

THE MONOTONE-LIGHT FACTORIZATION FOR 2-CATEGORIES VIA 2-PREORDERS

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ABSTRACT. It is shown that the reflection $2Cat \rightarrow 2Preord$ of the category of all 2-categories into the category of 2-preorders determines a monotone-light factorization system on $2Cat$ and that the light morphisms are precisely the 2-functors faithful on 2-cells with respect to the vertical structure. In order to achieve such result it was also proved that the reflection $2Cat \rightarrow 2Preord$ has stable units, a stronger condition than admissibility in categorical Galois theory, and that the 2-functors surjective both on horizontally and on vertically composable triples of 2-cells are the effective descent morphisms in $2Cat$.

1. Introduction

1.1. THE PROCESS OF STABILIZATION AND LOCALIZATION. The classical monotone-light factorization is due to S. Eilenberg [3] and G. T. Whyburn [14]. It consists of a pair $(\mathcal{E}', \mathcal{M}^*)$ of classes of morphisms in the category $CompHaus$ of compact Hausdorff spaces, such that any continuous map f of compact Hausdorff spaces factorizes as $f = m \circ e$, with $e \in \mathcal{E}'$ (the class of monotone maps, i.e., those who have totally disconnected fibres) and $m \in \mathcal{M}^*$ (the class of light maps, i.e., those whose fibres are connected).

This factorization system $(\mathcal{E}', \mathcal{M}^*)$ may be obtained from another (reflective) factorization system $(\mathcal{E}, \mathcal{M})$ on $CompHaus$ by a process of simultaneous stabilization and localization. That is: beginning with the reflector $I : CompHaus \rightarrow Stone$ (the category of Stone spaces), whose right adjoint H is a full inclusion, there is an associated (reflective) factorization system $(\mathcal{E}, \mathcal{M})$; then, we take the largest subclass \mathcal{E}' of \mathcal{E} which is stable under pullbacks, and take \mathcal{M}^* to be the class of morphisms which are obtained by localizing \mathcal{M} ($m \in \mathcal{M}^*$ if there is any surjective map p such that the pullback $p^*(m)$ along p is in \mathcal{M}).

The process explained in the last paragraph was studied in general in [1], beginning with a reflection $H \dashv I : \mathbf{C} \rightarrow \mathbf{X}$, and the surjective maps p used in the localization were generalized to morphisms $p : E \rightarrow B$ such that the pullback functor $p^* : \mathbf{C}/B \rightarrow \mathbf{C}/E$ is monadic, called effective descent morphisms (e.d.m.) in Grothendieck (basic-fibrational)

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descent theory.

In that paper [1] some examples were given in which this process actually achieves a new factorization system $(\mathcal{E}', \mathcal{M}^*)$ (non-trivial because $\mathcal{E}' \neq \mathcal{E} \Leftrightarrow \mathcal{M} \neq \mathcal{M}^*$). This is a quite rare phenomenon, since the pair $(\mathcal{E}', \mathcal{M}^*)$ usually fails to be a factorization system. When $(\mathcal{E}', \mathcal{M}^*)$ is in fact a factorization system (whose definition can be found in [4], and nicely exposed with other insights in [1]), obtained by simultaneous stabilization and localization, it is to be called monotone-light as in [1], paying tribute to that first example referred to above.

An interesting feature of this process is its connection to categorical Galois theory (see [5]). If the reflection $I \dashv H$ is semi-left-exact (in the sense of [2]), also called admissible in categorical Galois theory, then \mathcal{M}^* is the class of coverings in the sense of that theory (being $Spl(E, p)$ the full subcategory of \mathbf{C}/B , determined by the coverings over B split by the monadic extension (E, p) , the fundamental theorem of categorical Galois theory says that $Spl(E, p)$ has an algebraic description).

1.2. PAST AND PRESENT WORK. In [11], it was presented a new non-trivial example of the process above, for the reflection $Cat \rightarrow Preord$ of the category of all categories into the category of all preordered sets, where the coverings are the faithful functors.

Now, in this paper, it will be proved that also the reflection $2Cat \rightarrow 2Preord$ of the category of all 2-categories into the category of all 2-preorders has a non-trivial monotone-light factorization, where the coverings are the 2-functors which are faithful vertically with respect to 2-cells.

Notice that both reflections have stable units (in the sense of [2]; a stronger condition than semi-left-exactness), which is crucial to the proof in association with the fact that there are enough e.d.m. with domain in the subcategory.

The needed characterization of e.d.m. in $2Cat$ is given in this paper, obtained in a completely analogous way as the characterization of e.d.m. in Cat was done in [6]. These characterizations depend on the embedding of Cat and $2Cat$ in the obvious presheaf categories.

1.3. FUTURE WORK: PROSPECTS. This paper is intended to be a first step in showing that the monotone-light factorization in [11] can be extended to higher category theory.

In particular, we hope to achieve the same results for n -categories in general (and to ω -categories) in a similar way, embedding n -categories in categories of presheaves, if feasible.

We also believe, for the moment, that the good context for extending our results to higher category theory will be that of \mathcal{V} -categories. Remark that, considering $\mathcal{V} = Set$ the category of sets and then iterating one obtains n -categories. Of course, this opens the possibility of doing so for \mathcal{V} other than the category of sets. We would be very interested, for instance, to apply these future results to the following open problem: is there a monotone-light factorization for semigroups via semilattices (see [7] and [10])?

Remark finally that, because of the characterization given in this paper for the e.d.m. in $2Cat$ (cf. 4.1), we are driven to present the following conjecture about the nature of

e.d.m. for n -categories ($n \geq 1$), which may be helpful in related future work.

Conjecture: an n -functor is an e.d.m. in the category of all n -categories if and only if it is surjective both on vertically composable triples of n -cells, and on horizontally composable triples of k -cells, for every $k \in \{2, \dots, n\}$.

Similarly, we could present other obvious conjectures. Some concerning the classes of morphisms of categorical Galois theory characterized in this paper, and even of other important classes of morphisms not treated here, since we were not exhaustive.

2. The category of all 2-categories

Consider the category $2Cat$, with objects all 2-categories and whose morphisms are the 2-functors (see [9, §XII.3]). Its definition is going to be stated in a way that suits our purposes. In order to do so, some intermediate structures need to be defined first.

