mathcal{B}, A) \to 0 \text{ as } s \to +\infty, \text{ such that } dist(D, B) = \sup_{a\in D} \inf_{b\in B} \|a-b\|_F.$$

3. Regularized problems

In this section, we first present our approximation scheme. Then we proceed to prove the existence of a weak solution to our regularized problem. To design our regularized scheme, we consider

$$\begin{gathered}\alpha_r \text{ is of class } \mathcal{C}^1(\mathbb{R}) \text{ where } 0<\lambda<\alpha_r',\\ \alpha_r(0)=0,\ \alpha_r \longrightarrow \alpha \text{ in } \mathcal{C}_{loc}(\mathbb{R}) \text{ and } |\alpha_r|\leq|\alpha|,\\ f_r \text{ is of class } \mathcal{C}^\infty(\mathbb{R}),\\ f_r\to f, \text{in } L^1(Q) \text{ and a.e in } Q,\\ f_r \text{ satisfies } (H3)\,.\end{gathered} \tag{3.1}$$

The initial condition is regularized as in the proof of [11, Proposition 3, p. 761], that is,

$$v_{r,0}\in\mathcal{C}_c^\infty(\Omega) \text{ such that } v_{r,0}\to v_0 \text{ in } L^\infty(\Omega),\ \|v_{r,0}\|_{L^\infty(\Omega)}\leq\|v_0\|_{L^\infty(\Omega)}+1. \tag{3.2}$$

Our regularized problems are then given by

$$\begin{cases}\dfrac{\partial\alpha_r(v_r)}{\partial s}-\Delta_m^r v=\kappa\dfrac{f_r(v_r)}{(\int_\Omega f_r(v_r)dx)^2}, & \text{in} \quad Q=\Omega\times[0,M],\\ \alpha_r(v_{r,x}(0))=\alpha_r(v_{r,0}), & \text{in} \quad \Omega,\\ v_r=0, & \text{on} \quad \Gamma\times]0,M[,\end{cases} \tag{3.3}$$

where $\Delta_m^r v=\operatorname{div}\left(\left(|\nabla v|^2+r\right)^{\frac{m-2}{2}}\nabla v\right)$, $m\geq 2$.

Theorem 8. *Assume that hypotheses* $(H1)$–$(H3)$ *hold. Then there exists a solution to problem (3.3).*

The following lemma plays a key role in the proof of Theorem 8.

Lemma 9. *For all* $r>0$, *we have*

$$\| v_r \|_{L^\infty(Q)}\leq C(M,\| v_0 \|_{L^\infty(\Omega)}),$$

where $C(M,\| v_0 \|_{L^\infty(\Omega)})$ *is a positive constant.*

Proof. Multiplying the first equation of problem (3.3) by $\left[(\alpha_r(v_r)-\alpha_r(s_0))^+\right]^{p+1}$ (s_0 is a positive constant where $|v_r|>s_0$) and integrating over Ω, we get

$$\begin{aligned}&\int_\Omega\frac{\partial\alpha_r(v_r)}{\partial s}\left[(\alpha_r(v_r)-\alpha_r(s_0))^+\right]^{p+1}-\int_\Omega\Delta_m^r v_r\left[(\alpha_r(v_r)-\alpha_r(s_0))^+\right]^{p+1}\\ &=\int_\Omega\frac{\kappa\cdot f_r(v_r)}{\left(\int_\Omega f_r(v_r)dx\right)^2}\left[(\alpha_r(v_r)-\alpha_r(s_0))^+\right]^{p+1}.\end{aligned}$$

So, we have

$$\begin{aligned}\frac{1}{p+2}\int_\Omega\frac{\partial}{\partial s}\left[(\alpha_r(v_r)-\alpha_r(s_0))^+\right]^{p+2}&=\int_\Omega\Delta_m^r v_r\left[(\alpha_r(v_r)-\alpha_r(s_0))^+\right]^{p+1}\\ &+\int_\Omega\frac{\kappa\cdot f_r(v_r)}{\left(\int_\Omega f_r(v_r)dx\right)^2}\left[(\alpha_r(v_r)-\alpha_r(s_0))^+\right]^{p+1}.\end{aligned}$$

Then,

$$\begin{aligned}\frac{1}{p+2}\frac{\partial}{\partial s}\int_\Omega \left[(\alpha_r(v_r)-\alpha_r(s_0))^+\right]^{p+2} &= \int_\Omega \Delta_m^r v_r \left[(\alpha_r(v_r)-\alpha_r(s_0))^+\right]^{p+1} \\ &+ \int_\Omega \frac{\kappa\cdot f_r(v_r)}{\left(\int_\Omega f_r(v_r)dx\right)^2}\left[(\alpha_r(v_r)-\alpha_r(s_0))^+\right]^{p+1}.\end{aligned} \tag{3.4}$$

On the other hand, we have

$$\begin{aligned}&\int_\Omega \Delta_m^r v_r \left[(\alpha_r(v_r)-\alpha_r(s_0))^+\right]^{p+1} \\ &= \int_\Omega \operatorname{div}\left(\left(\mid \nabla v_r \mid^2 + r\right)^{\dfrac{m-2}{2}} \nabla v_r\right)\left[(\alpha_r(v_r)-\alpha_r(s_0))^+\right]^{p+1} \\ &= -(p+1)\int_\Omega \left(\left(\mid \nabla v_r \mid^2 + r\right)^{\dfrac{m-2}{2}} \mid \nabla v_r \mid^2\right)\alpha_r'(v_r)\left[(\alpha_r(v_r)-\alpha_r(s_0))^+\right]^{p} \\ &+ \int_{\partial\Omega} \left(\left(\mid \nabla v_r \mid^2 + r\right)^{\dfrac{m-2}{2}} \frac{\partial v_r}{\partial \nu}\right)\left[(\alpha_r(v_r)-\alpha_r(s_0))^+\right]^{p+1}.\end{aligned}$$

Since $\left(\mid \nabla v_r \mid^2 + r\right)^{\dfrac{m-2}{2}} \mid \nabla v_r \mid^2 \geq 0$ and $\alpha_r' > 0$, we get

$$\begin{aligned}&\frac{1}{p+2}\frac{\partial}{\partial s}\int_\Omega \left[(\alpha_r(v_r)-\alpha_r(s_0))^+\right]^{p+2} \\ &\leq \int_{\partial\Omega} \left((\mid \nabla v_r \mid^2 + r)^{\dfrac{m-2}{2}} \frac{\partial v_r}{\partial \nu}\right)\left[(\alpha_r(v_r)-\alpha_r(s_0))^+\right]^{p+1} \\ &\quad + \int_\Omega \frac{\kappa\cdot f_r(v)}{\left(\int_\Omega f_r(v)dx\right)^2}\left[(\alpha_r(v_r)-\alpha_r(s_0))^+\right]^{p+1}.\end{aligned} \tag{3.5}$$

