On a regular ψ-fractional Sturm-Liouville problem*

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Abstract

In this short paper, we consider a ψ-fractional Sturm-Liouville eigenvalue problem by using left ψ-Caputo and right ψ-Riemann-Liouville fractional derivatives. We study the main properties of the eigenfunctions and the eigenvalues of the associated fractional boundary problem.

Keywords: ψ-fractional derivatives; ψ-fractional Sturm-Liouville problem; Eigenvalues; Eigenfunctions.
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1 Introduction

The classical regular Sturm-Liouville (S-L) problem is associated with the real second-order linear ordinary differential equation of the form:

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)\,y(x) = \lambda\,w(x)\,y(x),$$

where $p(x), w(x) > 0$, and $p(x), p'(x), q(x), w(x)$ are continuous functions on (a, b). The function $w(x)$, sometimes denoted by $r(x)$, is called the weight or density function. The unknown function $y(x)$ is continuous and differentiable on (a, b). In addition, $y(x)$ is required to satisfy some boundary conditions of the form $c_1 y(a) + c_2 y'(a) = 0$ and $d_1 y(b) + d_2 y'(b) = 0$, with $c_1^2 + c_2^2 \neq 0$ and $d_1^2 + d_2^2 \neq 0$. Finding the values of λ, known as eigenvalues, for which there exists a non-trivial solution $y(x)$, called eigenfunction associated to λ, constitutes the classical S-L problem. Fractional versions of this problem were already considered in the literature (see e.g. [3, 4] and references therein). Motivated by the unification given by the ψ-fractional calculus we study a regular ψ-fractional S-L problem.

2 Preliminaries

In this section, we introduce the concepts related to the ψ-fractional calculus necessary to this work (for more details see [5] and references therein).

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Definition 2.1 *(cf. [5]) Let (a,b) be a finite or infinite interval on the real line $\mathbb{R}$ and $\alpha>0$. Also let ψ be a monotone increasing and positive function on (a,b), having a continuous derivative ψ' in (a,b). The left- and right-sided Riemnann-Liouville (RL) fractional integrals of a function f with respect to another function ψ on (a,b) are given respectively by*

$$I_{a^+}^{\alpha;\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)\,(\psi(x)-\psi(t))^{\alpha-1}\, f(t)\; dt, \qquad x>a, \tag{1}$$

$$I_{b^-}^{\alpha;\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t)\,(\psi(t)-\psi(x))^{\alpha-1}\, f(t)\; dt, \qquad x<b. \tag{2}$$

Assuming further that $f,\psi\in C^n(a,b)$, where $n=[\alpha]+1$, and $\psi'(x)\neq 0$, for all $x\in(a,b)$, the corresponding inverse operators, i.e., the left- and right-sided ψ-RL fractional derivatives of order $\alpha>0$, are defined respectively by

$$D_{a^+}^{\alpha;\psi} f(x) = \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n I_{a^+}^{n-\alpha;\psi} f(x) \quad \textit{and} \quad D_{b^-}^{\alpha;\psi} f(x) = \left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n I_{b^-}^{n-\alpha;\psi} f(x). \tag{3}$$

Moreover, the left- and right-sided ψ-Caputo fractional derivatives of order $\alpha>0$, are defined respectively by

$${}^C D_{a^+}^{\alpha;\psi} f(x) = I_{a^+}^{n-\alpha;\psi} \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n f(x) \quad \textit{and} \quad {}^C D_{b^-}^{\alpha;\psi} f(x) = I_{b^-}^{n-\alpha;\psi} \left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n f(x). \tag{4}$$

Using the concepts of ψ-RL and ψ-Caputo fractional derivatives we can encompass in one definition several fractional derivatives known in the literature In [5], the authors listed several fractional derivatives that can be obtained for specific choices of the function ψ such as Caputo, RL, Hadamard, Katugampola, Chen, Jumarie, Prabhakar, Erdélyi-Kober, Weyl, among others.

