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An integral boundary fractional model to the world population growth

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ABSTRACT

We consider a fractional differential equation of order α, $\alpha \in (2, 3]$, involving a ψ-Caputo fractional derivative subject to initial conditions on function and its first derivative and an integral boundary condition that depends on the unknown function. As an application, we investigate the world population growth. We find an order α and a function ψ for which the solution of our fractional model describes given real data better than available models.

1. Introduction

Fractional calculus has emerged as one of the most important interdisciplinary subjects in Mathematics, Biology and Engineering. It has been shown that differential equation models involving fractional derivatives describe certain physical phenomena better than traditional integer-order differential equation models [1,2]. Mathematical modeling is the process of describing a real world problem in mathematical terms, usually in the form of equations, and then using such equations both to help understand the original problem, and also to discover new features. Currently, applications and activities related to fractional calculus have appeared in many fields of science and engineering, such as in HIV modeling [3,4] and fluid dynamics [5]. Recently, Heymans and Podlubny have given some physical interpretation for the fractional spring-pot model, the Zener model, the Maxwell and the Voigt models [6].

In [7], Almeida introduced the so-called ψ-Caputo fractional derivative, which generalizes a large class of fractional derivatives, such as Caputo and Caputo–Hadamard. For some results and recent developments on initial and boundary value problems involving the ψ-Caputo fractional derivative see [8–10].

Many analytical and numerical methods have been proposed in the literature to describe the population growth model [11–15]. In this paper, we formulate a new mathematical model consisting of a fractional differential equation with an integral boundary condition. Then, we use it to describe the world population growth. Our results show that the ψ-Caputo derivative introduced by Almeida [16] is a suitable approach for modeling, describing better the world population growth than available models. More precisely, we determine the order of the fractional differential equation and the kernel ψ such that the solution of our model approximates well given real data. To the best of our knowledge, this is the first time a concrete application is given to a fractional differential equation with an integral boundary condition depending on the unknown function.

The structure of this paper is displayed as follows. In Section 2 we recall the notions and results that are necessary for the sequel. Our results are given in Section 3: we introduce the new model; we obtain an explicit formula for the exact solution of the problem; and we show the accuracy and efficiency of our model with respect to real data of world population from 1910 to 2010. Finally, we end the paper with Section 4 of conclusion.

2. Preliminaries

We briefly recall here the necessary definitions and results from fractional calculus theory.

Definition 1 (*The ψ-Riemann–Liouville Fractional Integral [17]*)**.** Let $\alpha > 0$, $f : [a, b] \longrightarrow \mathbb{R}$ be integrable and $\psi \in C^1([a, b])$ an increasing function such that $\psi'(t) \neq 0$, for all $t \in [a, b]$. The ψ-Riemann–Liouville fractional integral of f of order α is defined as follows:

$$I_{a^+}^{\alpha,\psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s) ds, \quad \alpha > 0, \tag{1}$$

where $\Gamma(\alpha)$ is the Gamma function.

Note that for $\psi(t) = t$ and $\psi(t) = \ln(t)$, Eq. (1) is reduced to the Riemann–Liouville and Hadamard fractional integrals.

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Definition 2 (*The ψ-Riemann–Liouville Fractional Derivative [17]*). Let $n \in \mathbb{N}$ and let $\psi, f \in C^n([a,b],\mathbb{R})$ be two functions such that ψ is increasing and $\psi'(t) \neq 0$ for all $t \in [a,b]$. The ψ-Riemann–Liouville fractional derivative of f of order α is given by

$$ {}^{RL}D_{a^+}^{\alpha,\psi} f(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a^+}^{n-\alpha,\psi} f(t) \tag{2}$$

$$ = \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_a^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1} f(s)ds, \tag{3}$$

where $n = [\alpha] + 1$.

