Non-symmetric Number Triangles Arising from Hypercomplex Function Theory in $\mathbb{R}^{n+1}$

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Abstract. The paper is focused on *intrinsic properties* of a one-parameter family of non-symmetric number triangles $\mathcal{T}(n)$, $n \geq 2$, which arises in the construction of *hyperholomorphic Appell polynomials*.

Keywords: Non-symmetric Pascal triangle · Clifford algebra · Recurrence relation

1 Introduction

A one-parameter family of non-symmetric Pascal triangles was considered in [8] and a set of its basic properties was proved. Such family arises from studies on generalized Appell polynomials in the framework of Hypercomplex Function Theory in $\mathbb{R}^{n+1}, n \geq 1$, (cf. [7]). If $n \geq 2$, it is given by the infinite triangular array, $\mathcal{T}(n)$, of rational numbers

$$T_s^k(n) = \binom{k}{s} \frac{(\frac{n+1}{2})_{k-s}(\frac{n-1}{2})_s}{(n)_k}, \ k = 1, 2, \ldots, ; s = 0, 1, \ldots, k, \tag{1}$$

where $(a)_r := a(a+1)\ldots(a+r-1)$, for any integer $r \geq 1$, is the Pochhammer symbol with $(a)_0 := 1, a \geq 0$. If $n = 1$, then the triangle degenerates to a unique column because $T_0^k(1) \equiv 1$ and, as usual $T_s^k(1) := 0, \ s > 0$.

The non-symmetric structure of this triangle $\mathcal{T}(n)$ is a consequence of the peculiarities of a non-commutative Clifford algebra $\mathcal{C}\ell_{0,n}$ frequently used in problems of higher dimensional Harmonic Analysis, like the solution of spinor systems as n-dimensional generalization of Dirac equations and their application in Quantum Mechanics and Quantum-Field Theory [6].

Hypercomplex Function Theory in $\mathbb{R}^{n+1}$ is a natural generalization of the classical function theory of one complex variable in the framework of Clifford Algebras. The case $n > 1$ extends the complex case to paravector valued functions

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of n hypercomplex non-commutative variables or, by duality, of one paravector valued variable. For more details we refer to the articles [6,8,9,11].

Important to notice that the use of non-commutative Clifford algebras $\mathcal{C}\ell_{0,n}$ causes interesting challenges for dealing with polynomials in the hypercomplex setting. In particular, this concerns a hypercomplex counterpart of the *Binomial Theorem* and, naturally, some type of a hypercomplex Pascal triangle. In fact, the non-symmetric $\mathcal{T}(n),\ n \geq 2$, plays just the role of the array of coefficients in a sequence of generalized hypercomplex Appell polynomials as binomial coefficients are playing in the binomial expansion of $(x+iy)^k$ (cf. [8]).

From the other side, the definition of hypercomplex Appell polynomials as adequate generalization of $z^k = (x+iy)^k$ together with the characteristic property

$$(z^k)' = kz^{k-1},\ k = 1, \ldots,\ \text{and}\ \ z^0 = 1,$$

formally allows to construct an analogue of the geometric series in hypercomplex setting. Naturally, for a full analogy to the complex case, this raises the question, if the kernel of the hypercomplex Cauchy integral theorem derived from Greens formula in $\mathbb{R}^{n+1}$ (see [6]) could be expanded in form of that hypercomplex geometric series. The answer is affirmative as it has been shown by methods of Hypercomplex Function Theory in [3], where the alternating sums of the $k-$th row's entries in $\mathcal{T}(n)$, i.e. the one-parameter family of rational numbers

$$c_k(n) := \sum_{s=0}^{k} (-1)^s T_s^k(n),\ k = 0, 1, \ldots, \tag{2}$$

plays an important role. Moreover, in [3] it has been shown that the $c_k(2)$ coincide exactly with numbers, for the first time used by Vietoris in [14] and playing an important role in the theory of orthogonal polynomials as well as questions of positivity of trigonometric sums (see [1]). Consequently, based on a suggestive closed representation of (2) the authors introduced in [4] the sequence of generalized Vietoris numbers as follows

