Strichartz estimates for structurally damped equations of space-time-fractional type

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Abstract: We obtain Strichartz estimates for a wide class of space-time-fractional evolution equations arising in applications towards the theory of structural damping plate equations. We focus on the case where the damping term $\mu(-\Delta)^{\frac{\sigma}{2}}\partial_t^{\alpha}u(t,x)$, involving a coefficient $\mu > -2$, is described in terms of the time fractional derivative ∂_t^{α} of order α taken in the Caputo-Djrbashian sense and of the fractional Laplacian $-(-\Delta)^{\frac{\sigma}{2}}$ of order σ in the space variable. Our results heavily rely on the study of the pseudo-differential representation for the resolvent $E_{\alpha,\beta}(-\lambda(-\Delta)^{\frac{\sigma}{2}}t^{\alpha})$ in terms of the Mittag-Leffler functions $E_{\alpha,\beta}$.

keywords: Fractional differential equations; Strichartz estimates; structural damping.

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1 Introduction

The investigation of Strichartz estimates dates back to the fundamental paper [7] of R. Strichartz on the Fourier restriction problem. Nowadays they have been revealing to be a cornerstone tool in the study of Schrödinger equations, wave equations and alike. In great extent, because they permit us to obtain sharp estimates for the size and the decay for the solutions of PDEs on mixed Lebesgue spaces $L_t^r L_x^q$ (cf. [5]). To study it, from a fractional calculus perspective, we briefly introduce the main mathematical objects, which allows us to study properly our model problem.

Let us denote by $E_{\alpha,\beta}$ ($\Re(\alpha), \Re(\beta) > 0$) the two-parameter Mittag-Leffler function, and by $-(-\Delta)^{\frac{\gamma}{2}}$ ($\gamma > 0$) the fractional Laplacian of order γ. Through the aid of the Fourier transform, $\mathcal{F}$, and its inverse, $\mathcal{F}^{-1}$, we define for a function φ with membership in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ s.t. $\widehat{\varphi} = \mathcal{F}\varphi$, the resolvent operator $\varphi \mapsto E_{\alpha,\beta}(-\lambda(-\Delta)^{\frac{\sigma}{2}}t^{\alpha})\varphi$ in terms of the Fourier inversion formula, valid for each $(t,x) \in [0,\infty) \times \mathbb{R}^n$:

$$E_{\alpha,\beta}(-\lambda(-\Delta)^{\frac{\sigma}{2}}t^{\alpha})\varphi(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} E_{\alpha,\beta}(-\lambda|\xi|^{\sigma}t^{\alpha})\widehat{\varphi}(\xi)e^{ix\cdot\xi}d\xi, \quad \Re(\lambda) > 0. \tag{1}$$

Our main goal is to report that the pseudo-differential representation (1) can be used to solve and to obtain Strichartz estimates for the class of evolution equations

$$\partial_t^{2\alpha}u(t,x) + \mu(-\Delta)^{\frac{\sigma}{2}}\partial_t^{\alpha}u(t,x) + (-\Delta)^{\sigma}u(t,x) = 0, \quad (t,x) \in \mathbb{R}^n \times (0,\infty). \tag{2}$$

We note that the space-time-fractional equation (2) seamlessly represents a time-fractional regularization for the class of structurally damped $\sigma-$evolution equations in the semi-critical case, studied by so many authors (cf. [4]), by replacing the first and second partial derivatives ∂_t and ∂_t^2 by the so-called Caputo-Djrbashian derivatives ∂_t^γ of order $\gamma = \alpha$ and $\gamma = 2\alpha$, respectively.

Such model problem is original motivated by the theory of structurally damped plate equations ($\sigma = 2\alpha = 2$) in a way that $(-\Delta)^\sigma$ shall be interpreted as a space-fractional regularization for the biharmonic operator Δ^2, whereas $\mu(-\Delta)^{\frac{\sigma}{2}}\partial_t^\alpha u(t,x)$ – appearing on the evolution equation (2) – as a space-time regularization for the strong damping term $-\mu\Delta\partial_t u(t,x)$. For further details, we refer to [3] and to the references therein.

2 Results and discussion

In order to obtain a solution representation for (2), we will take into account Laplace transform identities involving the Caputo-Djrbashian derivatives ∂_t^α resp. $\partial_t^{2\alpha}$ (cf. [6, Chapter 2]) and the Mittag-Leffler functions $E_{\alpha,\beta}$ as well (cf. [6, Chapter 1]). In the following theorems, the constants λ_+ and λ_- denote the algebraic roots of the polynomial $P(\lambda) = \lambda^2 - \mu\lambda + 1$.

Theorem 1 (Case I) *Let $0 < \alpha \leq \frac{1}{2}$, $\mu > -2$ and $u_0 \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$v_{\alpha,\sigma}(t,x) = \begin{cases} \dfrac{E_{\alpha,1}(-\lambda_+ (-\Delta)^{\frac{\sigma}{2}} t^\alpha)u_0(x)}{(\lambda_+)^2 - 1} + \dfrac{E_{\alpha,1}(-\lambda_- (-\Delta)^{\frac{\sigma}{2}} t^\alpha)u_0(x)}{(\lambda_-)^2 - 1} & , \ \mu \neq 2 \\ E_{\alpha,1}(-(-\Delta)^{\frac{\sigma}{2}} t^\alpha)u_0(x) + \dfrac{|\xi|^\sigma t^\alpha}{\alpha} E_{\alpha,\alpha}(-(-\Delta)^{\frac{\sigma}{2}} t^\alpha)u_0(x) & , \ \mu = 2. \end{cases}$$

solves the Cauchy problem for eq. (2) with initial condition $u(0,x) = u_0(x)$.