By using $(H3)$, we have

$$\begin{aligned}&\int_\Omega \frac{\kappa\cdot f_r(v_r)}{(\int_\Omega f_r(v_r)dx)^2}\left[(\alpha_r(v_r)-\alpha_r(s_0))^+\right]^{p+1} \\ &\leq \frac{\kappa}{(\sigma\cdot meas(\Omega))^2}\int_\Omega f_r(v_r)\left[(\alpha_r(v_r)-\alpha_r(s_0))^+\right]^{p+1}.\end{aligned}$$

Since f_r satisfies (H3), it yields

$$\begin{aligned}f_r\left(v_r(x,t)\right) &= f_r\left(v_r(x,t)\right)\chi_{\{v_r(x,t)\in supp(f)\}} + f_r\left(v_r(x,t)\right)\chi_{\{v_r(x,t)\notin supp(f)\}} \\ &\leq f_r\left(v_r(x,t)\right)\chi_{\{v_r(x,t)\in supp(f)\}}.\end{aligned}$$

If $v_r(x,t) \in supp(f)$, then it follows that $(v_r(x,t))_r$ is bounded. Thus, there exists a positive constant C_0 such that

$$\int_\Omega f_r(v_r)\left[(\alpha_r(v_r)-\alpha_r(s_0))^+\right]^{p+1} \le C_0 \int_\Omega \left[(\alpha_r(v_r)-\alpha_r(s_0))^+\right]^{p+1}.$$

Keeping that in mind, we have for a positive constant C_1 that

$$\frac{1}{p+2}\frac{\partial}{\partial s}\int_\Omega \left[(\alpha_r(v_r)-\alpha_r(s_0))^+\right]^{p+2} \le C_1 \int_\Omega \left[(\alpha(v_r)-\alpha_r(s_0))^+\right]^{p+1}. \tag{3.6}$$

From Hölder's inequality, there exists positive constants C_j, $j=2,3,4$, such that

$$\begin{aligned}\int_\Omega \left[(\alpha_r(v_r)-\alpha_r(s_0))^+\right]^{p+1} &\le (meas(\Omega))^{\frac{1}{p+1}} \cdot \left(\int_\Omega \left[(\alpha_r(v_r)-\alpha_r(s_0))^+\right]^{p+2}\right)^{\frac{p+1}{p+2}} \\ &\le C_2\left[z_p(s)\right]^{p+1},\end{aligned}$$

where $z_p(s) := \| (\alpha_r(v_r)-\alpha_r(s_0))^+ \|_{L^{p+2}(\Omega)}$. In view of (3.6), we have

$$\frac{1}{p+2}\frac{\partial}{\partial s}\int_\Omega \left[(\alpha_r(v_r)-\alpha_r(s_0))^+\right]^{p+2} \le C_3\left[z_p(s)\right]^{p+1}.$$

Then,

$$\frac{1}{p+2}\frac{\partial}{\partial s}\left[z_p(s)\right]^{p+2} \le C_3\left[z_p(s)\right]^{p+1}, \tag{3.7}$$

and hence

$$\frac{\partial}{\partial s}\left[z_p(s)\right] \le C_3,$$

from which it follows that

$$\left[z_p(s)-z_p(0)\right] \le C_3 M,$$

which implies

$$z_p(s) \le z_p(0) + C_3 M.$$

Letting p go to infinity, we obtain that

$$\| (\alpha_r(v_r)-\alpha_r(s_0))^+ \|_{L^\infty(\Omega)} \le C_4. \tag{3.8}$$

Now, let $u_r = -v_r$, and consider the following problem:

$$\begin{cases} \dfrac{\partial \tilde{\alpha}_r(u_r)}{\partial s} - \Delta_m^r u_r = \kappa \dfrac{\tilde{f}_r(u_r)}{\left(\int_\Omega \tilde{f}_r(v_r)dx\right)^2} =: \tilde{g}(u_r) & \text{in} \quad Q, \\ \tilde{\alpha}_r(u_{x,r}(0)) = \tilde{\alpha}_r(u_0) & \text{in} \quad \Omega, \\ u_r = 0 & \text{on} \quad \Gamma\times]0,M[, \end{cases} \tag{3.9}$$

where $\tilde{\alpha}_r(\tau) = -\alpha_r(-\tau)$, $\tilde{g}_r(\tau) = -g_r(-\tau)$ and $\tilde{f}_r(\tau) = -f_r(-\tau)$. Those functions satisfy the same conditions verified by α, g and f, respectively. The same reasoning done to get (3.8), shows that

$$\| (\tilde{\alpha}_r(u_r)-\tilde{\alpha}_r(s_0))^+ \|_{L^\infty(\Omega)} \le C_5, \tag{3.10}$$

which is equivalent to

$$\| \left(-\alpha_r(-v_r(s)) + \alpha_r(-s_0)\right)^+ \|_{L^\infty(\Omega)} \leq C_5.$$

From (3.8) and (3.10), we deduce that there exists a positive constant C such that

$$\| v_r(s) \|_{L^\infty(\Omega)} \leq C(M, \| v_0 \|_{L^\infty(\Omega)}), \quad \text{for all } s \in [0, M].$$

The lemma is proved.

Proof of Theorem 8. From Lemma 9 and hypotheses $(H1)$–$(H3)$, we conclude, from the classical results of Ladyzenskaya (see [17, pp. 457–459]), with the existence of a classical solution to the regularized problem (3.3).

4. Existence of a weak solution

Definition 10. We say that $v \in L^\infty(Q) \cap L^m\left(0, M; W^{1,m}(\Omega)\right) \cap L^\infty\left(t, M; W^{1,m}(\Omega)\right)$, $t > 0$, is a bounded weak solution of problem (1.1), if it satisfies the following identity:

$$\int_0^M \left\langle \frac{\partial \alpha(v)}{\partial s}, u \right\rangle - \int_Q |\nabla v|^{m-2} \nabla v \nabla u = \kappa \int_Q \frac{f(v)}{(\int_\Omega f(v) dx)^2} u, \tag{4.1}$$

for all $u \in \left(L^m\left(0, M; W^{1,m}(\Omega)\right) \cap L^\infty(Q)\right)$. Furthermore, if we have

$$u \in \left(W^{1,1}\left(0, M; L^1(\Omega)\right) \cap L^m\left(0, M; W^{1,m}(\Omega)\right)\right)$$

with $u(\cdot, M) = 0$, then

$$\int_0^M \left\langle \frac{\partial \alpha(v)}{\partial s}, u \right\rangle = -\int_0^M \int_\Omega [\alpha(v) - \alpha(v_0)] \, \partial_s u,$$

where the duality product is defined by $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{W^{-1,m'}(\Omega), W^{1,m}(\Omega)}$.