First, consider the category \mathbf{P} generated by the following *precategory diagram*,

$$\begin{array}{ccccc}
 & \xrightarrow{q} & & \xrightarrow{d} & \\
 P_2 & \xrightarrow{m} & P_1 & \xleftarrow{e} & P_0 \\
 & \xrightarrow{r} & & \xleftarrow{c} & \\
 & \xrightarrow{\quad} & & \xrightarrow{\quad} &
 \end{array}$$

in which

$d \circ e = 1_{P_0} = c \circ e$, $d \circ m = d \circ q$, $c \circ m = c \circ r$ and $c \circ q = d \circ r$, where 1_{P_0} stands for the identity morphism of P_0 (see [1, §4.1]).

A *precategory* is an object in the category of presheaves $\hat{\mathbf{P}} = Set^{\mathbf{P}}$, that is, any functor $P : \mathbf{P} \rightarrow Set$ to the category of sets.

If

$$\begin{array}{ccccc}
 & \xrightarrow{q'} & & \xrightarrow{d'} & \\
 Q_2 & \xrightarrow{m'} & Q_1 & \xleftarrow{e'} & Q_0 \\
 & \xrightarrow{r'} & & \xleftarrow{c'} & \\
 & \xrightarrow{\quad} & & \xrightarrow{\quad} &
 \end{array}$$

is another precategory diagram, then a triple (f_2, f_1, f_0) with $f_2 : P_2 \rightarrow Q_2$, $f_1 : P_1 \rightarrow Q_1$ and $f_0 : P_0 \rightarrow Q_0$, will be called a *precategory morphism diagram* provided the following equations hold: $f_0 \circ d = d' \circ f_1$, $f_0 \circ c = c' \circ f_1$, $f_1 \circ e = e' \circ f_0$, $f_1 \circ q = q' \circ f_2$, $f_1 \circ m = m' \circ f_2$, $f_1 \circ r = r' \circ f_2$.

Secondly, consider the category $2\mathbf{P}$ generated by the following *2-precategory diagram*,

$$\begin{array}{ccccc}
 & & \xrightarrow{hq \times hq} & & \xrightarrow{hd \times hd} \\
 & hvP_2 & \xrightarrow{hm \times hm} & vP_2 & \xleftarrow{he \times he} P_0 \\
 & & \xrightarrow{hr \times hr} & & \xleftarrow{hc \times hc} \\
 \begin{array}{c} vr \times vr \\ vm \times vm \\ vq \times vq \end{array} \downarrow & \downarrow & \downarrow & \begin{array}{c} vr \\ vm \\ vq \end{array} \downarrow & \downarrow 1_{P_0} \\
 & hP_2 & \xrightarrow{hm} & 2P_1 & \xleftarrow{he} P_0 \\
 & & \xrightarrow{hr} & & \xleftarrow{hc} \\
 \begin{array}{c} vc \times vc \\ ve \times ve \\ vd \times vd \end{array} \downarrow & \downarrow & \downarrow & \begin{array}{c} vc \\ ve \\ vd \end{array} \downarrow & \downarrow 1_{P_0} \\
 & P_2 & \xrightarrow{m} & P_1 & \xleftarrow{e} P_0 \\
 & & \xrightarrow{r} & & \xleftarrow{c}
 \end{array} \tag{2.1}$$

in which:

- each one of the three horizontal diagrams (upwards, P , hP and hvP) is a precategory diagram;
- each one of the three vertical diagrams (from the left to the right, vhP , vP and the trivial P_0) is a precategory diagram;
- $(vc \times vc, vc, 1_{P_0})$, $(ve \times ve, ve, 1_{P_0})$, $(vd \times vd, vd, 1_{P_0})$, $(vr \times vr, vr, 1_{P_0})$, $(vm \times vm, vm, 1_{P_0})$, $(vq \times vq, vq, 1_{P_0})$ are all six precategory morphism diagrams (equivalently, $(hq \times hq, hq, q)$, $(hm \times hm, hm, m)$, $(hr \times hr, hr, r)$, $(hd \times hd, hd, d)$, $(he \times he, he, e)$, $(hc \times hc, hc, c)$ are all six precategory morphism diagrams).

Notice that the names given to objects and morphisms in (2.1) are arbitrary, being so chosen in order to relate to the following last definition of section 2 (for instance, $vq \times vq$ will denote the morphism uniquely determined by a pullback diagram).

The category $2Cat$ of all 2-categories is the full subcategory of $2\hat{\mathbf{P}} = Set^{2\mathbf{P}}$, determined by its objects $C : 2\mathbf{P} \rightarrow Set$ such that the image by C of each horizontal and vertical precategory diagram in (2.1) is a category. That is, for instance, in the case of the bottom horizontal precategory diagram in (2.1):

the commutative square

$$\begin{array}{ccc}
 C(P_2) & \xrightarrow{Cq} & C(P_1) \\
 Cr \downarrow & & \downarrow Cc \\
 C(P_1) & \xrightarrow{Cd} & C(P_0)
 \end{array} \quad (2.2)$$

is a pullback diagram in Set ;

the associative and unit laws hold for the operation Cm , that is, the following respective diagrams commute in Set ,

$$\begin{array}{ccc}
 C(P_2) \times_{C(P_1)} C(P_2) & \xrightarrow{Cm \times Cq} & C(P_2) \\
 Cr \times Cm \downarrow & & \downarrow Cm \\
 C(P_2) & \xrightarrow{Cm} & C(P_1),
 \end{array} \quad (2.3)$$

$$\begin{array}{ccccc}
 C(P_0) \times_{C(P_0)} C(P_1) & \xrightarrow{Ce \times 1_{C(P_1)}} & C(P_2) & \xleftarrow{1_{C(P_1)} \times Ce} & C(P_1) \times_{C(P_0)} C(P_0) \\
 \downarrow pr_2 & & \downarrow Cm & & \downarrow pr_1 \\
 C(P_1) & \xrightarrow{1_{C(P_1)}} & C(P_1) & \xleftarrow{1_{C(P_1)}} & C(P_1).
 \end{array} \quad (2.4)$$

It would be a long and trivial calculation to check that there is an isomorphism between the category of all 2-categories (in the sense of [9, §XII.3]) and the full subcategory of $2\hat{\mathbf{P}}$

just defined. Notice that: the requirement that the horizontal composite of two vertical identities is itself a vertical identity is encoded in diagram (2.1) in the commutativity of the square $hm \circ (ve \times ve) = ve \circ m$; the interchange law, which relates the vertical and the horizontal composites of 2-cells, is encoded in diagram (2.1) in the commutativity of the square $vm \circ (hm \times hm) = hm \circ (vm \times vm)$.

