Log-majorization type inequalities

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Abstract. Several inequalities have been established in the context of Hilbert spaces operators or operators algebras. Our discussion will be limited to matrices. Important inequalities in mathematics and other sciences, such as Golden-Thompson inequality or von Neumann trace inequality, and extensions, are revisited. Our main goal is to emphasize the link between majorization theory and other relevant inequalities.

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1. Notation

$\mathbb{N}$	set of natural numbers
$\mathbb{N}_0$	set of nonnegative integer numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}_0^+$	set of nonnegative real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{R}^n$	vector space of real n-tuples
$\mathbb{C}^n$	vector space of complex n-tuples
$\lVert \cdot \rVert$	Euclidean norm; spectral norm or operator norm
$\vert\vert\vert \cdot \vert\vert\vert$	unitarily invariant norm
$\lVert \cdot \rVert_{(k)}$	Ky Fan k-norm
$\lVert \cdot \rVert_p$	Schatten p-norm
$\lVert \cdot \rVert_2$	Frobenius norm, Hilbert-Schmidt norm or Schur norm
$M_n(\mathbb{C})$	algebra of $n \times n$ complex matrices
$M_{m\times n}(\mathbb{C})$	vector space of $m \times n$ complex matrices
Ω_n	set of $n \times n$ doubly stochastic matrices
$A = (a_{ij})$	matrix A with entries a_{ij}
A^*	adjoint of a matrix A
A^T	transpose of a matrix A
$\overline{A}$	entrywise conjugate of A
$\vert A\vert$	unique positive semidefinite square root of A^*A
$A^{\wedge k}$	kth compound or kth antisymmetric tensor power of A
$\rho(A)$	spectral radius of A
$f(A)$	functional calculus applied to a function f
$A \geq 0$	a positive semidefinite matrix A
$A > 0$	a positive definite matrix A
$A \geq B$	$A - B \geq 0$
ReA (ImA)	real (imaginary) part of A
$\mathrm{tr}(A)$	trace of a matrix A
$\det(A)$	determinant of a matrix A
$\lambda_i(A)$	eigenvalue of A
$\lambda_1(A)$	largest eigenvalue of A if A is Hermitian
$s_i(A)$	singular value of A
$s_1(A)$	largest singular value of A
I_n	identity matrix of order n
$A \circ B$	Hadamard product of matrices A and B
$\vert x\vert$	absolute value vector $(\vert x_1\vert, \ldots, \vert x_n\vert)$
$x \prec y$	x is majorized by y
$x \prec_w y$	x is weakly majorized by y
$x \prec_{\log} y$	x is log-majorized by y
$x \prec_{w\log} y$	x is weakly log-majorized by y
$\sharp_\alpha$	α-weighted geometric mean for $\alpha \in [0,1]$
$\sharp$	geometric mean
σ	operator connection
$\sigma^\perp$	dual of an operator connection σ
f_σ	representing function of an operator connection σ

S_n	symmetric group of degree n
$S(A,B)$	Umegaki relative entropy
$X^{\sim}$	X or X^T
H_n	set of $n\times n$ Hermitian matrices
H_n^T	set of $n\times n$ symmetric matrices
$\operatorname{per} A$	permanent of a matrix A
J	Hermitian involutive matrix
$\sigma_J^{\pm}(A)$	set of eigenvalues with eigenvectors x, such that $x^*Jx=\pm 1$
$A\geq^J B$	$J(A-B)\geq 0$

2. Introduction

The concept of majorization was introduced by Hardy, Littlewood and Pólya [43]. Since then various majorizations were obtained for the eigenvalues and singular values of matrices and compact operators [72]. These majorizations are powerful devices for the derivation of several norm inequalities, as well as trace or determinant inequalities for matrices or operators. In this section, we review in a concise way the majorization theory used throughout this chapter.

Any vector $x=(x_1,\dots,x_n)\in\mathbb{R}^n$ is assumed to have its components sorted in non-increasing order, that is, $x_1\geq\cdots\geq x_n$.

Let $x,y\in\mathbb{R}^n$. We say that x is *majorized* by y and write $x\prec y$ if

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i, \qquad k=1,\dots,n, \tag{2.1}$$

and equality occurs in (2.1) for $k=n$. Further, if (2.1) holds, then x is said to be *weakly majorized* or *submajorized* by y and the notation $x\prec_w y$ is used. We remark that $x\prec y$ is equivalent to

$$\sum_{i=k}^{n} x_i \geq \sum_{i=k}^{n} y_n, \qquad k=1,\dots,n, \tag{2.2}$$

with equality in (2.2) for $k=1$. If (2.2) holds, then x is said to be *supermajorized* by y and we write $x\prec^w y$.

Naively, vector majorization means that one vector is more *disordered* than the other. For instance, a physics interpretation may be that x describes a more chaotic state than y, thinking of x_i as the probability of the system described by x being in state i.

Two important resources on the topic of majorization are [21, 72].

A square matrix with non-negative entries is called *doubly stochastic* if all its row and column sums are one. The class Ω_n of doubly stochastic matrices of order n is a convex set, whose extreme points are the permutation matrices as stated by the famous Birkhoff's Theorem [24]. In fact, there is a close relation between majorization and doubly stochastic matrices [72].

Proposition 2.1. *A matrix $A\in\Omega_n$ if and only if $Ax\prec x$ for all $x\in\mathbb{R}^n$.*

Proposition 2.2. *For $x, y \in \mathbb{R}^n$, the following statements are equivalent:*

i. $x \prec y$*;*
ii. *x is in the convex hull of all the vectors obtained by permutating the coordinates of y;*
iii. $x = A\,y$ *for some* $A \in \Omega_n$*.*

For any real valued function f defined on an interval, containing all the components of the real n-tuple x, we adopt the notation

$$f(x) = \big(f(x_1), \ldots, f(x_n)\big).$$

Proposition 2.3. *Let $x, y \in \mathbb{R}^n$ and f be a convex function on an interval containing all the components of x and y. Then*

i. *If $x \prec y$, then $f(x) \prec_w f(y)$.*
ii. *If $x \prec_w y$ and f is also non-decreasing, then $f(x) \prec_w f(y)$.*

Log-majorization can be defined as a multiplicative version of majorization. If $x, y \in \mathbb{R}^n$ have nonnegative components, $x \prec_{\log} y$ means that

$$\prod_{i=1}^{k} x_i \leq \prod_{i=1}^{k} y_i, \qquad k = 1, \ldots, n, \tag{2.3}$$

and equality occurs in (2.3) for $k = n$. If $x, y > 0$, i.e., all the components of x, y are positive, this is clearly equivalent to

$$\prod_{i=k}^{n} x_i \geq \prod_{i=k}^{n} y_n, \qquad k = 1, \ldots, n, \tag{2.4}$$

with equality in (2.4) for $k = 1$. If $x, y > 0$, then

$$x \prec_{\log} y \quad \Leftrightarrow \quad \log x \prec \log y,$$

this justifying the log-majorization terminology. When equality between the products of all the componentes of x and y is not required, the following parallel notations are used:

$$x \prec_{w\log} y \quad \text{for} \quad (2.3) \qquad \text{and} \qquad x \prec^{w\log} y \quad \text{for} \quad (2.4).$$

Proposition 2.4. *Let $x, y \in \mathbb{R}^n$ have all the components positive and f be a non-decreasing continuous function on an interval containing all the components of x, y, such that $f(\mathrm{e}^t)$ is convex. Then*

$$x \prec_{w\log} y \quad \Rightarrow \quad f(x) \prec_w f(y).$$

In particular, $f(t) = t$ in the previous proposition shows that the weak log-majorization $\prec_{w\log}$ is stronger than the weak majorization $\prec_w$.

3. Matrix majorization

If $A = (a_{ij}), B = (b_{ij})$ are $m \times n$ complex matrices, let $A \circ B = (a_{ij}b_{ij})$ be the Hadamard product of A and B. Let $M_n(\mathbb{C})$ be the algebra of n-square complex matrices and I_n be the identity matrix of order n. If $A \in M_n(\mathbb{C})$, then its eigenvalues are denoted by $\lambda_1(A), \dots, \lambda_n(A)$ and

$$\rho(A) = \max_{i=1,\dots,n} |\lambda_i(A)|$$

is the *spectral radius* of A. Further, considering the Euclidean norm $\|x\|$ of a vector $x \in \mathbb{C}^n$, let

$$\|A\| = \max_{\|x\|=1} \|Ax\|$$

be the *spectral norm* or *operator norm* of A. It is clear that

$$\rho(A) \le \|A\|. \tag{3.1}$$

If $A \in M_n(\mathbb{C})$ has real eigenvalues, denote by $\lambda(A)$ the n-tuple of eigenvalues of A arranged in non-increasing order:

$$\lambda_1(A) \ge \cdots \ge \lambda_n(A).$$

For $A \in M_n(\mathbb{C})$, the unique positive semidefinite square root of A^*A is denoted by $|A|$. The eigenvalues of $|A|$ are the singular values of A, which are arranged in the vector $s(A)$ as follows:

$$s_1(A) \ge \cdots \ge s_n(A).$$

A norm $|||\cdot|||$ in $M_n(\mathbb{C})$ is said to be *unitarily invariant* if $|||UAV||| = |||A|||$ for any $A, U, V \in M_n(\mathbb{C})$ with U, V unitary. Examples of unitarily invariant norms are the *Schatten p-norms* given by

$$\|A\|_p = \left(\sum_{i=1}^{n} s_i^p(A)\right)^{\frac{1}{p}} = \left(\operatorname{tr}|A|^p\right)^{\frac{1}{p}}, \qquad p \ge 1,$$

and the *Ky Fan k-norms* defined by

$$\|A\|_{(k)} = \sum_{i=1}^{k} s_i(A), \qquad k = 1, \dots, n,$$

including $\|A\| = s_1(A)$. The Schatten 2-norm

$$\|A\|_2 = \sqrt{\operatorname{tr}(A^*A)},$$

also called *Frobenius norm*, *Hilbert-Schmidt norm* or *Schur norm*, is the norm induced by to the *Frobenius* or *Hilbert-Schmidt inner product* in $M_n(\mathbb{C})$:

$$\langle A, B\rangle = \operatorname{tr}(B^*A).$$

The notion of majorization gives a mean for comparing two probability distributions or two density matrices, that is positive semidefinte matrices of trace one, using the eigenvalues, in an elegant way. It arises in fields like computer science, economics or quantum mechanics.

Important sources on majorization for eigenvalues and singular values of matrices are [21, 46, 47, 55, 72] and two survey articles of T. Ando [2, 3].

For simplicity, if $A, B \in M_n(\mathbb{C})$ have real eigenvalues, then $\lambda(A) \prec \lambda(B)$ and $\lambda(A) \prec_w \lambda(B)$ are abreviated to $A \prec B$ and $A \prec_w B$, respectively.

The main diagonal entries and the eigenvalues of a Hermitian matrix are related through majorization. This classical result due to I. Schur [84] can be briefly stated as follows.

Theorem 3.1. (Schur Majorization Theorem, 1923) *If $A \in M_n(\mathbb{C})$ is Hermitian, then $I_n \circ A \prec A$.*

In 1954, A. Horn [51] proved the converse, giving rise to the next fundamental result, which received considerable attention and led to generalizations in several directions.