Definition 3 (*The ψ–Caputo Fractional Derivative [17]*). Let $n \in \mathbb{N}$ and let $\psi, f \in C^n([a,b],\mathbb{R})$ be two functions such that ψ is increasing and $\psi'(t) \neq 0$ for all $t \in [a,b]$. The ψ-Caputo fractional derivative of f of order α is given by

$$ {}^{C}D_{a^+}^{\alpha,\psi} f(t) = I_{a^+}^{n-\alpha,\psi} \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n f(t) \tag{4}$$

$$ = \frac{1}{\Gamma(n-\alpha)} \int_a^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1} f_\psi^{[n]}(s)ds, \tag{5}$$

where $n = [\alpha] + 1$ and $f_\psi^{[n]}(t) := \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n f(t)$.

Theorem 4 (*See [17]*). *Let $f : [a,b] \longrightarrow \mathbb{R}$. The following holds:*

1. *If $f \in C([a,b])$, then*
   $${}^{C}D_{a^+}^{\alpha,\psi} I_{a^+}^{\alpha,\psi} f(t) = f(t).$$
2. *If $f \in C^{n-1}([a,b])$, then*
   $$I_{a^+}^{\alpha,\psi}\, {}^{C}D_{a^+}^{\alpha,\psi} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f_\psi^{[k]}(a)}{k!}(\psi(t)-\psi(a))^k.$$

3. Main results

We consider the following ψ-Caputo fractional differential equation of the form

$$\begin{cases} {}^{C}D^{\alpha,\psi} u(t) + y(t) = 0, \quad t \in [t_0, T], \\ u(t_0) = u_0, \; u'(t_0) = u_0^1, \quad u(T) = \lambda \int_{t_0}^T u(s)ds, \end{cases} \tag{6}$$

where ${}^{C}D^{\alpha,\psi}$ is the ψ-Caputo fractional derivative of order $2 < \alpha \leq 3$ and $\lambda > 0$.

3.1. Theoretical analysis

We begin by rewriting our linear fractional integral boundary value problem (6) as an equivalent integral equation. This gives an explicit formula (7) for $u(t)$. Such formula is useful to prove existence and uniqueness of solution as well as for approximation purposes.

Theorem 5 (*Existence and Uniqueness of Solution to (6) With Explicit Formula for u*). *For a given function $y \in C([a,b])$, the unique solution of the linear fractional boundary value problem (6) is given by*

$$\begin{aligned} u(t) = &-I_{t_0}^{\alpha,\psi} y(t) + \frac{(\psi(t)-\psi(t_0))^2}{\Delta^2} I_{t_0}^{\alpha,\psi} y(t)_{|t=T} \\ &+ u_0\left(1 - \frac{(\psi(t)-\psi(t_0))^2}{\Delta^2} + \left(\frac{\lambda(T-t_0)}{\gamma\Delta^2} - \frac{\delta_2\lambda}{\gamma\Delta^4}\right)(\psi(t)-\psi(t_0))^2\right) \\ &+ \frac{u_0^1}{\psi'(t_0)}\left((\psi(t)-\psi(t_0)) - \frac{(\psi(t)-\psi(t_0))^2}{\Delta} \right. \\ &\left. + \frac{\delta_1\lambda}{\gamma\Delta^2}(\psi(t)-\psi(t_0))^2 - \frac{\delta_2\lambda}{\gamma\Delta^3}(\psi(t)-\psi(t_0))^2\right) \\ &- \frac{\lambda}{\gamma\Delta^2}(\psi(t)-\psi(t_0))^2 \int_{t_0}^T \frac{Q(s)}{\Gamma(\alpha)} y(s)ds \\ &+ \frac{\lambda(\psi(t)-\psi(t_0))^2}{\gamma\Gamma(\alpha)\Delta^4} \int_{t_0}^T \delta_2\psi'(s)(\psi(T)-\psi(s))^{\alpha-1} y(s)ds, \end{aligned} \tag{7}$$

where

$$\Delta := (\psi(T)-\psi(t_0)) \neq 0, \; \delta_1 := \int_{t_0}^T (\psi(t)-\psi(t_0))dt,$$

$$\delta_2 := \int_{t_0}^T (\psi(t)-\psi(t_0))^2 dt, \quad Q(s) := \int_s^T \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}dt,$$

$$\gamma = \left(1 - \frac{\lambda\delta_2}{\Delta^2}\right).$$

Proof. Taking the ψ-Riemann–Liouville fractional integral of order α to the first equation of (6) and applying Theorem 4, we obtain that

$$u(t) = -I_{t_0}^{\alpha,\psi} y(t) + c_0 + c_1\left(\psi(t)-\psi(t_0)\right) + c_2\left(\psi(t)-\psi(t_0)\right)^2, \quad c_0, c_1, c_2 \in \mathbb{R}. \tag{8}$$

