



Gabor orthonormal bases, tiling and periodicity

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Abstract

We show that if the Gabor system $\{g(x - t)e^{2\pi i s x}\}$, $t \in T$, $s \in S$, is an orthonormal basis in $L^2(\mathbb{R})$ and if the window function g is compactly supported, then both the time shift set T and the frequency shift set S must be periodic. To prove this we establish a necessary functional tiling type condition for Gabor orthonormal bases which may be of independent interest.

Mathematics Subject Classification 42B10 · 42C15

1 Introduction

1.1. Let g be a function in $L^2(\mathbb{R}^d)$, and let T and S be two countable sets in \mathbb{R}^d . The system of functions

$$\mathbf{G}(g, T, S) := \{g(x - t)e^{2\pi i \langle s, x \rangle} : t \in T, s \in S\} \quad (1.1)$$

is called the *Gabor system generated by the window function g , the time shift set T and the frequency shift set S* . Gabor systems have been extensively studied in the context of signal processing and time-frequency analysis, see e.g. [4].

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There is a long-standing problem concerning the structure of the Gabor systems (1.1) that constitute an orthonormal basis in the space $L^2(\mathbb{R}^d)$. It is generally expected that orthonormal bases of the form (1.1) must have a certain rigid structure. This problem has been studied by different authors, see e.g. [3,6,15,16].

In particular, it was proved in [16, Theorem 1.1] that if $\mathbf{G}(g, T, S)$ is an orthonormal basis in $L^2(\mathbb{R})$ where the function g has compact support, and if the frequency shift set S is periodic, then the time shift set T must be periodic as well. In the present paper we improve this result by establishing that the periodicity of both the time and the frequency shift sets T and S follows without any a priori assumption on the structure of these sets. We will prove the following:

Theorem 1.1 *Let $\mathbf{G}(g, T, S)$ be an orthonormal basis in $L^2(\mathbb{R})$, where $g \in L^2(\mathbb{R})$ has compact support. Then both T and S must be periodic sets, i.e. they have the form*

$$T = a\mathbb{Z} + \{t_1, \dots, t_n\}, \quad S = b\mathbb{Z} + \{s_1, \dots, s_m\}. \tag{1.2}$$

The result is not true in dimensions two and higher, see e.g. [16, Example 3.1].

1.2. Recall that a set $\Lambda \subset \mathbb{R}^d$ is said to be *uniformly distributed* if there exists a number $D(\Lambda)$ satisfying

$$\frac{\#(\Lambda \cap (x + [0, r]^d))}{r^d} = D(\Lambda) + o(1), \quad r \rightarrow +\infty \tag{1.3}$$

uniformly with respect to $x \in \mathbb{R}^d$. In this case, $D(\Lambda)$ is called the *uniform density* of Λ .

It is known, see e.g. [16, Lemma 2.2], that if $\mathbf{G}(g, T, S)$ is an orthonormal basis in $L^2(\mathbb{R}^d)$ then both T and S must be uniformly distributed sets, and moreover their uniform densities are positive and finite numbers satisfying the relation

$$D(T)D(S) = 1. \tag{1.4}$$

Our approach to the proof of Theorem 1.1 is based on the following result, which will also be proved in this paper and which may be of independent interest:

Theorem 1.2 *For any orthonormal basis $\mathbf{G}(g, T, S)$ in $L^2(\mathbb{R}^d)$ we have*

$$\sum_{t \in T} |g(x - t)|^2 = D(T) \quad a.e. \tag{1.5}$$

and

$$\sum_{s \in S} |\widehat{g}(\xi - s)|^2 = D(S) \quad a.e. \tag{1.6}$$

Here \widehat{g} is the Fourier transform of g defined by $\widehat{g}(\xi) = \int g(x) \exp(-2\pi i \langle \xi, x \rangle) dx$.

The conclusions (1.5) and (1.6) can be stated in terms of translational tilings, i.e. the function $|g|^2$ tiles \mathbb{R}^d at a constant level $D(T)$ by translations with respect to the

set T , while $|\widehat{g}|^2$ tiles at level $D(S)$ with respect to S . For an account on the theory of tiling by translates of a function we refer the reader to [13] and the references therein.

In dimension $d = 1$, the conclusion (1.5) was obtained in [3, Theorem 3.2] in the special case where both T and S are arithmetic progressions. In [2, Theorem 4.4] this result was extended to the case where S is an arithmetic progression, but T is an arbitrary set in \mathbb{R} . Theorem 1.2 establishes that both (1.5) and (1.6) are valid in all dimensions d and without imposing any a priori assumptions on the structure of the time or the frequency shift sets T and S .