Theorem 2 (Case II) *Let $\frac{1}{2} < \alpha \leq 1$, $\mu > -2$ and $u_0, u_1 \in \mathcal{S}(\mathbb{R}^n)$, and set $v_{\alpha,\sigma}$ as in* **Theorem 1** *&*

$$w_{\alpha,\sigma}(t,x) = \begin{cases} t\ \dfrac{\lambda_+ E_{\alpha,2}(-\lambda_+ (-\Delta)^{\frac{\sigma}{2}} t^\alpha)u_1(x) - \lambda_- E_{\alpha,2}(-\lambda_- (-\Delta)^{\frac{\sigma}{2}} t^\alpha)u_1(x)}{\lambda_+ - \lambda_-} & , \ \mu \neq 2 \\ \dfrac{t}{\alpha}\left(E_{\alpha,1}(-(-\Delta)^{\frac{\sigma}{2}} t^\alpha)u_1(x) - (\alpha-1)E_{\alpha,2}(-(-\Delta)^{\frac{\sigma}{2}} t^\alpha)u_1(x)\right) & , \ \mu = 2, \end{cases}$$

Then, $u(t,x) = v_{\alpha,\sigma}(t,x) + w_{\alpha,\sigma}(t,x)$ solves the Cauchy problem for eq. (2) with initial data $u(0,x) = u_0(x)$ and $\partial_t u(0,x) = u_1(x)$.

Our next step is to derive Strichartz estimates for the solutions obtained in Theorem 1 and Theorem 2 from $L^p - L^q$ estimates of (1). Starting with the estimates obtained for $E_{\alpha,\beta}(-z)$ in [1, Proposition 4.] (case of $z > 0$) and in [6, Theorem 1.6] (case of $z \in \mathbb{C} \setminus \mathbb{R}$), we obtain the following proposition.

Proposition 3 *For every $0 < \alpha \leq 1$, $\sigma > 0$ and $\frac{n}{\sigma} < r < \infty$ one has*

$$\left(\int_0^\infty |E_{\alpha,\beta}(-\lambda\rho^\sigma t^\alpha)|^r \rho^{n-1} d\rho\right)^{\frac{1}{r}} \leq \frac{(\ C_{r,\sigma,n}(\alpha,\beta,\lambda)\ )^{\frac{1}{r}}}{\Gamma(\beta)}\ t^{-\frac{\alpha n}{r\sigma}},$$

where $C_{r,\sigma,n}(\alpha,\beta,\lambda)$ is a constant depending on the parameters $r, \sigma, n, \alpha, \beta$ and λ.

Afterwards, we obtain Proposition 4 by a direct application of Young's inequality to the convolution representation of (1):

$$E_{\alpha,\beta}(-\lambda(-\Delta)^{\frac{\sigma}{2}}t^\alpha)\varphi = \varphi * E_{\alpha,\beta}(-\lambda(-\Delta)^{\frac{\sigma}{2}}t^\alpha)\delta.$$

We follow [2, Proposition 2.1] to deduce L^r estimates for $E_{\alpha,\beta}(-\lambda(-\Delta)^{\frac{\sigma}{2}}t^\alpha)\delta$ in terms of the operator norm of the Hankel transform, $\widetilde{\mathcal{H}}_{\frac{n}{2}-1}$. Indeed, by substituting $\varphi(x) = \delta(x)$ on both sides of eq. (1) (i.e. $\widehat{\varphi}(\xi) = 1$), one readily gets

$$E_{\alpha,\beta}(-\lambda(-\Delta)^{\frac{\sigma}{2}}t^\alpha)\delta = \frac{1}{(2\pi)^{\frac{n}{2}}}\widetilde{\mathcal{H}}_{\frac{n}{2}-1}f, \text{ with } f(\rho) = E_{\alpha,\beta}(-\lambda\rho^\sigma t^\alpha).$$

Proposition 4 *Under the conditions of Proposition 3 and of [2, Proposition 2.1], the following* L^p-L^q *estimate fulfils for every* $\sigma > \frac{n}{2}$, $\max\left\{1, \frac{n}{\sigma}\right\} < r \leq 2$ *and* $1 \leq p, q \leq \infty$ *s.t.* $\frac{1}{r}+\frac{1}{p} = \frac{1}{q}+1$*:*

$$\left\|E_{\alpha,\beta}(-\lambda(-\Delta)^{\frac{\sigma}{2}}t^\alpha)\varphi\right\|_q \leq \frac{(\omega_{n-1}C_{r,\sigma,n}(\alpha,\beta,\lambda))^{\frac{1}{r}}}{(2\pi)^{\frac{n}{2}}\Gamma(\beta)}\, t^{-\frac{\alpha n}{r\sigma}}\, \|\varphi\|_p,$$

where ω_{n-1} *denotes the* $(n-1)-$*dimensional volume of the sphere* $\mathbb{S}^{n-1}$.
Moreover, $\varphi \in L^p(\mathbb{R}^n)$ *implies* $E_{\alpha,\beta}(-\lambda(-\Delta)^{\frac{\sigma}{2}}t^\alpha)\varphi \in L^q(\mathbb{R}^n)$ *whenever*

$$\max\left\{0, \frac{n-\sigma}{n}\right\} < \frac{1}{p} \leq 1 \quad \wedge \quad \frac{1}{p} - \frac{1}{2} \leq \frac{1}{q} < \frac{1}{p} - \max\left\{0, \frac{n-\sigma}{n}\right\}.$$

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