Feasible initial conditions for 2D discrete state-space systems ⋆

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Abstract: The aim of this contribution is to characterize the set of feasible initial conditions on a diagonal line in order to compute the solutions of a 2D discrete state-space system (defined over $\mathbb{Z}^2$) on a half-plane of the 2D grid. This characterization is given in terms of the system matrices for the state updating.

Keywords: 2D systems, state-space models, initial conditions.

1. INTRODUCTION

In this paper we study discrete 2D systems defined over the whole grid $\mathbb{Z}^2$ that are described by a Fornasini-Marchesini state-space model, Fornasini and Marchesini [1985].

In particular we consider the problem of determining the set of initial conditions for the (pseudo-) state that can be assigned along a "separating line" in $\mathbb{Z}^2$ in order to compute the state trajectories on a half-plane. This is a relevant issue in order to study structural properties such as stability, controllability and observability.

As is well-known, for systems defined over the first quadrant, the (local) state initial conditions may be freely assigned on the non-negative axes, i.e., on the points $(i, 0)$ and $(0, j)$ with $i, j = 0, 1, 2, \ldots$.

On the other hand, for systems defined over the discrete half-plane $\Pi_+ := \{(i,j) \in \mathbb{Z}^2 : i + j \geq 0\}$, the initial conditions for the state may be freely assigned on the (discrete) diagonal line $\mathcal{L}_0 := \{(i,j) \in \mathbb{Z}^2 : i + j = 0\}$.

However, for systems defined over the whole $\mathbb{Z}^2$ grid, it is not possible to assign arbitrary values for the state on the line $\mathcal{L}_0$ even if one is only interested in computing the values of the state on the half-plane Π_+. This is due to the fact that the values of the state on $\mathcal{L}_0$ must correspond to a state trajectory defined over $\mathbb{Z}^2$, i.e., also defined over the discrete half-plane $\Pi_- := \{(i,j) \in \mathbb{Z}^2 : i + j < 0\}$. When this is the case, we say that the state values on $\mathcal{L}_0$ are feasible initial conditions.

Here we determine the set of feasible initial conditions and express it in an easy way in terms of the matrices of the 2D Fornasini-Marchesini model.

The paper is organized as follows. In Section 2 we present the relevant preliminary definitions and results. Section 3 is devoted to the determination of the set of feasible initial conditions. Finally, Section 4 contains our concluding remarks.

2. PRELIMINARIES

We consider discrete 2D state-space systems described by the Fornasini-Marchesini model

$$\begin{cases} x(i+1,j+1) = A_0x(i,j+1) + A_1x(i+1,j) \\ \qquad\qquad +B_0u(i,j+1) + B_1u(i+1,j) & (1) \\ y(i,j) = Cx(i,j) + Du(i,j)\ , \quad (i,j) \in \mathbb{Z}^2 & (2) \end{cases}$$

where the *(local) state*, x, takes values on $\mathbb{R}^n$, and the *input*, u, and the *output*, y, take values on $\mathbb{R}^m$ and $\mathbb{R}^p$, respectively, Fornasini and Marchesini [1985]. Moreover, A_0, A_1, B_0, B_1, C and D are real matrices with appropriate sizes.

Introducing the *horizontal shift-operator*, σ_1, defined by $\sigma_1 w(i,j) = w(i+1,j)$, and the *vertical shift-operator*, σ_2, defined by $\sigma_2 w(i,j) = w(i,j+1)$, for any $\mathbb{Z}^2$-sequence w and every $(i,j) \in \mathbb{Z}^2$, equations (1) and (2) may be written as:

$$\begin{cases} \sigma_1\sigma_2 x = (A_0\sigma_2 + A_1\sigma_1)x + (B_0\sigma_2 + B_1\sigma_1)u & (3) \\ \qquad y = Cx + Du, & (4) \end{cases}$$

where, for simplicity, we left out the point (i, j).