Theorem 3.2. (Schur-Horn Theorem) *Let $x, y \in \mathbb{R}^n$. There exists a Hermitian matrix with prescribed diagonal entries and prescribed eigenvalues arranged, respectively, in x and y if and only if $x \prec y$.*

After this, Horn's subsequent work on the eigenvalues of sums of Hermitian matrices culminated in the inequalities conjectured in [53]. The solution to Horn's conjecture appeared in two papers, one by A. Klyachko [60] (1998) and the other one by A. Knutson and T. Tao [61] (1999).

Another relevant result in matrix majorization is due to Ky Fan [28].

Theorem 3.3. (Ky Fan Dominance Theorem, 1951) *Let $A, B \in M_n(\mathbb{C})$. Then the following are equivalent statements:*

- **i.** $|A| \prec_w |B|$;
- **ii.** $\|A\| \leq \|B\|$ *for any unitarily invariant norm* $\|\cdot\|$ *in* $M_n(\mathbb{C})$.

If $A, B \in M_n(\mathbb{C})$ have nonnegative eigenvalues, $A \prec_{\log} B$ stands for

$$\lambda(A) \prec_{\log} \lambda(B).$$

Abreviated notations for the weaker versions, involving either $\prec_{w\log}$ or $\prec^{w\log}$, are analogously used. Clearly, if A, B have positive eigenvalues, then

$$A \prec_{w\log} B \quad \Leftrightarrow \quad B^{-1} \prec^{w\log} A^{-1}.$$

Matrix log-majorization is a powerful tool for establishing trace, determinantal and matrix norm inequalities. For instance,

$$A \prec_{\log} B \quad \Rightarrow \quad \det(I_n + A) \leq \det(I_n + B).$$

On the other hand, some classical determinantal inequalities can find their majorization counterparts.

As usual, $A > 0$ means that A is a positive definite matrix and $A \geq B$ means that $A - B$ is a positive semidefinite matrix. Real-valued continuous functions f defined on a real interval Γ, such that

$$A \geq B \quad \Rightarrow \quad f(A) \geq f(B)$$

for all Hermitian $A, B \in M_n(\mathbb{C})$ with spectra in Γ and all $n \in \mathbb{N}$, are said to be *operator monotone* on Γ. A useful and fundamental tool for treating operator inequalities is *Löwner-Heinz inequality.* Löwner's original proof [69] used an integral representation for operator monotone functions and an alternative proof was given by Heinz [44]. It states that

$$A \geq B \geq 0 \quad \Rightarrow \quad A^{\alpha} \geq B^{\alpha}, \tag{3.2}$$

that is, $f(t) = t^{\alpha}$ is operator monotone on $\mathbb{R}_0^+$, for $\alpha \in [0,1]$. In general, (3.2) is not true for $\alpha > 1$.

For $k = 1, \ldots, n$ and $n_k = \binom{n}{k}$, the *kth compound* or *kth antisymmetric tensor power* of $A \in M_n(\mathbb{C})$ is the matrix $A^{\wedge k} \in M_{n_k}(\mathbb{C})$ with entries given by the minors $\det A(\mathbf{i}, \mathbf{j})$, where the index sets $\mathbf{i}, \mathbf{j} \subset \{1, \ldots, n\}$ have cardinality k and are lexicographically ordered. As usual, $A(\mathbf{i}, \mathbf{j})$ denotes the submatrix of A that lies in rows and columns indexed, respectively, by $\mathbf{i}, \mathbf{j}$. Some essential properties of these matrices [21] are listed below:

P1. $(AB)^{\wedge k} = A^{\wedge k} B^{\wedge k}$ (Binet-Cauchy formula);

P2. $\left(A^{\wedge k}\right)^* = (A^*)^{\wedge k}$;

P3. $\left(A^{\wedge k}\right)^r = (A^r)^{\wedge k}$, $r > 0$;

P4. $\left(A^{\wedge k}\right)^{-1} = (A^{-1})^{\wedge k}$ if A is invertible;

P5. $\lambda_{\mathbf{i}}\left(A^{\wedge k}\right) = \prod_{j=1}^{k} \lambda_{i_j}(A)$, where $\mathbf{i} = (i_1, \ldots, i_k)$ and $1 \leq i_1 < \cdots < i_k \leq n$;

P6. $\|A^{\wedge k}\| = s_1\left(A^{\wedge k}\right) = \prod_{j=1}^{k} s_j\,(A),\ k = 1, \ldots, n.$

Thus, a useful tool in log-majorization is provided by the next lemma.

Lemma 3.4. *Let $A, B \in M_n(\mathbb{C})$ have nonnegative eigenvalues. The following are equivalent:*

i. $A \prec_{\log} B$;

ii. $\lambda_1\left(A^{\wedge k}\right) \leq \lambda_1\left(B^{\wedge k}\right)$, $k = 1, \ldots, n$, *and* $\det(A) = \det(B)$.

A basic log-majorization in matrix theory is Weyl's relation between eigenvalues and singular values [98]. Let $|\lambda(A)|$ be the vector of the absolute values of the eigenvalues of $A \in M_n(\mathbb{C})$ arranged in non-increasing order of magnitude:

$$|\lambda_1(A)| \geq \cdots \geq |\lambda_n(A)|.$$

Theorem 3.5. (Weyl's Majorant Theorem, 1949) *If $A \in M_n(\mathbb{C})$, then*

$$|\lambda(A)| \prec_{\log} s(A). \tag{3.3}$$

Proof. Use properties **P5** and **P6**, after applying (3.1), that is,

$$\rho(A) = |\lambda_1(A)| \leq s_1(A).$$

to the kth antisymmetric tensor power of A, $k = 1, \dots, n$, and observe that

$$\left|\prod_{i=1}^{n} \lambda_i(A)\right| = |\det(A)| = \left(\overline{\det(A)}\det(A)\right)^{\frac{1}{2}} = \left(\det(A^*A)\right)^{\frac{1}{2}} = \det|A|$$

is the product of all the singular values of A. $\square$

In 1954, A. Horn proved the converse [52], that is, there exists a square matrix with prescribed eigenvalues and singular values arranged in vectors x and y if the log-majorization $|x| \prec_{\log} y$ is satisfied.

In the sequel, we illustrate the potential of using the previous antisymmetric tensor power technique, also called Weyl trick, by using Lemma 3.4 to derive some other log-majorization for expressions involving products and fractional matrix powers, having in mind that these “commute” with the kth antisymmetric tensor power. As a consequence, some known results will be revisited. Some classical inequalities for the trace and the determinant are meanwhile surveyed in the next section.

4. Trace and determinantal inequalities

The von Neumann’s trace inequality was first published in 1937 by von Neumann [96] with a complicated proof. Other proofs were given in 1959 and subsequently in 1975, based on doubly stochastic matrices, by Mirsky [76, 77]. However, these proofs only work in the finite dimensional case. A simple proof, which also extends to the infinite dimensional setting, was finally obtained in 1991 by R. D. Grigorieff [41].

Theorem 4.1. (von Neumann’s inequality, 1937) *Let $A, B \in M_n(\mathbb{C})$. Then*

$$|\mathrm{tr}(AB)| \leq \sum_{i=1}^{n} s_i(A)s_i(B)$$

and equality occurs if A, B share a joint set of singular vectors.

This result is an important tool with various applications in pure and applied mathematics. For instance, just to mention a few, it is useful in Schatten’s theory of cross spaces and in Ball’s approach of the equations of nonlinear elasticity. Inspired by this famous inequality, further singular value inequalities have been meanwhile derived, among them Horn’s multiplicative inequalities (see, e.g. [72]).

Theorem 4.2. (Horn, 1950) *If $A, B \in M_n(\mathbb{C})$, then*

$$s(AB) \prec_{\log} s(A) \circ s(B).$$

Proof. By the submultiplicativity of the operator norm, we have

$$s_1(AB) = \|AB\| \leq \|A\|\|B\| = s_1(A)s_1(B).$$

Apply the antisymmetric tensor power technique to the previous inequality, that is, replace A, B by their kth compounds, $k = 1, \ldots, n$, and use **P6**. Equality for $k = n$ is immediate by properties of the determinants. $\square$

Remark 4.3. For $A, B \in M_n(\mathbb{C})$, Horn and Weyl's log-majorizations stated before, the second applied to the product AB, imply the corresponding weak majorizations, so we easily find for $k = 1, \ldots, n$ that

$$\left|\sum_{i=1}^{k} \lambda_i(AB)\right| \leq \sum_{i=1}^{k} |\lambda_i(AB)| \leq \sum_{i=1}^{k} s_i(AB) \leq \sum_{i=1}^{k} s_i(A)s_i(B). \tag{4.1}$$

In particular, von Neumann trace inequality is obtained when $k = n$ in (4.1).

In 1958, Richter [81] proved a related trace inequality for the product of two Hermitian matrices. Other contributions in this vein are due to Marcus [70], Mirsky [76] and Theobald [88]. Ruhe [82] reobtained it under the more restrictive assumption of positive semidefiniteness of both matrices.

Theorem 4.4. *If $A, B \in M_n(\mathbb{C})$ are Hermitian, then*

$$\sum_{i=1}^{n} \lambda_i(A)\lambda_{n-i+1}(B) \leq \operatorname{tr}(AB) \leq \sum_{i=1}^{n} \lambda_i(A)\lambda_i(B). \tag{4.2}$$

We remark that the lower bound is an immediate consequence of the upper bound in (4.2) with the Hermitian matrix B replaced by $-B$, since $\lambda_i(-B) = -\lambda_{n-i+1}(B)$, $i = 1, \ldots, n$. Note that the previous inequality is a matrix version of the following classical rearrangement inequality [43]. Let S_n be the symmetric group of degree n of all permutations of $\{1, \ldots, n\}$.

Theorem 4.5. (Hardy-Littlewood-Pólya rearrangement inequality, 1929) *If $x, y \in \mathbb{R}^n$, then*

$$\sum_{i=1}^{n} x_i y_{n-i+1} \leq \sum_{i=1}^{n} x_i y_{\sigma(i)} \leq \sum_{i=1}^{n} x_i y_i$$

for any permutation $\sigma \in S_n$.

Having in mind that the trace of a matrix is the sum of the eigenvalues while the determinant is the product, we can think in "dual inequalities" in the sense of replacing sums by products and products by sums. In fact, the determinant of the sum of matrices has no simple relation with the determinants of the summands. We recall some inequalities in this avenue. We start with a remarkable result due to Fiedler [29], after the following remark.

Remark 4.6. A continuity argument will be repeatedly used, when possible, along the proof of some of the results, involving eigenvalues of Hermitian matrices. In such cases, we only need to prove the results for nonsingular matrices. Otherwise, we may replace in the inequalities each nonsingular matrix A by $A + \epsilon I_n$ for $\epsilon > 0$ and then take the limit as ϵ converges to 0.

Theorem 4.7. *If $A, B \in M_n(\mathbb{C})$ are Hermitian with eigenvalues $\alpha_1 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \cdots \geq \beta_n$, respectively, then*

$$\min_{\sigma \in S_n} \prod_{j=1}^{n} \left(\alpha_j + \beta_{\sigma(j)}\right) \leq \det(A+B) \leq \max_{\sigma \in S_n} \prod_{j=1}^{n} \left(\alpha_j + \beta_{\sigma(j)}\right).$$

If $\alpha_n + \beta_n \geq 0$, then the minimum is attained when σ is the identity permutation and the maximum is attained when $\sigma(j) = n - j + 1$, $j = 1, \dots, n$.