Remark 11. *Since α_r is an increasing function and $|\alpha_r| \leq |\alpha|$, then, by using Lemma 9, we also have that $(\alpha_r(v_r))_r$ is bounded.*

Our plan is to derive now enough *a priori* estimates needed in the sequel.

Lemma 12. *For all $r > 0$, we have*

$$||v_r||_{L^m\left(0,M;W^{1,m}(\Omega)\right)} \leq C_6, \tag{4.2}$$

where C_6 is a positive constant independent of r.

Proof. Multiplying the first equation of (3.3) by v_r and integrating, we get

$$\int_\Omega \frac{\partial \alpha_r(v_r)}{\partial s} v_r - \int_\Omega \Delta_m^r v_r v_r = \int_\Omega \kappa \frac{f_r(v_r)}{(\int_\Omega f_r(v_r) dx)^2} v_r. \tag{4.3}$$

Applying (2.2), we obtain that

$$\int_\Omega \frac{\partial \alpha_r(v_r)}{\partial s} v_r = \int_\Omega \frac{\partial \left[\Psi^*\left(\alpha_r(v_r)\right)\right]}{\partial s}.$$

On another hand, by using Green's formula, we get

$$\int_\Omega \Delta_m^r v_r v_r = -\int_\Omega \left(|\nabla v_r|^2 + r\right)^{\frac{m-2}{2}} \nabla v_r \nabla v_r + \int_{\partial\Omega} \left(|\nabla v_r|^2 + r\right) \frac{\partial v_r}{\partial \nu} \cdot v_r.$$

Substituting into (4.3), we get

$$\int_\Omega \frac{\partial \alpha_r(v_r)}{\partial s} v_r + \int_\Omega \left(|\nabla v_r|^2 + r\right)^{\frac{m-2}{2}} |\nabla v_r|^2 = \int_\Omega \frac{\kappa \cdot f_r(v_r)}{\left(\int_\Omega f_r(v_r)dx\right)^2} v_r - \int_{\partial\Omega} \left(|\nabla v_r|^{m-2} + r\right) \frac{\partial v_r}{\partial \nu} \cdot v_r,$$

$$\int_\Omega \left(|\nabla v_r|^2 + r\right)^{\frac{m-2}{2}} |\nabla v_r|^2 = \int_\Omega \frac{\kappa \cdot f_r(v_r)}{\left(\int_\Omega f_r(v_r)dx\right)^2} v_r - \int_\Omega \frac{\partial \left[\Psi^*\left(\alpha_r(v_r)\right)\right]}{\partial s} - \int_{\partial\Omega} \left(|\nabla v_r|^{m-2} + r\right) \frac{\partial v_r}{\partial \nu} \cdot v_r.$$

Then, using the boundary conditions, we have

$$\int_0^M \int_\Omega \left(|\nabla v_r|^2 + r\right)^{\frac{m-2}{2}} |\nabla v_r|^2 = \int_0^M \int_\Omega \frac{\kappa \cdot f_r(v_r)}{\left(\int_\Omega f_r(v_r)dx\right)^2} v_r - \int_0^M \int_\Omega \frac{\partial \left[\Psi^*\left(\alpha_r(v_r)\right)\right]}{\partial s}. \tag{4.4}$$

From Remark 2, we know that $\left(\Psi^*\left(\alpha_r(v_r)\right)\right)_r$ is bounded. With the aid of hypothesis $(H3)$ and Lemma 9, there exists a positive constant C_7 such that

$$\begin{aligned}
&\int_0^M \int_\Omega \kappa \frac{f_r(v_r)}{\left(\int_\Omega f_r(v_r)dx\right)^2} v_r - \int_0^M \int_\Omega \frac{\partial \left[\Psi^*\left(\alpha_r(v_r)\right)\right]}{\partial s} \\
&\leq \int_0^M \int_\Omega \kappa \frac{f_r(v_r)}{\left(\int_\Omega f_r(v_r)dx\right)^2} v_r \\
&\quad - \int_\Omega \Psi^*\left(\alpha_r(v_r(\cdot, M))\right) + \int_\Omega \Psi^*\left(\alpha_r(v_r(\cdot, 0))\right) \\
&\leq \frac{\kappa}{\left(\sigma \cdot meas(\Omega)\right)^2} \int_0^M \int_\Omega f_r(v_r) \cdot |v_r| \\
&\quad + 2 \cdot \max\left\{ \left| \int_\Omega \Psi^*\left(\alpha_r(v_r(\cdot, M))\right) \right|, \left| \int_\Omega \Psi^*\left(\alpha_r(v_r(\cdot, 0))\right) \right| \right\} \leq C_7.
\end{aligned}$$

It yields that

$$\int_0^M \int_\Omega |\nabla v_r|^m \leq \int_0^M \int_\Omega \left(|\nabla v_r|^2 + r\right)^{\frac{m-2}{2}} |\nabla v_r|^2 \leq C_7.$$

We deduce that $v_r \in L^m\left(0, M; W^{1,m}(\Omega)\right)$.

Remark 13. *Inequality (4.2), combined with Young's inequality, imply that*

$$\left(\left(| \nabla v_r |^2 + r\right)^{\frac{m-2}{2}} \nabla v_r\right)_r$$

is bounded in $L^{m'}\left(0, M; W^{1,m'}(\Omega)\right)$.

A further upper bound for v_r is established in the following lemma.

Lemma 14. *For all* $r, s > 0$, *there exist positive constants* $C(t)$, $C(t, M)$, *and* $C_1(t, M)$, *such that the following inequalities hold:*

$$||v_r(s)||_{W^{1,m}(\Omega)} \leq C(t), \quad \text{for all } s \geq t, \tag{4.5}$$

$$\int_t^M \int_\Omega \alpha_r'(v_r) \left(\frac{\partial v_r}{\partial s}\right)^2 \leq C(t, M), \tag{4.6}$$

$$\int_t^M \int_\Omega \left(\frac{\partial \alpha_r(v_r)}{\partial s}\right)^2 \leq C_1(t, M). \tag{4.7}$$

Proof. Multiplying the first equation of problem (3.3) by $\frac{\partial v_r}{\partial s}$, and integrating, we obtain that

$$\int_\Omega \frac{\partial \alpha_r(v_r)}{\partial s} \frac{\partial v_r}{\partial s} - \int_\Omega \Delta_m^r v_r \frac{\partial v_r}{\partial s} = \int_\Omega \kappa \frac{f_r(v_r)}{(\int_\Omega f_r(v_r)dx)^2} \frac{\partial v_r}{\partial s}. \tag{4.8}$$