⋆ This work is supported by The Center for Research and Development in Mathematics and Applications (CIDMA) through the Portuguese Foundation for Science and Technology (FCT - Fundação para a Ciência e a Tecnologia), references UIDB/04106/2020 and UIDP/04106/2020 and by Base Funding - UIDB/00147/2020 and Programmatic funding - UIDP/00147/2020 of the Systems and Technologies Center - SYSTEC - funded by national funds through the FCT/MCTES (PIDDAC)

Since we are only interested in studying the (local) state trajectories, we concentrate on equation (3). Note that, due to the fact that the system defined by (3) is shift-invariant, this equation is equivalent to:

$$\sigma_1 x = (A_0 + A_1\sigma)x + (B_0 + B_1\sigma)u, \tag{5}$$

where $\sigma := \sigma_2^{-1}\sigma_1$, which we shall write as:

$$\sigma_1 x = A(\sigma)x + B(\sigma)u, \tag{6}$$

with $A(\sigma) := A_0 + A_1\sigma$ and $B(\sigma) := B_0 + B_1\sigma$. The system described by (6) will be denoted by $\Sigma(A(\sigma), B(\sigma))$.

The reason why the variable x is called a local state is that, contrary to what happens in the 1D case, the knowledge of $x(i,j)$ (together with the input values) does not allow to compute its "future" values. For instance, in order to compute the values of x on the half-plane $\Pi_+ := \{(i,j) \in \mathbb{Z}^2 : i + j \geq 0\}$, all the values of x on the diagonal line $\mathcal{L}_0 := \{(i,j) \in \mathbb{Z}^2 : i + j = 0\}$ (together with the input values on Π_+) must be available. Based on this information, equation (6) allows computing all the local state values on the line $\mathcal{L}_1 := \{(i,j) \in \mathbb{Z}^2 : i + j = 1\}$, and so on. Thus, the lines $\mathcal{L}_k := \{(i,j) \in \mathbb{Z}^2 : i + j = k\}$ can be viewed as "propagation fonts" for the state-space trajectories.

The infinite sequence of values of the local states on a line $\mathcal{L}_k$, $X_k := \big(x(i, k-i)\big)_{i\in\mathbb{Z}}$, is called the *global state*. Analogously, the (infinite) sequence of values of the inputs on a line $\mathcal{L}_k$, $U_k := \big(u(i, k-i)\big)_{i\in\mathbb{Z}}$, is called the *global input*. The set of all $\mathbb{R}^n$-valued sequences $\mathcal{X} := \{\xi : \mathbb{Z} \to \mathbb{R}^n\} =: (\mathbb{R}^n)^{\mathbb{Z}}$ constitutes the *global state-space*. Note that, here, the role of the global state is similar to the one of the state in the 1D case.

It obviously follows from (6) that the updating equation for the global state is

$$X_{k+1} = A(\sigma)X_k + B(\sigma)U_k, \;\; k \in \mathbb{Z}, \tag{7}$$

where the shift σ operates along the lines $\mathcal{L}_k$. The system defined by equation (7) is denoted by $\Sigma_g(A(\sigma), B(\sigma))$.

3. FEASIBLE INITIAL CONDITIONS

As mentioned in the Introduction, contrary to what happens for systems defined over the half-plane Π_+, for systems defined over the whole grid $\mathbb{Z}^2$ it may happen that not all combinations of local states on a line $\mathcal{L}_k$ (or equivalently, due to shift-invariance, on the line $\mathcal{L}_0$) are possible for the solutions of (6). In terms of the global state, this means that it is not guaranteed that every global state $X^* \in \mathcal{X}$ is "visited" by a global state trajectory of the system $\Sigma_g(A(\sigma), B(\sigma))$ defined by (7).

Definition 3.1. Let $\Sigma_g(A(\sigma), B(\sigma))$ be defined by (7), and consider the corresponding global state space $\mathcal{X}$. A global state $X^* \in \mathcal{X}$ is said to be *feasible* if there exist a solution (X, U) of (7) and $k^* \in \mathbb{Z}$ such that $X_{k^*} = X^*$.

Remark 3.1. Clearly, as mentioned above, due to shift-invariance, the requirement that there exists $k^* \in \mathbb{Z}$ such that $X_{k^*} = X^*$ can be replaced by the condition that $X_0 = X^*$.

Definition 3.2. Let $\Sigma(A(\sigma), B(\sigma))$ be a 2D state-space system defined by (6), $\Sigma_g(A(\sigma), B(\sigma))$ be the system that describes the corresponding global state evolution and $\mathcal{X}$ be the associated global state-space. $\mathcal{X}$ and $\Sigma_g(A(\sigma), B(\sigma))$ are said to be *trim* if every global state $X^* \in \mathcal{X}$ is feasible. In this case, the system $\Sigma(A(\sigma), B(\sigma))$ is said to be *globally-trim*.