Proof. If A and B commute, they are simultaneously unitarily diagonalizable and the result easily follows. Otherwise, there exists $U \in M_n(\mathbb{C})$ unitary, such that

$$\det(A+B) = \det(A_0 + U^* B_0 U),$$

where A_0, B_0 are the diagonal forms of A, B. Since the unitary group is compact and the determinant is continuous, $\det(A_0 + V^* B_0 V)$ attains its maximum and minimum values for some unitary matrix $V \in M_n(\mathbb{C})$. Take

$$U_\epsilon = \mathrm{e}^{\mathrm{i}\epsilon S} = I_n + \mathrm{i}\epsilon S + O(\epsilon^2),$$

where ϵ is a small quantity and $S \in M_n(\mathbb{C})$ is Hermitian. Assuming that $A_0 + V^* B_0 V$ is nonsingular and calculating

$$\det(A_0 + U_\epsilon^* V^* B_0 V U_\epsilon)$$

to the first order in ϵ, it can be easily shown that $V^* B_0 V$ commutes with the inverse of $A_0 + V^* B_0 V$. Thus $V^* B_0 V$ commutes with A_0 and the theorem follows. If $A_0 + V^* B_0 V$ is singular, then the result follows by a limiting argument. □

Remark 4.8. A natural generalization of Fiedler's Theorem would be the following. If $A, B \in M_n(\mathbb{C})$ are normal matrices with eigenvalues $\alpha_1, \dots, \alpha_n$, $\beta_1, \dots, \beta_n$, respectively, then $\det(A+B)$ lies in the convex hull of the products

$$\prod_{j=1}^{n} \left(\alpha_j + \beta_{\sigma(j)}\right), \quad \sigma \in S_n.$$

This is *Marcus-de Oliveira conjecture* [71, 79], a longstanding open problem.

Concerning more general multiplicative inequalities, involving singular values of matrices, we state Gel'fand-Naimark Theorem (see, e.g. [47, 72]).

Theorem 4.9. (Gel'fand-Naimark, 1950) *For $A, B \in M_n(\mathbb{C})$,*

$$\prod_{j=1}^{k} s_{i_j}(A)\, s_{n-i_j+1}(B) \leq \prod_{j=1}^{k} s_j(AB), \qquad k = 1, \dots, n,$$

equivalently,

$$\prod_{j=1}^{k} s_{i_j}(AB) \leq \prod_{j=1}^{k} s_j(A)\, s_{i_j}(B), \qquad k = 1, \dots, n,$$

for $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, with equality for $k = n$.

The next result [9, 21] has a simple proof, using majorization theory.

Theorem 4.10. *If $A, B \geq 0$ have eigenvalues $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$, respectively, then*

$$\prod_{j=1}^{n} \left(a_j^2 + b_j^2\right) \leq |\det(A + iB)|^2 \leq \prod_{j=1}^{n} \left(a_j^2 + b_{n-j+1}^2\right).$$

Proof. We may assume $A, B > 0$. We easily find that

$$\begin{aligned} |\det(A + iB)|^2 &= \det(A)^2 \det\left(I_n + (A^{-1}B)^2\right) \\ &= \prod_{j=1}^{n} a_j^2 \prod_{j=1}^{n} \left(1 + \lambda_j^2(A^{-1}B)\right) \end{aligned}$$

and

$$\lambda_j(A^{-1}B) = s_j^2\left(A^{-\frac{1}{2}}B^{\frac{1}{2}}\right), \qquad j = 1, \ldots, n.$$

By Gel'fand-Naimark Theorem with A, B replaced by $B^{\frac{1}{2}}, A^{-\frac{1}{2}}$, respectively,

$$\prod_{j=1}^{k} s_{n-j+1}(A^{-\frac{1}{2}})s_j(B^{\frac{1}{2}}) \leq \prod_{j=1}^{k} s_j(A^{-\frac{1}{2}}B^{\frac{1}{2}}) \leq \prod_{j=1}^{k} s_j(A^{-\frac{1}{2}})s_j(B^{\frac{1}{2}})$$

hold for $k = 1, \ldots, n$, with equality for $k = n$. Clearly,

$$s_{n-j+1}^2(A^{-\frac{1}{2}})s_j^2(B^{\frac{1}{2}}) = \frac{b_j}{a_j}, \qquad s_j^2(A^{-\frac{1}{2}})s_j^2(B^{\frac{1}{2}}) = \frac{b_j}{a_{n-j+1}}, \qquad j = 1, \ldots, n.$$

Thus, the previous singular values inequalities are equivalent to

$$\left(\frac{b_1}{a_1}, \ldots, \frac{b_n}{a_n}\right) \prec_{\log} \lambda(A^{-1}B) \prec_{\log} \left(\frac{b_1}{a_n}, \ldots, \frac{b_n}{a_1}\right).$$

Since the function $f(x) = \log(1 + x^2)$ is a continuous increasing function on $(0, \infty)$, such that $f(\mathrm{e}^t)$ is convex in t, by Proposition 2.4 applied to the previous log-majorization, we obtain

$$\sum_{j=1}^{n} \log\left(1 + \frac{b_j^2}{a_j^2}\right) \leq \sum_{j=1}^{n} \log\left(1 + \lambda_j^2(A^{-1}B)\right) \leq \sum_{j=1}^{n} \log\left(1 + \frac{b_{n-j+1}^2}{a_j^2}\right).$$

Thus,

$$\prod_{j=1}^{n} \left(1 + \frac{b_j^2}{a_j^2}\right) \leq \prod_{j=1}^{n} \left(1 + \lambda_j^2(A^{-1}B)\right) \leq \prod_{j=1}^{n} \left(1 + \frac{b_{n-j+1}^2}{a_j^2}\right)$$

and the result easily follows. □

Remark 4.11. If one of the two matrices in Theorem 4.10 is not positive semidefinite, the lower bound is not necessarily true. Indeed, let A, B be Hermitian matrices with eigenvalues $a_1 = 1$, $a_2 = -1$ and $b_1 = 2$, $b_2 = 1$, respectively. As Marcus-de Oliveira conjecture holds for $n = 2$, then $\det(A + iB)$ is in the line segment with endpoints $-3 - i$ and $-3 + i$, consequently, we have

$$3 \leq |\det(A + iB)| \leq \sqrt{10}.$$

If Theorem 4.10 was true, the lower bound of $|\det(A+iB)|$ would be $\sqrt{10}$, but in the previous case the lower bound 3 is attained.

Now, considering the Cartesian decomposition $A = \operatorname{Re} A + i \operatorname{Im} A$ of $A \in M_n(\mathbb{C})$, where

$$\operatorname{Re} A = \frac{A + A^*}{2} \qquad \text{and} \qquad \operatorname{Im} A = \frac{A - A^*}{2i}$$

are Hermitian matrices, the next corollary is easy to derive.

Corollary 4.12. *If $A \in M_n(\mathbb{C})$ is such that $\operatorname{Re} A > 0$ then*

$$\det |\operatorname{Re} A| \leq |\det A| \,.$$

We remark that related inequalities are surveyed in [101, Section 3.4].

5. Golden-Thompson inequality and Araki's log-majorization

For matrices A, B that commute, the following identity holds:

$$\mathrm{e}^{A+B} = \mathrm{e}^{A}\mathrm{e}^{B}.$$

In the noncommutative case, the situation is not so simple.

Let H, K be Hermitian matrices of the same order. It is obvious that

$$\det\left(\mathrm{e}^{H+K}\right) = \det\left(\mathrm{e}^{H}\right) \det\left(\mathrm{e}^{K}\right).$$

Furthermore, the following remarkable trace inequality, motivated by considerations in statistical mechanics, states a relation between e^{H+K} and $\mathrm{e}^{H}\mathrm{e}^{K}$, even when these matrices do not commute.

Theorem 5.1. (Golden-Thompson inequality, 1965) *If $H, K \in M_n(\mathbb{C})$ are Hermitian, then*

$$\operatorname{tr}\left(\mathrm{e}^{H+K}\right) \leq \operatorname{tr}\left(\mathrm{e}^{H}\mathrm{e}^{K}\right). \tag{5.1}$$

Nowadays, (5.1) is a basic tool in quantum statistical mechanics. Some historical aspects and its applications in random matrix theory are collected in [30], including a not previously published proof due to Dyson. Golden [40] proved (5.1) for positive semidefinite matrices and observed that it may be used to get lower bounds for the Helmholtz free energy by partitioning the Hamiltonian. C. J. Thompson [89] showed (5.1) for Hermitian matrices, independently of the semidefiniteness condition, and applied it to obtain an upper bound for the *partition function* of an antiferromagnetic chain, that is, for $z = \operatorname{tr}\left(\mathrm{e}^{-\beta H}\right)$, where H is the Hamiltonian of the physical system and $\beta = 1/(k_B T)$, with k_B the Boltzmann constant and T the absolute temperature. Symanzik [86] obtained (5.1) for particular selfadjoint Hilbert space operators, in that context showing that the classical partition function is an upper bound for the corresponding quantum partition function. Some discussion on Golden-Thompson inequality can also be found in the expository blog post by Terence Tao [87].

The direct extension of the Golden-Thompson inequality to three or more matrices fails. Then if any two of the Hermitian matrices H, K, L commute,

$$\operatorname{tr}\left(e^{H+K+L}\right) \leq \left|\operatorname{tr}\left(e^{H}e^{K}e^{L}\right)\right|$$

obviously holds, but this is not true in general as the next counter-example, due to C. J. Thompson [90], shows.

Example. Consider the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

the real vector $a = (a_1, a_2, a_3)$ and the matrix $A = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3$. Then

$$e^{A} = \cosh\|a\|\, I_2 + \frac{\sinh\|a\|}{\|a\|}\, I_2.$$

Here $\|a\|$ is the Euclidean norm of a. For $\epsilon \in \mathbb{R}\setminus\{0\}$, let

$$H = \epsilon\sigma_1, \qquad K = \epsilon\sigma_2, \qquad L = \epsilon(\sigma_3 - \sigma_2 - \sigma_1).$$

In this case,

$$\operatorname{tr}\left(e^{H+K+L}\right) = 2\cosh\epsilon$$

and, after some calculations, we find

$$\left|\operatorname{tr}\left(e^{H}e^{K}e^{L}\right)\right| = 2\cosh\epsilon\left(1 - \frac{\epsilon^4}{12} + O(\epsilon^6)\right).$$

Therefore, for ϵ small enough, we have

$$\left|\operatorname{tr}\left(e^{H}e^{K}e^{L}\right)\right| < \operatorname{tr}\left(e^{H+K+L}\right).$$

Nevertheless, there is a nontrivial generalization of Golden-Thompson inequality to a triple of Hermitian matrices by Lieb [67], as well as recent multivariate versions [42, 85].

We notice in passing the interesting related result due to R. C. Thompson [91]: if H, K are Hermitian matrices, then

$$e^{iH}e^{iK} = e^{i(UHU^* + VKV^*)},$$

for some unitary matrices U, V. This result has application in quantum computing. We observe that Thompson's result was obtained before the long-standing Horn's conjecture on eigenvalues of sums of Hermitian matrices has been solved (see [20] for more details).