Definition 1. *Let $n \in \mathbb{N}$. The generalized Vietoris number sequence is defined by $\mathcal{V}(n) := (c_k(n))_{k\geq 0}$ with*

$$c_k(n) := \frac{\left(\frac{1}{2}\right)_{\lfloor \frac{k+1}{2} \rfloor}}{\left(\frac{n}{2}\right)_{\lfloor \frac{k+1}{2} \rfloor}}. \tag{3}$$

Remark 1. A general analysis of the coefficients in the sequence $\mathcal{V}(n)$ reveals the following picture. For $n = 1$ all elements are identically equal to 1 for all $k \geq 0$. For a fixed $n > 1$ the floor function in the representation of the coefficients (3) implies a repetition of the coefficient with an even index $k = 2m$ following after that with an odd index, i.e. $c_{2m-1}(n) = c_{2m}(n)$. The value of the next following odd-indexed coefficient is decreasing by a variable factor given by the formula

$$c_{2m+1}(n) = \frac{2m+1}{2m+n} c_{2m}(n). \tag{4}$$

For readers, not so much interested in the hypercomplex origins of $\mathcal{T}(n)$ or $\mathcal{V}(n)$, we finish our introduction recalling some formulas from [4]. They show the specific interrelationship of $\mathcal{V}(n)$ with the dimension of the space of homogeneous polynomials in n variables of degree k. This became evident in [4] where Vietoris numbers (3), contrary to [1], have been studied by real methods without directly relying on Jacobi polynomials $P_k^{(\alpha,\beta)}$. In fact, we found that the Taylor series expansion of the rational function

$$\frac{1}{(1-t)^{\gamma}(1+t)^{\delta}}$$

in the open unit interval with the special values

$$\gamma = \frac{n+1}{2} \text{ and } \delta = \frac{n-1}{2}$$

implies the following series development

$$\frac{1}{(1-t)^{\frac{n+1}{2}}(1+t)^{\frac{n-1}{2}}} = \sum_{k=0}^{\infty} \frac{(n)_k}{k!} c_k(n) t^k. \tag{5}$$

Formula (5) shows that generalized Vietoris numbers (3) appear also in coefficients of a special power series multiplied by a term that characterizes the space of homogeneous polynomials in higher dimension. Indeed,

$$\frac{(n)_k}{k!} = \binom{n+k-1}{k} = \dim \mathcal{H}_k(\mathbb{R}^n),$$

where $\mathcal{H}_k(\mathbb{R}^n)$ is the space of homogeneous polynomials in n variables of degree k.

As mentioned before,

$$T_0^k(1) \equiv 1 \text{ and } T_s^k(1) = 0, \text{ if } 0 < s \leq k.$$

and for $n = 1$ formula (5) is the sum of the ordinary geometric series due to the special values of γ and δ. Since the geometric series in one real respectively one complex variable coincide (in the latter case (5) has to be considered as a Taylor series expansion in the unit complex disc) our hypercomplex approach contains the complex case as particular case $\mathbb{R}^{1+1} \cong \mathbb{C}$.

The important role of ordinary Vietoris numbers, corresponding to $n = 2$ in (3), in the theory of orthogonal polynomials is discussed in [1]. The paper of Ruschewey and Salinas [12] shows the relevance of the celebrated result of Vietoris for a complex function theoretic result in the context of subordination of analytic functions.

In this paper we deal with interesting properties of $\mathcal{T}(n)$. In particular, in Sect. 2 we present results concerning sums over the entries of the rows of $\mathcal{T}(n)$.

The consideration of the main diagonal, in Sect. 3, includes results on series over its entries and also a recurrence relation which has a certain counterpart with the ordinary Pascal triangle in the limit case $n \to \infty$. Additionally, in Sect. 4 some relations to *Jacobsthal numbers* have been found.

2 Sums over Entries of the Rows in the Non-symmetric Number Triangles

The general structure of the triangle in Fig. 1 can easily be recognized by using the relations between its entries indicated by the arrows in different directions (see [8, Theorems 3.1–3.3]). In this section we want to complete and extend some other properties.