Remark 3.2. This nomenclature is inspired by the one used in Willems [1991].

Clearly, the updating equation (7) allows computing the values of the global states X_k, with $k > 0$, given the global inputs U_k on Π_+ and an arbitrary initial global state X_0. So, forward updating does not arise any problems; the non-feasibility of a global state X^* can only follow from the fact that there are no values for the global state X_k and for the global input U_k, with $k < 0$, that are able to "produce" the global state $X_0 = X^*$.

Writing equation (7) as

$$X_{k+1} = [A(\sigma) \;\; B(\sigma)] \begin{bmatrix} X_k \\ U_k \end{bmatrix}, \tag{8}$$

and taking into account that a polynomial (matrix) shift-operator $R(\sigma) : \big(\mathbb{R}^{l_1}\big)^{\mathbb{Z}} \to \big(\mathbb{R}^{l_2}\big)^{\mathbb{Z}}$ is surjective if and only if the corresponding polynomial matrix $R(z)$ (with entries in the ring $\mathbb{R}[z]$ of polynomials in z with real coefficients) has full row rank over $\mathbb{R}[z]$, it is not difficult to obtain the following result.

Proposition 3.1. $\Sigma(A(\sigma), B(\sigma))$ is globally-trim if and only if the polynomial matrix $[A(z) \;\; B(z)]$ has full row rank (over $\mathbb{R}[z]$).

Remark 3.3. Since the number of rows of $[A(z) \;\; B(z)]$ is equal to the local state-space dimension n, the necessary and sufficient condition of the proposition is obviously equivalent to saying that rank$\,[A(z) \;\; B(z)] = n$.

In the case where rank$\,[A(z) \;\; B(z)] < n$, $\Sigma(A(\sigma), B(\sigma))$ fails to be globally-trim, meaning that not every global state $X^* \in \mathcal{X}$ is feasible, or, in other words, not all initial conditions $(x(i,-i))_{i\in\mathbb{Z}}$ assigned on $\mathcal{L}_0$ are feasible.

Definition 3.3. Let $\Sigma(A(\sigma), B(\sigma))$ be the 2D state-space system defined by (6), and let $\mathcal{X}$ be the corresponding global state-space. We define the *globally-trim subspace* of $\Sigma(A(\sigma), B(\sigma))$ as the largest subspace $\mathcal{T}$ of $\mathcal{X}$ such that for all $X^* \in \mathcal{T}$ there exists a solution (X, U) of (7) (and hence a solution (x, u) of (6)) such that $X_0 = X^*$ (or, equivalently, $(x(i,-i))_{i\in\mathbb{Z}} = X^*$).

Note that, although Proposition 3.1 states that the global trimness of a system $\Sigma(A(\sigma), B(\sigma))$ corresponds to the surjectivity of the operator $[A(\sigma)\ B(\sigma)]$, the globally-trim subspace $\mathcal{T}$ of the system does not necessarily coincide with the image of this operator, $\operatorname{im}[A(\sigma)\ B(\sigma)]$. This is illustrated in the following simple example.

Example 3.1. Let $\Sigma(A(\sigma), B(\sigma))$ be a 2D state-space system with

$$A(\sigma) = \begin{bmatrix} 0 & 0 \\ \sigma+1 & 0 \end{bmatrix} \quad , \quad B(\sigma) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and global state-space $\mathcal{X} = \left(\mathbb{R}^2\right)^{\mathbb{Z}^2}$. Clearly

$$\operatorname{im}[A(\sigma)\ B(\sigma)] = \operatorname{im} \begin{bmatrix} 0 & 0 & 0 \\ \sigma+1 & 0 & 0 \end{bmatrix}$$

$$= \operatorname{im} \begin{bmatrix} 0 \\ \sigma+1 \end{bmatrix} = \operatorname{im} \begin{bmatrix} 0 \\ 1 \end{bmatrix} ,$$

since the operator $[A(\sigma)\ B(\sigma)]$ acts on $\left(\mathbb{R}^2\right)^{\mathbb{Z}^2} \times \mathbb{R}^{\mathbb{Z}^2}$. Therefore, the global state $X^* = (x(i,-i))_{i\in\mathbb{Z}}$ such that $x(i,-i) \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for $i \in \mathbb{Z}$ belongs to $\operatorname{im}[A(\sigma)\ B(\sigma)]$. Suppose now that X^* is feasible, i.e., (in this case where there are no inputs) suppose that there exists a global state system trajectory X_k ($k \in \mathbb{Z}$) such that $X_0 = X^*$. Then:

$$X_0 = \begin{bmatrix} 0 & 0 \\ \sigma+1 & 0 \end{bmatrix} X_{-1}$$

for some global state $X_{-1} = \begin{bmatrix} x_1(i,-1-i) \\ x_2(i,-1-i) \end{bmatrix}, i \in \mathbb{Z}$, which is equivalent to:

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \sigma+1 \end{bmatrix} x_1(i,-1-i) \ \text{ and } \ x_2(i,-1-i) \in \mathbb{R},\ i \in \mathbb{Z}.$$

But this implies that $x_1(i,-1-i) \not\equiv 0$. Hence $X_{-1} \notin \operatorname{im}[A(\sigma)\ B(\sigma)] = \operatorname{im} A(\sigma)$ and, therefore, there is no global state X_{-2} such that:

$$X_{-1} = A(\sigma)X_{-2}.$$

Consequently, there is no global state trajectory X_k, $k \in \mathbb{Z}$, such that $X_0 = X^*$. Thus $X^* \in \operatorname{im}[A(\sigma)\ B(\sigma)]$ but $X^* \notin \mathcal{T}$, showing that $\mathcal{T} \neq \operatorname{im}[A(\sigma)\ B(\sigma)]$. □

In order to characterize the globally-trim subspace $\mathcal{T}$ of a system $\Sigma(A(\sigma), B(\sigma))$, recall that $X^* \in \mathcal{T}$ if and only if there exists a sequence (X_{-k}, U_{-k}), $k \in \mathbb{N}$, of global states and global inputs such that:

$$\begin{aligned} X^* &= X_0 = A(\sigma)X_{-1} + B(\sigma)U_{-1} \\ &\quad X_{-p+1} = A(\sigma)X_{-p} + B(\sigma)U_{-p}\ ,\ p = 2,3,4,\ldots. \end{aligned}$$

This yields:

$$X^* \in \operatorname{im}\left[A^p(\sigma)\ A^{p-1}(\sigma)B(\sigma)\ \ldots\ A(\sigma)B(\sigma)\ B(\sigma)\right], \forall p \in \mathbb{N}. \tag{9}$$

In order to analyse (9), we introduce the following lemmas.

Lemma 3.1. Let $A(z)$ and $B(z)$ be two polynomial matrices of sizes $n \times n$ and $n \times m$, respectively, and, for $p \in \mathbb{N}$, denote by $R_p(\sigma)$ the shift-operator defined by:

$$R_p(\sigma) := \left[A^p(\sigma)\ A^{p-1}(\sigma)B(\sigma)\ \ldots\ A(\sigma)B(\sigma)\ B(\sigma)\right]. \tag{10}$$

Then, $\operatorname{im} R_{p+1}(\sigma) \subseteq \operatorname{im} R_p(\sigma)$.

Proof. Clearly,

$$\begin{aligned} &\operatorname{im}\left[A^{p+1}(\sigma)\ A^p(\sigma)B(\sigma)\ \ldots\ A(\sigma)B(\sigma)\ B(\sigma)\right] \\ &= \operatorname{im}\left[A^p(\sigma)\left[A(\sigma)\ B(\sigma)\right]\right] + \operatorname{im}\left[A^{p-1}(\sigma)B(\sigma)\ \ldots\ B(\sigma)\right] \\ &= A^p(\sigma)\operatorname{im}\left[A(\sigma)\ B(\sigma)\right] + \operatorname{im}\left[A^{p-1}(\sigma)B(\sigma)\ \ldots\ B(\sigma)\right] \\ &\subseteq \operatorname{im} A^p(\sigma) + \operatorname{im}\left[A^{p-1}(\sigma)B(\sigma)\ \ldots\ B(\sigma)\right] \\ &= \operatorname{im}\left[A^p(\sigma)\ A^{p-1}(\sigma)B(\sigma)\ \ldots\ B(\sigma)\right], \end{aligned}$$

as $A^p(\sigma)\operatorname{im}[A(\sigma)\ B(\sigma)] \subseteq \operatorname{im} A^p(\sigma)$ □

Thus,

$$\operatorname{im} R_1(\sigma) \supseteq \operatorname{im} R_2(\sigma) \supseteq \ldots \supseteq \operatorname{im} R_p(\sigma) \supseteq \operatorname{im} R_{p+1}(\sigma) \supseteq \ldots$$

is a decreasing sequence of subspaces. The next lemma states that this sequence has a lower bound (in the sense of subspace inclusion).