Several trace inequalities may be strengthened in the set up of majorization. This is the case of the Golden-Thompson inequality. In fact, it was proved by Lenard [66] and Thompson [90] that

$$e^{H+K} \prec_w e^{\frac{H}{2}}e^{K}e^{\frac{H}{2}}$$

holds for Hermitian matrices H, K.

Closely related, Araki [7] obtained a log-majorization presented in the next theorem that extends the *Lieb-Thirring trace inequality*:

$$\operatorname{tr}(AB)^r \leq \operatorname{tr}\left(A^rB^r\right), \qquad r \in \mathbb{N},$$

for $A, B \geq 0$, firstly used to derive inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian [68].

Theorem 5.2. (Araki's log-majorization, 1990) *Let* $A, B \geq 0$. *Then*

$$(A^{\frac{1}{2}}BA^{\frac{1}{2}})^r \prec_{\log} A^{\frac{r}{2}}B^rA^{\frac{r}{2}}, \qquad r \geq 1, \tag{5.2}$$

or equivalently

$$(A^{\frac{q}{2}}B^qA^{\frac{q}{2}})^{\frac{1}{q}} \prec_{\log} \left(A^{\frac{p}{2}}B^pA^{\frac{p}{2}}\right)^{\frac{1}{p}}, \quad 0 < q \leq p. \tag{5.3}$$

Proof. It is clear that $(A^{\frac{1}{2}}BA^{\frac{1}{2}})^r$ and $A^{\frac{r}{2}}B^rA^{\frac{r}{2}}$ have the same determinant. Assuming A invertible, let us prove that

$$\lambda_1((A^{\frac{1}{2}}BA^{\frac{1}{2}})^r) \leq \lambda_1(A^{\frac{r}{2}}B^rA^{\frac{r}{2}}). \tag{5.4}$$

To do so we may prove that $A^{\frac{r}{2}}B^rA^{\frac{r}{2}} \leq I_n$ implies that $A^{\frac{1}{2}}BA^{\frac{1}{2}} \leq I_n$, because both sides of (5.4) have the same order of homogeneity for A and B, so that we can multiply A, B by a positive constant. Since $B^r \leq A^{-r}$, for $r \geq 1$, then Löwner-Heinz inequality implies $B \leq A^{-1}$. If A is not invertible, by a continuity argument, we obtain (5.4). By properties **P1** and **P3**, then

$$(A^{\wedge k})^{\frac{r}{2}}(B^{\wedge k})^r(A^{\wedge k})^{\frac{r}{2}} = (A^{\frac{r}{2}}B^rA^{\frac{r}{2}})^{\wedge k},$$

$$\big((A^{\wedge k})^{\frac{1}{2}}(B^{\wedge k})(A^{\wedge k})^{\frac{1}{2}}\big)^r = \big((A^{\frac{1}{2}}BA^{\frac{1}{2}})^r\big)^{\wedge k}.$$

Hence, (5.2) follows from (5.4) applied to the matrices $A^{\wedge k}$, $B^{\wedge k}$, $k = 1, \ldots, n$, using Lemma 3.4.

For $0 < p \leq q$, we may replace A and B by A^q, B^q and take $r = p/q$ in (5.2) so that (5.3) follows. □

Araki's log-majorization readily implies the next trace inequality.

Corollary 5.3. (Araki-Lieb-Thirring inequality) *If* $A, B \geq 0$, $r \geq 1$ *and* $s > 0$, *then*

$$\mathrm{tr}\big(A^{\frac{1}{2}}BA^{\frac{1}{2}}\big)^{rs} \leq \mathrm{tr}\big(A^{\frac{r}{2}}B^rA^{\frac{r}{2}}\big)^s.$$

The next extension of Golden-Thompson inequality is now easy to derive, as observed by Ando and Hiai [5].

Corollary 5.4. *If* $H, K \in M_n(\mathbb{C})$ *are Hermitian and* $p > 0$, *then*

$$\mathrm{e}^{H+K} \prec_{\log} \big(\mathrm{e}^{\frac{pH}{2}}\mathrm{e}^{pK}\mathrm{e}^{\frac{pH}{2}}\big)^{\frac{1}{p}}. \tag{5.5}$$

Proof. Consider $A = \mathrm{e}^H$ and $B = \mathrm{e}^K$ in (5.3). Further, have in mind the continuous parameter version of Lie-Trotter formula [5, Lemma 1.6]:

$$\lim_{q\to 0}\big(\mathrm{e}^{\frac{qH}{2}}\mathrm{e}^{qK}\mathrm{e}^{\frac{qH}{2}}\big)^{\frac{1}{q}} = \mathrm{e}^{H+K}$$

and the result follows. □

Let H, K be Hermitian matrices. If $p = 1$, then (5.5) can be written as

$$e^{H+K} \prec_{\log} e^{H} e^{K}, \tag{5.6}$$

since $e^{\frac{H}{2}} e^{K} e^{\frac{H}{2}}$ and $e^{H} e^{K}$ have the same eigenvalues. From the previous results, we can see that Golden-Thompson inequality is strengthened to

$$\left\| e^{H+K} \right\| \leq \left\| \left(e^{\frac{pH}{2}} e^{pK} e^{\frac{pH}{2}} \right)^{\frac{1}{p}} \right\|, \qquad p > 0,$$

for any unitarily invariant norm, and the right hand side decreases to the left hand side as p converges to 0.

6. Ando-Hiai Inequality

The axiomatic theory of operator connections was developed by F. Kubo and T. Ando [62]. A *matrix connection of order* n is a binary operation σ on the cone of positive semidefinite matrices in $M_n(\mathbb{C})$, satisfying for any $A, B, C, D, A_k, B_k \geq 0$:

C1. (*joint monotonicity*) $A \leq C$ and $B \leq D \;\Rightarrow\; A\,\sigma\,B \leq C\,\sigma\,D$;

C2. (*transformer inequality*) $X^*(A\,\sigma\,B)X \leq (X^*AX)\,\sigma\,(X^*BX)$ for any $X \in M_n(\mathbb{C})$;

C3. (*joint continuity from above*) $A_k \downarrow A$ and $B_k \downarrow B \;\Rightarrow\; A_k\,\sigma\,B_k \downarrow A\,\sigma\,B$,

where $A_k \downarrow A$ means that $A_1 \geq A_2 \geq \cdots A_k \geq \cdots$ and A_k converges strongly to A as $k \to \infty$. An *operator connection* is a matrix connection of every order $n \in \mathbb{N}$. An *operator mean* is an operator connection σ, satisfying the *normalization property* $I_n\,\sigma\,I_n = I_n$. For instance, for $\alpha \in [0, 1]$,

$$A \nabla_\alpha B = (1-\alpha)A + \alpha B \qquad \text{and} \qquad A !_\alpha B = \left((1-\alpha)A^{-1} + \alpha B^{-1}\right)^{-1}$$

are the *weighted arithmetic* and *harmonic means*, respectively; $A w_l B = A$ and $A w_r B = B$ are the left and right *trivial operator means*, respectively.

Kubo and Ando proved that there is a one-to-one correspondence between operator connections and operator monotone functions on $\mathbb{R}_0^+$.

Theorem 6.1. [62] *For each operator connection σ, there exists a unique operator monotone function $f \colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$, satisfying*

$$f(t)\, I_n = I_n\,\sigma\,(tI_n), \qquad t > 0,$$

and for $A, B > 0$ the formula

$$A\,\sigma\,B \;=\; A^{\frac{1}{2}}\, f\big(A^{-\frac{1}{2}} B\, A^{-\frac{1}{2}}\big)\, A^{\frac{1}{2}}$$

holds, with the right hand side defined via functional calculus, and extended to $A, B \geq 0$ as follows

$$A\,\sigma\,B \;=\; \lim_{\epsilon \to 0^+} (A + \epsilon I_n)\,\sigma\,(B + \epsilon I_n).$$

Let $\alpha \in [0,1]$. In particular, associated with the operator monotone function $f(t) = t^\alpha$, the *α-weighted geometric mean* is

$$A \sharp_\alpha B \,=\, A^{\frac12}\big(A^{-\frac12} B\, A^{-\frac12}\big)^\alpha A^{\frac12}.$$

It is easy to see that $A\sharp_\alpha B = B\sharp_{1-\alpha} A$ and when A commutes with B, then $A\sharp_\alpha B = A^{1-\alpha}B^\alpha$. The *geometric mean*, simply denoted by $\sharp$, is the mean corresponding to $f(t) = t^{\frac12}$. It is the unique positive semidefinite solution of the Riccati equation $XA^{-1}X = B$, also characterized [80] as

$$A \sharp B \,=\, \max\left\{X \in H_n :\ \begin{bmatrix} A & X \\ X & B \end{bmatrix} \ge 0\right\}. \tag{6.1}$$

Further, there is a unitary matrix U such that $A\sharp B = A^{\frac12} U B^{\frac12}$.

Ando and Hiai [5] proved the following interesting result, concerning the weighted geometric mean.

Theorem 6.2. (Ando-Hiai inequality, 1994) *For $A, B \ge 0$ and $\alpha \in [0,1]$,*

$$A^r \sharp_\alpha B^r \prec_{\log} (A\sharp_\alpha B)^r, \qquad r \ge 1, \tag{6.2}$$

or, equivalently,

$$\big(A^p \sharp_\alpha B^p\big)^{\frac1p} \prec_{\log} \big(A^q \sharp_\alpha B^q\big)^{\frac1q}, \qquad 0 < q \le p. \tag{6.3}$$

Proof. If $1 \le r \le 2$, then $r = 1+\epsilon$ with $\epsilon \in [0,1]$. Suppose

$$A\sharp_\alpha B \le I_n. \tag{6.4}$$

Let $C = A^{-\frac12} B A^{-\frac12}$. By continuity, we may assume that A, B are invertible. It follows from (6.4) that $A \le C^{-\alpha}$. By Löwner-Heinz inequality, we have $A^\epsilon \le C^{-\alpha\epsilon}$. In this case,

$$\begin{aligned} A^r \sharp_\alpha B^r &= A^{\frac12}\big(A^\epsilon \sharp_\alpha \big((CAC) \sharp_{1-\epsilon} C\big)\big) A^{\frac12} \\ &\le A^{\frac12}\big(C^{-\alpha\epsilon} \sharp_\alpha \big((C^{2-\alpha} \sharp_{1-\epsilon} C\big)\big) A^{\frac12} \\ &= A^{\frac12} C^\alpha A^{\frac12} = A\sharp_\alpha B, \end{aligned}$$

by the joint monotonicity of the weighted geometric means $\sharp_\alpha$ and $\sharp_{1-\epsilon}$. Therefore, $A^r\sharp_\alpha B^r \le I_n$. We have just proved that $\lambda_1(A\sharp_\alpha B) \le 1$ implies $\lambda_1(A^r\sharp_\alpha B^r) \le 1$. Thus,

$$\lambda_1(A^r \sharp_\alpha B^r) \le \lambda_1(A\sharp_\alpha B)^r$$

and applying the antisymmetric tensor power trick, having also in mind that

$$\det\big(A^r\sharp_\alpha B^r\big) = \det(A\sharp_\alpha B)^r,$$

then (6.2) holds for $1 \le r \le 2$. If $r > 2$, then $r = 2^m s$ for $m \in \mathbb{N}$ and $1 \le s \le 2$. By repeated use of the above case, we find that

$$A^{2^m s}\sharp_\alpha B^{2^m s} \prec_{\log} \Big(A^{2^{m-1}s}\sharp_\alpha B^{2^{m-1}s}\Big)^2 \prec_{\log} \cdots \prec_{\log} \big(A^s\sharp_\alpha B^s\big)^{2^m} \prec_{\log} (A\sharp_\alpha B)^{2^m s}.$$

This proves that (6.2) also holds for $r > 2$. Now, for $0 < q \le p$, the result easily follows. ☐

The following corollary complements the previous log-majorizations of Golden-Thompson type [5].