3. Internal categories and limits

In section 2, if the category *Set* of sets is replaced by any category \mathcal{C} with pullbacks, then one obtains the definition of $2Cat(\mathcal{C})$, the category of internal 2-categories in \mathcal{C} .

In this section 3, the goal is to show that the category of all 2-categories $2Cat$ is closed under limits in the presheaves category $\hat{2}\mathbf{P} = Set^{2\mathbf{P}}$. The following Lemmas 3.1 and 3.2 give some well known facts about limits of internal categories, which will translate into internal 2-categories, and finally into 2-categories in the special case of $\mathcal{C} = Set$.

In what follows, $Cat(\mathcal{C})$ will denote the category of internal categories in \mathcal{C} , that is, the full subcategory of the category of functors $\mathcal{C}^{\mathbf{P}}$, determined by all the functors $C : \mathbf{P} \rightarrow \mathcal{C}$ such that the diagram (2.2) is a pullback diagram in \mathcal{C} and the diagrams (2.3) and (2.4) commute in \mathcal{C} (\mathbf{P} is of course the category defined in section 2).

3.1. LEMMA. *Let \mathcal{C} be a category with pullbacks.*

Then, $Cat(\mathcal{C})$ is closed under pullbacks in $\mathcal{C}^{\mathbf{P}}$, where pullbacks exist and are calculated pointwise.

3.2. LEMMA. *Let \mathcal{C} be a category with pullbacks.*

If \mathbf{I} is a discrete category (that is, a set) and \mathcal{C} has all limits $\mathbf{I} \rightarrow \mathcal{C}$, then $Cat(\mathcal{C})$ is closed under all limits $\mathbf{I} \rightarrow Cat(\mathcal{C})$ in $\mathcal{C}^{\mathbf{P}}$, where limits $\mathbf{I} \rightarrow \mathcal{C}^{\mathbf{P}}$ exist and are calculated pointwise.

3.3. COROLLARY. *If \mathcal{C} has all limits then $2Cat(\mathcal{C})$ is closed under limits in the functor category $\mathcal{C}^{2\mathbf{P}}$, where all limits exist and are calculated pointwise.*

In particular, for $\mathcal{C} = Set$, $2Cat$ is closed under limits in $\hat{2}\mathbf{P} = Set^{2\mathbf{P}}$.

PROOF. The proof follows from the fact that limits are calculated pointwise in $\mathcal{C}^{2\mathbf{P}}$, and that a category with pullbacks and all products has all limits, and from Lemmas 3.1 and 3.2. ■

4. Effective descent morphisms in $2Cat$

Consider again the category of all categories *Cat* and its full inclusion in the category of precategories $\hat{\mathbf{P}} = Set^{\mathbf{P}}$.

A functor $p : \mathbf{E} \rightarrow \mathbf{B}$ is an effective descent morphism (e.d.m.)¹ in *Cat* if and only if it is surjective on composable triples of morphisms. The proof of this statement can be

¹Also called a *monadic extension* in categorical Galois theory.

found in [6, Proposition 6.2]. In a completely analogous way, the class of effective descent morphisms in $2Cat$ is going to be given in the following Proposition 4.1.

4.1. PROPOSITION. *A 2-functor $2p : 2\mathbf{E} \rightarrow 2\mathbf{B}$ is an e.d.m. in the category of all 2-categories $2Cat$ if and only if it is surjective both on horizontally composable triples of 2-cells and on vertically composable triples of 2-cells.*

PROOF.

(if:)

Let $2p : 2\mathbf{E} \rightarrow 2\mathbf{B}$ be surjective on triples of composable 2-cells (both horizontally and vertically). Then, $2p$ is an e.d.m. in $2\hat{\mathbf{P}} = Set^{2\mathbf{P}}$, since the effective descent morphisms in a category of presheaves are simply those surjective pointwise (which, of course, is implied by either surjectivity on triples of composable 2-cells). Hence, the following instance of [6, Corollary 3.9] can be applied:

if $2p : 2\mathbf{E} \rightarrow 2\mathbf{B}$ in $2Cat$ is an e.d.m. in $2\hat{\mathbf{P}} = Set^{2\mathbf{P}}$ then $2p$ is an e.d.m. in $2Cat$ if and only if, for every pullback square

$$\begin{array}{ccc}
 2\mathbf{D} & \longrightarrow & 2A \\
 \downarrow & & \downarrow \\
 2\mathbf{E} & \xrightarrow{2p} & 2\mathbf{B}
 \end{array} \tag{4.1}$$

in $2\hat{\mathbf{P}} = Set^{2\mathbf{P}}$ such that $2\mathbf{D}$ is in $2Cat$, then also $2A$ is in $2Cat$.

Since the pullback square (4.1) is calculated pointwise (cf. Corollary 3.3), it induces six other pullback squares in $\hat{\mathbf{P}} = Set^{\mathbf{P}}$, corresponding to the three rows P, hP and hvP , and the three columns vhP, vP and P_0 , in the 2-precategory diagram (2.1).

The fact that $2p$ is surjective on triples of composable 2-cells (both horizontally and vertically) implies that its six restrictions (to the six rows and columns $2\mathbf{E}(P), 2\mathbf{E}(hP), 2\mathbf{E}(hvP), 2\mathbf{E}(vhP), 2\mathbf{E}(vP)$ and $2\mathbf{E}(P_0)$) are surjective on triples of composable morphisms in Cat , as it is easy to check. Hence, these six restrictions are effective descent morphisms in Cat . Therefore, $2A$ must always be a 2-category, provided so is $2\mathbf{D}$.

(only if:)

Let's now prove the converse, that if $2p : 2\mathbf{E} \rightarrow 2\mathbf{B}$ is not surjective on vertically composable triples or on horizontally composable triples of 2-cells, then it is not an effective descent morphism.

Suppose first that $2p : 2\mathbf{E} \rightarrow 2\mathbf{B}$ is not surjective on horizontally composable triples of 2-cells, that is, there is a triple

$$\begin{array}{ccccc}
 0 & \xrightarrow{\quad} & 1 & \xrightarrow{\quad} & 2 & \xrightarrow{\quad} & 3 \\
 & \Downarrow \beta_{01} & & \Downarrow \beta_{12} & & \Downarrow \beta_{23} & \\
 0 & \xrightarrow{\quad} & 1 & \xrightarrow{\quad} & 2 & \xrightarrow{\quad} & 3
 \end{array}$$

in $2\mathbf{B}$, and there is no triple

$$\begin{array}{ccccc}
 0 & \xrightarrow{\quad} & 1 & \xrightarrow{\quad} & 2 & \xrightarrow{\quad} & 3 \\
 & \Downarrow \varepsilon_1 & & \Downarrow \varepsilon_2 & & \Downarrow \varepsilon_3 & \\
 0 & \xrightarrow{\quad} & 1 & \xrightarrow{\quad} & 2 & \xrightarrow{\quad} & 3
 \end{array}$$

in $2\mathbf{E}$, such that $2p(\varepsilon_1) = \beta_{01}$, $2p(\varepsilon_2) = \beta_{12}$ and $2p(\varepsilon_3) = \beta_{23}$.