Since

$$\int_\Omega \frac{\partial \alpha_r(v_r)}{\partial s} \frac{\partial v_r}{\partial s} = \int_\Omega \alpha_r'(v_r) \left(\frac{\partial v_r}{\partial s}\right)^2,$$

the equality (4.8) becomes

$$\int_\Omega \alpha_r'(v_r) \left(\frac{\partial v_r}{\partial s}\right)^2 - \int_\Omega \Delta_m^r v_r \frac{\partial v_r}{\partial s} = \int_\Omega \kappa \frac{f_r(v_r)}{(\int_\Omega f_r(v_r)dx)^2} \frac{\partial v_r}{\partial s}.$$

By applying Green's formula, we get

$$\int_\Omega \alpha_r'(v_r) \left(\frac{\partial v_r}{\partial s}\right)^2 + \frac{1}{m} \frac{\partial}{\partial s} \int_\Omega \left(| \nabla v_r |^2 + r\right)^{\frac{m}{2}} = \int_\Omega \kappa \frac{f_r(v_r)}{(\int_\Omega f_r(v_r)dx)^2} \frac{\partial v_r}{\partial s}. \tag{4.9}$$

Let $G_r(v_r) := \int_0^{v_r} g_r(s) ds$ and $g_r(s) := \frac{f_r(s)}{(\int_\Omega f_r(s)dx)^2}$. By using the boundedness of v_r and (3.1), we have $\frac{\partial G_r(v_r)}{\partial s} \leq C_8$. Then, it yields that

$$\int_\Omega g_r(v_r) \frac{\partial v_r}{\partial s} \leq C_8 \cdot meas(\Omega).$$

With this in mind, we derive

$$\int_\Omega \alpha_r'(v_r)\left(\frac{\partial v_r}{\partial s}\right)^2 + \frac{1}{m}\frac{\partial}{\partial s}\int_\Omega \left(\mid \nabla v_r \mid^2 + r\right)^{\frac{m}{2}} \leq C_9. \tag{4.10}$$

Then,

$$\frac{1}{m}\frac{\partial}{\partial s}\int_\Omega \left(\mid \nabla v_r \mid^2 + r\right)^{\frac{m}{2}} \leq C_9 \tag{4.11}$$

and, by using Gronwall's Lemma 3, we get

$$\int_\Omega |\nabla v_r|^m \leq \frac{1}{m}\int_\Omega \left(\mid \nabla v_r \mid^2 + r\right)^{\frac{m}{2}} \leq C_{10}. \tag{4.12}$$

According to Poincaré's inequality, it follows that

$$||v_r(s)||_{W^{1,m}(\Omega)} \leq C(t), \quad \text{for all } s \geq t.$$

This, combined with inequality (4.10), yields

$$\begin{aligned} &\int_t^M \int_\Omega \alpha_r'(v_r)\left(\frac{\partial v_r}{\partial s}\right)^2 + \frac{1}{m}\int_\Omega \left(\mid \nabla v_r(\cdot, M) \mid^2 + r\right)^{\frac{m}{2}} \\ &\leq \frac{1}{m}\int_\Omega \left(\mid \nabla v_r(\cdot, t) \mid^2 + r\right)^{\frac{m}{2}} + C_9\,(M-t)\,. \end{aligned} \tag{4.13}$$

Now, add (4.12) to (4.13), to obtain

$$\int_t^M \int_\Omega \alpha_r'(v_r)\left(\frac{\partial v_r}{\partial s}\right)^2 + \frac{1}{m}\int_\Omega \left(\mid \nabla v_r(\cdot, M) \mid^2 + r\right)^{\frac{m}{2}} \leq C(t, M).$$

As a consequence, we have

$$\int_t^M \int_\Omega \alpha_r'(v_r)\left(\frac{\partial v_r}{\partial s}\right)^2 \leq C(t, M).$$

Since α is a locally Lipschitzian function, then there exists a positive constant L such that $\alpha_r' \leq L$. Hence, we get

$$\int_t^M \int_\Omega \left(\frac{\partial \alpha_r(v_r)}{\partial s}\right)^2 \leq L\int_t^M \int_\Omega \alpha_r'(v_r)\left(\frac{\partial v_r}{\partial s}\right)^2 \leq C_1(t, M).$$

The proof is complete.

Theorem 15. *Assume that hypotheses* $(H1)$–$(H3)$ *hold. Then there exists a weak bounded solution to problem (3.3).*

Proof. To achieve the proof of Theorem 15, we need to pass to the limit in problem (3.3). By virtue of Lemma 9, there exists a subsequence, still denoted $(v_r)_r$, such that

$$v_r \longrightarrow v \text{ weakly star in } L^\infty(Q).$$

Note from estimate (4.2) that

$$v_r \longrightarrow v \text{ weakly in } L^m\left(0, M; W^{1,m}(\Omega)\right).$$

Since $(v_r)_r$ is bounded in $L^\infty\left(t, M; W^{1,m}(\Omega)\right)$, then

$$v_r \longrightarrow v \text{ weakly star in } L^\infty\left(t, M; W_0^{1,m}(\Omega)\right).$$

Under the hypotheses of f_r, we have $f_r \longrightarrow f$ a.e. This, together with Vitali's theorem (see [20]), implies the convergence to $f(v)$ in $L^1(Q)$. Applying Green's formula,

$$\left|\int_0^M \int_\Omega \Delta_m^r v_r u\right| \leq \left|\int_\Omega \left(\mid \nabla v_r \mid^2 + r\right)^{\dfrac{m-2}{2}} \nabla v_r \nabla u\right|, \text{ for } u \in L^m\left(0, M; W_0^{1,m}(\Omega)\right).$$

By using Remark 13, the right-hand side of this inequality is bounded. Then there exists $\vartheta \in L^{m'}\left(0, M; W^{-1,m'}(\Omega)\right)$ such that

$$\Delta_m^r v_r \longrightarrow \vartheta \text{ weakly in } L^{m'}\left(0, M; W^{-1,m'}(\Omega)\right).$$

A classical argument (see [5]), asserts that $\vartheta = \Delta_m v$.