Corollary 6.3. *If H, K are Hermitian matrices and $\alpha \in [0,1]$, then*

$$\left(\mathrm{e}^{pH} \sharp_\alpha \mathrm{e}^{pK}\right)^{\frac{1}{p}} \prec_{\log} \mathrm{e}^{(1-\alpha)H+\alpha K}, \quad p > 0.$$

Proof. Consider (6.3) applied to $A = \mathrm{e}^H$ and $B = \mathrm{e}^K$, then use

$$\lim_{q\to 0} \left(\mathrm{e}^{qH} \sharp_\alpha \mathrm{e}^{qK}\right)^{\frac{1}{q}} = \mathrm{e}^{(1-\alpha)H+\alpha K},$$

that is, the Lie-Trotter like formula for the weighted geometric mean obtained in [45, Lemma 3.3]. □

The corresponding inequality for unitarily invariant norms holds, with the left hand side norm decreasing to the right hand side as p converges to 0.

A celebrated development of Löwner-Heinz inequality established by T. Furuta [35] is the next order preserving operator inequality, which was motivated by a previous conjecture by Chan and Kwong [25].

Theorem 6.4. (Furuta inequality, 1987) *Let $A \ge B \ge 0$. Then*

$$A^{\frac{p+r}{q}} \ge \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1}{q}} \quad \textit{and} \quad \left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}}\right)^{\frac{1}{q}} \ge B^{\frac{p+r}{q}} \tag{6.5}$$

hold for $r \ge 0$, $p \ge 0$ and $q \ge 1$ with $(1+r)q \ge p+r$.

The case of p, q, r all equal to 2 in (6.5) affirmatively answers Chan and Kwong's conjecture:

$$A \ge B \ge 0 \quad \Rightarrow \quad A^2 \ge \left(A\,B^2 A\right)^{\frac{1}{2}}.$$

Furuta and many other researchers refined and generalized (6.5) and applied these results to produce new inequalities [36].

The *essential part* of Furuta inequality is the case $q = \frac{p+r}{1+r}$, which can be formulated for invertible A, using the weighted geometric mean, as follows:

$$A \ge B \ge 0 \quad \Rightarrow \quad A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \le A, \qquad p \ge 1,\ r \ge 0. \tag{6.6}$$

Fujii and Kamei [33] showed that Ando-Hiai inequality is equivalent to Furuta inequality. Next, we show this direct implication. Indeed, let $A \ge B > 0$ and $p \ge 1$. Firstly, if $0 \le r \le 1$, then $A^r \ge B^r$ by Löwner-Heinz inequality. Consequently,

$$A^{-r} \sharp_{\frac{r}{p+r}} B^p \le B^{-r} \sharp_{\frac{r}{p+r}} B^p = I_n.$$

On the other hand, for $r > 1$, observe that $A^{-1} \le B^{-1}$ yields

$$A^{-1} \sharp_{\frac{r}{p+r}} B^{\frac{p}{r}} \le B^{-1} \sharp_{\frac{r}{p+r}} B^{\frac{p}{r}} = I_n$$

and so Ando-Hiai inequality implies that

$$A^{-r} \sharp_{\frac{r}{p+r}} B^p \le I_n.$$

Therefore, for any $r \ge 0$ we find that

$$A^{-r} \sharp_{\frac{1+r}{p+r}} B^p = \left(A^{-r} \sharp_{\frac{r}{p+r}} B^p\right) \sharp_{\frac{1}{p}} B^p \le I_n \sharp_{\frac{1}{p}} B^p = B \le A,$$

that is, the *essential part* of Furuta inequality (6.6) holds. The remaining part follows readily from Löwner-Heinz inequality.

Extensions of Furuta inequality and Ando-Hiai log-majorization were given by Furuta [37] and afterwards by other authors. Nowadays, the multivariate geometric mean as settled in [22, 78], following a Riemannian geometric approach, is often called the *Karcher mean* [63]. It is also called *Cartan mean* and *Riemannian mean*. Extensions of Ando-Hiai inequality to the Karcher mean and generalized Karcher mean are due to Yamazaki [99, 100]. Other Ando-Hiai type inequalities have meanwhile been obtained. See the recent works by Hiai, Seo and Wada [49, 50], Kian, Moslehian and Seo [57, 58, 59] and references therein.

7. BLP and Matharu-Aujla inequalities

Using the previous techniques of Ando and Hiai, Bebiano, Lemos and Providência [12, Theorem 2.1] obtained the next log-majorization of Araki's type.

Theorem 7.1. (BLP inequality, 2005) *For $A, B \geq 0$,*

$$A^{\frac{1+q}{2}} B^q A^{\frac{1+q}{2}} \prec_{\log} A^{\frac{1}{2}} \left(A^{\frac{r}{2}} B^r A^{\frac{r}{2}}\right)^{\frac{q}{r}} A^{\frac{1}{2}}, \qquad 0 < q \leq r. \tag{7.1}$$

Proof. The equality of the determinants is clear. It is enough to prove that

$$\lambda_1\left(A^{\frac{1+q}{2}} B^q A^{\frac{1+q}{2}}\right) \leq \lambda_1\left(A^{\frac{1}{2}} \left(A^{\frac{r}{2}} B^r A^{\frac{r}{2}}\right)^{\frac{q}{r}} A^{\frac{1}{2}}\right) \tag{7.2}$$

when A is invertible, otherwise a continuity argument is used. Assume that

$$A^{\frac{1}{2}} \left(A^{\frac{r}{2}} B^r A^{\frac{r}{2}}\right)^{\frac{q}{r}} A^{\frac{1}{2}} \leq I_n,$$

that is,

$$A^{-1} \geq \left(A^{\frac{r}{2}} B^r A^{\frac{r}{2}}\right)^{\frac{q}{r}} \geq 0.$$

By Furuta inequality, since $r > 0$, $\frac{r}{q} \geq 1$ and $(1+r)\frac{r}{q} \geq \frac{r}{q} + r$, we find

$$A^{-(1+q)} = A^{-\frac{r/q+r}{r/q}} \geq \left(A^{-\frac{r}{2}} \left(A^{\frac{r}{2}} B^r A^{\frac{r}{2}}\right)^{\frac{q}{r}\frac{r}{q}} A^{-\frac{r}{2}}\right)^{\frac{q}{r}} = B^q,$$

that is,

$$A^{\frac{1+q}{2}} B^q A^{\frac{1+q}{2}} \leq I_n$$

and then (7.2) holds. Using Lemma 3.4, the result follows from (7.2) replacing A, B by $A^{\wedge k}$, $B^{\wedge k}$, respectively, for $k = 1, \ldots, n$, by properties **P1**, **P3**, **P5**. □

For convenience of notation, for $\alpha \in \mathbb{R}$, let

$$A \natural_\alpha B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$$

which is just the α-weighted geometric mean of $A, B \geq 0$ when $\alpha \in [0, 1]$.

Corollary 7.2. *If $A, B > 0$ and $1 \leq \alpha \leq 2$, then*

$$A^{1-\alpha} B^\alpha \prec_{\log} A \natural_\alpha B.$$

Proof. The case $\alpha = 1$ is trivial. For $\alpha > 1$, let $q = \alpha - 1$, $r = 1$, replace A, B by B, A^{-1}, respectively, in Theorem 7.1 and note that $B \natural_{1-\alpha} A = A \natural_\alpha B$. $\square$

A. Matsumoto, R. Nakamoto and M. Fujii [75, Theorem 1] proved that

$$\big\|A^{\frac{s+q}{2}} B^q A^{\frac{s+q}{2}}\big\| \;\le\; \big\|A^{\frac{s}{2}} (A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{q}{p}} A^{\frac{s}{2}}\big\|, \qquad 0 < q \le p, \quad s \ge 0, \tag{7.3}$$

for $A, B \ge 0$, which reduces to Araki-Cordes inequality [31] if $s = 0$. They also proved (7.3) with the reverse inequality sign if $0 \le s \le p \le q$ and $p > 0$ [75, Theorem 2]. Furuta [39, Corollary 3.1 iii.] obtained a norm inequality, that yields the reverse of (7.3) for $0 \le q \le p$ and $-s \ge q$ (see [64, p.28]). These norm inequalities can be restated as follows.

Theorem 7.3. *Let $A, B \ge 0$. If $0 < q \le p$ and $s \ge 0$, then*

$$A^{\frac{s+q}{2}} B^q A^{\frac{s+q}{2}} \prec_{\log} A^{\frac{s}{2}} \big(A^{\frac{p}{2}} B^p A^{\frac{p}{2}}\big)^{\frac{q}{p}} A^{\frac{s}{2}}. \tag{7.4}$$

If either $0 \le s \le p \le q$ and $p > 0$ or $0 \le q \le p$ and $-s \ge q$, then (7.4) *holds with reversed log-majorization.*

Araki's log-majorization is obtained if $s = 0$ and BLP inequality if $s = 1$.

Corollary 7.4. *If $A > 0$, $B \ge 0$ and $\alpha \ge 2$, then*

$$A \natural_\alpha B \prec_{\log} A^{1-\alpha} B^\alpha.$$

Proof. Let $q = \alpha - 1$, $p = s = 1$ and replace A, B by B, A^{-1}, respectively, in Theorem 7.3. $\square$

Clearly, if $\alpha = 2$, the matrices in both hand sides of the log-majorizations given in Corollary 7.2 and Corollary 7.4 have the same eigenvalues.

The *Umegaki relative entropy* [92] of the density matrices A, B is

$$S(A, B) \;=\; \operatorname{tr}\,(A\,(\log A - \log B)).$$

Fujii and Kamei [32] introduced the variant

$$\hat{S}(A|B) \;=\; A^{\frac{1}{2}} \log\,(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})\, A^{\frac{1}{2}}.$$

A logarithmic trace inequality [64] is now presented.

Theorem 7.5. *Let $A, B > 0$. If $q, s \ge 0$, then*

$$\operatorname{tr}\,(A^s\,(\log A^q + \log B^q)) \;\le\; \operatorname{tr}\,\Big(A^s \log\,(A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{q}{p}}\Big), \qquad p > 0, \tag{7.5}$$

and the left hand side converges to the right hand side as p converges to 0.