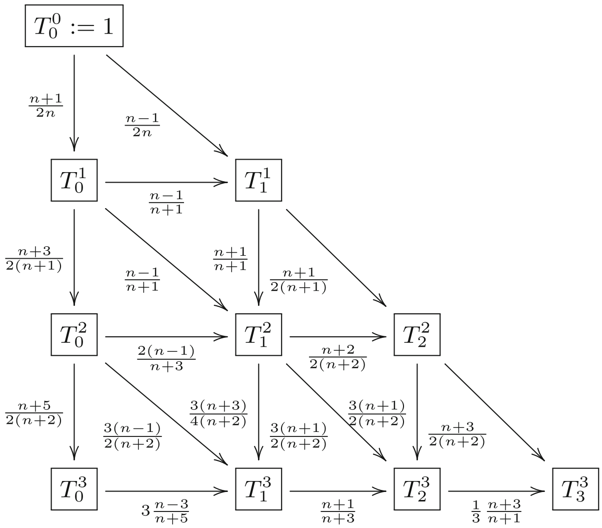

Fig. 1. Relations between the first triangle elements

The first results that we mention in this section are relations between (1) and the corresponding binomial coefficients. We start by listing a set of properties already presented and proved in [8], but which will be useful to prove new results.

I. Relation between adjacent elements in the $s-$th column ([8], Theorem 3.1):

$$T_s^{k+1}(n) = \frac{(k+1)(n+2k-2s+1)}{2(k-s+1)(n+k)}\, T_s^k(n), \tag{6}$$

for $k = 0, 1, \ldots;\ s = 0, \ldots, k$;

II. Relation between adjacent "diagonal" elements ([8], Theorem 3.2):

$$T_{s+1}^{k+1}(n) = \frac{(k+1)(n+2s-1)}{2(s+1)(n+k)}\, T_s^k(n), \tag{7}$$

for $k = 0, 1, \ldots;\ s = 0, \ldots, k$;

III. Relation between adjacent elements in the $k-$th row ([8], Theorem 3.3):

$$T_{s+1}^k(n) = \frac{(k-s)(n+2s-1)}{(s+1)(n+2k-2s-1)}\, T_s^k(n), \tag{8}$$

for $k = 0, 1, \ldots ;\ s = 0, \ldots, k-1$;

IV. Relation between adjacent elements in the $k-$th row and an element in the $(k-1)-$th row ([8], Theorem 3.5):

$$(k-s)T_s^k(n) + (s+1)T_{s+1}^k(n) = kT_s^{k-1}(n), \tag{9}$$

for $k = 1, 2, \ldots ; s = 0, \ldots, k-1$;

V. Partition of the unit ([8], Theorem 3.7):

$$\sum_{s=0}^{k} T_s^k(n) = 1. \tag{10}$$

Following the accompanying factors in Fig. 1 along the corresponding paths, the identity (10) does not come as a surprise. Its proof relies on the definition (1) and the well known Chu-Vandermonde convolution identity for the Pochhammer symbols

$$(x+y)_m = \sum_{r=0}^{m} \binom{m}{r} (x)_r (y)_{m-r}. \tag{11}$$

Remark 2. We observe that, considering $n \to \infty$ in (1), the entries of $\mathcal{T}(n)$ converge to

$$T_s^k(\infty) = 2^{-k}\binom{k}{s}. \tag{12}$$

Because the sum of all binomial coefficients in row k of the ordinary Pascal triangle equals 2^k, (12) relates the limit case of the triangle in Fig. 1 with an *ordinary normalized* Pascal triangle whose row sum is constantly 1. The next formula [5, Theorem 1], is an analogue of a property of binomial coefficients multiplied by the counting index s in their sum which can be obtained by simply differentiation of the corresponding binomial formula.

Proposition 1. *For* $k = 0, 1, 2, \ldots$

$$\sum_{s=0}^{k} sT_s^k(n) = \frac{k(n-1)}{2n}.$$

We prove now some new properties of the triangle $\mathcal{T}(n)$. The next is another analogue of a property of binomial coefficients that can also be obtained by differentiation of the corresponding binomial formula. In this case, the proof uses some not so trivial relations between binomial coefficients and the Pochhammer symbol.

