Journal of Dynamical and Control Systems
https://doi.org/10.1007/s10883-022-09616-y

On the Structure of Singular Points of a Solution to Newton's Least Resistance Problem

Alexander Plakhov[1,2]

Received: 29 March 2022 / Revised: 29 March 2022 / Accepted: 10 September 2022
© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2022

Abstract
We consider the following problem stated in 1993 by Buttazzo and Kawohl (*Math Intell* 15:7–12, 1993): minimize the functional $\int\int_\Omega (1 + |\nabla u(x,y)|^2)^{-1} dxdy$ in the class of concave functions $u : \Omega \to [0,M]$, where $\Omega \subset \mathbb{R}^2$ is a convex domain and $M > 0$. It generalizes the classical minimization problem, which was initially stated by I. Newton in 1687 in the more restricted class of radial functions. The problem is not solved until now; there is even nothing known about the structure of singular points of a solution. In this paper we, first, solve a family of auxiliary 2D least resistance problems and, second, apply the obtained results to study singular points of a solution to our original problem. More precisely, we derive a necessary condition for a point being a ridge singular point of a solution and prove, in particular, that all ridge singular points with horizontal edge lie on the top level and zero level sets.

Keywords Newton's problem of least resistance · Convex geometry · Singular points of a convex body

Mathematics Subject Classification (2010) 52A15 · 26B25 · 49Q10

1 Introduction

Isaac Newton in his *Principia* [10] considered the following minimization problem. A solid body moves with constant velocity in a sparse medium. Collisions of the medium particles with the body are perfectly elastic. The absolute temperature of the medium is zero, so as the particles are initially at rest. The medium is extremely rare, so that mutual interactions of the particles are neglected. As a result of body-particle collisions, the drag force acting on the body is created. This force is usually called *resistance*.

✉ Alexander Plakhov
plakhov@ua.pt

1 Center for R&D in Mathematics and Applications, Department of Mathematics, University of Aveiro, Aveiro, Portugal

2 Institute for Information Transmission Problems, Moscow, Russia

Published online: 21 September 2022

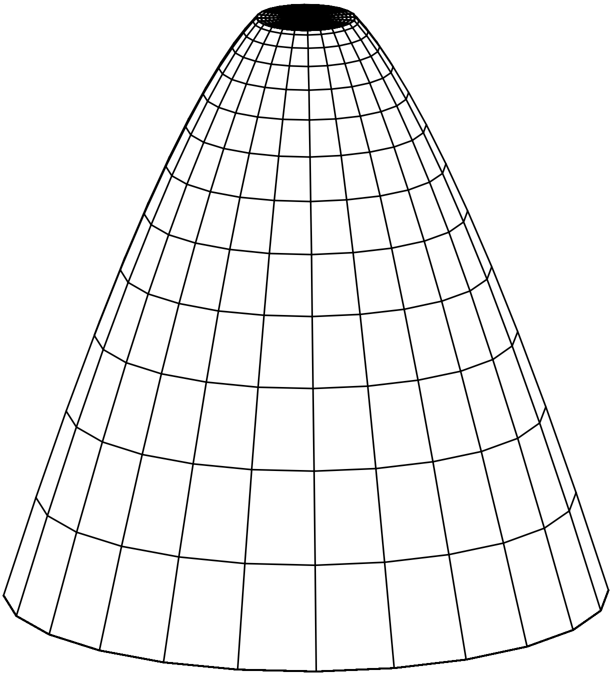


Fig. 1 A solution to the rotationally symmetric Newton problem

The problem is: given a certain class of bodies, find the body in this class with the smallest resistance. Newton considered the class of convex bodies that are rotationally symmetric and have fixed length along the direction of motion and fixed maximal width.

In modern terms the problem can be formulated as follows. Let a reference system x_1,x_2,z be connected with the body and the z-axis coincide with the symmetry axis of the body. We assume that the particles move downward along the z-axis. Let the upper part of the body's surface be the graph of a concave radially symmetric function $z = u(x_1, x_2) = \varphi\left(\sqrt{x_1^2 + x_2^2}\right)$, $x_1^2 + x_2^2 \le L^2$; then the resistance equals

$$2\pi\rho v^2 \int_0^L \frac{1}{1 + \varphi'(r)^2} rdr,$$

where the density of the medium ρ and the scalar velocity of the body v are assumed to be constant. The problem is to minimize the resistance in the class of convex monotone decreasing functions $\varphi : [0, L] \to \mathbb{R}$ satisfying $0 \le \varphi \le M$. Here M and L are the parameters of the problem: M is length of the body and $2L$ is its maximal width.

Newton gave a geometric description of the solution to the problem and did not explain how the solution was obtained. The optimal function bounds a convex body that looks like a truncated cone with slightly inflated lateral boundary. An optimal body, corresponding to the case when the length is equal to the maximal width, is shown in Fig. 1.

Starting from the pioneering paper by Buttazzo and Kawohl [4], the problem of minimal resistance has been studied in various classes of (generally) nonsymmetric and/or (generally) nonconvex bodies; see, e.g., [1–3, 5–8, 11–13, 17, 19].

In this paper we consider the generalization of the original Newton's problem to the class of convex bodies without the assumption of axial symmetry. The problem is as follows: Minimize

$$F(u) = \int\int_{\Omega} \frac{1}{1 + |\nabla u(x_1, x_2)|^2} dx_1 dx_2 \tag{1}$$

in the class of functions

$$\mathfrak{C}_M = \{u : \Omega \to \mathbb{R} : 0 \le u \le M, \ u \text{ is concave}\}.$$

Here $\Omega \subset \mathbb{R}^2$ is a compact convex set with nonempty interior int(Ω), and $M > 0$ is the parameter of the problem.