Proof. The log-majorization (7.4) implies the trace inequality

$$\operatorname{tr}\,(A^s A^q B^q) \;\le\; \operatorname{tr}\big(A^s (A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{q}{p}}\big), \qquad 0 \le q \le p, \quad s \ge 0,$$

occuring trace equality when $q = 0$. Taking the derivatives of the right and left hand sides of the previous inequality at $q = 0$, observing that

$$\frac{d}{dq}\big(A^q B^q\big)\Big|_{q=0} \;=\; \log A + \log B, \tag{7.6}$$

$$\frac{d}{dq}\left((A^{\frac{p}{2}}B^pA^{\frac{p}{2}})^{\frac{q}{p}}\right)\Big|_{q=0} = \log\left(A^{\frac{p}{2}}B^pA^{\frac{p}{2}}\right)^{\frac{1}{p}}, \quad p>0, \tag{7.7}$$

yields a trace inequality. Multiplying both hand sides of the obtained trace inequality by q provides (7.5). By the parametric Lie-Trotter formula, we may see that (7.7) converges to (7.6) as p converges to 0. □

The case $q=s=1$ in Theorem 7.5 is due to Hiai and Petz [45]. It was later complemented in [5]. Using relative entropy terminology, Theorem 7.5 for $q=s$, replacing B by B^{-1}, may be written in the condensed form

$$S(A^s,B^s) \leq -\frac{s}{p}\operatorname{tr}\big(\hat{S}(A^p|B^p)A^{s-p}\big), \qquad s\geq 0, \quad p>0,$$

this providing an upper bound for the relative entropy $S(A,B)$ when $s=1$.

Fujii, Nakamoto and Tominaga [34] improved BLP inequality as follows.

Theorem 7.6. *If $A,B\geq 0$, $p\geq 1$, $q\geq 0$, then*

$$\left\|A^{\frac{1+q}{2}}B^{1+q}A^{\frac{1+q}{2}}\right\|^{\frac{p+q}{p(1+q)}} \leq \left\|A^{\frac{1}{2}}\left(A^{\frac{q}{2}}B^{q+p}A^{\frac{q}{2}}\right)^{\frac{1}{p}}A^{\frac{1}{2}}\right\|.$$

The next log-majorization [73] is obtained from Furuta inequality too.

Theorem 7.7. (Matharu-Aujla inequality, 2012) *Let $A,B\geq 0$ and $0\leq\alpha\leq 1$, then*

$$A\sharp_\alpha B \prec_{\log} A^{1-\alpha}B^\alpha.$$

Proof. If $\alpha=0$ or $\alpha=1$, the result is trivial. Let $0<\alpha<1$. Clearly, $A\sharp_\alpha B$ and $A^{1-\alpha}B^\alpha$ have the same determinant. Let us prove that

$$\lambda_1(A\sharp_\alpha B)\leq \lambda_1(A^{1-\alpha}B^\alpha). \tag{7.8}$$

If A is invertible and $\lambda_1(A^{1-\alpha}B^\alpha)\leq 1$, then

$$A^{-(1-\alpha)}\geq B^\alpha.$$

By Furuta inequality with $p=q=\frac{1}{\alpha}\geq 1$ and $r\geq 0$, we find

$$\left(A^{-(1-\alpha)}\right)^{1+\alpha r} \geq \left(\left(A^{-(1-\alpha)}\right)^{\frac{r}{2}}B\left(A^{-(1-\alpha)}\right)^{\frac{r}{2}}\right)^{\alpha}.$$

Taking $r=\frac{1}{1-\alpha}$ yields

$$A^{-1}\geq \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\alpha},$$

so that $\lambda_1(A\sharp_\alpha B)\leq 1$ holds. If A is not invertible, by a continuity argument, (7.8) is obtained. Using Lemma 3.4, the result follows from (7.8) replacing A,B by $A^{\wedge k}$, $B^{\wedge k}$, respectively, for $k=1,\dots,n$, by properties **P1**, **P3**, **P5**. □

Furuta considered other operator inequalities implying the generalized BLP inequality [38] as well as Matharu-Aujla inequality [39].

8. Inequalities for operator connections

In this section, some inequalities involving operators conections are presented. For that purpose, we recall that the *dual* of a nonzero operator connection σ is the operator connection $\sigma^{\perp}$ defined by

$$A\,\sigma^{\perp} B \;=\; \left(B^{-1}\sigma\, A^{-1}\right)^{-1}$$

for $A, B > 0$ and extended by continuity to $A, B \geq 0$ as usual. Its representing function satisfies

$$f_{\sigma^{\perp}}(t) \;=\; t/f_{\sigma}(t), \qquad t > 0.$$

In the sequel, for $C, X \in M_n(\mathbb{C})$ the condensed notation $X^{\sim}$ stands for X or X^T, whereas $C \in H_n^{\sim}$ means that if the symbol $^{\sim}$ is omitted along the stated result, then C is assumed Hermitian, and if $^{\sim}$ acts as the transpose along the result, then C is assumed symmetric.

Theorem 8.1. *Let $A, B \geq 0$ and $C \in H_n^{\sim}$. If the representing functions of the nonzero operator connections σ, τ, ρ satisfy $f_{\sigma}^2 \leq f_{\tau} f_{\rho}$, then*

$$s_1\big((A\,\tau^{\perp} B)^{\frac{1}{2}} C^{*}(A\,\sigma\, B)^{\sim} C\,(A\,\rho^{\perp} B)^{\frac{1}{2}}\big) \;\leq\; \lambda_1(A\,C^{*}B^{\sim}C). \tag{8.1}$$

Proof. For C Hermitian, there exists U unitary, such that $U^{*}CU = D$ is real diagonal and it is enough to prove that

$$s_1\big((A\,\tau^{\perp} B)^{\frac{1}{2}} D(A\,\sigma\, B)D\,(A\,\rho^{\perp} B)^{\frac{1}{2}}\big) \;\leq\; \lambda_1(ADBD),$$

since we may replace A, B by $U^{*}AU, U^{*}BU$, respectively, and apply the transformer inequality. If C is symmetric, by Takagi's factorization [54, Corollary 4.4.4], there exist V unitary and D diagonal with the singular values of C in its main diagonal, such that $C = V D\, V^T$. In this case, we need to show that

$$s_1\big((A\,\tau^{\perp} B)^{\frac{1}{2}} D(A\,\sigma\, B)^{T} D\,(A\,\rho^{\perp} B)^{\frac{1}{2}}\big) \;\leq\; \lambda_1(ADB^{T}D)$$

from which the result follows, replacing A, B by $V^{T}A\overline{V}, V^{T}B\overline{V}$, respectively.

Thus, assuming D to be a real diagonal matrix, we will check that

$$\lambda_1(ADB^{\sim}D) \leq 1 \quad \Rightarrow \quad s_1\big((A\,\tau^{\perp} B)^{\frac{1}{2}} D\,(A\,\sigma\, B)^{\sim} D\,(A\,\rho^{\perp} B)^{\frac{1}{2}}\big) \;\leq\; 1. \tag{8.2}$$

Firstly, let $A, B > 0$. If $\lambda_1(ADB^{\sim}D) \leq 1$, equivalently, $\lambda_1(DA^{\sim}DB) \leq 1$, then $DA^{\sim}D \leq B^{-1}$ and $DB^{\sim}D \leq A^{-1}$. By the transformer inequality **C2** and the joint monotonicity **C1**, we find

$$\begin{aligned} D\,(A\,\rho^{\perp} B)^{\sim} D = D\left(A^{\sim}\rho^{\perp} B^{\sim}\right) D \;&\leq (DA^{\sim}D)\,\rho^{\perp}(DB^{\sim}D) \\ &\leq B^{-1}\rho^{\perp}A^{-1} \\ &= (A\,\rho\, B)^{-1} \end{aligned}$$

Analogously, $D\,(A\,\tau^{\perp} B)^{\sim} D \;\leq\; (A\,\tau\, B)^{-1}$. Under the hypothesis, we see that

$$A^{\frac{1}{2}} f_{\sigma}(M)\big(f_{\rho}(M)\big)^{-1} f_{\sigma}(M)\, A^{\frac{1}{2}} \;\leq\; A^{\frac{1}{2}} f_{\tau}(M)\, A^{\frac{1}{2}} \;=\; A\,\tau\, B, \tag{8.3}$$

where $M = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. Therefore

$$\lambda_1\big((A\,\sigma\, B)(A\,\rho\, B)^{-1}(A\,\sigma\, B)(A\,\tau\, B)^{-1}\big) \;\leq\; 1. \tag{8.4}$$

Moreover,

$$s_1\big((A\,\tau^{\perp} B)^{\frac{1}{2}} D\,(A\,\sigma\, B)^{\sim} D\,(A\,\rho^{\perp} B)^{\frac{1}{2}}\big)$$

is equal to the square root of

$$\lambda_1\big((A\,\sigma\, B)\, D\,(A\,\rho^{\perp} B)^{\sim} D\,(A\,\sigma\, B)\, D\,(A\,\tau^{\perp} B)^{\sim} D\big). \tag{8.5}$$

Now, it is clear that (8.5) is not greater than (8.4) and the implication (8.2) holds. If $A, B \geq 0$, by a continuity argument, the result is obtained. □

If the nonzero operator connections σ, τ, ρ satisfy $f_\sigma^2 \geq f_\tau f_\rho$, then (8.1) holds with each connection replaced by its dual [65].

Applying Theorem 8.1 for $A, B \geq 0$, $C \in H_n^{\sim}$ and $\sigma \leq \tau = \rho$ yields

$$\lambda_1\big((A\,\tau^{\perp} B)\, C^*(A\,\sigma\, B)^{\sim} C\big) \;\leq\; \lambda_1(A\, C^* B^{\sim} C). \tag{8.6}$$

If $\tau = \sigma$ and $\sim$ is absent, this was observed in [64, Theorem 2.1] for $C \geq 0$.

Corollary 8.2. *If $A, B \geq 0$, $C \in H_n^{\sim}$ and σ is a nonzero operator connection, then*

$$\begin{aligned}(A\,\sharp\, B)\, C^*(A\,\sharp\, B)^{\sim}\, C \prec_{\log} &\left|(A\,\sigma^{\perp} B)^{\frac{1}{2}}\, C^*(A\,\sharp\, B)^{\sim}\, C\,(A\,\sigma\, B)^{\frac{1}{2}}\right| \\ &\prec_{\log} (A\,\sigma^{\perp} B)\, C^*(A\,\sigma\, B)^{\sim}\, C.\end{aligned}$$

Proof. We can see that

$$\lambda_1\big((A\,\sharp\, B)\, C^*(A\,\sharp\, B)^{\sim} C\big) \leq s_1\big(A^{\frac{1}{2}}\, C^*(A\,\sharp\, B)^{\sim} C\, B^{\frac{1}{2}}\big) \leq \lambda_1(A\, C^* B^{\sim} C). \tag{8.7}$$

The last inequality in (8.7) is the case τ, ρ as the trivial operator means w_l, w_r and $\sigma = \sharp$ in Theorem 8.1. The first inequality in (8.7) follows after taking square roots of the obtained eigenvalues from the case $\tau = \sigma = \sharp$ with $^{\sim}$ deleted in (8.6), then replacing C by $C^*(A\,\sharp\, B)^{\sim} C$. Applying Weyl's trick to (8.7) and observing the equality of the determinants of the matrices involved, a log-majorization is obtained. Next, replace A by $A\,\sigma^{\perp} B$, B by $A\,\sigma\, B$ in that log-majorization and use the identity $(A\,\sigma^{\perp} B)\,\sharp\,(A\,\sigma\, B) = A\,\sharp\, B$. □