Substituting $t = t_0$ in (8), and using the first boundary condition of (6), we get $c_0 = u_0$. Differentiating (8), we get

$$u'(t) = -\frac{d}{dt}(I_{t_0}^{\alpha,\psi} y(t)) + c_1\psi'(t) + 2c_2\psi'(t)(\psi(t)-\psi(t_0)).$$

Using the fact that $u'(t_0) = u_0^1$, we find that $c_1 = u_0^1/\psi'(t_0)$. Therefore,

$$u(t) = -I_{t_0}^{\alpha,\psi} y(t) + u_0 + \frac{u_0^1}{\psi'(t_0)}(\psi(t)-\psi(t_0)) + c_2(\psi(t)-\psi(t_0))^2. \tag{9}$$

Now, the condition $u(T) = \lambda\int_{t_0}^T u(s)ds$ implies that

$$c_2 = \frac{1}{\Delta^2} I_{t_0}^{\alpha,\psi} y(t)_{|t=T} + \frac{\lambda}{\Delta^2}\int_{t_0}^T u(s)ds - \frac{u_0}{\Delta^2} - \frac{u_0^1}{\Delta\psi'(t_0)}.$$

Hence,

$$\begin{aligned} u(t) = &-I_{t_0}^{\alpha,\psi} y(t) + \frac{(\psi(t)-\psi(t_0))^2}{\Delta^2} I_{t_0}^{\alpha,\psi} y(t)_{|t=T} + u_0\left(1 - \frac{(\psi(t)-\psi(t_0))^2}{\Delta^2}\right) \\ &+ \frac{u_0^1}{\psi'(t_0)}\left((\psi(t)-\psi(t_0)) - \frac{(\psi(t)-\psi(t_0))^2}{\Delta}\right) \\ &\times \frac{\lambda\left(\psi(t)-\psi(t_0)\right)^2}{\Delta^2}\int_{t_0}^T u(s)ds. \end{aligned} \tag{10}$$

Let $A = \int_{t_0}^T u(s)ds$. Then, from (10), we deduce that

$$\begin{aligned} \int_{t_0}^T u(t)dt = &-\frac{1}{\Gamma(\alpha)}\int_{t_0}^T\int_{t_0}^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}y(s)dsdt \\ &+ \frac{1}{\Delta^2\Gamma(\alpha)}\int_{t_0}^T\int_{t_0}^T (\psi(t)-\psi(t_0))^2\psi'(s)(\psi(T)-\psi(s))^{\alpha-1}y(s)dsdt \\ &+ u_0\left((T-t_0) - \frac{\int_{t_0}^T\left(\psi(t)-\psi(t_0)\right)^2 dt}{\Delta^2}\right) \\ &+ \frac{u_0^1}{\psi'(t_0)}\left(\int_{t_0}^T(\psi(t)-\psi(t_0))dt - \frac{\int_{t_0}^T\left(\psi(t)-\psi(t_0)\right)^2 dt}{\Delta}\right) \\ &+ \frac{\lambda}{\Delta^2}\int_{t_0}^T\left(\psi(t)-\psi(t_0)\right)^2 dt\ A. \end{aligned}$$