Consider the 2-category $h\mathbf{4}$ generated by the following diagram,

$$\begin{array}{ccccc}
 & \xrightarrow{\quad} & & \xrightarrow{\quad} & & \xrightarrow{\quad} & \\
 0 & \Downarrow \alpha_{01} & 1 & \Downarrow \alpha_{12} & 2 & \Downarrow \alpha_{23} & 3 . \\
 & \xrightarrow{\quad} & & \xrightarrow{\quad} & & \xrightarrow{\quad} &
 \end{array}$$

Then, consider also the presheaf $h\mathbf{4}^{n.a.} : 2\mathbf{P} \rightarrow Set$, which has the same objects (0-cells), the same morphisms (1-cells), and the same six non-identity 2-cells α_{ij} , $0 \leq i < j \leq 3$ as $h\mathbf{4}$, plus a new 2-cell $\alpha'_{03} : 0 \rightarrow 3$, with the horizontal composition of 2-cells being as follows: $\alpha_{12} \circ \alpha_{01} = \alpha_{02}$, $\alpha_{23} \circ \alpha_{12} = \alpha_{13}$, $\alpha_{23} \circ \alpha_{02} = \alpha_{03}$ and $\alpha_{13} \circ \alpha_{01} = \alpha'_{03}$ (with $\alpha'_{03} \neq \alpha_{03}$, which destroys the horizontal associativity). Hence, one can call $h\mathbf{4}^{n.a.}$ a horizontally non-associative 2-category, in the sense that the images of the precategory diagrams P, vP are categories, the image of the commutative square $hc \circ hq = hd \circ hr$ is a pullback diagram, the unit laws hold for the image of hP , but the image of hm is not associative (cf. diagram (2.1)).

Now, if one takes $2A = h\mathbf{4}^{n.a.}$ and $2A \rightarrow 2\mathbf{B}$ to be the morphism φ determined by $\varphi\alpha_{ij} = \beta_{ij}$, $0 \leq i < j \leq 3$, and $\varphi\alpha'_{03} = \beta_{03}$, then the pullback $2\mathbf{D}$ is a 2-category; and therefore $2p$ is not an effective descent morphism, because it is not surjective on horizontal triples. In fact, $2\mathbf{D}$ would not be a 2-category only if there was a triple of 2-cells

$$\begin{array}{ccccc}
 & \xrightarrow{\quad} & & \xrightarrow{\quad} & & \xrightarrow{\quad} & \\
 & \Downarrow (\varepsilon_1, \alpha_1) & & \Downarrow (\varepsilon_2, \alpha_2) & & \Downarrow (\varepsilon_3, \alpha_3) & \\
 & \xrightarrow{\quad} & & \xrightarrow{\quad} & & \xrightarrow{\quad} &
 \end{array}$$

in $2\mathbf{D}$ such that

$$((\varepsilon_3, \alpha_3) \circ (\varepsilon_2, \alpha_2)) \circ (\varepsilon_1, \alpha_1) \neq (\varepsilon_3, \alpha_3) \circ ((\varepsilon_2, \alpha_2) \circ (\varepsilon_1, \alpha_1)),^2$$

which would imply $\alpha_1 = \alpha_{01}$, $\alpha_2 = \alpha_{12}$ and $\alpha_3 = \alpha_{23}$, and so, contrary to the hypothesis, $2p$ would be surjective on horizontally composable triples of 2-cells.

The proof that if $2p : 2\mathbf{E} \rightarrow 2\mathbf{B}$ is not surjective on vertically composable triples of 2-cells, then it is not an effective descent morphism, can be made “mutatis mutandis” using the 2-category $v\mathbf{4}$ in the following Example 4.2. ■

4.2. EXAMPLE. It is obvious that the coproduct of 2-categories is just the disjoint union, as for categories.

Let $v\mathbf{4}$ and $h\mathbf{4}$ be the 2-categories generated by the following two diagrams, respectively:

²Since pullbacks are calculated componentwise in a category of presheaves, and that the proof of Lemma 3.1 is made by considering the associative and unit laws independently (cf. above (2.2), (2.3) and (2.4)).

$$\begin{array}{c}
 \longrightarrow \\
 \Downarrow \\
 0 \longrightarrow 1 \\
 \Downarrow \\
 \longrightarrow \\
 \Downarrow \\
 \longrightarrow
 \end{array}
 \quad ; \quad
 \begin{array}{c}
 \longrightarrow \\
 \Downarrow \\
 0 \longrightarrow 1 \\
 \longrightarrow \\
 \Downarrow \\
 \longrightarrow
 \end{array}
 \longrightarrow
 \begin{array}{c}
 \longrightarrow \\
 \Downarrow \\
 1 \longrightarrow 2 \\
 \longrightarrow \\
 \Downarrow \\
 \longrightarrow
 \end{array}
 \longrightarrow
 \begin{array}{c}
 \longrightarrow \\
 \Downarrow \\
 2 \longrightarrow 3 \\
 \longrightarrow \\
 \Downarrow \\
 \longrightarrow
 \end{array}
 .$$

Consider, for each 2-category $2\mathbf{B}$, the 2-category

$$2\mathbf{E} = \left(\coprod_{i \in I} v\mathbf{4}\right) + \left(\coprod_{j \in J} h\mathbf{4}\right),$$

such that I is the set of all vertically composable triples of 2-cells in $2\mathbf{B}$, and J is the set of all horizontally composable triples of 2-cells in $2\mathbf{B}$.

Then, there is an e.d.m. $2p : 2\mathbf{E} \rightarrow 2\mathbf{B}$ which projects the corresponding copy of $v\mathbf{4}$ and $h\mathbf{4}$ to every $i \in I$ and every $j \in J$, respectively.