Proposition 2.2 *Let $\alpha,\beta>0$ and $f\in C([a,b])$. Then the following relations hold*

$$\begin{aligned} &I_{a^+}^{\alpha;\psi} I_{a^+}^{\beta;\psi} = I_{a^+}^{\alpha+\beta;\psi}, && I_{b^-}^{\alpha;\psi} I_{b^-}^{\beta;\psi} = I_{b^-}^{\alpha+\beta;\psi},\\ &D_{a^+}^{\alpha;\psi} I_{a^+}^{\alpha;\psi} f(x) = f(x), && D_{b^-}^{\alpha;\psi} I_{b^-}^{\alpha;\psi} f(x) = f(x),\\ &{}^C D_{a^+}^{\alpha;\psi} I_{a^+}^{\alpha;\psi} f(x) = f(x), && {}^C D_{b^-}^{\alpha;\psi} I_{b^-}^{\alpha;\psi} f(x) = f(x). \end{aligned}$$

Moreover, for $f\in C^n([a,b])$, $n=[\alpha]+1$, and putting $f_\psi^{[k]}(x)=\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^k f(x)$, we have

$$\begin{aligned} {}^C D_{a^+}^{\alpha;\psi} f(x) &= D_{a^+}^{\alpha;\psi}\left[f(x)-\sum_{k=0}^{n-1}\frac{f_\psi^{[k]}(a)}{k!}(\psi(x)-\psi(a))^k\right],\\ I_{a^+}^{\alpha;\psi}\ {}^C D_{a^+}^{\alpha;\psi} f(x) &= f(x)-\sum_{k=0}^{n-1}\frac{f_\psi^{[k]}(a)}{k!}(\psi(x)-\psi(a))^k. \end{aligned}$$

Proposition 2.3 *For $\alpha\in(0,1)$, the fractional differential operators (3) and (4) satisfy the following ψ-fractional integration by parts:*

$$\int_a^b f(x) D_{a^+}^{\alpha;\psi} g(x)\psi'(x)\,dx = \int_a^b {}^C D_{b^-}^{\alpha;\psi} f(x) g(x)\psi'(x)\,dx + f(x) I_{a^+}^{1-\alpha;\psi} g(x)\Big|_{x=a}^{x=b}, \tag{5}$$

$$\int_a^b f(x) D_{b^-}^{\alpha;\psi} g(x)\psi'(x)\,dx = \int_a^b {}^C D_{a^+}^{\alpha;\psi} f(x) g(x)\psi'(x)\,dx - f(x) I_{b^-}^{1-\alpha;\psi} g(x)\Big|_{x=a}^{x=b}. \tag{6}$$

For more properties of these ψ-fractional operators see [5] and references therein.

3 Regular ψ-fractional Sturm-Liouville problem

We consider the following ψ-fractional S-L equation

$$-D_{b^-}^{\alpha;\psi}\left(p(x)\ {}^C D_{a^+}^{\alpha;\psi} f(x)\right) + q(x) f(x) = \lambda\, w(x)\, f(x) \tag{7}$$

subject to the following boundary conditions

$$c_1\, f\,(a) + c_2\, I_{b^-}^{1-\alpha;\psi} \left(p(x)\ {}^C D_{a^+}^{\alpha;\psi} f(x)\right) \Big|_{x=a} = 0, \quad c_1^2 + c_2^2 \neq 0, \tag{8}$$

$$d_1\, f\,(b) + d_2\, I_{b^-}^{1-\alpha;\psi} \left(p(x)\ {}^C D_{a^+}^{\alpha;\psi} f(x)\right) \Big|_{x=b} = 0, \quad d_1^2 + d_2^2 \neq 0, \tag{9}$$

where $\alpha \in (0,1)$, $x \in [a,b]$, $p(x), w(x) > 0$, for all $x \in [a,b]$, and p, q, w are real-valued continuous functions in $[a,b]$. Our problem consists in finding the values of λ such that the boundary-value problem (7)-(9) has a non-trivial solution. We denote by $\mathcal{L}^{\alpha;\psi}$ the ψ-fractional S-L operator associated to problem (7) given by

$$\mathcal{L}^{\alpha;\psi} := -D_{b^-}^{\alpha;\psi} \left(p(x)\, {}^C D_{a^+}^{\alpha;\psi}\right) + q(x).$$

As suggested by the ψ-fractional integration by parts of Proposition 2.3 we will work in the weighted space $L_1([a,b], \psi'(x)dx)$. Using (6) and the boundary conditions (8)-(9) one can prove the following theorem.

Theorem 3.1 *The eigenvalues of our ψ-fractional problem (7)-(9) are real. Moreover, the eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the sensity function $w(x)$ on $[a,b]$, that is*

$$\int_a^b f_{\lambda_1}(x) f_{\lambda_2}(x) w(x)\, \psi'(x)\, dx, \qquad \lambda_1 \neq \lambda_2.$$

Our ψ-fractional S-L equation with boundary conditions can be written in integral form as the next proposition shows.