Lemma 3.2. With the same notation and assumptions on $A(z)$ and $B(z)$ as in Lemma 3.1:

$$\operatorname{im} R_n(\sigma) \subseteq \operatorname{im} R_p(\sigma),\ \forall p \in \mathbb{N}. \tag{11}$$

Proof. It immediately follows from Lemma 3.1 that the inclusion in (11) is satisfied for all $p \leq n$. Let then $p > n$; it follows from the Cayley-Hamilton theorem for polynomial matrices $A(z)$, see Fragulis [1995], that:

$$\begin{aligned} &\operatorname{im}\left[A^p(\sigma)\ A^{p-1}(\sigma)B(\sigma)\ \ldots\ B(\sigma)\right] \\ &= \operatorname{im} A^p(\sigma) + \operatorname{im}\left[A^{p-1}(\sigma)B(\sigma)\ \ldots\ B(\sigma)\right] \\ &= \operatorname{im} A^p(\sigma) + \operatorname{im}\left[A^{n-1}(\sigma)B(\sigma)\ \ldots\ B(\sigma)\right]. \end{aligned} \tag{12}$$

Now, in order to obtain the desired result, it remains to prove that

$$\operatorname{im} A^p(\sigma) = \operatorname{im} A^n(\sigma),\ \forall p > n. \tag{13}$$

For this purpose, we start by noting that (for $r \in \mathbb{N}$):

$$\operatorname{im} A^{r+1}(\sigma) = \operatorname{im} A^r(\sigma) \Rightarrow \operatorname{im} A^q(\sigma) = \operatorname{im} A^r(\sigma),\ \forall q \geq r. \tag{14}$$

Indeed, if $\operatorname{im} A^{r+1}(\sigma) = \operatorname{im} A^r(\sigma)$, then $A(\sigma)\big(\operatorname{im} A^{r+1}(\sigma)\big) = A(\sigma)\big(\operatorname{im} A^r(\sigma)\big)$, which implies that $\operatorname{im} A^{r+2}(\sigma) = \operatorname{im} A^{r+1}(\sigma) = \operatorname{im} A^r(\sigma)$. Now, a simple induction procedure shows that implication (14) holds true. Therefore, (13) will follow if we prove that there exists indeed $r \in \mathbb{N}$ such that $r \leq n$ and $\operatorname{im} A^{r+1}(\sigma) = \operatorname{im} A^r(\sigma)$.

Let $\rho_k := \operatorname{rank} A^k(z)$ (over $\mathbb{R}[z]$). Since $\rho_{k+1} \leq \rho_k$ and $\rho_k \leq n$, there must exist an $r \leq n$ such that $\rho_{r+1} = \rho_r$.

Moreover, because rank $A^r(z) = \rho_r$, there exists a unimodular [1] polynomial matrix $U(z)$ such that:

$$U(z)A^r(z) = \begin{bmatrix} F(z) \\ \hline 0 \end{bmatrix},$$

where $F(z)$ has ρ_r rows and has full row rank. This implies that:

$$U(z)A^{r+1}(z) = \begin{bmatrix} F(z) \\ \hline 0 \end{bmatrix} A(z) = \begin{bmatrix} F(z)A(z) \\ \hline 0 \end{bmatrix},$$

and, since:

$$\begin{aligned} \operatorname{rank}\left(U(z)A^{r+1}(z)\right) &= \operatorname{rank} A^{r+1}(z) \\ &= \rho_{r+1} = \rho_r = \operatorname{rank} A^r(z), \end{aligned}$$

the polynomial matrix $F(z)A(z)$ (which, like $F(z)$, has ρ_r rows) has full row rank.