As another option, let

$$2\mathbf{E} = \coprod_{k \in I \cup J} vh\mathbf{4},$$

with $vh\mathbf{4}$ the 2-category³ generated by the following diagram,

$$\begin{array}{c}
 \longrightarrow \\
 \Downarrow \\
 0 \longrightarrow 1 \\
 \longrightarrow \\
 \Downarrow \\
 \longrightarrow
 \end{array}
 \longrightarrow
 \begin{array}{c}
 \longrightarrow \\
 \Downarrow \\
 1 \longrightarrow 2 \\
 \longrightarrow \\
 \Downarrow \\
 \longrightarrow
 \end{array}
 \longrightarrow
 \begin{array}{c}
 \longrightarrow \\
 \Downarrow \\
 2 \longrightarrow 3 \\
 \longrightarrow \\
 \Downarrow \\
 \longrightarrow
 \end{array}
 .$$

5. The reflection of 2-categories into 2-preorders has stable units and a monotone-light factorization

Let $2Preord$ be the full subcategory of $2Cat$ determined by the objects $C : 2\mathbf{P} \rightarrow Set$ such that Cvd and Cvc are jointly monic (cf. diagram (2.1)), that is,

$$\begin{array}{ccc}
 & \xrightarrow{Cvq} & \xrightarrow{Cvd} \\
 C(vP_2) & \xrightarrow{Cvm} & C(2P_1) & \xleftarrow{Cve} & C(P_1) \\
 & \xrightarrow{Cvr} & & \xleftarrow{Cvc} & \\
 & \longrightarrow & & \longrightarrow &
 \end{array}
 \quad (5.1)$$

³Remark that $v\mathbf{4}$, $h\mathbf{4}$ and $vh\mathbf{4}$ are really 2-preorders as defined just below at the beginning of the following section 5.

is a preordered set.

There is a reflection

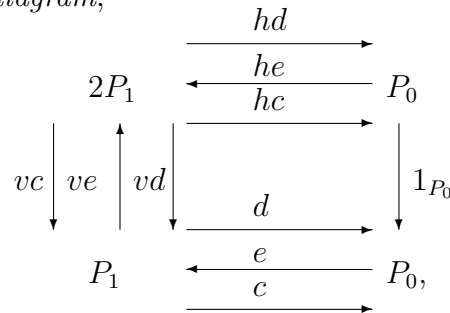
$$H \vdash I : 2Cat \longrightarrow 2Preord, \quad a \begin{array}{c} \xrightarrow{f} \\ \Downarrow \theta \\ \xrightarrow{g} \end{array} b \mapsto a \begin{array}{c} \xrightarrow{f} \\ \Downarrow \leq \\ \xrightarrow{g} \end{array} b, \quad (5.2)$$

which identifies all 2-cells which have the same domain and codomain for the vertical composition. That is, the reflector I takes the middle vertical category $C(vP)$ (cf. diagram (5.1)) to its image by the well known reflection $Cat \rightarrow Preord$ from categories into preordered sets (see [11]).

Many of the results in [13] are going to be stated again, with small improvements in their presentation⁴, in order to prove that the reflection $H \vdash I : 2Cat \rightarrow 2Preord$ has stable units (in the sense of [2]).

5.1. GROUND STRUCTURE. Consider the adjunction $H \vdash I : 2Cat \rightarrow 2Preord$, described just above in (5.2), with unit $\eta : 1_{2Cat} \rightarrow HI$.

- $2Cat$ has pullbacks (in fact, it has all limits - see Corollary 3.3).
- H is a full inclusion of $2Preord$ in $2Cat$, that is, I is a reflection of a category with pullbacks into a full subcategory.
- Consider also the forgetful functor $U : 2Cat \rightarrow 2RGrph$, where $2RGrph$ is the presheaves category $Set^{2\mathbf{G}}$, with $2\mathbf{G}$ the category generated by the following 2-reflexive graph diagram,



satisfying the same equations as in the 2-precATEGORY diagram (2.1).

- \mathcal{E} denotes the class of all morphisms $(2g_1, g_1, g_0) : G \rightarrow H$ of $2RGrph$ which are bijections on objects and on arrows, and surjections on 2-cells (that is, $g_0 : G(P_0) \rightarrow H(P_0)$ and $g_1 : G(P_1) \rightarrow H(P_1)$ are bijections, and $2g_1 : G(2P_1) \rightarrow H(2P_1)$ is a surjection).

⁴The reader could easily bring these small improvements to the paper [13]. In fact, although they are stated here in the particular case of the reflection from $2Cat$ into $2Preord$, they are completely general.

- $\mathcal{T} = \{T\}$ is a singular set, with T the 2-preorder generated by the diagram

$$\begin{array}{ccc}
 & \xrightarrow{h} & \\
 a & \Downarrow \leq & a' \text{ (5.3),} \\
 & \xrightarrow{h'} &
 \end{array}$$

that is, a 2-preorder with two objects, two non-identity arrows and only one non-identity (both horizontally and vertically) 2-cell.

Then, the following four conditions are satisfied.

- (a) U preserves pullbacks (in fact, it preserves all limits).
- (b) \mathcal{E} is pullback stable in $2RGrph$, and if $g' \circ g$ is in \mathcal{E} so is g' , provided g is in \mathcal{E} .⁵
- (c) Every map $U\eta_C : U(C) \rightarrow UHI(C)$ belongs to \mathcal{E} , $C \in 2Cat$ (this is also obvious).
- (d) ⁶Let $g : N \rightarrow M$ be any morphism of $2Preord$ such that $UHg : UH(N) \rightarrow UH(M)$ is in \mathcal{E} .

If,

there is one morphism $f : A \rightarrow UH(N)$ of $2RGrph$ in \mathcal{E} such that,

for all morphisms $c : T \rightarrow M$ in $2Preord$ (T as defined in (5.3)),

there is a commutative diagram as below

$$\begin{array}{ccccc}
 A \times_{UH(M)} UH(T) & \xrightarrow{pr_2} & UH(T) & & \\
 pr_1 \downarrow & & \swarrow & & \downarrow UHc \\
 A & \xrightarrow{f} & UH(N) & \xrightarrow{UHg} & UH(M)
 \end{array} \tag{5.4}$$

then

$g : N \rightarrow M$ is an isomorphism in $2Preord$.

⁵In [13], it was also demanded in (b) that \mathcal{E} is closed under composition, which is not needed. We take this opportunity to correct that redundancy in [13].

⁶This item is rephrased from [13], in a way that seems to us now more easily understandable. Remark also that the diagram (5.4) is simplified, suppressing one morphism $UH(T) \rightarrow UH(T)$, which can be the identity. We take this opportunity to correct that other redundancy.