Combining (4.5) and the smoothness of function α_r, yields the boundedness of the sequence $(\alpha_r(v_r))_r$ in $L^\infty\left(t, M; W^{1,m}(\Omega)\right)$. On the other hand, by using (4.7), we deduce that $\left(\dfrac{\partial \alpha_r(v_r)}{\partial s}\right)_r$ is bounded in $L^2\left(t, M; L^2(\Omega)\right)$, for all $t > 0$. Aubin's lemma (see [25]) allows us to claim that $(\alpha_r(v_r))_r$ is relatively compact in $C\left(]0, M[; L^1(\Omega)\right)$. Therefore, $\alpha_r(v_r) \longrightarrow \delta$ strongly in $C\left(]0, M[; L^1(\Omega)\right)$. Hence, in an entirely similar manner as in [5, p. 1048], it can be handled that $\delta = \alpha(v)$. For the continuous of the solution at point $s = 0$, we proceed as in [4]. From Lemma 14, we deduce that $\alpha_r(v_r) \longrightarrow \alpha(v)$ strongly in $C\left([0, M]; L^1(\Omega)\right)$.

Let us consider $v_0 \in L^\infty(\Omega)$ and take a smooth sequence $(v_{r,0})$ satisfying (3.2). Hence, $(v_{r,0})$ is bounded and convergent to v_0 in $L^1(\Omega)$. Then, thanks to the dominate convergence theorem, we have $\alpha\left(v_{r,0}\right) \longrightarrow \alpha\left(v_0\right)$ in $L^1(\Omega)$. Now, we deal with initial data $v_0 \in C^1(\bar{\Omega})$. Choosing the sequence $(v_{r,0})$ bounded in the space $W^{1,m}(\Omega)$ and verifying hypothesis (3.2), the corresponding $\alpha(v_r)$ are continuous at $s = 0$. Furthermore, we have

$$\begin{aligned} \|\alpha(v(s)) - \alpha(v(0))\|_{L^1(\Omega)} \leq \|\alpha(v(s)) - \alpha\left(v_r(s)\right)\|_{L^1(\Omega)} + \|\alpha\left(v_r(s)\right) - \alpha\left(v_{r,0}\right)\|_{L^1(\Omega)} \\ + \|\alpha\left(v_{r,0}\right) - \alpha\left(v_0\right)\|_{L^1(\Omega)}. \end{aligned} \tag{4.14}$$

In view of Lemma 16, we have

$$\begin{aligned} \|\alpha(v(s)) - \alpha(v(0))\|_{L^1(\Omega)} \leq e^{Ks} \|\alpha\left(v_0\right) - \alpha\left(v_{r,0}\right)\|_{L^1(\Omega)} \\ + \|\alpha\left(v_r(s)\right) - \alpha\left(v_{r,0}\right)\|_{L^1(\Omega)} + \|\alpha\left(v_{r,0}\right) - \alpha\left(v_0\right)\|_{L^1(\Omega)}. \end{aligned} \tag{4.15}$$

As s goes to 0 of (4.15), all terms of the right hand side of (4.15) tend to 0. Then, we deduce that $\alpha(v) \in C\left([0,M]; L^1(\Omega)\right)$. Finally, letting $r \longrightarrow 0$ in (3.3), we obtain the existence of a weak bounded solution.

5. Uniqueness of solution

To prove the uniqueness of the solution, we need to impose some further hypothesis. We assume that there exists a positive constant L_2 such that

$$| f(u) - f(v) | \leq L_2 \mid \alpha(u) - \alpha(v) \mid . \tag{5.1}$$

Lemma 16. *Let v and u be two solutions of problem (1.1) with initial data v_0 and u_0, respectively. Then, the following inequality holds:*

$$\|\alpha(v(s)) - \alpha(u(s))\|_{L^1(\Omega)} \leq e^{Ks} \|\alpha(v_0) - \alpha(u_0)\|_{L^1(\Omega)}, \tag{5.2}$$

where K is a positive constant.

Proof. The proof is similar to the one in [8].

For the proof of our next result, we need the following lemma.

Lemma 17 (Tartar's inequality [26])**.** *If $a, b \in \mathbb{R}^N$, then*

$$\left[|a|^{m-2}a - |b|^{m-2}b\right] \cdot (a-b) \geq C(m) \begin{cases} |a-b|^m, & \text{if } m \geq 2, \\ \frac{|a-b|^2}{(|a|+|b|)^{2-m}}, & \text{if } 1 < m < 2, \end{cases} \tag{5.3}$$

for all $m > 1$, where $C(m) = 2^{2-m}$ when $m \geq 2$ and $C(m) = m-1$ when $1 < m < 2$.

Lemma 18. *Let us consider two solutions v and u of problem (1.1) with initial data v_0 and u_0, respectively, such that $v_0 = u_0$. Then, $v = u$ in Q.*

Proof. For a small positive μ, let

$$H_\mu(Y) = \min\left\{1, \max\left\{\frac{Y}{\mu}, 0\right\}\right\}, \text{ for all } Y \in \mathbb{R}.$$

We use $H_\mu(v-u)$ as a test function. Multiplying the first equation of problem (1.1), corresponding to u and v, by $H_\mu(v-u)$ and subtracting the two equations, we derive that

$$\begin{aligned} &\int_0^s \int_\Omega \frac{\partial}{\partial s}(\alpha(v) - \alpha(u)) H_\mu(v-u) - \int_0^s \int_\Omega (\Delta_m v - \Delta_m u) H_\mu(v-u) \\ &= \int_0^s \int_\Omega \kappa \frac{f(v)}{(\int_\Omega f(v)dx)^2} H_\mu(v-u) - \int_0^s \int_\Omega \kappa \frac{f(u)}{(\int_\Omega f(u)dx)^2} H_\mu(v-u). \end{aligned} \tag{5.4}$$

Using Green's formula and taking into account the boundary conditions, we obtain that

$$\int_0^s \int_\Omega (\Delta_m v) H_\mu(v-u) = -\int_0^s \int_\Omega \mid \nabla v \mid^{m-2} \nabla v \cdot \nabla(v-u) \cdot H'_\mu(v-u). \tag{5.5}$$

We easily check that

$$\int_0^s \int_\Omega (\Delta_m u)\, H_\mu(v-u) = -\int_0^s \int_\Omega |\nabla u|^{m-2} \nabla u \cdot \nabla(v-u) \cdot H'_\mu(v-u). \tag{5.6}$$

From (5.5) and (5.6), it follows that

$$\begin{aligned}&\int_0^s \int_\Omega (\Delta_m v - \Delta_m u) \cdot H_\mu(v-u)\\ &= -\int_0^s \int_\Omega \left[|\nabla v|^{m-2} \nabla v - |\nabla u|^{m-2} \nabla u \right] \nabla(v-u) \cdot H'_\mu(v-u).\end{aligned}$$