Proposition 2. *For* $k = 0, 1, 2, \ldots$

$$\sum_{s=0}^{k} s^2 T_s^k(n) = \frac{k(k+1)(n-1)}{4n}.$$

Proof. Recalling that $s\binom{k}{s} = k\binom{k-1}{s-1}$ and $\left(\frac{n-1}{2}\right)_s = \frac{n-1}{2}\left(\frac{n+1}{2}\right)_{s-1}$, from (1) we can write

$$\sum_{s=0}^{k} s^2 T_s^k(n) = \sum_{s=0}^{k} s^2 \binom{k}{s} \frac{\left(\frac{n+1}{2}\right)_{k-s}\left(\frac{n-1}{2}\right)_s}{(n)_k}$$
$$= \frac{k(n-1)}{2(n)_k} \sum_{s=1}^{k} s \binom{k-1}{s-1} \left(\frac{n+1}{2}\right)_{k-s} \left(\frac{n+1}{2}\right)_{s-1}.$$

From $s\left(\frac{n+1}{2}\right)_{s-1} = \left(\frac{n+1}{2}\right)_s - \left(\frac{n-1}{2}\right)_s$, we obtain

$$\sum_{s=0}^{k} s^2 T_s^k(n) = \frac{k(n-1)}{2(n)_k} \sum_{s=1}^{k} \binom{k-1}{s-1} \left[\left(\tfrac{n+1}{2}\right)_{k-s}\left(\tfrac{n+1}{2}\right)_s - \left(\tfrac{n+1}{2}\right)_{k-s}\left(\tfrac{n-1}{2}\right)_s\right]$$
$$= \frac{k(n-1)}{2(n)_k} \left[\tfrac{n+1}{2}\alpha_s^k(n) - \tfrac{n-1}{2}\beta_s^k(n)\right],$$

with

$$\alpha_s^k(n) = \sum_{s=1}^{k} \binom{k-1}{s-1} \left(\frac{n+1}{2}\right)_{k-s} \left(\frac{n+3}{2}\right)_{s-1} \tag{13}$$

and

$$\beta_s^k(n) = \sum_{s=1}^{k} \binom{k-1}{s-1} \left(\frac{n+1}{2}\right)_{k-s} \left(\frac{n+1}{2}\right)_{s-1}. \tag{14}$$

Using in (13) and (14) the Chu-Vandermonde convolution identity (11) with $x = \frac{n+3}{2}, y = \frac{n+1}{2}$ and $x = y = \frac{n+1}{2}$, respectively, we get

$$\alpha_s^k(n) = (n+2)_{k-1} \text{ and } \beta_s^k(n) = (n+1)_{k-1}.$$

Therefore

$$\sum_{s=0}^{k} s^2 T_s^k(n) = \frac{k(n-1)}{4(n)_k} [(n+1)(n+2)_{k-1} - (n-1)(n+1)_{k-1}]$$
$$= \frac{k(n-1)}{4(n)_k}(n+1)_{k-1}(k+1) = \frac{k(k+1)(n-1)}{4n}.$$

□

Remark 3. In the limit case $n \to \infty$, the equalities of the Proposition 1 and Proposition 2 become the well known identities $\sum_{s=0}^{k} s\binom{k}{s} = 2^{k-1}k$ and $\sum_{s=0}^{k} s^2\binom{k}{s} = 2^{k-2}k(k+1)$, respectively ([2, p. 14]) .

Alternating sums involving (1) can be connected to the Vietoris sequence introduced in Definition 1, as follows:

Proposition 3. *For* $k = 0, 1, 2, \ldots,$

$$\sum_{s=0}^{k} (-1)^s s T_s^k(n) = \begin{cases} \frac{1-n}{2} c_k(n), & \textit{if } k \textit{ is odd} \\ 0, & \textit{if } k \textit{ is even.} \end{cases}$$

Proof. Noting that

$$\sum_{s=0}^{k} (-1)^s s T_s^k(n) = \sum_{s=0}^{k-1} (-1)^{s+1} (s+1) T_{s+1}^k(n),$$

the result follows from

$$\sum_{s=0}^{k-1} (-1)^s (s+1) T_{s+1}^k(n) = \begin{cases} \frac{1}{2} \frac{k(n-1)}{n+k-1} c_{k-1}(n), & \text{if } k \text{ is odd} \\ 0, & \text{if } k \text{ is even} \end{cases}$$

(cf. [[8], Theorem 3.9]), and (4). □

The following proposition shows again the connection with the Pascal triangle, this time with the central binomial coefficients. The result follows immediately from (6).