It remains to show that the statement in (d) is true, which is trivial, since if $g : N \rightarrow M$ is in \mathcal{E} , seen as a morphism of $2RGrph$, then g must be an isomorphism in $2Preord$ by the uniqueness of the 2-cells in N and in M .

5.2. STABLE UNITS. Using the fact that a *ground structure* holds (which guarantees the validity of Theorems 2.1 and 2.2 in [13]), it will be possible to show that $H \vdash I : 2Cat \rightarrow 2Preord$ is an admissible reflection in the sense of categorical Galois theory (cf. [5]) or, equivalently, semi-left-exact in the sense of [2]. Furthermore, it will be shown, always using the results in [13], that the reflection $H \vdash I : 2Cat \rightarrow 2Preord$ satisfies the stronger condition of having stable units.

5.3. DEFINITION. Consider any morphism $\mu : T \rightarrow HI(C)$ from $T (\in \mathcal{T};$ cf. (5.3)), for some $C \in 2Cat$.

The connected component of the morphism μ is the pullback $C_\mu = C \times_{HI(C)} T$ in the following pullback square

$$\begin{array}{ccc} C_\mu & \xrightarrow{\pi_2^\mu} & T \\ \pi_1^\mu \downarrow & & \downarrow \mu \\ C & \xrightarrow{\eta_C} & HI(C), \end{array} \quad (5.5)$$

where η_C is the unit morphism of C in the reflection $H \vdash I : 2Cat \rightarrow 2Preord$, and T is identified with $H(T)$.

5.4. THEOREM. The reflection $H \vdash I : 2Cat \rightarrow 2Preord$ is semi-left-exact.

PROOF. According to Theorem 2.1 in [13], one has to show that $I\pi_2^\mu : I(C_\mu) \rightarrow I(T)$ is an isomorphism, for every connected component C_μ .

$$\text{If } \mu(a \xrightarrow{h} a') = c \xrightarrow{k} c' \quad \text{then,} \\ \mu(a \xrightarrow{h'} a') = c \xrightarrow{k'} c'$$

since $U\eta_C \in \mathcal{E}$ (identity on objects and morphisms, and surjection on 2-cells), the pullback C_μ is the 2-category generated by the diagram

$$\begin{array}{ccc} & \xrightarrow{(k, h)} & \\ (c, a) & \xrightarrow{\Downarrow (\theta_r, \leq)} & (c', a') \\ & \xrightarrow{(k', h')} & \end{array}$$

with $\theta_r \in Hom_{C(vP)}(k, k') = \{\theta_r \mid r \in R\}$, the set R indexing all the 2-cells θ_r in C with vertical domain $k : c \rightarrow c'$ and vertical codomain $k' : c \rightarrow c'$.

Hence, $I(C_\mu) \cong T$. ■

5.5. THEOREM. The reflection $H \vdash I : 2Cat \rightarrow 2Preord$ has stable units.

PROOF. According to Theorem 2.2 in [13], one has to show that $I(C_\mu \times_T D_\nu) \cong T$, for every pair of connected components C_μ, D_ν , where $C_\mu \times_T D_\nu$ is the pullback object in any pullback of the form

$$\begin{array}{ccc} C_\mu \times_T D_\nu & \xrightarrow{p_2} & D_\nu \\ p_1 \downarrow & & \downarrow \pi'_2 \\ C_\mu & \xrightarrow{\pi_2^\mu} & T \end{array} ,$$

where π_2^μ and π'_2 are the second projections in pullback diagrams of the form (5.5).

According to the previous Theorem 5.4, one can suppose (up to isomorphism) that

$$C_\mu = c \begin{array}{c} \xrightarrow{k} \\ \Downarrow \theta_r \\ \xrightarrow{k'} \end{array} c', \quad r \in R, \quad \text{and} \quad D_\nu = d \begin{array}{c} \xrightarrow{l} \\ \Downarrow \delta_s \\ \xrightarrow{l'} \end{array} d',$$

$s \in S$ (the identity morphisms and the identity 2-cells are not displayed); the sets R and S indexing respectively all the 2-cells θ_r in C with vertical domain $k : c \rightarrow c'$ and vertical codomain $k' : c \rightarrow c'$, and all the 2-cells δ_s in D with vertical domain $l : d \rightarrow d'$ and vertical codomain $l' : d \rightarrow d'$.

Hence, $C_\mu \times_T D_\nu = (c, d) \begin{array}{c} \xrightarrow{(k, l)} \\ \Downarrow (\theta_r, \delta_s) \\ \xrightarrow{(k', l')} \end{array} (c', d') \quad , \quad (r, s) \in R \times S$, and so it is obvious that $I(C_\mu \times_T D_\nu) \cong a \begin{array}{c} \xrightarrow{h} \\ \Downarrow \leq \\ \xrightarrow{h'} \end{array} a'$. ■

5.6. MONOTONE-LIGHT FACTORIZATION FOR 2-CATEGORIES VIA 2-PREORDERS.

5.7. THEOREM. *The reflection $H \vdash I : 2Cat \rightarrow 2Preord$ does have a monotone-light factorization.*

PROOF. The statement is a consequence of the central result of [1] (cf. Corollary 6.2 in [12]), because $H \vdash I$ has stable units (cf. Theorem 5.5) and for every $2\mathbf{B} \in 2Cat$ there is an e.d.m. $2p : 2\mathbf{E} \rightarrow 2\mathbf{B}$ with $2\mathbf{E} \in 2Preord$ (cf. Example 4.2). ■

In the following section 6, it will be proved that the monotone-light factorization system is not trivial. That is, it does not coincide with the reflective factorization system associated to the reflection of $2Cat$ into $2Preord$.

6. Vertical and stably-vertical 2-functors

In this section, it will be given a characterization of the class of vertical morphisms \mathcal{E}_I in the reflective factorization system $(\mathcal{E}_I, \mathcal{M}_I)$, and of the class of the stably-vertical mor-

phisms $\mathcal{E}'_I (\subseteq \mathcal{E}_I)$ ⁷ in the monotone-light factorization system $(\mathcal{E}'_I, \mathcal{M}_I^*)$, both associated to the reflection $2Cat \rightarrow 2Preord$. Then, since \mathcal{E}'_I is a proper class of \mathcal{E}_I , one concludes that $(\mathcal{E}'_I, \mathcal{M}_I^*)$ is a non-trivial monotone-light factorization system.