By using Lemma 17, it follows that

$$\int_0^s \int_\Omega (\Delta_m v - \Delta_m u) \cdot H_\mu(v-u) \leq 0.$$

Hence,

$$\begin{aligned}\int_0^s \int_\Omega \frac{\partial}{\partial s} (\alpha(v)-\alpha(u))\, H_\mu(v-u) \leq &\int_0^s \int_\Omega \frac{\partial}{\partial s} (\alpha(v)-\alpha(u))\, H_\mu(v-u)\\ &- \int_0^s \int_\Omega (\Delta_m v - \Delta_m u)\, H_\mu(v-u).\end{aligned} \tag{5.7}$$

Recalling (5.4) and (5.7), we get

$$\int_0^s \int_\Omega \frac{\partial}{\partial s} (\alpha(v)-\alpha(u)) \cdot H_\mu(v-u) \quad \leq \quad \int_0^s \int_\Omega \gamma(x) \cdot H_\mu(v-u), \tag{5.8}$$

where

$$\gamma(x) := \kappa \frac{f(v)}{(\int_\Omega f(v)dx)^2} - \kappa \frac{f(u)}{(\int_\Omega f(u)dx)^2},$$

$$\begin{aligned}\gamma(x) \cdot \chi_{\{v-u>0\}} = &\kappa f(u) \frac{\int_\Omega [f(u)-f(v)]\, \mathrm{d}x \int_\Omega [f(u)+f(v)]\, \mathrm{d}x}{\left(\int_\Omega f(u)dx\right)^2 \left(\int_\Omega f(v)dx\right)^2} \cdot \chi_{\{v-u>0\}}\\ &+ \kappa \frac{f(v)-f(u)}{\left(\int_\Omega f(v)dx\right)^2} \cdot \chi_{\{v-u>0\}}.\end{aligned}$$

Adding this to (5.1),

$$\begin{aligned}\gamma(x) \cdot \chi_{\{v-u>0\}} \leq &\kappa L_2 \frac{\int_\Omega (\alpha(v)-\alpha(u))\, \mathrm{d}x \left(\int_\Omega (f(v)+f(u))\, \mathrm{d}x\right)}{\left(\int_\Omega f(u)dx\right)^2 \left(\int_\Omega f(v)dx\right)^2} f(u) \cdot \chi_{\{v-u>0\}}\\ &+ \kappa L_2 \frac{(\alpha(v)-\alpha(u))}{\left(\int_\Omega f(v)\, \mathrm{d}x\right)^2} \cdot \chi_{\{v-u>0\}}.\end{aligned} \tag{5.9}$$

On the other hand, we have

$$\begin{aligned}
&\kappa \cdot L_2 \int_0^s \int_\Omega \frac{\left(\int_\Omega (\alpha(v)-\alpha(u))\ \mathrm{d}x\right)\left(\int_\Omega (f(v)+f(u))\ \mathrm{d}x\right)}{\left(\int_\Omega f(u)dx\right)^2\left(\int_\Omega f(v)dx\right)^2} f(u)\cdot \chi_{\{v-u>0\}} \\
&\quad\leq 2\kappa\cdot L_2\cdot meas(\Omega)\cdot \sup_{a\in \mathrm{supp}(f)} f(a) \int_0^s\int_\Omega \frac{\left(\int_\Omega (\alpha(v)-\alpha(u))\ \mathrm{d}x\right)}{\left(\int_\Omega f(u)dx\right)^2\left(\int_\Omega f(v)dx\right)^2} f(u) \\
&\quad\leq 2\kappa\cdot L_2\cdot meas(\Omega)\cdot \left(\sup_{a\in \mathrm{supp}(f)} f(a)\right)^2 \int_0^s\int_\Omega \frac{\left(\int_\Omega (\alpha(v)-\alpha(u))\ \mathrm{d}x\right)}{\left(\int_\Omega f(u)dx\right)^2\left(\int_\Omega f(v)dx\right)^2}.
\end{aligned} \tag{5.10}$$

Since

$$\begin{aligned}
\int_\Omega (\alpha(v)-\alpha(u))\ \mathrm{d}x &= \int_\Omega (\alpha(v)-\alpha(u))\cdot \chi_{\{v-u>0\}}\ \mathrm{d}x \\
&\quad + \int_\Omega (\alpha(v)-\alpha(u))\cdot \chi_{\{v-u\leq 0\}}\ \mathrm{d}x,
\end{aligned}$$

and α is an increasing function, we get that

$$\int_\Omega (\alpha(v)-\alpha(u))\ \mathrm{d}x \leq \int_\Omega (\alpha(v)-\alpha(u))\cdot \chi_{\{v-u>0\}}\ \mathrm{d}x \leq \int_\Omega (\alpha(v)-\alpha(u))^+\ \mathrm{d}x. \tag{5.11}$$

Keeping in mind (5.9)–(5.11) and hypothesis $(H3)$ on f, it follows that

$$\begin{aligned}
&\int_0^s\int_\Omega \gamma(x)\cdot \chi_{\{v-u>0\}}\ \mathrm{d}x\ \mathrm{d}t \\
&\leq \frac{\kappa\cdot L_2}{(meas(\Omega)\sigma)^2}\int_0^s\int_\Omega (\alpha(v)-\alpha(u))^+\ \mathrm{d}x\ \mathrm{d}t \\
&\quad + \frac{2\kappa\cdot L_2\cdot meas(\Omega)}{(meas(\Omega)\cdot\sigma)^4}\cdot \left(\sup_{a\in \mathrm{supp}(f)} f(a)\right)^2 \int_0^s\int_\Omega \left(\int_\Omega (\alpha(v)-\alpha(u))^+\ \mathrm{d}x\right) \\
&\leq \left(\frac{\kappa L_2}{(meas(\Omega)\sigma)^2} + \frac{2\kappa\cdot L_2(meas(\Omega))^2}{(meas(\Omega)\sigma)^4}\left(\sup_{a\in \mathrm{supp}(f)} f(a)\right)^2\right)\int_0^s\int_\Omega (\alpha(v)-\alpha(u))^+\,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{5.12}$$

On the another hand, when we tend μ to zero, we get

$$\int_0^s\int_\Omega \frac{\partial}{\partial s}(\alpha(v)-\alpha(u))\, H_\mu(v-u) \longrightarrow \int_0^s\int_\Omega \frac{\partial}{\partial s}(\alpha(v)-\alpha(u))\cdot \chi_{\{v-u>0\}}.$$

We also have that

$$\int_0^s\int_\Omega \gamma(x)\cdot H_\mu(v-u) \longrightarrow \int_0^s\int_\Omega \gamma(x)\cdot \chi_{\{v-u>0\}}.$$

This, combined with (5.8) and (5.12), yields the existence of a positive constant C_{11} such that

$$\int_\Omega (\alpha(v)-\alpha(u))^+ \leq C_{11}\cdot \int_0^s\int_\Omega (\alpha(v)-\alpha(u))^+. \tag{5.13}$$

Applying the usual Gronwall's lemma, we get $\alpha(v)\leq\alpha(u)$. Knowing that α is an increasing function, it follows, in particular, that $\alpha(v)=\alpha(u)$ in $\{v-u>0\}$. Keeping this and (5.3) in mind, we obtain that $\nabla(v-u)=0$ in $\{\mu>v-u>0\}$. Hence, $\max\{0,\min\{v-u,\mu\}\}=C_{12}$, where C_{12} is a positive constant. We deduce that $v \leq u$ in Q. Interchanging the role of v and u, the proof of uniqueness is finished.