Using Fubini's theorem, and the definition of $Q(s)$, we obtain that

$$\begin{aligned} A = &-\frac{1}{\gamma}\int_{t_0}^T \frac{Q(s)}{\Gamma(\alpha)}y(s)ds + \frac{1}{\Gamma(\alpha)\Delta^2\gamma}\int_{t_0}^T \delta_2\psi'(s)(\psi(T)-\psi(t_0))^{\alpha-1}y(s)ds \\ &+ \frac{u_0}{\gamma}\left((T-t_0) - \frac{\delta_2}{\Delta^2}\right) + \frac{u_0^1}{\gamma\psi'(t_0)}\left(\delta_1 - \frac{\delta_2}{\Delta}\right). \end{aligned}$$

Replacing this relation into (10), we arrive to the expression (7), which completes the proof. □

3.2. Application and discussion

In this section, we consider an application of our fractional differential equation with ψ-Caputo fractional derivatives to the world

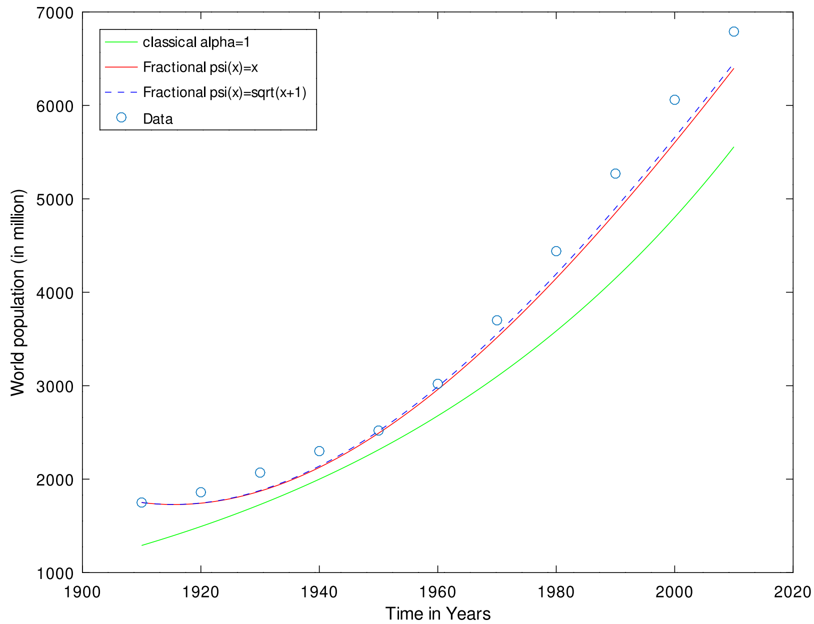

Fig. 1. Experimental data vs classical model vs fractional models.

Table 1
World population from year 1910 until 2010.

Year	1910	1920	1930	1940	1950	1960	1970	1980	1990	2000	2010
Population	1750	1860	2070	2300	2520	3020	3700	4440	5270	6060	6790

Table 2
Errors and gains: classical solution (12) versus (7) with $\alpha = 2.5$.

Model	Error	Gain $\mathcal{G}$
Classical	0, 173404	–
$\psi(t) = t$	0, 054079	69%
$\psi(t) = \sqrt{t+1}$	0, 046932	73%

population growth. The growth of a population has certain limits due to environmental restrictions, such as food availability, competition with other species, competition for territory, etc. We provide an order α and a function ψ for which the solutions of the fractional model (6) correspond to given real data. For the data source, we use the one available on [18]. We study the growth of world population from the year 1910 up to 2010, measured every 10 years, that is, at given 11 values: see Table 1.

The classical population growth model is given by the differential equation

$$u'(t) = \Lambda u(t), \tag{11}$$

where Λ is the population growth rate. The solution to (11) is

$$u(t) = c_0 e^{c_1 t} \tag{12}$$

and the best fit of (12) to the data of Table 1, in the sense of the least squares method, is given by $u(t) = 10^{-9} e^{0,0146t}$.