Thus:

$$\operatorname{im}\left(U(\sigma)A^r(\sigma)\right) = U(\sigma)\operatorname{im} A^r(\sigma) = \operatorname{im}\begin{bmatrix} F(\sigma) \\ \hline 0 \end{bmatrix} = \begin{bmatrix} (\mathbb{R}^r)^{\mathbb{Z}} \\ \hline 0 \end{bmatrix}$$

implying that

$$\operatorname{im} A^r(\sigma) = U^{-1}(\sigma)\begin{bmatrix} (\mathbb{R}^r)^{\mathbb{Z}} \\ \hline 0 \end{bmatrix}. \tag{15}$$

Analogously:

$$\begin{aligned} \operatorname{im}\left(U(\sigma)A^{r+1}(\sigma)\right) &= U(\sigma)\operatorname{im} A^{r+1}(\sigma) \\ &= \operatorname{im}\begin{bmatrix} F(\sigma)A(\sigma) \\ \hline 0 \end{bmatrix} = \begin{bmatrix} (\mathbb{R}^r)^{\mathbb{Z}} \\ \hline 0 \end{bmatrix} \end{aligned}$$

which yields:

$$\operatorname{im} A^{r+1}(\sigma) = U^{-1}(\sigma)\begin{bmatrix} (\mathbb{R}^r)^{\mathbb{Z}} \\ \hline 0 \end{bmatrix}. \tag{16}$$

Comparing (15) and (16) we conclude that $\operatorname{im} A^{r+1}(\sigma) = \operatorname{im} A^r(\sigma)$, and hence (13) indeed holds true. This completes the proof of the lemma. □

Using Lemmas 3.1 and 3.2 as well as condition (9), which is necessary and sufficient for a global state X^* to belong to the globally-trim subspace $\mathcal{T}$, we obtain the desired characterization for $\mathcal{T}$ in terms of the system matrices.

Theorem 3.1. Let $\Sigma(A(\sigma), B(\sigma))$ be a 2D state-space system described by (6). Then, the corresponding globally-trim subspace $\mathcal{T}$ is given by:

$$\mathcal{T} = \operatorname{im} R_n(\sigma)$$

with $R_n(z) := \left[A^n(z)\ A^{n-1}(z)B(z)\ \ldots\ A(z)B(z)\ B(z)\right]$.

Remark 3.4. Note that $\mathcal{T}$ can also be written as

$$\mathcal{T} = \operatorname{im} A^n(\sigma) + \operatorname{im}\left[B(\sigma)\ A(\sigma)B(\sigma)\ \ldots\ A^{n-1}(\sigma)B(\sigma)\right],$$

where $\left[B(\sigma)\ A(\sigma)B(\sigma)\ \ldots\ A^{n-1}(\sigma)B(\sigma)\right]$ is the *global reachability matrix* associated with $\Sigma(A(\sigma), B(\sigma))$, cf. Fornasini and Marchesini [1985]. Moreover, is easily follows from our previous considerations that $\mathcal{T}$ is the smallest $A(\sigma)$-invariant subspace of $(\mathbb{R}^n)^{\mathbb{Z}}$ that contains both $\operatorname{im} A^n(\sigma)$ and $\operatorname{im} B(\sigma)$.

Clearly, the globally-trim subspace $\mathcal{T}$ is the set of all *feasible initial conditions*, $X_0 = (x(i,-i))_{i\in\mathbb{Z}}$, that can be assigned on the line $\mathcal{L}_0$ in order to compute a state solution of (5) (or (1)) on the half-plane Π_+.

Example 3.2. Let $\Sigma(A(\sigma), B(\sigma))$ be a 2D state-space system such that:

$$A(\sigma) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \sigma+1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma-1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma+2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } B(\sigma) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \sigma+3 \end{bmatrix}.$$

The globally-trim subspace for this system is

$$\begin{aligned} \mathcal{T} &= \operatorname{im} A^5(\sigma) + \operatorname{im}\left[B(\sigma)\ A(\sigma)B(\sigma)\ A^2(\sigma)B(\sigma)\ A^3(\sigma)B(\sigma)\ A^4(\sigma)B(\sigma)\right] \\ &= \operatorname{im}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (\sigma-1)^5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \operatorname{im}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & (\sigma+2)(\sigma+3) & 0 & 0 & 0 \\ \sigma+3 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \operatorname{im}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (\sigma-1)^5 & 0 & 0 \\ 0 & 0 & (\sigma+2)(\sigma+3) \\ 0 & \sigma+3 & 0 \end{bmatrix} = \operatorname{im}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \end{aligned}$$

where the last equality results from the fact that the images are taken over $\mathbb{R}^{\mathbb{Z}}$ rather than over the ring $\mathbb{R}[\sigma]$. Similar to what happens in Example 3.1, here $\mathcal{T} \neq \operatorname{im}\left[A(\sigma)\ B(\sigma)\right]$; moreover

$$\mathcal{T} \neq \operatorname{im}\left[B(\sigma)\ A(\sigma)B(\sigma)\ \ldots\ A^4(\sigma)B(\sigma)\right],$$

which is known as the global reachability subspace of $\Sigma(A(\sigma), B(\sigma))$, see Fornasini and Marchesini [1985]. □