Surprisingly enough, this problem is still poorly understood. It is known that there exists at least one solution [3, 9]. Let u be a solution; then $u\rfloor_{\partial\Omega} = 0$ [14] and at any regular point $x = (x_1, x_2)$ of u we have either $|\nabla u(x)| \ge 1$, or $|\nabla u(x)| = 0$ [3]. Moreover, if the set $L = \{x : u(x) = M\} \subset \mathbb{R}^2$ has nonempty interior then we have $\lim_{\substack{x \notin L \\ x \to \bar{x}}} |\nabla u(x)| = 1$ for almost all $\bar{x} \in \partial L$ [15]. If u is regular in an open set $\mathcal{U} \subset \Omega$, then the surface $\operatorname{graph}\left(u\rfloor_{\mathcal{U}}\right) = \{(x, u(x)) : x \in \mathcal{U}\}$ does not contain extreme points of the convex body[1]

$$C_u = \{(x, z) : x \in \Omega, \ 0 \le z \le u(x)\},$$

and therefore, is developable [16].

This paper is devoted to studying singular points of an optimal function u or, equivalently, singular points of the corresponding convex body $C = C_u$.

A singular point r_0 on the boundary of a convex body C is called *conical point*, if the tangent cone to C at r_0 is not degenerate, and *ridge point*, if the tangent cone degenerates into a dihedral angle (see, e.g., [18]). In this paper we consider ridge points, postponing the study of conical points to the future.

Let $r_0 = (x, u(x))$, $x \in \text{int}(\Omega)$ be a ridge point of $C = C_u$ and e_1 and e_2 be the outward normals to the faces of the corresponding dihedral angle. Introduce some additional notation. Let l be the edge of this angle, and denote by $\theta \in [0, \pi/2)$ the angle between l and the x-plane. Draw a plane orthogonal to l, that is, parallel to e_1 and e_2. The angle between the plane and the z-axis is θ.

Take an orthonormal basis f_1, f_2 in this plane, so as f_1 is horizontal and the z-coordinate of f_2 is positive. Let e_1 and e_2 form the angles φ_1 and φ_2 with f_2 counted in a certain direction, $-\pi/2 < \varphi_2 < \varphi_1 < \pi/2$. In appropriate coordinates x_1, x_2 on the x-plane, each vector $e_i, i = 1, 2$ takes the form

$$e_i = (-\sin\varphi_i, \cos\varphi_i \sin\theta, \cos\varphi_i \cos\theta).$$

The triple of angles θ, φ_1, and φ_2 uniquely defines a dihedral angle, up to motions of the x-plane.

The following Theorem 1 is the main result of this paper. It establishes a necessary condition for a ridge singular point of an optimal body.

Theorem 1 Let θ, φ_1, and φ_2 be the angles associated with a ridge point of an optimal body.

(a) If $\theta \in [\pi/4, \pi/2)$ then φ_1 and φ_2 satisfy the inequalities

[1] A convex body is a convex compact set with nonempty interior.

$$2\varphi_1 + \varphi_2 \le \pi/2, \qquad \varphi_1 + 2\varphi_2 \ge -\pi/2;$$

the corresponding set of admissible points (φ_1,φ_2) is the triangle (i) in Fig. 5.

(b) If $\theta \in (0,\pi/4)$ then

$$2\varphi_1 + \varphi_2 \le \pi/2, \quad \varphi_1 + 2\varphi_2 \ge -\pi/2, \quad |\varphi_1| \ge \varphi_*, \quad |\varphi_2| \ge \varphi_*,$$

where $\varphi_* = \arccos\left(\frac{1/\sqrt{2}}{\cos\theta}\right)$. The corresponding set is shown lightgray in Fig. 6a.

(c) If $\theta = 0$ then either $(\varphi_1,\varphi_2) = (\pi/4,0)$, or $(\varphi_1,\varphi_2) = (0,-\pi/4)$, or φ_1 and φ_2 satisfy

$$2\varphi_1 + \varphi_2 \le \pi/2, \quad \varphi_1 + 2\varphi_2 \ge -\pi/2, \quad \varphi_1 \ge \pi/4, \quad \varphi_2 \le -\pi/4.$$

The corresponding set is the union of two points and a quadrangle shown lightgray in Fig. 6b.

Remark 1 Observe that almost all points on the boundary $\partial\mathscr{L}$ of the upper level set (ULS) $\mathscr{L} = \{(x, u(x)) : u(x) = M\}$ are ridge points with horizontal edge. The rest of the points on $\partial\mathscr{L}$ are (finitely or countably many) conical points.

If $\mathscr{L}$ has nonempty interior, each ridge point on $\partial\mathscr{L}$ corresponds to one of the points $(\pi/4,0)$ and $(0,-\pi/4)$; see Fig. 6b. If, otherwise, $\mathscr{L}$ is a line segment, each ridge point on $\partial\mathscr{L}$ corresponds to a point of the quadrangle shown lightgray in Fig. 6b. There are no other ridge points with horizontal edge.

Let us formulate this observation as a corollary of the theorem.

Corollary 1 All ridge points with horizontal edge of an optimal body lie on the boundary of the upper level set $C_u \cap \{z = M\}$ and on the boundary of the zero level set $C_u \cap \{z = 0\}$.