Proposition 3.2 *Let $\alpha > 1/2$ and f_λ be an eigenfunction associated to the eigenvalue λ. On the space $C([a,b])$, the $\psi-$fractional S-L problem is equivalent to the integral equation*

$$f_\lambda(x) = -I_{a^+}^{\alpha;\psi} \Big(\frac{1}{p\,(x)}\, I_{b^-}^{\alpha;\psi} F_\lambda\,(f) \Big) + A\,(x) \int_a^b F_\lambda\,(f)\; dx + B\,(x) \left(I_{a^+}^{\alpha;\psi} \Big(\frac{1}{p\,(x)}\, I_{b^-}^{\alpha;\psi} F_\lambda\,(f) \Big) \right) \Bigg|_{x=b} \tag{10}$$

where the coefficients $A\,(x)$ and $B\,(x)$ are given by

$$A\,(x) \;=\; \frac{c_2}{\Delta} \left(d_2 + d_1 \Big(Y - \; I_{a^+}^{\alpha;\psi} \Big(\frac{(\psi\,(b) - \psi\,(x))^{\alpha-1}}{p\,(x)\; \Gamma\,(\alpha)} \Big) \Big) \right),$$

$$B\,(x) \;=\; \frac{d_1}{\Delta} \left(c_1\, I_{a^+}^{\alpha;\psi} \Big(\frac{(\psi\,(b) - \psi\,(x))^{\alpha-1}}{p\,(x)\; \Gamma\,(\alpha)} \Big) - c_2 \right),$$

with $\Delta = c_1(d_1 Y + d_2) - c_2 d_1 \neq 0$, $Y = I_{a^+}^{\alpha;\psi} \Big(\frac{(\psi(b)-\psi(x))^{\alpha-1}}{p(x)\,\Gamma(\alpha)} \Big) \Big|_{x=b}$, and $F_\lambda(f) = q(x) f_\lambda(x) - \lambda w(x) f_\lambda(x)$.

Sketch of the proof: Using composition rules we can write the equation (7) as follows:

$$D_{b^-}^{\alpha;\psi} \left(p(x)\ {}^C D_{a^+}^{\alpha;\psi}\right) \Big[f_\lambda(x) + I_{a^+}^{\alpha;\psi} \Big(\frac{1}{p(x)} I_{b^-}^{\alpha;\psi} F_\lambda(f) \Big) \Big] = 0. \tag{11}$$

On the space $C([a,b])$ a general solution of (11) is given by

$$f_\lambda(x) + I_{a^+}^{\alpha;\psi} \Big(\frac{1}{p(x)} I_{b^-}^{\alpha;\psi} F_\lambda(f) \Big) = \xi_1 + \xi_2\, I_{a^+}^{\alpha;\psi} \Big(\frac{(\psi(b) - \psi(x))^{\alpha-1}}{p(x)\, \Gamma(\alpha)} \Big), \quad \xi_1, \xi_2 \in \mathbb{R}. \tag{12}$$

From [3, Eq. (34)] the fractional integral that is multiplied by ξ_2 gives a continuous function on $[a,b]$ only for $1/2 < \alpha < 1$. From (12) and using composition rules we obtain:

$$I_{b^-}^{1-\alpha;\psi} \left(p(x)\ {}^C D_{a^+}^{\alpha;\psi} f_\lambda(x) \right) = \xi_2 - I_{b^-}^{1;\psi} F_\lambda(f). \tag{13}$$

Finally, from (13) and (12) and using the boundary conditions (8)-(9) we can relate the coefficients ξ_1, ξ_2 with the values $c_j, d_j, j = 1,2$ in the boundary conditions to obtain (10).

Our final theorem gives the conditions under which the eigenfunctions exist and are unique. The proof follows similar arguments as the proof of [3, Thm. 9].

Theorem 3.3 *Let $\alpha > 1/2$ and assume that $\Delta \neq 0$. Then our ψ-fractional problem has unique continuous eigenfunctions f_λ if the fixed point condition holds*

$$\|q(x) + \lambda \omega(x)\| < \frac{m_p}{\|\varphi(x)\| + \|B(x)\| \varphi(b) + m_p \|A(x)\| (\psi(b) - \psi(a))},$$

where $\varphi(x) = I_{a^+}^{\alpha;\psi} \Big(\frac{(\psi(b)-\psi(x))^{\alpha-1}}{p(x)\,\Gamma(\alpha)} \Big)$ and $m_p = \min_{x\in[a,b]} |p(x)|$.

4 Conclusion

In this work we showed how to use the ψ-fractional calculus to study a ψ-fractional S-L problem.

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