6. Existence of an absorbing set and the universal attractor

In this section we prove the existence of an universal attractor by first proving the existence of an absorbing set. To this end, let us consider $(S(s))_{s\geq 0}$ a continuous semigroup generated by problem (1.1) such that

$$\begin{array}{rll} S(s): & L^\infty(\Omega) & \to L^\infty(\Omega) \\ & v_0 & \to \alpha(v(s)), \end{array} \tag{6.1}$$

where v is the bounded weak solution of problem (1.1). By using Theorem 8, the map (6.1) is well defined. Now, let us formulate the second main result in this paper.

Theorem 19. *For $m > 2$, $(S(s))_{s\geq 0}$ possesses an universal attractor, which is bounded in $W_0^{1,m}(\Omega)$.*

In order to prove Theorem 19, we first show the following result.

Lemma 20. *Under assumptions $(H1)$–$(H3)$, there exists a positive constant ρ such that*

$$\| v(s) \|_{L^\infty(\Omega)} \leq \rho, \quad \textit{for all} \;\; s > 0.$$

Proof. Multiplying the first equation of (1.1) by $|\alpha(v)|^p \alpha(v)$, and integrating over Ω, we obtain that

$$\int_\Omega \frac{\partial \alpha(v)}{\partial s} |\alpha(v)|^p \alpha(v) - \int_\Omega \Delta_m v \cdot |\alpha(v)|^p \alpha(v) = \kappa \int_\Omega \frac{f(v)}{(\int_\Omega f(v) dx)^2} |\alpha(v)|^p \alpha(v).$$

Then,

$$\frac{1}{p+2} \frac{\partial}{\partial s} \int_\Omega |\alpha(v)|^{p+2} - \int_\Omega \Delta_m v \cdot |\alpha(v)|^p \alpha(v) = \kappa \int_\Omega \frac{f(v)}{(\int_\Omega f(v) dx)^2} |\alpha(v)|^p \alpha(v).$$

Applying Green's formula, and using the boundary conditions, we get

$$\frac{1}{p+2} \frac{\partial}{\partial s} \int_\Omega |\alpha(v)|^{p+2} + (p+1) \int_\Omega |\nabla v|^m \alpha'(v) |\alpha(v)|^p = \kappa \int_\Omega \frac{f(v)}{(\int_\Omega f(v) dx)^2} |\alpha(v)|^p \alpha(v). \tag{6.2}$$

On the other hand, since $\alpha'(v) \geq \lambda$, we have

$$\int_\Omega |\nabla v|^m \alpha'(v) |\alpha(v)|^p \geq \lambda \int_\Omega |\nabla v|^m |\alpha(v)|^p, \quad \text{in} \;\; [0, M].$$

Now, we discuss two cases.

Case 1. If $|\nabla v| \geq |\alpha(v)|$, then

$$\int_\Omega |\nabla v|^m \alpha'(v) |\alpha(v)|^p \geq \lambda \int_\Omega |\alpha(v)|^{m+p}. \tag{6.3}$$

Case 2. If $|\nabla(v)| \leq |\alpha(v)|$, we get

$$\int_\Omega |\nabla v|^m \alpha'(v) |\alpha(v)|^p \geq \lambda \int_\Omega |\nabla v|^m |\alpha(v)|^p \geq \lambda \int_\Omega |\nabla v|^{m+p}.$$

By using Poincaré's inequality, we derive that

$$\int_\Omega |\nabla v|^m \alpha'(v)\, |\alpha(v)|^p \geq \lambda \cdot C_{13} \int_\Omega |v|^{m+p}, \text{ for a positive constant } C_{13}.$$

The smoothness of the function α implies

$$\int_\Omega |\nabla v|^m \alpha'(v)\, |\alpha(v)|^p \geq \frac{\lambda \cdot C_{13}}{L_1} \int_\Omega |\alpha(v)|^{m+p}, \tag{6.4}$$

where L_1 is the Lipshitzity constant of function α. Recall from (6.2) $-$ (6.4) that

$$\frac{1}{p+2}\frac{\partial}{\partial s}\int_\Omega |\alpha(v)|^{p+2} + \min\left\{\frac{\lambda \cdot C_{13}}{L_1}, \lambda\right\} \cdot \int_\Omega |\alpha(v)|^{m+p} \;\leq\; \kappa \int_\Omega \frac{f(v)}{\left(\int_\Omega f(v)dx\right)^2} |\alpha(v)|^p \alpha(v).$$

It is easy to check that

$$\frac{1}{p+2}\frac{\partial}{\partial s}\int_\Omega |\alpha(v)|^{p+2} + \min\left\{\frac{\lambda \cdot C_{13}}{L_1}, \lambda\right\} \cdot \int_\Omega |\alpha(v)|^{m+p} \leq C_{14} \int_\Omega |\alpha(v)|^{p+1},$$

for a positive constant C_{14}.

Set $z_p(s) := \| \alpha(v) \|_{L^{p+2}(\Omega)}$ and $C_{15} := \min\left\{\frac{\lambda \cdot C_{13}}{L_1}, \lambda\right\}$. Making use of Hölder's inequality and the continuous embedding of $L^{m+p}(\Omega)$ in $L^{p+2}(\Omega)$, we obtain that

$$\frac{\partial z_p(s)}{\partial s} \left(z_p(s)\right)^{p+1} + C_{15}\left(z_p(s)\right)^{m+p} \leq C_{14}\left(z_p(s)\right)^{p+1}.$$

It follows that

$$\frac{\partial z_p(s)}{\partial s} + C_{15}\left(z_p(s)\right)^{m-1} \leq C_{14}. \tag{6.5}$$

This puts us in a position to employ Ghidaglia's Lemma 4, to get

$$z_p(s) \leq \left(\frac{C_{14}}{C_{15}}\right)^{\frac{1}{m-1}} + \frac{1}{\left(C_{15}\,(m-2)\,s\right)^{\frac{1}{m-2}}} := \rho_s. \tag{6.6}$$

Letting p going to infinity, we obtain that

$$\| \alpha(v) \|_{L^\infty(\Omega)} \leq C(\eta)$$

for all $s \geq \eta > 0$. This implies

$$\| v(s) \|_{L^\infty(\Omega)} \leq \max\left(| \alpha^{-1}(C(\eta)) |, | \alpha^{-1}(-C(\eta)) | \right). \tag{6.7}$$

Let us consider $\rho := \max\left(| \alpha^{-1}(C(\eta)) |, | \alpha^{-1}(-C(\eta)) | \right)$ as the radius of the ball centered at 0. This ball is an absorbing set in $L^\infty(\Omega)$.