Now, consider our fractional model (6) with $t_0 = 1910$, $T = 2010$, $u_0 = 1750$, $u_0^1 = -7.9487$ and $\lambda = 0.019206$. We obtained these values using GNU Octave as follows:

```
% 1) Step 1: we define the real data

% Real Data World population

TT  = [1910 1920 1930 1940 1950 1960 1970 1980 1990 2000 2010];
TU = [1750 1860 2070 2300 2520 3020 3700 4440 5270 6060 6790];

% Parameters of the model that we know directly from data

t0 = TT(1);
T = TT(end);
u0 = TU(1);

% 2) Step 2: we define a polynomial that fits well the data

p = polyfit(TT,TU,3);
tt = linspace(t0,T,100);
pp = polyval(p,tt);
%figure
%plot(tt,pp)
%hold on
%plot(TT,TU,"o")
%hold off

% 3) Step 3: we define the other parameters from data and the polynomial


u = @(t) polyval(p,t);

A = integral(u,t0,T);
lambda = TU(end)/A   % 0.019206
p1 = polyder(p);
u10 = polyval(p1,t0) % -7.9487
```

We show that the solution of our model approximates well the data of Table 1.

Given data consisting of r points, $(t_0, u_0), \ldots, (t_r, u_r)$, we approximate these values by the solution $t \mapsto u(t)$ of some theoretical model. In our case, the form u is known, being given by (7), but it depends on ψ and α. For each approximation $u(t_i)$ of u_i the relative error is defined by $d_i := |u_i - u(t_i)|/|u_i|$, for $i = 0, \ldots, r$, while the average of the total relative error is given by

$$E_{rel} = \sum_{i=0}^{r} \frac{d_i}{r+1}. \tag{13}$$

Here we observe that the solution of our fractional model is closer to the data than the solution of the classical model and also better than the solution to the model proposed in [11]. To see that, we define the gain $\mathcal{G}$ of the efficiency of our model as in [11], comparing the relative error (13) of the classical model, $E_{classical}$, with the relative error of our fractional model:

$$\mathcal{G} = \left| \frac{E_{classical} - E_{fractional}}{E_{classical}} \right|.$$

For the classical model (11), the value of the relative error is

$$E_{classical} \approx 17.34\%. \tag{14}$$

Now, we give functions ψ and an order α for which the respective relative errors are smaller. If we choose $\alpha = 5/2$ and $\psi(t) = t$, that is, if we deal with the classical Caputo fractional derivative, then for all $t \in [t_0, T]$ we have the explicit solution

$$u(t) = 1,64452 \times 10^7 - 24396t + 12,0275t^2 - 0,0019697t^3,$$

with $E_{fractional} \approx 5,41\%$ with an efficiency gain of

$$\mathcal{G} = \left| \frac{0,173404 - 0,054079}{0,173404} \right| \approx 68.81\%.$$

However, the model of [11] has a better gain of 71%. This fact motivate us to look for a different ψ for which the gain is higher than 71%. This is indeed possible by taking $\psi(t) = \sqrt{t+1}$: the classical error (14) of 17.34% is decreased to just $4,7\%$ with a gain of efficiency of

$$\left| \frac{0,173404 - 0,046932}{0,173404} \right| \approx 72.94\%.$$

Therefore, with this kernel, we have improved the gain efficiency of 69% of model (6) with $\psi(t) = t$ and the gain efficiency of 71% of the model [11]. In Table 2 we summarize the obtained results.

With the help of GNU Octave, in Fig. 1 we give the curves obtained from the fractional model (6) with $\psi(t) = t$ and $\psi(t) = \sqrt{t+1}$, together with the data of Table 1.

4. Conclusion

We have treated the population growth problem modeling it by a fractional differential equation subject to an integral boundary condition. First we proved the existence and uniqueness of solution to the proposed model. Our proof is constructive and an explicit formula for the solution is given, which depends on the fractional order α and an arbitrary smooth function ψ. By using real data, we have proved that our mathematical model with the ψ-Caputo fractional derivative is more accurate than the classical model and the fractional model proposed in [11].

CRediT authorship contribution statement

Om Kalthoum Wanassi: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Data curation, Writing – original draft, Writing – review & editing, Visualization. **Delfim F.M. Torres:** Conceptualization, Methodology, Software, Formal analysis, Investigation, Data curation, Writing – original draft, Writing – review & editing, Visualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

All data is given in the paper.

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