Proposition 4.

$$T_k^{2k}(n) = T_k^{2k-1}(n).$$

Remark 4. We note that, letting $n \to \infty$ in the assertion of Proposition 4, the identity $\binom{2k}{k} = 2\binom{2k-1}{k}$ is obtained.

3 Recurrences and Series over the Entries in the Main Diagonal

The richness and beauty of the Pascal triangle and the already mentioned connections with the triangle $\mathcal{T}(n)$ motivated the search for new structures and patterns within $\mathcal{T}(n)$.

The family of sequences we are going to consider contains the main diagonal elements of the triangle (see Fig. 2), where we use the abbreviation

$$\mathcal{T}_k(n) := T_k^k(n) = \frac{\left(\frac{n-1}{2}\right)_k}{(n)_k}, \; k = 0, 1, 2, \ldots; n = 2, 3, \ldots. \tag{15}$$

Observe that this sequence can be written in terms of the generalized Vietoris sequence (3), as

$$\mathcal{T}_k(n) = \frac{c_{2k}(2n)}{c_{2k}(n-1)} = \frac{c_{2k-1}(2n)}{c_{2k-1}(n-1)}.$$

The next property shows how an element in the main diagonal can be obtained by simply subtracting two consecutive elements in the first column.

						$\mathcal{T}_k(n)$
T_0^0						1
T_0^1	T_1^1					$\frac{n-1}{2n}$
T_0^2	T_1^2	T_2^2				$\frac{n-1}{4n}$
T_0^3	T_1^3	T_2^3	T_3^3			$\frac{(n-1)(n+3)}{8n(n+2)}$
T_0^4	T_1^4	T_2^4	T_3^4	T_4^4		$\frac{(n-1)(n+5)}{16n(n+2)}$
T_0^5	T_1^5	T_2^5	T_3^5	T_4^5	T_5^5	$\frac{(n-1)(n+5)(n+7)}{32n(n+2)(n+4)}$

Fig. 2. The sequence $(\mathcal{T}_k(n))_{k\geq 0}$ as main diagonal in $\mathcal{T}(n)$

Proposition 5.

$$\mathcal{T}_k(n) = T_0^{k-1}(n) - T_0^k(n).$$

Proof. The repeated use of (8), relating consecutive elements in the same row, allows to write

$$\mathcal{T}_k(n) = \frac{n-1}{n+2k-1} T_0^k(n),$$

while (6) gives

$$T_0^k(n) = \frac{n+2k-1}{2n+2k-2} T_0^{k-1}(n).$$

The result follows at once, by combining these two last identities. □

Considering now some series built of elements of the main diagonal, we use (15) and connect this representation formula with special values of the hypergeometric function ${}_2F_1\left(a,b;c;z\right)$. Gauss' hypergeometric series is defined by

$${}_2F_1\left(a,b;c;z\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}$$

on the disk $|z| < 1$ and by analytic continuation elsewhere. In particular on the circle $|z| = 1$, Gauss' series converges absolutely when $\mathrm{Re}(c-a-b) > 0$. Therefore, choosing $a = 1$, $b = \frac{n-1}{2}$, $c = n$ (we recall that $n \geq 2$) and $z = \pm 1$, it follows immediately from (15)

$$\sum_{k=0}^{\infty} (-1)^k \mathcal{T}_k(n) = {}_2F_1\left(1, \tfrac{n-1}{2}; n; -1\right) \tag{16}$$

and

$$\sum_{k=0}^{\infty} \mathcal{T}_k(n) = {}_2F_1\left(1, \tfrac{n-1}{2}; n; 1\right). \tag{17}$$

It seems to be of interest to notice that the last relation admits an explicit evaluation.