Remark 21. *Existence of an absorbing set in $W^{1,m}(\Omega)$ is obtained due to inequality (4.5) together with the lower semi-continuity of the norm. It yields that*

$$||v(s)||_{W^{1,m}(\Omega)} \leq C(t) := \rho_t, \quad \text{for all } \ s \geq t.$$

Then the ball $B(0, \rho_t)$ is an absorbing set in $W^{1,m}(\Omega)$.

Now, in order to prove Lemma 23 below, we show that the solution of problem (1.1) is Hölder continuous. To this end, we set $\alpha(v) := w$ and we add the following assumptions:

$(H4)$ α is a strict increasing function and $\alpha^{-1} \in \mathcal{C}^1(\mathbb{R})$;

$(H5)$ $i)$ $\left(\alpha^{-1}(w)\right)'$ is degenerate in the neighborhood of zero and there exists $z \in [-\eta_0, \eta_0]$, η_0 a positive constant, such that

$$\beta_0 |z|^{k_0} \leq \left(\alpha^{-1}(w)\right)' \leq \beta_1 |z|^{k_1} \tag{6.8}$$

for positive constants β_j and k_j, $j = 0, 1$;

$ii)$ there exists two positive constants e_0 and e_1 such that

$$e_0 \leq \left(\alpha^{-1}(w)\right)' \leq e_1, \tag{6.9}$$

$$\frac{\partial w}{\partial s} - \operatorname{div}\left(\left|\left(\alpha^{-1}(w)\right)'\right|^{m-2} \cdot \left(\alpha^{-1}(w)\right)' |\nabla w|^{m-2} \nabla w\right) = \kappa \frac{f(\alpha^{-1}(w))}{\left(\int_\Omega f(\alpha^{-1}(w))dx\right)^2}, \tag{6.10}$$

$$w = 0, \tag{6.11}$$

for all $z \in]-\infty, -\eta_0[\bigcup]\eta_0, +\infty[$.

Identifying (6.10) with (1) in the paper [27], and using hypotheses (H3)–(H5), we can apply the following theorem.

Theorem 22 (See [27])**.** *Suppose that Theorem 8 holds. Then, under assumptions (H3)–(H5), the solution of problem (1.1) is Hölder continuous.*

In the following Lemma we prove that the operator $(S(s))_{s\geq 0}$ is uniformly compact for s large enough.

Lemma 23. *If B is a bounded set, then*

$$\bigcup_{s \geq s_0} S(s)B$$

is relatively compact for any $s \geq s_0$.

Proof. We can derive from Lemma 9 that the set $\bigcup_{s\geq s_0} S(s)B$ is bounded in $L^\infty(\Omega)$. Furthermore, the approximation solution is uniformly bounded. We are in position to invoke Theorem 22 and, consequently, we deduce, by Ascoli–Arzelà theorem, that the set $\bigcup\limits_{s\geq s_0} S(s)B$ is relatively compact.

Proof of Theorem 19. We have to prove that $(S(s))_{s\geq 0}$ related to problem (1.1) possesses an universal attractor. We consider the following ω-limit:

$$\omega(B_0) := \{v \in L^\infty(\Omega) : \exists s_n \to +\infty,\ \exists v_n \in B_0 \text{ such that } S(s_n)\, v_n \to v \text{ in } L^\infty(\Omega)\},$$

where $B_0 := \overline{S(t)B}^{L^\infty(\Omega)}$ for some $t > 0$. We apply Lemma 1.1 in [21] to get that $\omega(B_0)$ is a nonempty compact invariant set. Then the first condition of Definition 7 holds. For the second condition of Definition 7, we proceed by absurd. Assume that A does not attract each bounded set in $L^\infty(\Omega)$. Then there exists a bounded set B, not attracted by A, and there exists $s_n \to \infty$ and $\epsilon > 0$ such that

$$dist\,(S(s_n)B, A) \geq \frac{\epsilon}{2}, \tag{6.12}$$

from whence follows that, for every n, there exists $d_n \in B$ such that

$$dist\,(S(s_n)d_n, A) \geq \frac{\epsilon}{2}. \tag{6.13}$$

Knowing that B_0 is an absorbing set for B (a bounded set), there exists s such that $s \geq s_1$, where s_1 is a positive constant, and we have $S(s)B \subset B_0$. Since $s_n \to \infty$, then $s_n \geq s_1$ for large enough n and $S(s_n)B \subset B_0$. As a consequence, we have

$$S(s_n)d_n \in B_0. \tag{6.14}$$

On the other hand, recall from Lemma 23 that $\bigcup\limits_{s\geq s_0} S(s)B_0$ is relatively compact. Consequently, the sequence $(S(s_n)d_n)_n$ is also relatively compact. So, there exists a subsequence such that

$$S(s_n)d_n \longrightarrow \ell \in L^\infty(\Omega), \text{ as } s_n \longrightarrow \infty.$$

With the semi-group propriety, we have

$$\lim_{n\longrightarrow\infty} S(s_n)d_n = \lim_{n\longrightarrow\infty} S(s_n - s_1)S(s_1)d_n = \lim_{n\longrightarrow\infty} S(s'_n)d'_n = \ell, \tag{6.15}$$

where $s'_n := s_n - s_1$ and $d'_n := S(s_1)d_n$. We infer that

$$\omega(B_0) := \{v :\ \exists s_n,\ d_n \text{ such that } S(s_n)d_n \longrightarrow v\}. \tag{6.16}$$

In view of the fact that $d'_n \in B_0$, then s'_n and d'_n play the role of s_n and d_n, respectively, in (6.16). Keeping this and (6.15) in mind, we obtain that $\ell \in \omega(B_0) = A$. Then $dist\,(\ell, A) = 0 < \frac{\epsilon}{2}$. This is in contradiction with inequality (6.13). Hence, A is the universal attractor.

7. Conclusions and perspectives

In this paper, we proved existence and uniqueness of a bounded weak solution in Sobolev spaces for a non-local thermistor problem in the presence of triply nonlinear terms. We also proved the existence of the global attractor. As future work, we plan to study the regularity of the global attractor, the stability of the solution, and the optimal control for the thermistor problem (1.1).

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