Corollary 8.3. *If $A, B \geq 0$, $C \in H_n^{\sim}$ and $0 \leq \alpha \leq \beta \leq 1$, then*

$$\left|(A\sharp_{1-\alpha} B)^{\frac{1}{2}}\, C^*(A\,\sharp_{\frac{\beta}{2}}\, B)^{\sim}\, C\,(A\,\sharp_{1+\alpha-\beta} B)^{\frac{1}{2}}\right| \prec_{\log} A\, C^* B^{\sim}\, C. \tag{8.8}$$

Proof. If $\sigma = \sharp_{\frac{\beta}{2}}$, $\tau = \sharp_\alpha$, $\rho = \sharp_{\beta-\alpha}$ in Theorem 8.1, since $\beta - \alpha \in [0, 1]$, then

$$s_1\big((A\,\sharp_{1-\alpha} B)^{\frac{1}{2}}\; C^*(A\,\sharp_{\frac{\beta}{2}}\, B)^{\sim}\, C\,(A\,\sharp_{1+\alpha-\beta} B)^{\frac{1}{2}}\big) \;\leq\; \lambda_1(A\, C^* B^{\sim}\, C).$$

Replace A, B, C by their kth compounds and apply properties **P1-P6**. The determinants of the matrices in both hand sides of (8.8) are equal. □

Remark 8.4. Let $A, B \geq 0$, $C \in H_n^{\sim}$ and $\alpha \in [0, 1]$. If $\sigma = \sharp_\alpha$ in Corollary 8.2 and $\beta = 1$ in Corollary 8.3, then

$$\left|(A\sharp_{1-\alpha} B)^{\frac{1}{2}}\, C^*(A\,\sharp\, B)^{\sim}\, C\,(A\,\sharp_\alpha B)^{\frac{1}{2}}\right|$$

is log-majorized by $(A\,\sharp_{1-\alpha}B)\,C^*(A\,\sharp_\alpha B)^\sim C$ and $A\,C^*B^\sim C$, respectively, being these two matrices related as follows:

$$(A\,\sharp_{1-\alpha}B)\,C^*(A\,\sharp_\alpha B)^\sim C\ \prec_{\log}\ A\,C^*B^\sim C \tag{8.9}$$

as a consequence of applying Weyl's trick to (8.6) when $\sigma=\tau=\sharp_\alpha$. In particular, this implies the next trace inequality observed by Bhatia, Lim and Yamazaki [23]:

$$\operatorname{tr}\big((A\sharp_{1-\alpha}B)(A\sharp_\alpha B)\big)\le \operatorname{tr}(AB).$$

The question on whether it is possible to extend (8.9) to

$$(A\,\sigma^\perp B)\,C^*(A\,\sigma\,B)^\sim C\ \prec_{\log}\ A\,C^*B^\sim C$$

for other operator connections σ naturally arises.

Theorem 8.5. *If $A,B\ge 0$ and $r\in\mathbb{N}_0$, then*

$$(A\sharp B)^{r+1}\prec_{\log}\ |A^{\frac12}(A\sharp B)^rB^{\frac12}|\ \prec_{\log}\ (AB)^{\frac{r+1}{2}}.$$

Proof. We can check that

$$\lambda_1(A\,\sharp\,B)^{2(r+1)}\ \le\ \lambda_1\big(A\,(A\,\sharp\,B)^rB\,(A\,\sharp\,B)^r\big)\ \le\ \lambda_1(AB)^{r+1}. \tag{8.10}$$

The first inequality in (8.10) follows from (8.7) when the symbol $\sim$ is absent and $C=(A\sharp B)^r$. Concerning the second, if $A>0$ and $\lambda_1(AB)^{r+1}\le 1$, then $B\le A^{-1}$ implies

$$(A\,\sharp\,B)\,B\,(A\,\sharp\,B)\ \le\ (A\sharp\,B)\,A^{-1}(A\,\sharp\,B)\ =\ B\ \le\ A^{-1}.$$

By induction on $r\in\mathbb{N}_0$, we easily prove that $(A\,\sharp\,B)^rB\,(A\,\sharp\,B)^r\le A^{-1}$, so

$$\lambda_1\big(A\,(A\,\sharp\,B)^rB\,(A\,\sharp\,B)^r\big)\ \le\ 1.$$

Thus, the last inequality in (8.10) is true. By continuity, it remains valid for $A\ge 0$. After applying Weyl's trick to (8.10), the obtained log-majorization implies the log-majorization between the corresponding square roots. $\square$

If $A,B\ge 0$, the last log-majorization in Theorem 8.5 and $AB\ \prec_{\log}\ |AB|$ which follows readily from Weyl's Majorant Theorem yield

$$\prod_{i=1}^k s_i\big(A^{\frac12}(A\,\sharp\,B)^rB^{\frac12}\big)\ \le\ \prod_{i=1}^k s_i^{\frac{r+1}{2}}(AB),\qquad k=1,\dots,n,$$

for $r\in\mathbb{N}_0$. If $r=1$, these inequalities were obtained by Zou [102].

Conjecture 8.6. *If $A,B\ge 0$ then*

$$\big|A^\alpha(A\,\sharp_\alpha B)B^{1-\alpha}\big|\ \prec_{\log}\ |AB|$$

for all $\alpha\in[0,1]$.

9. Ando and Visick's inequalities for the Hadamard product

In this section, Ando and Visick's inequalities [4, 94] for the Hadamard product of positive definite matrices, which settled affirmatively Bapat and Johnson's conjecture [56], are revisited and weighted interpolations are presented.

We recall that a map $\Phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ is called *positive* if $A \geq 0$ implies $\Phi(A) \geq 0$ and it is called *unital* if $\Phi(I_m) = I_n$.

Lemma 9.1. [1] *If $\Phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ is a unital positive linear map and f is operator monotone on $\mathbb{R}_0^+$, then*

$$f(\Phi(A)) \geq \Phi(f(A)), \qquad A \geq 0.$$

The proof presented below of Ando and Visick's inequalities follows Ando's arguments [4]. We state these results in the following condensed form, where $^\sim$ is either omitted or acts as the transpose.

Theorem 9.2. *If $A, B > 0$, then $A \circ B \prec^{w\log} AB^\sim$, that is,*

$$\prod_{i=k}^{n} \lambda_i(A \circ B) \geq \prod_{i=k}^{n} \lambda_i(AB^\sim), \qquad k = 1, \ldots, n. \tag{9.1}$$

Proof. There exits a unital positive linear map Φ such that $\Phi(X \otimes Y) = X \circ Y$ for all $X, Y \in M_n(\mathbb{C})$. For $A, B > 0$,

$$\log(A \otimes B) = \log A \otimes I_n + I_n \otimes \log B.$$

Then $H = \log A$, $K = \log B$ are Hermitian and

$$\Phi\big(\log(A \otimes B)\big) = H \circ I_n + I_n \circ K = I_n \circ (H + K).$$

Using Lemma 9.1 with $f(t) = \log t$, $t > 0$, we have

$$\log\big(\Phi(A \otimes B)\big) = \log(A \circ B) \geq I_n \circ (H + K).$$

By Schur Majorization Theorem, $I_n \circ (H + K) \prec H + K$ holds, as $H + K$ is Hermitian. Therefore,

$$\sum_{i=k}^{n} \lambda_i(\log(A \circ B)) \geq \sum_{i=k}^{n} \lambda_i(I_n \circ (H + K)) \geq \sum_{i=k}^{n} \lambda_i(H + K)$$

for $k = 1, \ldots, n$. It follows that

$$\begin{aligned}\prod_{i=k}^{n} \lambda_i(A \circ B) &\geq \prod_{i=k}^{n} \mathrm{e}^{\lambda_i(I_n \circ (H+K))} \geq \prod_{i=k}^{n} \mathrm{e}^{\lambda_i(H+K)} = \prod_{i=k}^{n} \lambda_i\big(\mathrm{e}^{H+K}\big)\\ &\geq \prod_{i=k}^{n} \lambda_i(\mathrm{e}^H \mathrm{e}^K) = \prod_{i=k}^{n} \lambda_i(AB), \qquad k = 1, \ldots, n.\end{aligned}$$

The last inequality is a consequence of the Golden-Thompson type log-majorization (5.6). Then (9.1) with $\sim$ deleted is proved.

Since K and K^T have the same diagonal entries, $I_n \circ (H + K)$ may be replaced by $I_n \circ (H + K^T)$ in the proof above. In such case, $\mathrm{e}^H \mathrm{e}^K$ is replaced by $\mathrm{e}^H \mathrm{e}^{K^T} = AB^T$ and (9.1) with $\sim$ acting as the transpose is fullfilled. $\square$

Remark 9.3. For $A, B > 0$, by the Lie-Trotter formula, we have

$$\lim_{p\to 0} \left(A^p B^p\right)^{\frac{1}{p}} = \mathrm{e}^{\log A + \log B}$$

and a Lie-Trotter type formula for the Hadamard product [97] is

$$\lim_{p\to 0} \left(A^p \circ B^p\right)^{\frac{1}{p}} = \mathrm{e}^{I_n \circ (\log A + \log B)}$$

(see also [95, Theorem 1]). According to the previous proof, we can write

$$A \circ B \prec^{w\log} \lim_{p\to 0} \left(A^p \circ B^p\right)^{\frac{1}{p}} \prec^{w\log} \lim_{p\to 0} \left(A^p (B^{\sim})^p\right)^{\frac{1}{p}} \prec_{\log} AB^{\sim}.$$

Moreover, for $A, B > 0$ and $r > 0$, Visick [94] obtained

$$\sum_{i=k}^{n} \lambda_i (A \circ B)^{-r} \le \sum_{i=k}^{n} \lambda_i \left(AB^{\sim}\right)^{-r}, \qquad k = 1, \dots, n,$$

and deduced Theorem 9.2 from it. In fact, this is equivalent to Theorem 9.2 as shown by Bebiano and Perdigão [10]. One of the implications is a trivial consequence of the following limit:

$$\lim_{r\to 0} \frac{\lambda^{-r} - 1}{r} = -\log \lambda, \qquad \lambda > 0.$$

To prove the other, note that (9.1) implies

$$(A \circ B)^{-1} \prec_{w\log} (AB^{\sim})^{-1}.$$

Considering the function $f(t) = \log(1 + \epsilon\, \mathrm{e}^{rt})$, which is convex and increasing for $t > 0$, with $\epsilon, r > 0$, by Proposition 2.3 ii, we obtain

$$\prod_{i=k}^{n} \left(1 + \epsilon\, \lambda_i (A \circ B)^{-r}\right) \le \prod_{i=k}^{n} \left(1 + \epsilon\, \lambda_i (AB^{\sim})^{-r}\right), \quad k = 1, \dots, n.$$

The implication follows, because

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon} \left(\prod_{i=k}^{n} \left(1 + \epsilon\, \lambda_i^{-r}\right) - 1 \right) = \sum_{i=k}^{n} \lambda_i^{-r}.$$

Theorem 9.4. *Let $A, B > 0$ and $D \in M_n(\mathbb{C})$ be a diagonal matrix, assumed real when $^{\sim}$ is absent. If $\alpha \in [0, 1]$, then*

$$\begin{aligned} \prod_{i=k}^{n} \lambda_i\big((A \circ B)\, |D|^2\big) &\ge \prod_{i=k}^{n} \lambda_i\big((A\sharp B)\, \overline{D}\, (A\sharp B)^{\sim} D\big) \\ &\ge \prod_{i=k}^{n} \lambda_i\big((A\sharp_{1-\alpha} B)\, \overline{D}\, (A\sharp_{\alpha} B)^{\sim} D\big) \\ &\ge \prod_{i=k}^{n} \lambda_i\big(A\, \overline{D}\, B^{\sim} D\big), \qquad k = 1, \dots, n, \end{aligned} \tag{9.2}$$

equality occurring for $k = 1$ in the last two inequalities.