Proposition 6. *The series built of all elements of the main diagonal is convergent and its sum is independent of the parameter n, i.e.*

$$\sum_{k=0}^{\infty} \mathcal{T}_k(n) = 2.$$

Proof. We recall Gauss' identity

$${}_2F_1\left(a, b; c; 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}(c-a-b) > 0,$$

and rewrite result (17) in the form

$$\sum_{k=0}^{\infty} \mathcal{T}_k(n) = \frac{\Gamma(n)\Gamma(\frac{n-1}{2})}{\Gamma(n-1)\Gamma(\frac{n+1}{2})}.$$

Then the sum of the considered series can simply be read off by using the basic properties of the Gamma function, $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(n) = (n-1)!$. □

The next result about the evaluation of the alternating series (16) for $n = 2$, uses the relation of $\mathcal{T}_k(2)$ to the celebrated Catalan numbers $\mathcal{C}_k = \frac{1}{k+1}\binom{2k}{k}$, which already was mentioned in [5].

Proposition 7. *For the particular case of $\mathcal{T}_k(2)$, the alternating series* (16) *converges and its sum is given by*

$$\sum_{k=0}^{\infty} (-1)^k \mathcal{T}_k(2) = 2(\sqrt{2} - 1).$$

Proof. According to (15), we have

$$\mathcal{T}_k(2) = \frac{1}{2^{2k}} \mathcal{C}_k. \tag{18}$$

Applying the Catalan identity

$$\sum_{k=0}^{\infty} \frac{(-1)^k \mathcal{C}_k}{4^k} = 2(\sqrt{2} - 1),$$

which follows from the generating function of Catalan numbers $g(x) = \frac{1-\sqrt{1-4x}}{2x}$, with $x = -\frac{1}{4}$ (see e.g. [13]), we recognize the result as identical with the Catalan identity. □

It is worth noting that all the elements of the triangle $\mathcal{T}(2)$ are related to Catalan numbers. It is enough to use (18) together with the relation (9) in order to obtain any element of the triangle as a linear combination of Catalan numbers.

Before arriving to another group of general but more intrinsic properties of the sequence $\mathcal{T}_k(n)$ we prove an auxiliary relation of the entries in a row of the triangle $\mathcal{T}(n)$ which culminates in a binomial transform between $\mathcal{T}_k(n)$ and the sequence of the first column. For arbitrary integers $n \geq 2$, an element $T_s^k(n)$ of the triangle $\mathcal{T}(n)$ is related to the numbers in the main diagonal $\mathcal{T}_m(n)$, $m = s, s+1, \ldots, k$ in the following particular way.

Proposition 8. *For $k = 0, 1, 2, \ldots$ and $r = 0, \ldots, k$, we have*

$$T_{k-r}^k(n) = (-1)^r \binom{k}{r} \sum_{s=0}^{r} \binom{r}{s} (-1)^s \mathcal{T}_{k-s}(n).$$

Proof. If $r = 0$, the statement is obvious.

Assume the statement is true for $r = i$, i.e.,

$$T_{k-i}^k(n) = (-1)^i \binom{k}{i} \sum_{s=0}^{i} \binom{i}{s} (-1)^s \mathcal{T}_{k-s}(n). \tag{19}$$

Using $s = k - (i+1)$ in the relation (9), we obtain

$$(i+1) T_{k-(i+1)}^k(n) + (k-i) T_{k-i}^k(n) = k T_{k-1-i}^{k-1}(n).$$

Combining this relation with the induction hypothesis (19), we get

$$\begin{aligned} T_{k-(i+1)}^k(n) &= -\frac{k-i}{i+1} \binom{k}{i} (-1)^i \sum_{s=0}^{i} \binom{i}{s} (-1)^s \mathcal{T}_{k-s}(n) \\ &\quad + \frac{k}{i+1} \binom{k-1}{i} (-1)^i \sum_{s=0}^{i} \binom{i}{s} (-1)^s \mathcal{T}_{k-1-s}(n) \\ &= (-1)^{i+1} \binom{k}{i+1} \Big[\binom{i}{0} \mathcal{T}_k(n) + \binom{i}{i} (-1)^{i+1} \mathcal{T}_{k-(i+1)}(n) \Big] \\ &\quad + (-1)^{i+1} \binom{k}{i+1} \Big[\sum_{s=1}^{i} \left(\binom{i}{s} + \binom{i}{s-1} \right) (-1)^s \mathcal{T}_{k-s}(n) \Big] \\ &= (-1)^{i+1} \binom{k}{i+1} \sum_{s=0}^{i+1} (-1)^s \binom{i+1}{s} \mathcal{T}_{k-s}(n). \end{aligned}$$