Remark 2 Numerical simulation [19] indicates that if Ω is a unit circle and $M \lesssim 1.5$ then the ULS of an optimal body is a regular polygon and the lateral surface of the body is foliated by line segments and planar triangles. Moreover, a part of the surface near ULS is foliated by horizontal segments. See Fig. 2, where the ULS is a square. It follows from Corollary 1 that all points of this part of surface, except for endpoints of the segments, are regular.

Further, there are several singular curves on the lateral surface joining the endpoints of the polygon with points of the base $\partial\Omega \times \{0\}$. Each point of such a curve is a ridge point and satisfies $\theta \neq 0, \varphi_1 = -\varphi_2$. Thus, we come to the following conjecture.

Conjecture 1 If Ω is a unit circle and the ULS has nonempty interior then all singular points outside ULS are ridge points and satisfy $\theta \neq 0, \varphi_1 = -\varphi_2$.

On the other hand, if the ULS is a line segment or Ω is not a circle, the structure of singular points on the lateral surface is unclear.

2 Surface Area Measure of Convex Bodies

Here we provide some information concerning the surface area measure of convex bodies and representation of the resistance in terms of surface area measure, which will be needed later on.

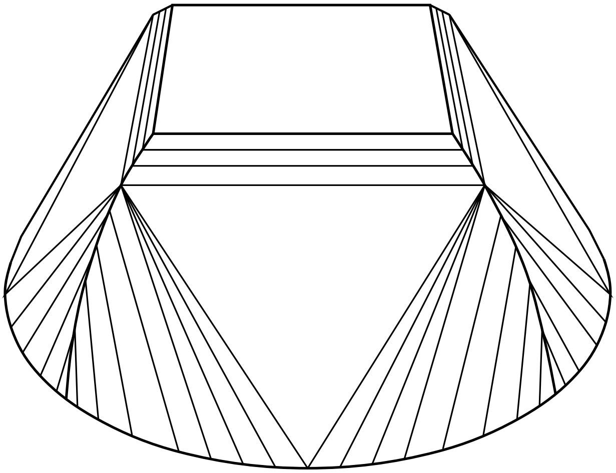


Fig. 2 An optimal body with the upper level set being a square

Let C be a convex body in $\mathbb{R}^d$. Denote by n_r the outward normal to C at a regular point $r \in \partial C$. The surface measure of C is the Borel measure ν_C in S^{d-1} defined by

$$\nu_C(A) = \mathrm{Leb}(\{r \in \partial C : n_r \in A\})$$

for any Borel set $A \subset S^2$, where Leb means the $(d-1)$-dimensional Lebesgue measure on ∂C. We will only need the cases $d = 2$ and 3.

It is well known that the surface area measure satisfies the equation

$$\int_{S^{d-1}} n d\nu_C(n) = \vec{0}.$$

In a similar way one defines the measure induced by a Borel subset of ∂C.

The functional $F(u)$ in (1) can be represented in terms of surface area measure as $F(u) = \mathcal{F}(C_u)$, where

$$\mathcal{F}(C) = \int_{S^2} (n_3)_+^3 d\nu_C(n).$$

Here $n = (n_1, n_2, n_3)$ and $z_+ = \max\{z, 0\}$ means the positive part of a real number z.

A similar formula holds in the 2D case. Let $u : [a, b] \to \mathbb{R}$ be a concave function and C_u be the convex body bounded above by the graph of u and below by the segment joining the points $(a, u(a))$ and $(b, u(b))$. Then we have $F(u) = \mathcal{F}(C_u)$, where

$$F(u) = \int_a^b \frac{1}{1 + u'^2(x)} dx$$

and

$$\mathcal{F}(C) = \int_{S^1} (n_3)_+^3 d\nu_C(n),$$

with $n = (n_1, n_3)$.

Let μ_C be the push-forward measure of ν_C under the map from S^1 to $(-\pi,\pi]$ given by $(-\sin\varphi, \cos\varphi) \mapsto \varphi$. Then the latter formula can be rewritten as

$$\mathcal{F}(C) = \int_{-\pi/2}^{\pi/2} (\cos\varphi)^3 d\mu_C(\varphi).$$

3 2D Problems of Minimal Resistance

Here we state and solve some auxiliary 2-dimensional problems of minimal resistance. They will be used in the next section.

The direct generalization of Newton's problem to the 2D case is as follows: given $M > 0$, minimize the integral $\int_0^1 (1 + u'^2(x))^{-1} dx$ in the class of concave functions $u : [0,1] \to \mathbb{R}$ such that $u(0) = M$, $u(1) = 0$ and $u'(x) \le 0$. This problem and its solution were stated in [4]; the solution is

$$u(x) = \begin{cases} \min\{M, 1 - x\}, & \text{if } M < 1, \\ M(1 - x), & \text{if } M \ge 1. \end{cases}$$

We will consider the problem in a slightly different form. Suppose that we are given 4 real numbers $x_0 > 0, z_0, k_1, k_2$ such that $k_2 < z_0/x_0 < k_1$ and $x_0^2 + z_0^2 = 1$, and denote $K_0 = z_0/x_0$.

Problem. Minimize the integral

$$\int_0^{x_0} \frac{1}{1 + u'^2(x)} dx \tag{2}$$

in the class $\mathfrak{C}_{K_0,k_1,k_2}$ of concave functions $u : [0, x_0] \to \mathbb{R}$ such that $u(0) = 0$, $u(x_0) = z_0$, and $k_2 \le u'(x) \le k_1$.