As an immediate consequence of Theorem 1.2 we obtain:

Corollary 1.3 *If $\mathbf{G}(g, T, S)$ is an orthonormal basis in $L^2(\mathbb{R}^d)$ then both g and \widehat{g} must belong to $L^\infty(\mathbb{R}^d)$, and moreover we have*

$$\|g\|_{L^\infty(\mathbb{R}^d)} \leq \sqrt{D(T)}, \quad \|\widehat{g}\|_{L^\infty(\mathbb{R}^d)} \leq \sqrt{D(S)}. \tag{1.7}$$

The proofs of Theorems 1.1 and 1.2 are given in Sects. 2 and 3 below. In the last Sect. 4 we give, as another application of Theorem 1.2, a new proof of a result obtained in [6] on the structure of the Gabor orthonormal bases $\mathbf{G}(g, T, S)$ with a nonnegative window function g .

2 Gabor orthonormal bases and tiling

In this section we prove Theorem 1.2. We assume that $g \in L^2(\mathbb{R}^d)$, that T, S are two countable sets in \mathbb{R}^d , and that the system $\mathbf{G}(g, T, S)$ is an orthonormal basis in $L^2(\mathbb{R}^d)$. We will show that conditions (1.5) and (1.6) must be satisfied.

Proof of Theorem 1.2 Since the system (1.1) is an orthonormal basis in $L^2(\mathbb{R}^d)$ then, by Parseval’s equality, for any $f \in L^2(\mathbb{R}^d)$ we have

$$\sum_{s \in S} \sum_{t \in T} \left| \int_{\mathbb{R}^d} f(x) \overline{g(x-t)} e^{-2\pi i \langle s, x \rangle} dx \right|^2 = \|f\|^2. \tag{2.1}$$

Fix the function f and for each $t \in T$ define the function

$$h_t(x) := f(x)g(x-t) \tag{2.2}$$

which is in $L^1(\mathbb{R}^d)$. By applying (2.1) to the function $\overline{f(x)} e^{2\pi i \langle \xi, x \rangle}$ in place of $f(x)$, we obtain

$$\sum_{s \in S} \sum_{t \in T} |\widehat{h}_t(\xi - s)|^2 = \|f\|^2, \quad \xi \in \mathbb{R}^d. \tag{2.3}$$

In what follows we will assume that f is not the zero function. Define

$$H(\xi) := \|f\|^{-2} \sum_{t \in T} |\widehat{h}_t(\xi)|^2, \tag{2.4}$$

then the condition (2.3) can be stated as

$$\sum_{s \in S} H(\xi - s) = 1, \quad \xi \in \mathbb{R}^d. \tag{2.5}$$

In other words, the function H tiles \mathbb{R}^d at a constant level 1 by translations with respect to the set S . We will now use the following known result:

Theorem 2.1 [11] *Let H be a nonnegative measurable function on \mathbb{R}^d , and let S be a countable set in \mathbb{R}^d . If the tiling condition (2.5) holds then S has a finite uniform density satisfying $D(S) = (\int H)^{-1}$.*

(Strictly speaking, this was proved in [11, Section 2] in dimension $d = 1$, under the extra assumption that $H \in L^1(\mathbb{R})$, and for a weaker notion of density (i.e. not the uniform density). The full statement of Theorem 2.1 can be established by appropriate adjustments to the proof in [11]).

In the present case the frequency shift set S is known to have positive uniform density, that is, $D(S) > 0$. Hence Theorem 2.1 implies that the function H must be in $L^1(\mathbb{R}^d)$. Moreover, using (1.4) we conclude that $\int H = D(T)$.

On the other hand, using (2.2), (2.4) and Plancherel’s equality we obtain

$$\|f\|^2 \int_{\mathbb{R}^d} H(\xi) d\xi = \sum_{t \in T} \int_{\mathbb{R}^d} |\widehat{h}_t(\xi)|^2 d\xi = \sum_{t \in T} \int_{\mathbb{R}^d} |h_t(x)|^2 dx \tag{2.6}$$

$$= \int_{\mathbb{R}^d} |f(x)|^2 \sum_{t \in T} |g(x - t)|^2 dx. \tag{2.7}$$

We thus arrive at the equality

$$\int_{\mathbb{R}^d} |f(x)|^2 \sum_{t \in T} |g(x - t)|^2 dx = D(T) \|f\|^2 \tag{2.8}$$

which holds for an arbitrary nonzero $f \in L^2(\mathbb{R}^d)$. This implies that (1.5) must be true.