That is, the statement also holds for $r = i+1$. Hence, by induction, the result is achieved. □

Remark 5. For the case $r = k$, the equality of Proposition 8 is equivalent to

$$T_0^k(n) = \sum_{s=0}^{k} (-1)^s \binom{k}{s} \mathcal{T}_s(n), \tag{20}$$

showing that the sequence $\left(T_0^k(n)\right)_{k\geq 0}$, formed by the elements of the first column of the triangle $\mathcal{T}(n)$ is the binomial transform of the main diagonal sequence $(\mathcal{T}_k(n))_{k\geq 0}$. For completeness, we mention also the corresponding inverse binomial transform given by

$$\mathcal{T}_k(n) = \sum_{s=0}^{k} (-1)^s \binom{k}{s} T_0^s(n).$$

The binomial transform (20) in some sense completes another relation between both partial sequences of entries of $\mathcal{T}(n)$, namely Proposition 5, where one element of the main diagonal is expressed as a difference of two elements of the first column.

The next property is of another type and could be considered as an intrinsic property of $\mathcal{T}_k(n)$. The proof relies on both, Proposition 5 and relation (20).

Proposition 9. *For* $k = 0, 1, 2, \ldots$

$$\sum_{s=0}^{k} (-1)^s \binom{k+1}{s} \mathcal{T}_{s+1}(n) = \begin{cases} 2\mathcal{T}_{k+2}(n), & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

Proof. Observe that

$$\begin{aligned}
\sum_{s=0}^{k} (-1)^s \binom{k+1}{s} \mathcal{T}_{s+1}(n) &= \sum_{s=0}^{k} (-1)^s \left(\binom{k+2}{s+1} - \binom{k+1}{s+1} \right) \mathcal{T}_{s+1}(n) \\
&= \sum_{s=1}^{k+1} (-1)^{s-1} \binom{k+2}{s} \mathcal{T}_s(n) + \sum_{s=1}^{k+1} (-1)^s \binom{k+1}{s} \mathcal{T}_s(n) \\
&= -\sum_{s=0}^{k+2} (-1)^s \binom{k+2}{s} \mathcal{T}_s(n) + \mathcal{T}_0(n) + (-1)^{k+2} \mathcal{T}_{k+2}(n) \\
&\quad + \sum_{s=0}^{k+1} (-1)^{s-1} \binom{k+1}{s} \mathcal{T}_s(n) - \mathcal{T}_0(n).
\end{aligned}$$

Combining this with (20) we obtain

$$\sum_{s=0}^{k} (-1)^s \binom{k+1}{s} \mathcal{T}_{s+1}(n) = -T_0^{k+2} + (-1)^{k+2} \mathcal{T}_{k+2}(n) + T_0^{k+1}.$$

Finally, the use of Proposition 5 allows to write

$$\sum_{s=0}^{k}(-1)^s\binom{k+1}{s}\mathcal{T}_{s+1}(n) = (1+(-1)^k)\mathcal{T}_{k+2}(n)$$

and the result is proved. □

Proposition 10. *For any integer* $n \geq 2$, $\mathcal{T}_k(n)$ *is a positive decreasing sequence convergent to zero.*

Proof. From (7), we obtain

$$\mathcal{T}_{k+1}(n) = \tfrac{2k+n-1}{2k+2n}\mathcal{T}_k(n) < \mathcal{T}_k(n)$$

and

$$\mathcal{T}_k(n) = \tfrac{\Gamma(n)}{\Gamma(\frac{n-1}{2})}\tfrac{\Gamma(k+\frac{n-1}{2})}{\Gamma(k+n)} \underset{k\to\infty}{\longrightarrow} 0.$$

□

Let $(D_k(n))_{k\geq 0}$ be the sequence consisting of alternating partial sums of the main diagonal elements, i.e.