The problem can be interpreted as minimizing the resistance in the class of planar convex bodies that are contained in the triangle ABC and contain the points $A = (0,0)$ and $B = (x_0,z_0)$. The slopes of the sides AC and BC are k_1 and k_2, respectively; see Fig. 3. It is assumed that there is a flow incident on the body moving downward in the z-direction.

Lemma 1 The solution u to problem (2) is unique.

(i) Let $-\sqrt{1 + k_1^2} \le k_1 + k_2 \le \sqrt{1 + k_2^2}$. Then the derivative $u'(x)$ takes two values, k_1 and k_2, and therefore, $u(x) = \min\{k_1 x, z_0 + k_2(x - x_0)\}$, and the graph of u is the broken line ACB.

(ii) Let $k_1 \le -1/\sqrt{3}$ or $k_2 \ge 1/\sqrt{3}$. Then $u(x) = K_0 x$, and therefore, the graph of u is the segment AB.

(iii) Let $k_2 < 1/\sqrt{3}$, $k_1 > \sqrt{1 + k_2^2} - k_2$. Then

  (a) if $k_2 < K_0 < \sqrt{1 + k_2^2} - k_2$ then $u'(x)$ takes two values, k_2 and $\sqrt{1 + k_2^2} - k_2$, and the graph of u is the broken line $AC''B$, where C'' is a point on CB;

  (b) if $\sqrt{1 + k_2^2} - k_2 \le K_0 < k_1$ then $u(x) = K_0 x$.

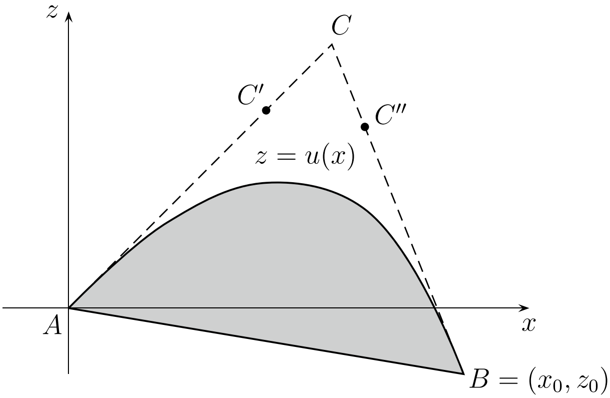


Fig. 3 The triangle ABC and a typical convex body from the chosen class

(iv) Let $k_1 > -1/\sqrt{3}$, $k_2 < -\sqrt{1+k_1^2} - k_1$. Then

(a) if $-\sqrt{1+k_1^2} - k_1 < K_0 < k_1$ then $u'(x)$ takes the values k_1 and $-\sqrt{1+k_1^2} - k_1$, and the graph of u is the broken line $AC'B$, where C' is a point on AC;

(b) if $k_2 < K_0 \le -\sqrt{1+k_1^2} - k_1$ then $u(x) = K_0 x$.

Proof Denote by $p(\xi)$ the restriction of the function $1/(1+\xi^2)$ on the segment $[k_2,k_1]$, and by $\bar{p}(\xi)$, the maximal convex function on $[k_2,k_1]$ satisfying $\bar{p}(\xi) \le p(\xi)$. In other words, the epigraph of $\bar{p}$ is the convex hull of the epigraph of p.

One easily checks that the function $1/(1+\xi^2)$ is convex on the intervals $(-\infty, -1/\sqrt{3}]$ and $[1/\sqrt{3}, +\infty)$ and concave on the interval $[-1/\sqrt{3}, 1/\sqrt{3}]$. Note that the graph of $\bar{p}$ cannot contain two strictly convex arcs separated by a line segment; otherwise the line containing this segment is tangent to the graph of $z = 1/(1+\xi^2)$ at two points, which is impossible.

Likewise, the graph of $\bar{p}$ does not contain two line segments separated by a (strictly convex) arc of graph(p). Indeed, otherwise the endpoints of each segment bound a part of graph(p) containing at least one point where $p'' < 0$. This means that there exist points $\xi_1 < \xi_2 < \xi_3$ such that $p''(\xi_1) < 0, p''(\xi_2) > 0, p''(\xi_3) < 0$, which is impossible.

Therefore there are 4 possibilities for graph($\bar{p}$):

(i) it is the line segment joining the endpoints of graph(p);
(ii) it coincides with graph(p);
(iii) it is the union of a line segment on the left and a part of graph(p) on the right;
(iv) vice versa: it is the union of a part of graph(p) on the left and a line segment on the right.

The case (ii) is realized, if and only if the intervals $[k_2,k_1]$ and $(-1/\sqrt{3}, 1/\sqrt{3})$ do not intersect, that is, either $k_2 < k_1 \le -1/\sqrt{3}$ or $1/\sqrt{3} \le k_2 < k_1$. Let us consider conditions for realization of cases (iii) and (iv).

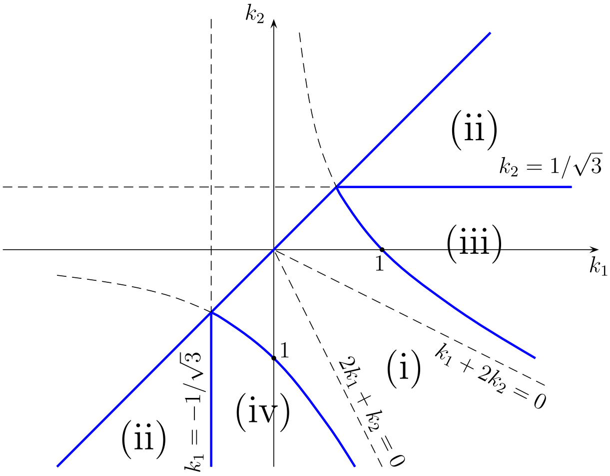


Fig. 4 The domains (i), (ii), (iii), (iv).