Proof. Let $D \in M_n(\mathbb{C})$ be a diagonal matrix. Then

$$D(A \circ B)\overline{D} \geq D\big((A \sharp B) \circ (A \sharp B)\big)\overline{D} = \big(D(A \sharp B)\overline{D}\big) \circ (A \sharp B)$$

and replacing A, B in Theorem 9.2 by $D(A \sharp B)\overline{D}$, $A \sharp B$, respectively, yields

$$(A \circ B)|D|^2 \prec^{w\log} (D(A \sharp B)\overline{D}) \circ (A \sharp B) \prec^{w\log} (A \sharp B)\overline{D}(A \sharp B)^{\sim} D.$$

Further, if $C = D \in H_n^{\sim}$ in Corollary 8.2 with $\sigma = \sharp_\alpha$, $\alpha \in [0,1]$, and in (8.9), the result is obtained. □

If $^{\sim}$ is deleted and $D = I_n$, then (9.2) was previously given by Ando [4, Theorem 2] and, in this case, the remaining inequalities were obtained by Hiai and Lin [48]. The complete version in Theorem 9.4 is derived in [65]. The inequalities in (9.2) hold for $A, B > 0$, $k = 1, \ldots, n$ and D diagonal, but

$$\prod_{i=k}^{n} \lambda_i\big((A \circ B)\,|D|^2\big) \geq \prod_{i=k}^{n} \lambda_i((A\sharp B)\,D^*(A\sharp B)^{\sim} D) \tag{9.3}$$

does not remain true, in general, when D is replaced by any Hermitian matrix.

Example. Consider

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 1+i \\ 1-i & -3 \end{bmatrix}.$$

In this case, $A\sharp B = B^{\frac{1}{2}}$ and (9.3) with $\sim$ absent does not hold, because

$$\lambda_2((A \circ B)\,D^2) \approx 3.783 \leq \lambda_2\big(B^{\frac{1}{2}}D\big)^2 \approx 4.095.$$

10. Indefinite inequalities

The permanent of $A = (a_{ij}) \in M_n(\mathbb{C})$ is denoted and defined as

$$\operatorname{per} A = \sum_{\sigma \in S_n} \prod_{j=1}^{n} a_{j\sigma(j)}, \quad \sigma \in S_n.$$

Although permanents and determinants have similar definitions and share some common properties, they exibit substancial differences, such as the non-multiplicativity of the permanent.

In 1926, van der Waerden raised a question [93] and motivated a conjecture: the minimum of the permanent of a n-square doubly stochastic matrix is $\frac{n!}{n^n}$ and equality occurs when every entry of the matrix equals $\frac{1}{n}$. This conjecture attracted the attention of mathematicians all over the world, although it remained open for more than fifty years. The proof of this famous conjecture by G. P. Egoritjev [26] in 1981, also proved by Falikman [27], is based on an inequality for permanents, which is a special case of a result of A. D. Alexandroff on positive definite quadratic forms. In what follows we write $\operatorname{per} A = \operatorname{per}(a_1, \ldots, a_n)$ with a_i the ith column of A.

Theorem 10.1. (Alexandroff permanental inequality) *For* $a_1, \ldots, a_n \in \mathbb{R}^n$,

$$\operatorname{per}(a_1, \ldots, a_{n-1}, b)^2 \geq \operatorname{per}(a_1, \ldots, a_{n-1}, a_{n-1})\operatorname{per}(a_1, \ldots, b, b),$$

with equality if and only if $b = \lambda a_{n-1}$ *for some constant* λ.

This inequality resembles Schwartz inequality, but the direction of the inequality sign is reversed. The reason is the following. Taking the permanent as the inner product in $\mathbb{R}^n$:

$$\langle x, y\rangle = \operatorname{per}(a_1, \ldots, a_{n-2}, x, y),$$

the space $\mathbb{R}^n$ is no longer *Euclidean* but *Lorentzian*, accordingly as the length of the vector $(x_1, \ldots, x_n)$ is

$$x_1^2 + \cdots + x_n^2 \qquad \text{or} \qquad x_1^2 - x_2^2 \cdots - x_{n-1}^2 - x_n^2.$$

That is, we are dealing now with a so called *indefinite inner product space.*

In this section, we present miscellaneous indefinite inequalities obtained in this set up. First, we introduce some definitions and notations.

Let J be a Hermitian involutive matrix, that is, $J^* = J$ and $J^2 = I_n$. Consider $\mathbb{C}^n$ endowed with the indefinite inner product induced by J, given by $[x, y] = y^*Jx$ for all $x, y \in \mathbb{C}^n$. Let $A^\sharp = JA^*J$. A matrix $A \in M_n(\mathbb{C})$ is said to be *J-Hermitian* if $A = A^\sharp$, that is, if JA is Hermitian. These matrices appear in several problems of relativistic quantum mechanics and quantum physics. Let $A, B \in M_n(\mathbb{C})$ be J-Hermitian and consider $A \geq^J B$ defined by

$$[Ax, x] \geq^J [Bx, x], \quad x \in \mathbb{C}^n,$$

which means that $J(A - B) \geq 0$. A matrix $A \in M_n(\mathbb{C})$ is called a *J-contraction* if $I_n \geq^J A^\sharp A$. It is well known that the eigenvalues of a J-Hermitian matrix $A \in M_n(\mathbb{C})$ may not be real, nevertheless its spectrum is symmetric relative to the real axis. If A is J-Hermitian and $I_n \geq^J A$, then all the eigenvalues of A are real. In fact, in this case, $I_n - A$ is the product of the Hermitian matrix J and a positive semidefinite matrix. If A is a J-contraction, by a Theorem of Potapov-Ginzburg [8, Chapter 2, Section 4], then all the eigenvalues of $A^\sharp A$ are nonnegative. Sano [83, Theorem 2.6] obtained the next indefinite version of Löwner-Heinz inequality.

Theorem 10.2. (Löwner inequality of indefinite type, 2007) *If* $A, B \in M_n(\mathbb{C})$ *are J-Hermitian matrices with nonnegative eigenvalues,* $I_n \geq^J A \geq^J B$ *and* $0 < \alpha < 1$, *then the J-Hermitian powers* A^α, B^α *are well defined and*

$$I_n \geq^J A^\alpha \geq^J B^\alpha.$$

The case $\alpha = \frac{1}{2}$ in Theorem 10.2 is due to Ando [6, Theorem 6], being the cases $\alpha = 0$ and $\alpha = 1$ trivially satisfied. Motivated by these results, the Furuta inequality of indefinite type in (10.1) and (10.2) was established by Sano [83, Theorem 3.4] and Bebiano *et al.* [14, Theorem 2.1], respectively.

Theorem 10.3. (Furuta inequality of indefinite type) *Let $A, B \in M_n(\mathbb{C})$ be J-Hermitian with nonnegative spectra, $\mu I_n \geq^J A \geq^J B$ (or $A \geq^J B \geq^J \mu I_n$) for some $\mu > 0$. Then for each $r \geq 0$,*

$$A^{\frac{p+r}{q}} \geq^J \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1}{q}} \tag{10.1}$$

and

$$\left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq^J B^{\frac{p+r}{q}} \tag{10.2}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

In particular, Löwner-Heinz inequality of indefinite type is recovered by Theorem 10.3 for $r = 0$.

In order to present the indefinite version of Theorem 4.4 obtained in [13, Corollary 1.2], assume $(r, n-r)$ to be the inertia of J and $0 < r < n$. Without loss of generality, we may consider

$$J = I_r \oplus -I_{n-r}, \qquad 0 < r < n.$$

For an arbitrary J-Hermitian matrix $A \in M_n(\mathbb{C})$, we denote by $\sigma_J^{\pm}(A)$ the set of eigenvalues of A with eigenvectors x, such that $x^* J x = \pm 1$. We say that A is *J-unitarily diagonalizable* if every eigenvalue of A belongs to either $\sigma_J^{+}(A)$ or to $\sigma_J^{-}(A)$. In this case, $\sigma_J^{+}(A)$ and $\sigma_J^{-}(A)$ have r and $n-r$ eigenvalues, respectively. Consider a J-Hermitian matrix A, whose eigenvalues $\alpha_1 \geq \cdots \geq \alpha_r$ belong to $\sigma_J^{+}(A)$ and $\alpha_{r+1} \geq \cdots \geq \alpha_n$ belong to $\sigma_J^{-}(A)$. In this case, the eigenvalues of A are said to *not interlace* if either $\alpha_r > \alpha_{r+1}$ or $\alpha_n > \alpha_1$, otherwise, they are said to *interlace.*

Theorem 10.4. *Let $J = I_r \oplus -I_{n-r}$, $0 < r < n$, and $A, C \in M_n(\mathbb{C})$ be nonscalar J-Hermitian and J-unitarily diagonalizable matrices with eigenvalues $\alpha_1 \geq \cdots \geq \alpha_r$ ($c_1 \geq \cdots \geq c_r$) in $\sigma_J^{+}(A)$ ($\sigma_J^{+}(C)$) and $\alpha_{r+1} \geq \cdots \geq \alpha_n$ ($c_{r+1} \geq \cdots \geq c_n$) in $\sigma_J^{-}(A)$ ($\sigma_J^{-}(C)$). If the eigenvalues of A and C do not interlace, then statements* i. *and* ii. *hold.*

i. *If $(\alpha_k - \alpha_l)(c_{k'} - c_{l'}) < 0$ for all $1 \leq k, k' \leq r$, $r+1 \leq l, l' \leq n$, then*

$$\operatorname{tr}(CA) \leq \sum_{i=1}^{n} c_i \alpha_i.$$

ii. *If $(\alpha_k - \alpha_l)(c_{k'} - c_{l'}) > 0$ for all $1 \leq k, k' \leq r$, $r+1 \leq l, l' \leq n$, then*

$$\sum_{i=1}^{r} c_i \alpha_{r-i+1} + \sum_{i=r+1}^{n} c_i \alpha_{n+r-i+1} \leq \operatorname{tr}(CA).$$

Several other inequalities of indefinite type have been studied. For instance, just to mention a few, we refer some spectral inequalities for the trace of the exponential or the logarithmic of J-Hermitian matrices [15], operator inequalities associated with Furuta inequality of indefinite type [16], a reversed Heinz-Kato-Furuta inequality [17] and indefinite versions of some determinantal inequalities [19], including a Fiedler-type theorem for the determinant of J-positive matrices [18].

Recently, Matharu, Malhotra and Moslehian [74] defined a J-mean associated with a positive matrix monotone function f on $(0, \infty)$, such that $f(1) = 1$, for J-Hermitian matrices with spectra in $(0, \infty)$. Fundamental properties of this J-mean, such as the power monotonicity and an indefinite version of Ando-Hiai inequality [74, Theorem 3.11] were obtained.

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