Consider a 2-functor $f : A \rightarrow B$, which is obviously determined by the three functions $f_0 : A(P_0) \rightarrow B(P_0)$, $f_1 : A(P_1) \rightarrow B(P_1)$ and $2f_1 : A(2P_1) \rightarrow B(2P_1)$ (cf. diagram (2.1)), so that we may make the identification $f = (2f_1, f_1, f_0)$.

6.1. PROPOSITION. *A 2-functor $f = (2f_1, f_1, f_0) : A \rightarrow B$ belongs to the class \mathcal{E}_I of vertical 2-functors if and only if the following two conditions hold:*

1. f_0 and f_1 are bijections;
2. for every two elements h and h' in $A(P_1)$, if $Hom_{B(vP)}(f_1h, f_1h')$ is nonempty then so is $Hom_{A(vP)}(h, h')$.

PROOF. The 2-functor $f = (2f_1, f_1, f_0)$ belongs to \mathcal{E}_I if and only if If is an isomorphism (cf. [1, §3.1]), that is, If_0, If_1 , and $I2f_1$ are bijections. Since $If_0 = f_0$ and $If_1 = f_1$, the fact that $f \in \mathcal{E}_I$ implies and is implied by (1) and (2) is trivial. ■

6.2. PROPOSITION. *A 2-functor $f = (2f_1, f_1, f_0) : A \rightarrow B$ belongs to the class \mathcal{E}'_I of stably-vertical 2-functors if and only if the following two conditions hold:*

1. f_0 and f_1 are bijections;
2. for every two elements h and h' in $A(P_1)$, f induces a surjection $Hom_{A(vP)}(h, h') \rightarrow Hom_{B(vP)}(f_1h, f_1h')$ (f is a “full functor on 2-cells”).

PROOF. As every pullback $g^*(f) = \pi_1 : C \times_B A \rightarrow C$ in $2Cat$ of f along any 2-functor $g : C \rightarrow B$ is calculated pointwise, and $(2f_1, f_1) : A(vP) \rightarrow B(vP)$ is a stably-vertical functor for the reflection $Cat \rightarrow Preord$, that is, f_1 is a bijection and $(2f_1, f_1)$ is a full functor (cf. Propositions 4.4 and 3.2 in [11]), then (1) and (2) imply that $g^*(f)$ belongs to \mathcal{E}'_I (cf. last Proposition 6.1).

Hence, $f \in \mathcal{E}'_I$ if (1) and (2) hold.

If $f \in \mathcal{E}'_I$, then $f \in \mathcal{E}_I$ ($\mathcal{E}'_I \subseteq \mathcal{E}_I$), and therefore (1) holds.

Suppose now that (2) does not hold, so that there is $\theta : f_1h \rightarrow f_1h'$ not in the image

of f , and consider the 2-category C generated by the diagram $b \begin{array}{ccc} & \xrightarrow{f_1h} & \\ & \Downarrow \theta & \\ & \xrightarrow{f_1h'} & b' \end{array}$, and let g be the inclusion of C in B . Then, $C \times_B A \cong b \begin{array}{ccc} & \xrightarrow{f_1(h)} & \\ & \xrightarrow{f_1(h')} & b' \end{array}$, with no non-identity 2-cells, and so $g^*(f)$ is not in \mathcal{E}_I .

⁷ \mathcal{E}'_I is the largest subclass of \mathcal{E}_I stable under pullbacks. The terminologies “vertical morphisms” and “stably-vertical morphisms” were introduced in [8].

Hence, if $f \in \mathcal{E}'_I$ then (1) and (2) must hold. ■

It is evident that \mathcal{E}'_I is a proper class of \mathcal{E}_I , therefore the monotone-light factorization system $(\mathcal{E}'_I, \mathcal{M}_I^*)$ is non-trivial ($\neq (\mathcal{E}_I, \mathcal{M}_I)$).

7. Trivial coverings for 2-categories via 2-preorders

A 2-functor $f : A \rightarrow B$ belongs to the class \mathcal{M}_I of trivial coverings (with respect to the reflection $H \vdash I : 2Cat \rightarrow 2Preord$) if and only if the following square

$$\begin{array}{ccc}
 A & \xrightarrow{\eta_A} & I(A) \\
 f \downarrow & & \downarrow If \\
 B & \xrightarrow{\eta_B} & I(B)
 \end{array} \quad (7.1)$$

is a pullback diagram, where η_A and η_B are unit morphisms for the reflection $H \vdash I : 2Cat \rightarrow 2Preord$ (cf. [2, Theorem 4.1]).

Since the pullback (as any limit) is calculated pointwise in $2Cat$ (cf. Corollary 3.3), then $f \in \mathcal{M}_I$ if and only if the following seven squares are pullbacks, corresponding to the seven pointwise components of η_A and of η_B (cf. diagram (2.1)):

$$\begin{array}{ccc}
 A(P_i) & \xrightarrow{\eta_{A(P_i)}} & I(A)(P_i) \\
 f_{P_i} \downarrow & (D_i) & \downarrow If_{P_i} \\
 B(P_i) & \xrightarrow{\eta_{B(P_i)}} & I(B)(P_i)
 \end{array} \quad (i = 0, 1, 2);$$

$$\begin{array}{ccc}
 A(2P_1) & \xrightarrow{\eta_{A(2P_1)}} & I(A)(2P_1) \\
 f_{2P_1} \downarrow & (2D) & \downarrow If_{2P_1}; \\
 B(2P_1) & \xrightarrow{\eta_{B(2P_1)}} & I(B)(2P_1)
 \end{array}$$

$$\begin{array}{ccc}
 A(vP_2) & \xrightarrow{\eta_{A(vP_2)}} & I(A)(vP_2) \\
 f_{vP_2} \downarrow & (vD) & \downarrow If_{vP_2}; \\
 B(vP_2) & \xrightarrow{\eta_{B(vP_2)}} & I(B)(vP_2)
 \end{array}$$

$$\begin{array}{ccc}
 A(hP_2) & \xrightarrow{\eta_{A(hP_2)}} & I(A)(hP_2) \\
 f_{hP_2} \downarrow & (hD) & \downarrow If_{hP_2}; \\
 B(hP_2) & \xrightarrow{\eta_{B(hP_2)}} & I(B)(hP_2)
 \end{array}$$

$$\begin{array}{ccc}
 A(hvP_2) & \xrightarrow{\eta_{A(hvP_2)}} & I(A)(hvP_2) \\
 f_{hvP_2} \downarrow & (hvD) & \downarrow If_{hvP_2} \\
 B(hvP_2) & \xrightarrow{\eta_{B(hvP_2)}} & I(B)(hvP_2)
 \end{array}$$

The three first squares (D_i) ($i = 0, 1, 2$) are pullbacks since $\eta_{A(P_i)}$ and $\eta_{B(P_i)}$ are identity maps for $i = 0, 1, 2$ (cf. diagram (2.1) and the definition of the reflection $H \vdash I :$

$2Cat \rightarrow 2Preord$ in (5.2)).