The necessary and sufficient conditions for graph($\bar{p}$) to be the union of a line segment projecting to $[k_2, \bar{k}]$ and a part of graph(p) projecting to $[\bar{k}, k_1]$ are that $k_2 < 1/\sqrt{3}$, $\bar{k} > 1/\sqrt{3}$, and the straight line through $(k_2, p(k_2))$ and $(\bar{k}, p(\bar{k}))$ is tangent to graph(p) at the point $(\bar{k}, p(\bar{k}))$. The condition of tangency looks as follows:

$$\frac{p(\bar{k}) - p(k_2)}{\bar{k} - k_2} = p'(\bar{k}).$$

Using that $p(\xi) = 1/(1 + \xi^2)$, after a simple algebra one obtains the equation $\bar{k}^2 + 2k_2\bar{k} - 1 = 0$, which has two solutions. The solution $\bar{k} = -k_2 - \sqrt{1 + k_2^2} < 0 < 1/\sqrt{3}$ does not serve, therefore we have $\bar{k} = -k_2 + \sqrt{1 + k_2^2}$. One easily sees that $\bar{k} > 1/\sqrt{3}$.

Thus, the conditions (iii) for graph($\bar{p}$) be composed of a line segment on the left and a strictly convex part on the right are:

$$k_2 < 1/\sqrt{3}, \qquad k_1 > -k_2 + \sqrt{1 + k_2^2}.$$

In a similar way one obtains that the conditions (iv) for graph($\bar{p}$) to be the union of a strictly convex part on the left and line segment on the right are:

$$k_1 > -1/\sqrt{3}, \qquad k_2 < -k_1 - \sqrt{1 + k_1^2}.$$

The resting part corresponding to case (i) is described by the inequalities

$$-\sqrt{1 + k_1^2} \le k_1 + k_2 \le \sqrt{1 + k_2^2};$$

see Fig. 4.

In the case (i) the value $\int_0^{x_0} \bar{p}(u'(x))dx$ is constant for all $u \in \mathfrak{C}_{K_0,k_1,k_2}$. We have the inequality $\bar{p} \le p$, with the equality being attained only at k_1 and k_2. Therefore,

$$\int_0^{x_0} \bar{p}(u'(x))dx \le \int_0^{x_0} p(u'(x))dx,$$

and the equality is attained *iff* u' takes only the values k_1 and k_2. Thus, claim (i) of the lemma is proved.

In the case (ii) the function p is strictly convex, and by Jensen's inequality,

$$\int_0^{x_0} p(u'(x))dx \ge x_0 p(K_0),$$

with the equality being attained *iff* $u'(x) = K_0$ for all x. Thus, claim (ii) is also proved.

In the case (iii) the function $\bar{p}$ is linear on $[k_2, \bar{k}]$ and strictly convex on $[\bar{k}, k_1]$, with $\bar{k} = -k_2 + \sqrt{1 + k_2^2}$. Besides, $\bar{p} = p$ on $\{k_2\} \cup [\bar{k}, k_1]$, and $\bar{p} < p$ on $(k_2, \bar{k})$. We have

$$\int_0^{x_0} p(u'(x))dx \ge \int_0^{x_0} \bar{p}(u'(x))dx \ge x_0 \bar{p}(K_0).$$

The former inequality becomes equality *iff* $u'(x) \in \{k_2\} \cup [\bar{k}, k_1]$ for all x. In the case (iii)(a), $k_2 < K_0 < \bar{k}$, the latter inequality becomes equality *iff* $u' \in [k_2, \bar{k}]$. In the case (iii)(b), $\bar{k} \le K_0 < k_1$, the latter inequality becomes equality *iff* $u' = K_0$. We conclude that the integral $\int_0^{x_0} p(u'(x))dx$ takes its minimal value $x_0 \bar{p}(K_0)$, if u' takes the values k_2 and $\bar{k}$ in the case (iii)(a), and if $u' = K_0$ in the case (iii)(b). Claim (iii) is proved.

The proof of claim (iv) is completely analogous to the proof of claim (iii). Lemma 1 is proved. □

One can reformulate these results in terms of the surface measure.

Denote $\varphi_0 = \arctan K_0$, $\varphi_1 = \arctan k_1$, $\varphi_2 = \arctan k_2$. We have $\varphi_2 < \varphi_0 < \varphi_1$. We shall use the notation $e_\varphi = (-\sin\varphi, \cos\varphi) \in S^1$. The 2-dimensional convex body bounded below by the segment AB and above by the graph of u induces the measure $-\delta_{\varphi_0} + \mu_u$, where the measure $\mu = \mu_u$ is supported on $[\varphi_2,\varphi_1]$ and satisfies the relation

$$\int_{\varphi_2}^{\varphi_1} e_\varphi d\mu(\varphi) = e_{\varphi_0}. \tag{3}$$

Denote by $\mathscr{M}_{\varphi_0,\varphi_1,\varphi_2}$ the set of measures μ on $[\varphi_2,\varphi_1]$ satisfying (3). It is well known that there is a one-to-one correspondence between the set of measures $\mathscr{M}_{\varphi_1,\varphi_2,\varphi_0}$ and the set of functions $\mathfrak{C}_{K_0,k_1,k_2}$.