Next we use the fact that the Fourier transform maps an orthonormal basis in $L^2(\mathbb{R}^d)$ onto another orthonormal basis, from which it follows that the system $\mathbf{G}(\widehat{g}, S, T)$ is also an orthonormal basis in $L^2(\mathbb{R}^d)$. So we may apply the same considerations above to the system $\mathbf{G}(\widehat{g}, S, T)$ in place of $\mathbf{G}(g, T, S)$, which implies that (1.6) must also be true and completes the proof of Theorem 1.2. □

3 Gabor orthonormal bases and periodicity

Next we prove Theorem 1.1, which states that if the system $\mathbf{G}(g, T, S)$ is an orthonormal basis in $L^2(\mathbb{R})$ and the function g has compact support, then both T and S must be periodic sets. This result is true only in dimension $d = 1$ and does not extend to higher dimensions, see e.g. [16, Example 3.1].

Proof of Theorem 1.1 First we establish the periodicity of the time shift set T . Define the function $\varphi(x) := D(T)^{-1}|g(x)|^2$ which is in $L^1(\mathbb{R})$. Due to Theorem 1.2 we know that condition (1.5) must hold, i.e. φ tiles \mathbb{R} at a constant level 1 by translations with respect to the set T . We now invoke another result from [11].

Theorem 3.1 [11] *Let φ be a nonnegative, compactly supported function in $L^1(\mathbb{R})$. If φ tiles at level one by translations with respect to some set $T \subset \mathbb{R}$ then T must be a disjoint union of finitely many arithmetic progressions, namely*

$$T = \bigcup_{j=1}^N (a_j\mathbb{Z} + b_j) \tag{3.1}$$

where a_j, b_j are real numbers and $a_j > 0$.

In the present case the function g has compact support, and hence the same is true for φ and Theorem 3.1 applies. We conclude that the time shift set T is of the form (3.1). On the other hand it is well-known that T must be a *uniformly discrete* set, i.e. there is $\delta > 0$ such that $|t' - t| \geq \delta$ for any two distinct points $t, t' \in T$. (This follows from the orthogonality of the two functions $g(x - t)e^{2\pi isx}$ and $g(x - t')e^{2\pi isx}$, where s is any element of the frequency shift set S .) Hence using Kronecker’s theorem it follows that all the ratios a_i/a_j must be rational numbers. In turn this implies that T is a periodic set with the structure as in (1.2) (see e.g. [11, p. 670]).

Next we establish the periodicity of the frequency shift set S . For this purpose we will use the condition (1.6) from Theorem 1.2, which can be stated by saying that the function $\psi(\xi) := D(S)^{-1}|\widehat{g}(\xi)|^2$ tiles \mathbb{R} at level 1 by translations with respect to S . We observe however that Theorem 3.1 does not apply to this tiling since the Fourier transform \widehat{g} is not a compactly supported function, being analytic (this can be seen as a version of the classical uncertainty principle). We also note that there do exist nonperiodic tilings of \mathbb{R} by translates of a nonnegative $L^1(\mathbb{R})$ function of unbounded support, see [12].

To address this issue we will use a different result obtained in [8,12]. To state the result we first recall that a set $S \subset \mathbb{R}$ is said to have *finite local complexity* if S can be enumerated as a sequence $\{s_n\}$, $n \in \mathbb{Z}$, such that $s_n < s_{n+1}$ and the successive differences $s_{n+1} - s_n$ take only finitely many different values. The following result establishes that translational tilings of finite local complexity must be periodic, even if the function which tiles does not have compact support:

Theorem 3.2 [8,12] *Let $S \subset \mathbb{R}$ have finite local complexity. If a function $\psi \in L^1(\mathbb{R})$ tiles at level one by translations with respect to S , then S must be a periodic set, namely, it has the form $S = b\mathbb{Z} + \{s_1, \dots, s_m\}$.*

It would therefore suffice to show that in our case the set S has finite local complexity. It is easy to verify that S has finite local complexity if and only if S has bounded gaps and the set of differences $S - S$ is a discrete closed set in \mathbb{R} . Since we have $D(S) > 0$ then S indeed has bounded gaps. For any fixed $t \in T$, the orthogonality of the system $\{g(x - t)e^{2\pi isx} : s \in S\}$ implies that the set $(S - S) \setminus \{0\}$ is contained in the zero set of the

Fourier transform of the $L^1(\mathbb{R})$ function $|g|^2$. But since $|g|^2$ is a compactly supported function, its Fourier transform is the restriction to \mathbb{R} of a nonzero entire function and hence the set of its real zeros is a discrete closed set in \mathbb{R} . This establishes that S has finite local complexity, and thus the periodicity of S follows from Theorem 3.2. \square

4 Application to a conjecture of Liu and Wang

There is an interesting conjecture due to Youming Liu and Yang Wang [16] which concerns the structure of the Gabor orthonormal bases $\mathbf{G}(g, T, S)$ such that the window function g has compact support.

