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ABSTRACT

We comment on the validity of Noether's theorem and on the conclusions of Anerot *et al.* [J. Math. Phys. **61**(11), 113502 (2020)].

In Ref. 1, Anerot *et al.* concluded that the Noether-type theorem proved by Bartosiewicz and Torres in Ref. 2 is true along the Euler–Lagrange extremals under the assumption that the Euler–Lagrange extremals are also satisfying the second Euler–Lagrange equation: see Ref. 1, bottom of p. 9. While this is true, the authors of Ref. 1 seemed not aware that (i) such a conclusion is not new (see Ref. 5) and that (ii) in the class of C^2-functions where Noether's theorem is proved, all Euler–Lagrange extremals do satisfy the second Euler–Lagrange equation (sometimes also called the DuBois–Reymond condition): see also Ref. 5 and the references therein.

Indeed, it has been shown in Ref. 5 that while Noether's theorem is valid along every C^2 Euler–Lagrange extremal (and every C^2 Euler–Lagrange extremal is also a DuBois–Reymond extremal), Noether's result is not true when one enlarges the class C^2 of admissible functions to be the class *Lip* of Lipschitz functions, for which the Euler–Lagrange equation is still valid. Moreover, it has been proved in Ref. 5 that one can still prove a version of Noether's theorem in the class of *Lip* admissible functions, but in that case, we need to restrict the set of Euler–Lagrange extremals to those that also satisfy the DuBois–Reymond condition (the second Euler–Lagrange equation). Hence, the conclusion of Ref. 1 that Noether's theorem is only true for Euler–Lagrange extremals that also satisfy the second Euler–Lagrange equation is not a new result and does not invalidate, *per se*, the Noether theorem proved in Ref. 2: for the class of smooth admissible functions considered by Noether's theorem, the second Euler–Lagrange equation is always satisfied along the Euler–Lagrange extremals.

However, the main message of Ref. 1 is that the result of Ref. 2 is not correct. To show that, an example is given. While the example is also not new since it was borrowed from Ref. 2, we fully agree that the example deserves indeed some comments to make things clear. Unfortunately, the analysis of Ref. 1 has some inconsistencies that do not help to clarify things: in the simulations of Ref. 1, the authors considered the initial condition $x(0) = x_0 = 0$ and a nonstandard condition $\Delta x_0 = 0.1$ that has never been considered in the literature of the calculus of variations under Noether's theorem.

Since Ref. 1 is very technical and not all readers may be familiar with the language of the calculus of variations on time scales,[3] to make things completely clear and understandable, we restrict our discussion here to the classical case when the time scale is the set of real numbers. This allow us to easily explain the discrepancies between Noether's theorem and that particular example. In our opinion, such discrepancies are intrinsic to the nature of the calculus of variations and do not depend on the theory of time scales.

The example of the calculus of variations given in Ref. 2, and recalled in Ref. 1, has a Lagrangian given by $L(t,x,v) = \frac{x^2}{t} + tv^2$, defined in a time interval $t \in [a,b]$ that does not include $t = 0$. This means that for this Lagrangian, the Euler–Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial v}(t,x(t),x'(t)) = \frac{\partial L}{\partial x}(t,x(t),x'(t))$$

reduces to

$$\frac{d}{dt}(tx'(t)) = \frac{x(t)}{t},$$

from which one concludes that the Euler–Lagrange extremals have the form

$$x(t) = \frac{c_1}{t} + c_2 t,$$

where the constants c_1 and c_2 are determined from the boundary conditions of the problem. For example, if one considers the (nonstandard) boundary conditions $x(0) = 0$ and $x'(0) = 0.1$ considered in Ref. 1, then the Euler–Lagrange extremal is obtained with $c_1 = 0$ and $c_2 = 0.1$, that is, the Euler–Lagrange extremal is the function $x(t) = 0.1t$. However, for standard boundary conditions $x(a) = \alpha$ and $x(b) = \beta$ of the calculus of variations with $0 < a < b$, as considered in Ref. 2, the Euler–Lagrange extremal is given by

$$x(t) = \frac{ab(b\alpha - a\beta)}{(b^2 - a^2)t} + \frac{b\beta - a\alpha}{b^2 - a^2}t.$$

What is important to mention here is that $x(t)$ is an absolutely continuous function, which is not Lipschitz (and in particular does not belong to the class C^2 where Noether's theorem is proved). We also recall that while the Euler–Lagrange equation can still be proved in the class of Lipschitz functions, the Euler–Lagrange equation (and thus also Noether's theorem) is not valid in the class of absolutely continuous functions: see, e.g., Ref. 4 and the references therein.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

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