Now problem (2) can be reformulated as follows.

Problem. Given $-\pi/2 \le \varphi_2 < \varphi_0 < \varphi_1 \le \pi/2$, minimize the integral

$$\Phi(\mu) = \int_{\varphi_2}^{\varphi_1} (\cos\varphi)^3 d\mu(\varphi) \tag{4}$$

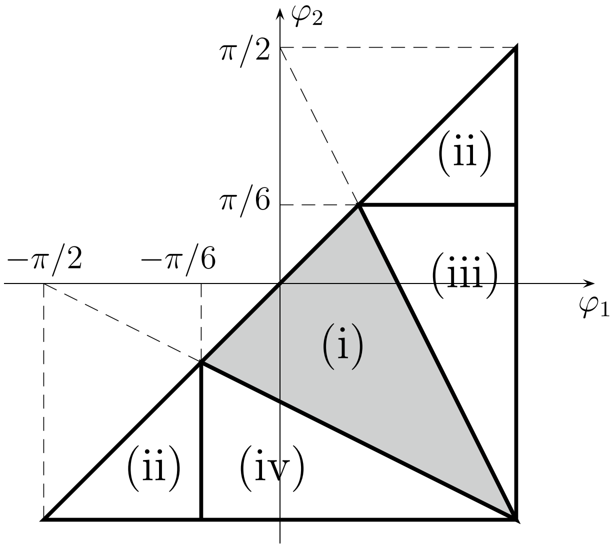
in the class of measures $\mathscr{M}_{\varphi_0,\varphi_1,\varphi_2}$.

Choose $\lambda_1 > 0$ and $\lambda_2 > 0$ so as $\lambda_1 e_{\varphi_1} + \lambda_2 e_{\varphi_2} = e_{\varphi_0}$.

Define the values $\bar{\lambda}_1$ and $\bar{\lambda}_2$ in the two particular cases, which are indicated as (iii)(a) and (iv)(a) in the following Lemma 2 and cannot occur simultaneously.

If $\varphi_2 < \pi/6$, $2\varphi_1 + \varphi_2 > \pi/2$, and $\varphi_2 < \varphi_0 < (\pi - 2\varphi_2)/4$ (case (iii)(a)), choose $\bar{\lambda}_1 > 0$ and $\bar{\lambda}_2 > 0$ so as

Fig. 5 The domains (i), (ii), (iii), (iv) in terms of φ_1 and φ_2



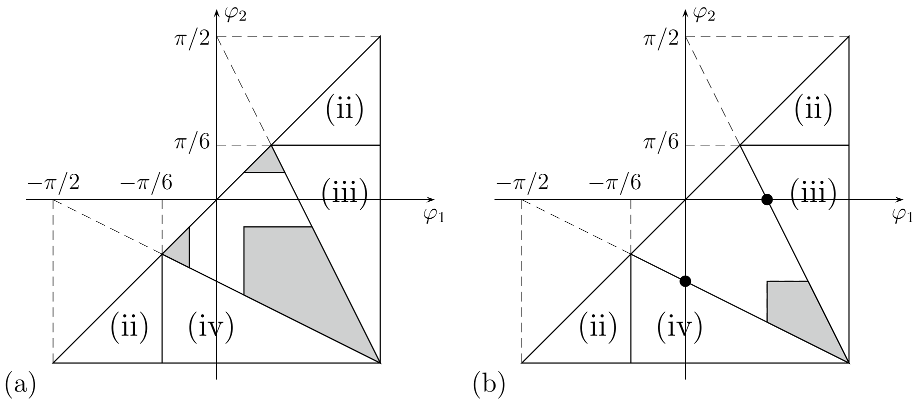


Fig. 6 (**a**) $\theta \neq 0$; (**b**) $\theta = 0$

$$\bar{\lambda}_1 e_{(\pi-2\varphi_2)/4} + \bar{\lambda}_2 e_{\varphi_2} = e_{\varphi_0}.$$

If $\varphi_1 > -\pi/6$, $\varphi_1 + 2\varphi_2 < -\pi/2$, and $-(\pi + 2\varphi_1)/4 < \varphi_0 < \varphi_1$ (case (iv)(a)), choose $\bar{\lambda}_1 > 0$ and $\bar{\lambda}_2 > 0$ so as

$$\bar{\lambda}_1 e_{\varphi_1} + \bar{\lambda}_2 e_{-(\pi+2\varphi_1)/4} = e_{\varphi_0}.$$

The following Lemma 2 is just a reformulation of Lemma 1 in terms of the angles φ_0, φ_1, φ_2.

Lemma 2 The solution μ to problem (4) is unique.

(i) Let $2\varphi_1 + \varphi_2 \leq \pi/2$ and $\varphi_1 + 2\varphi_2 \geq -\pi/2$; then $\mu = \lambda_1 \delta_{e_{\varphi_1}} + \lambda_2 \delta_{e_{\varphi_2}}$.
(ii) Let $\varphi_1 \leq -\pi/6$ or $\varphi_2 \geq \pi/6$; then $\mu = \delta_{e_{\varphi_0}}$.
(iii) Let $\varphi_2 < \pi/6$ and $2\varphi_1 + \varphi_2 > \pi/2$; then

   (a) if $\varphi_2 < \varphi_0 < (\pi - 2\varphi_2)/4$ then $\mu = \bar{\lambda}_1 \delta_{e_{(\pi-2\varphi_2)/4}} + \bar{\lambda}_2 \delta_{e_{\varphi_2}}$.
   (b) if $(\pi - 2\varphi_2)/4 \leq \varphi_0 < \varphi_1$ then $\mu = \delta_{e_{\varphi_0}}$.