Remark 3.5. An alternative way to determine $\mathcal{T}$ would be to apply behavioral approach techniques, Willems [1991], Zerz [2000]. For this purpose, Equation (6) is written as $H(\sigma_1, \sigma)w = 0$ with $H(\sigma_1, \sigma) = [\sigma_1 I_n - A(\sigma) \mid -B(\sigma)]$ and $w = \left[x^\top\ u^\top\right]^\top$; then, w is a solution of $H(\sigma_1, \sigma)w = 0$ if and only if $r(\sigma_1, \sigma)w = 0$ for every row $r(z_1, z)$ belonging to the $\mathbb{R}[z_1, z]$-module generated by the rows of $H(z_1, z)$, $RM(H(z_1, z))$, Zerz [2000]. Using computer algebra methods, it is possible to determine the submodule RM^* formed by the rows of $RM(H(z_1, z))$ of the form $r(z_1, z) = [r(z) \mid 0]$, where $r(z)$ has n columns, as well as a matrix $[T(z) \mid 0]$ that generates RM^*. The feasible initial conditions, X^*, for the global state are the ones such that $T(\sigma)X^* = 0$. This is equivalent to saying that $\mathcal{T} = \ker T(\sigma)$ (with $T(\sigma)$ viewed as a polynomial matrix shift operator acting on $(\mathbb{R}^n)^{\mathbb{Z}}$), Napp and Rocha [2010].

[1] Recall that a square polynomial matrix $U(z)$ is said to be unimodular if it is invertible as an element of the ring of polynomial matrices with the same size as $U(z)$.

However, here we chose not to follow this approach as it does not explicitly give $\mathcal{T}$ in terms of the system matrices $A(\sigma)$ and $B(\sigma)$.

Remark 3.6. As already mentioned, due to shift-invariance, $\mathcal{T}$ is also the set of feasible states on each line $\mathcal{L}_k$, $k \in \mathbb{Z}$. This has implications in the way the concept of global reachability should be defined. Indeed, for systems $\Sigma(A(\sigma), B(\sigma))$ defined over Π_+, a global state $X^* \in (\mathbb{R}^n)^{\mathbb{Z}}$ (where n is the local state-space dimension) is said to be reachable if there is a solution (X, U) of the associated global state-space system $\Sigma_g(A(\sigma), B(\sigma))$ such that $X_0 = 0$ and $X_{k^*} = X^*$ for some $k^* \in \mathbb{N}$. However, for systems $\Sigma(A(\sigma), B(\sigma))$ defined over the whole grid $\mathbb{Z}^2$, it makes more sense to define global reachability taking into account that not all global states $X^* \in (\mathbb{R}^n)^{\mathbb{Z}}$ are feasible. This question will be addressed in our future work.

4. CONCLUDING REMARKS

In this paper we have considered discrete 2D systems, defined over the whole grid $\mathbb{Z}^2$, described by a Fornasini-Marchesini model with state updating matrices $A(\sigma) = A_0 + A_1\sigma$ and $B(\sigma) = B_0 + B_1\sigma$. For such systems, not all global states $X^* \in (\mathbb{R}^n)^{\mathbb{Z}}$ are feasible, in the sense that they are “visited” by a global state trajectory. Thus, we defined the globally-trim subspace $\mathcal{T}$ as the set of all feasible global states and gave a characterization of $\mathcal{T}$ in terms of the matrices $A(\sigma)$ and $B(\sigma)$.

Our approach is more intuitive than the alternative behavioral approach, which does not explicitly express $\mathcal{T}$ in terms of the system matrices.

Possible consequences of the non-feasibility of all global states on the definition of global reachability for systems defined over $\mathbb{Z}^2$ will be investigated in our future work.

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