To state the conjecture we first recall that if $\Omega \subset \mathbb{R}^d$ is a set of positive and finite measure and if $\Lambda \subset \mathbb{R}^d$ is a countable set, then (Ω, Λ) is called a *tiling pair* if the family of sets $\Omega + \lambda$ ($\lambda \in \Lambda$) forms a partition of \mathbb{R}^d up to measure zero; and that (Ω, Λ) is called a *spectral pair* if the system of exponential functions

$$E(\Lambda) := \{e^{2\pi i \langle \lambda, x \rangle} : \lambda \in \Lambda\} \quad (4.1)$$

forms an orthogonal basis in the space $L^2(\Omega)$.

Conjecture 4.1 [16] *Let $\mathbf{G}(g, T, S)$ be an orthonormal basis in $L^2(\mathbb{R}^d)$ such that the function g has compact support. Then the following three conditions must hold:*

- (i) $|g(x)| = (\text{mes } \Omega)^{-1/2} \mathbf{1}_\Omega(x)$ a.e. for some measurable set $\Omega \subset \mathbb{R}^d$;
- (ii) (Ω, T) is a tiling pair;
- (iii) (Ω, S) is a spectral pair.

It is easy to verify, see e.g. [16, Lemma 3.1], that any Gabor system $\mathbf{G}(g, T, S)$ satisfying the conditions (i), (ii) and (iii) above forms an orthonormal basis in $L^2(\mathbb{R}^d)$. The Liu-Wang conjecture asserts that if g has compact support, then these sufficient conditions are also necessary for the system $\mathbf{G}(g, T, S)$ to be an orthonormal basis.

Conjecture 4.1 is known to hold in several special cases. In [15] the conjecture was proved in dimension one and in the case where $T = S = \mathbb{Z}$. A similar result is true also in \mathbb{R}^d . In [16, Theorem 1.2] the conjecture was proved in dimension one without any assumption on the structure of the sets T and S but assuming that the set $\Omega = \{x : g(x) \neq 0\}$ coincides a.e. with an interval in \mathbb{R} . In [6] the conjecture was proved in the special case where $g \in L^2(\mathbb{R}^d)$ is a nonnegative function (in this special case, the conjecture is true even if g is not assumed to have compact support).

Theorem 1.2 allows us to give a new proof of the latter result, namely:

Theorem 4.2 [6] *Let g be a nonnegative function in $L^2(\mathbb{R}^d)$. If $\mathbf{G}(g, T, S)$ is an orthonormal basis, then the conclusions (i), (ii) and (iii) in Conjecture 4.1 hold.*

Proof Let $\Omega = \{x : g(x) \neq 0\}$. Since g is a nonnegative function, the orthogonality of the system $\{g(x-t)e^{2\pi i \langle s, x \rangle} : t \in T\}$ where s is any fixed element of S , implies that the sets $\Omega + t$ ($t \in T$) must be pairwise disjoint up to measure zero. (This is the only place in the proof where the nonnegativity of g is in fact used; the rest of the argument is valid for an arbitrary g .) But the sets $\Omega + t$ are respectively the supports

of the functions $|g(x - t)|^2$, $t \in T$, so it follows from the tiling condition (1.5) that (Ω, T) is a tiling pair and that $|g(x)|^2 = D(T)$ a.e. on Ω . In turn this implies that $D(T) \text{mes}(\Omega) = \int |g(x)|^2 dx = 1$ and hence (i) and (ii) are established.

The fact that (Ω, T) is a tiling pair implies that for any fixed $t \in T$, the system $\{g(x - t)e^{2\pi i(s,x)} : s \in S\}$ is an orthonormal basis in the space $L^2(\Omega + t)$. But since the weight function $g(x - t)$ has constant modulus a.e. on its support $\Omega + t$, this implies that also the unweighted system of exponentials $E(S)$ must be an orthogonal basis in $L^2(\Omega + t)$. This establishes (iii) and completes the proof. \square

We conclude the paper by mentioning the references [1,5,7,10,14] where other questions in the spirit of the Liu-Wang conjecture are studied. Similar questions in the finite group setting are considered in [9].

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