$$D_k(n) := \sum_{s=0}^{k}(-1)^s\mathcal{T}_s(n).$$

Proposition 11. *The sequence* $(D_k(n))_{k\geq 0}$, *satisfies the recurrence relation*

$$(n+1)D_{k+1}(n) - 2(k+n+1)D_{k+2}(n) + (2k+n+1)D_k(n) = 0$$

with initial conditions

$$D_0(n) = 1;\ D_1(n) = \frac{n+1}{2n}.$$

Proof. First we note that, for each $n \in \mathbb{N}$,

$$(-1)^k\mathcal{T}_{k+1}(n) = D_k(n) - D_{k+1}(n) \tag{21}$$

and

$$D_{k+2}(n) = D_k(n) + (-1)^{k+1}\mathcal{T}_{k+1}(n) + (-1)^{k+2}\mathcal{T}_{k+2}(n). \tag{22}$$

Applying (21) and (7) to (22), we obtain

$$D_{k+2}(n) = D_{k+1}(n) + \frac{n+2k+1}{2(n+k+1)}(D_k(n) - D_{k+1}(n)),$$

which leads to the result. □

The first elements of the sequences $(D_k(n))_{k\geq 0}$, for some values of n, are shown in Fig. 3.

n	$D_k(n)$
2	$1, \frac{3}{4}, \frac{7}{8}, \frac{51}{64}, \frac{109}{128}, \frac{415}{512} \dots$
3	$1, \frac{2}{3}, \frac{5}{6}, \frac{11}{15}, \frac{4}{5}, \frac{79}{105} \dots$
4	$1, \frac{5}{8}, \frac{13}{16}, \frac{45}{64}, \frac{99}{128}, \frac{1485}{2048} \dots$
...	...
$\boldsymbol{r}$	$1, \frac{r+1}{2r}, \frac{3r+1}{4r}, \frac{(r+1)(5r+7)}{8r(r+2)}, \frac{11r^2+28r+9}{16r(r+2)}, \frac{(1+r)(107+112r+21r^2)}{32r(2+r)(4+r)}, \dots$
...	...
∞	$1, \frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{11}{16}, \frac{21}{32} \dots$

Fig. 3. First elements of $(D_k(n))_{k\geq 0}$; $n = 2, 3, 4, r, \infty$.

4 A Relation to the Sequence of Jacobsthal Numbers

In the limit case $n \to \infty$, $\mathcal{T}_k(\infty) = 2^{-k}$ and the rational numbers $D_k(\infty)$ are *weighted* terms of the Jacobsthal sequence $(J_k)_{k\geq 0}$ given explicitly by

$$0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, \dots$$

or generated by its Binet form

$$J_k = \frac{1}{3}\left(2^k - (-1)^k\right), \quad k = 0, 1, 2, \dots$$

(cf. [10, page 447]). Next result shows the relation between $D_k(\infty)$ and the sequence of Jacobsthal numbers.

Proposition 12. *Consider the Jacobsthal sequence* $(J_k)_{k\geq 0}$. *Then*

$$D_k(\infty) = \frac{1}{2^k} J_{k+1}, \quad k = 0, 1, 2, \dots.$$

Proof. The result follows by observing that

$$\begin{aligned} D_k(\infty) &= \sum_{s=0}^{k} (-1)^s \mathcal{T}_s(\infty) = \sum_{s=0}^{k} (-2)^{-s} \\ &= \frac{2}{3}\left(1 - (-2)^{-(k+1)}\right) = \frac{1}{2^k}\frac{1}{3}\left(2^{k+1} - (-1)^{k+1}\right) \\ &= \frac{1}{2^k} J_{k+1}, \quad k = 0, 1, 2, \dots. \end{aligned}$$

□

Considering Proposition 12, the recurrence satisfied by the Jacobsthal sequence,

$$J_k = J_{k-1} + 2J_{k-2}, \quad k = 2, 3, \dots$$
$$J_0 = 0; \quad J_1 = 1,$$

is transformed into a simple recurrence of order 2 with constant coefficients for the elements $D_k(\infty)$. Indeed, it holds

$$D_k(\infty) = \frac{1}{2}\left(D_{k-1}(\infty) + D_{k-2}(\infty)\right), \quad k = 2, 3, \dots$$
$$D_0(\infty) = 1; \quad D_1(\infty) = \frac{1}{2}.$$

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