(iv) Let $\varphi_1 > -\pi/6$, $\varphi_1 + 2\varphi_2 < -\pi/2$; then

   (a) if $-(\pi + 2\varphi_1)/4 < \varphi_0 < \varphi_1$ then $\mu = \bar{\lambda}_1 \delta_{e_{\varphi_1}} + \bar{\lambda}_2 \delta_{e_{-(\pi+2\varphi_1)/4}}$;
   (b) if $\varphi_2 < \varphi_0 \leq -(\pi + 2\varphi_1)/4$ then $\mu = \delta_{e_{\varphi_0}}$.

See Fig. 5.

4 Proof of Theorem 1

Recall that $r_0 = (x, u(x)), x \in \mathrm{int}(\Omega)$ is a ridge point of $C = C_u$ and e_1 and e_2 are the outward normals to the faces of the corresponding dihedral angle. Choose a vector e on the smaller arc of the great circle in S^2 through e_1 and e_2; we have $|e| = 1$ and $e = \lambda_1 e_1 + \lambda_2 e_2$ for some $\lambda_1 > 0, \lambda_2 > 0$.

Let θ be the angle between a plane parallel to e_1 and e_2 and the z-axis. In appropriate coordinates x_1, x_2 on the x-plane the vectors e_1, e_2, and e have the form

$$e_i = (-\sin\varphi_i, \cos\varphi_i \sin\theta, \cos\varphi_i \cos\theta), \quad i = 1,\ 2, \quad \varphi_2 < \varphi_1;$$

$$e = (-\sin\varphi_0, \cos\varphi_0 \sin\theta, \cos\varphi_0 \cos\theta) \quad \text{for some} \quad \varphi_2 < \varphi_0 < \varphi_1.$$

Take $t > 0$ and consider the convex body

$$C_t = C \cap \{r : \langle r - r_0, e\rangle \geq -t\};$$

it is the piece of C cut off by the plane with the normal vector e at the distance t from r_0. Here $\langle\cdot,\cdot\rangle$ means the scalar product. The body C_t is bounded by the planar domain

$$B_t = C \cap \{r : \langle r - r_0, e\rangle = -t\}$$

and the convex surface

$$S_t = \partial C \cap \{r : \langle r - r_0, e\rangle \geq -t\}; \tag{5}$$

that is, $\partial C_t = B_t \cup S_t$.

Let ν_{S_t} be the measure induced by the surface S_t. It is proved in Theorem 2 in [15] that there exists at least one weak partial limit of the normalized measure $\frac{1}{|B_t|}\nu_{S_t}$ as $t \to 0$. In other words, there exists at least one sequence $t_i \to 0^+$ as $i \to \infty$ such that $\frac{1}{|B_{t_i}|}\nu_{S_{t_i}}$ weakly converges to a measure ν_* as $i \to \infty$. Moreover, the support of the limiting measure ν_* is contained in the smaller arc of the great circle through the vectors e_1 and e_2 and contains these vectors. Additionally, one has

$$\int_{S^2} n d\nu_*(n) = e.$$

Since the body C is optimal, the following inequality holds

$$\lim_{i\to\infty} \frac{1}{|B_{t_i}|}\big[\mathcal{F}(C) - \mathcal{F}(C_{t_i})\big] = \int_{S^2} (n_3)^3 d\nu_*(n) - (e_3)^3 \le 0. \tag{6}$$

Let μ_* be the push-forward measure of ν_* under the map from the great circle through e_1 and e_2 to $(-\pi,\pi]$ defined by $(-\sin\varphi, \cos\varphi\sin\theta, \cos\varphi\cos\theta) \mapsto \varphi$. The support of μ_* is contained in $[\varphi_2,\varphi_1]$ and contains the points φ_2 and φ_1, therefore μ_* is not an atom of the form δ_φ, $\varphi \in (\varphi_2,\varphi_1)$. Additionally, one has

$$\int_{\varphi_2}^{\varphi_1} (-\sin\varphi, \cos\varphi\sin\theta, \cos\varphi\cos\theta) d\mu_*(\varphi) = (-\sin\varphi_0, \cos\varphi_0\sin\theta, \cos\varphi_0\cos\theta).$$

It follows that $\mu_* \in \mathscr{M}_{\varphi_0,\varphi_1,\varphi_2}$.

Using that $n_3 = \cos\varphi\cos\theta$ and $e_3 = \cos\varphi_0\cos\theta$, the inequality in (6) can be rewritten in terms of μ_* as

$$\int_{\varphi_2}^{\varphi_1} (\cos\varphi\cos\theta)^3 d\mu_*(\varphi) \le (\cos\varphi_0\cos\theta)^3.$$

Using the notation

$$\Phi(\mu) = \int_{\varphi_2}^{\varphi_1} (\cos\varphi)^3 d\mu(\varphi),$$

we obtain the inequality

$$\Phi(\mu_*) \le \Phi(\delta_{\varphi_0}). \tag{7}$$

By Lemma 2, in the cases (ii), (iii), and (iv) one can choose $\varphi_0 \in (\varphi_2,\varphi_1)$ so as the unique minimum of Φ is attained at δ_{φ_0}, and so, $\Phi(\mu_*) > \Phi(\delta_{\varphi_0})$, in contradiction with (7). Thus, only the case (i) can be realized.