Notice that if diagram (2.1) is restricted to the (vertical) precategory diagram vP , one obtains from (7.1) the following square in Cat , with unit morphisms of the reflection of all categories into preorders $Cat \rightarrow Preord$ (cf. [11]),

$$\begin{array}{ccc}
 A(vP) & \xrightarrow{\eta_{A(vP)}} & I(A)(vP) \\
 f_{vP} \downarrow & & \downarrow If_{vP} \\
 B(vP) & \xrightarrow{\eta_{B(vP)}} & I(B)(vP).
 \end{array} \quad (7.2)$$

It is known (cf. [11, Proposition 3.1]) that this square is a pullback in Cat if and only if, for every two objects h and h' in $A(P_1)$ with $Hom_{A(2P_1)}(h, h')$ nonempty, the map

$$Hom_{A(2P_1)}(h, h') \rightarrow Hom_{B(2P_1)}(f_1h, f_1h')$$

induced by f is a bijection.

A necessary condition for the 2-functor f to be a trivial covering was just stated; the following Lemma 7.1 will help to show that this necessary condition is also sufficient in next Proposition 7.2.

7.1. LEMMA. *Consider the following commutative parallelepiped*

$$\begin{array}{ccccccc}
 & & q^A & & c^A & & \\
 & & \xrightarrow{\quad} & & \xrightarrow{\quad} & & \\
 A_2 & \xrightarrow{\quad} & A_1 & \xrightarrow{\quad} & A_0 & & \\
 \downarrow f_2 & \searrow \eta_{A,2} & \downarrow f_1 & \searrow \eta_{A,1} & \downarrow f_0 & \searrow \eta_{A,0} & \\
 & & I(A)_2 & \xrightarrow{\quad} & I(A)_1 & \xrightarrow{\quad} & I(A)_0 \\
 & & \downarrow If_2 & & \downarrow If_1 & & \downarrow If_0 \\
 & & B_2 & \xrightarrow{\quad} & B_1 & \xrightarrow{\quad} & B_0 \\
 & & \downarrow \eta_{B,2} & \searrow \eta_{B,1} & \downarrow \eta_{B,0} & & \\
 & & I(B)_2 & \xrightarrow{\quad} & I(B)_1 & \xrightarrow{\quad} & I(B)_0,
 \end{array} \quad (7.3)$$

where the five squares $c^A q^A = d^A r^A$, $c^B q^B = d^B r^B$, $Ic^A Iq^A = Id^A Ir^A$, $If_0 \eta_{A,0} = \eta_{B,0} f_0$ and $If_1 \eta_{A,1} = \eta_{B,1} f_1$ are pullbacks.

Then, the square $If_2 \eta_{A,2} = \eta_{B,2} f_2$ is also a pullback.⁸

PROOF. The proof is obtained by an obvious diagram chase. ■

7.2. PROPOSITION. A 2-functor $f : A \rightarrow B$ is a trivial covering for the reflection $H \vdash I : 2Cat \rightarrow 2Preord$ (in notation, $f \in \mathcal{M}_I$) if and only if, for every two objects h and h' in $A(P_1)$ with $Hom_{A(2P_1)}(h, h')$ nonempty, the map

$$Hom_{A(2P_1)}(h, h') \rightarrow Hom_{B(2P_1)}(f_1 h, f_1 h')$$

induced by f is a bijection.

PROOF. In the considerations just above, it was showed that the statement warrants that the squares (2D) and (vD) are pullbacks, adding to the fact that (D_0) , (D_1) and (D_2) are all three pullbacks.

Then, (hD) and (hvD) must also be pullbacks according to Lemma 7.1. ■

8. Coverings for 2-categories via 2-preorders

A 2-functor $f : A \rightarrow B$ belongs to the class \mathcal{M}_I^* of coverings (with respect to the reflection $H \vdash I : 2Cat \rightarrow 2Preord$) if there is some effective descent morphism (also called monadic extension in categorical Galois theory) $p : C \rightarrow B$ in $2Cat$ with codomain B such that the pullback $p^*(f) : C \times_B A \rightarrow C$ of f along p is a trivial covering ($p^*(f) \in \mathcal{M}_I$).

The following Lemma 8.1 can be found in [11, Lemma 4.2], in the context of the reflection of categories into preorders, but for 2-categories via 2-preorders the proof is exactly the same, since the same conditions hold (cf. Theorem 5.5 and Example 4.2). The next Proposition 8.2 characterizes the coverings for 2-categories via 2-preorders.

8.1. LEMMA. A 2-functor $f : A \rightarrow B$ in $2Cat$ is a covering (for the reflection $H \vdash I : 2Cat \rightarrow 2Preord$) if and only if, for every 2-functor $\varphi : X \rightarrow B$ over B from any 2-preorder X , the pullback $X \times_B A$ of f along φ is also a 2-preorder.

8.2. PROPOSITION. A 2-functor $f : A \rightarrow B$ in $2Cat$ is a covering (for the reflection $H \vdash I : 2Cat \rightarrow 2Preord$) if and only if it is faithful vertically with respect to 2-cells, that is, for every pair of morphisms g and g' , the map

$$Hom_{A(2P_1)}(g, g') \rightarrow Hom_{B(2P_1)}(f_1 g, f_1 g')$$

induced by f is an injection.

⁸The notation used in diagram (7.3) is arbitrary, being so chosen in order to make the application of Lemma 7.1 in this section more easily understandable.

PROOF. Consider again the 2-preorder T generated by the diagram $a \begin{array}{c} \xrightarrow{h} \\ \Downarrow \leq \\ \xrightarrow{h'} \end{array} a'$.

If f is not faithful vertically with respect to 2-cells, then, by including T in B , one could obtain a pullback $T \times_B A$ that is not a preorder.

Therefore, f is not a covering, by the previous Lemma 8.1.

Reciprocally, consider any 2-functor $\varphi : X \rightarrow B$ such that X is a 2-preorder.

If f is faithful (vertically with respect to 2-cells), then the pullback $X \times_B A$ is a 2-preorder, given the nature of X . Hence, f is a covering, by the previous Lemma 8.1. ■

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