Consider three possible cases for θ.

(a) $\theta \in [\pi/4,\pi/2)$. Since the case (i) in Lemma 2 is realized, we have

$$2\varphi_1 + \varphi_2 \le \pi/2 \quad \text{and} \quad \varphi_1 + 2\varphi_2 \ge -\pi/2.$$

Claim (a) of Theorem 1 is proved.

(b) $\theta \in (0,\pi/4)$. Additionally to the inequalities in (a), we use that the slope of the surface of an optimal body at any regular point is either equal to 0, or greater than or equal to $\pi/4$ (Theorem 2.3 in [3]). It follows that the angle between e_i, $i = 1,2$ and the z-axis is either 0, or $\ge \pi/4$. Since $\theta \neq 0$, these angles cannot be equal to zero. It follows that

$$\cos\varphi_1\cos\theta \le 1/\sqrt{2} \quad \text{and} \quad \cos\varphi_2\cos\theta \le 1/\sqrt{2},$$

hence

$$|\varphi_1| \geq \varphi_*, \quad |\varphi_2| \geq \varphi_*, \quad \text{where } \varphi_* = \arccos\left(\frac{1/\sqrt{2}}{\cos\theta}\right) < \frac{\pi}{4}.$$

The admissible set is shown light gray in Fig. 6a. It is the disjoint union of a quadrangle and two triangles, if $\theta \geq \arccos\sqrt{2/3}$, and a quadrangle, if $0 < \theta < \arccos\sqrt{2/3}$. Claim (b) of Theorem 1 is proved.

(c) $\theta = 0$. Each angle between e_i, $i = 1,2$ and the z-axis is either equal to 0, or greater than or equal to $\pi/4$. It follows that either $\varphi_i = 0$ or $|\varphi_i| \geq \pi/4$, $i = 1,2$. Taking into account the inequalities of case (a), one concludes that (φ_1, φ_2) either coincides with one of the points $(\pi/4, 0)$ and $(0, -\pi/4)$, or lies in the domain $2\varphi_1 + \varphi_2 \leq \pi/2$, $\varphi_1 + 2\varphi_2 \geq -\pi/2$, $\varphi_1 \geq \pi/4$, $\varphi_2 \leq -\pi/4$; see Fig. 6b. Claim (c) of Theorem 1 is proved.

Acknowledgements This work was supported by the Center for Research and Development in Mathematics and Applications (CIDMA) through the Portuguese Foundation for Science and Technology (FCT), within projects UIDB/04106/2020 and UIDP/04106/2020.

Data Availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

References

1. Akopyan A, Plakhov A. Minimal resistance of curves under the single impact assumption. SIAM J Math Anal 2015;47:2754–2769.
2. Brock F, Ferone V, Kawohl B. A symmetry problem in the calculus of variations. Calc Var 1996;4:593–599.
3. Buttazzo G, Ferone V, Kawohl B. Minimum problems over sets of concave functions and related questions. Math Nachr 1995;173:71–89.
4. Buttazzo G, Kawohl B. On Newton's problem of minimal resistance. Math Intell 1993;15:7–12.
5. Buttazzo G, Guasoni P. Shape optimization problems over classes of convex domains. J Convex Anal 1997;4(2):343–351.
6. Comte M, Lachand-robert T. Newton's problem of the body of minimal resistance under a single-impact assumption. Calc Var Partial Differ Equ 2001;12: 173–211.
7. Comte M, Lachand-robert T. Existence of minimizers for Newton's problem of the body of minimal resistance under a single-impact assumption. J Anal Math 2001;83:313–335.
8. Lachand-Robert T, Oudet E. Minimizing within convex bodies using a convex hull method. SIAM J Optim 2006;16:368–379.
9. Marcellini P. Nonconvex integrals of the Calculus of Variations. Proceedings of Methods of Nonconvex Analisys, Lecture Notes in Math. 1990;1446:16–57.
10. Newton I. Philosophiae naturalis principia mathematica. London: Streater; 1687.
11. Plakhov A. Billiards and two-dimensional problems of optimal resistance. Arch Ration Mech Anal 2009;194:349–382.
12. Plakhov A. Problems of minimal resistance and the Kakeya problem. SIAM Rev 2015;57:421–434.
13. Plakhov A. Exterior billiards. Systems with impacts outside bounded domains. New York: Springer; 2012. xiv+284 pp. ISBN: 978-1-4614-4480-0.
14. Plakhov A. A note on Newton's problem of minimal resistance for convex bodies. Calc Var Partial Differ Edu 2020;59:167.
15. Plakhov A. On generalized Newton's aerodynamic problem. Trans Moscow Math Soc 2021;82:217–226.
16. Plakhov A. Method of nose stretching in Newton's problem of minimal resistance. Nonlinearity 2021;34:4716–4743.
17. Plakhov A, Tchemisova T. Problems of optimal transportation on the circle and their mechanical applications. J Diff Eqs 2017;262:2449–2492.
18. Pogorelov AV. 1973. Extrinsic geometry of convex surfaces. Providence R.I.: American Mathematical Society (AMS).

19. Wachsmuth G. The numerical solution of Newton's problem of least resistance. Math Program A 2014;147:331–350.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.