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Research article

Variational problems of variable fractional order involving arbitrary kernels

Ricardo Almeida*

Center for Research and Development in Mathematics and Applications (CIDMA), Department of Mathematics, University of Aveiro, 3810–193 Aveiro, Portugal

* **Correspondence:** Email: ricardo.almeida@ua.pt.

Abstract: The aim of this work is to study several problems of the calculus of variations, where the dynamics of the state function is given by a generalized fractional derivative. This derivative combines two well-known concepts: fractional derivative with respect to another function and fractional derivative of variable order. We present the Euler–Lagrange equation, which is a necessary condition that every optimal solution of the problem must satisfy. Other problems are also studied: with integral and holonomic constraints, with higher order derivatives, and the Herglotz variational problem.

Keywords: fractional calculus; calculus of variations; variable-order fractional derivative
Mathematics Subject Classification: 26A33, 49L99

1. Introduction

The concept of fractional calculus, or arbitrary order calculus, is an extension of the standard calculus, where derivatives and integrals of non-integer order are used (see e.g., [20]). This theory was originated from a question formulated in an exchange of correspondence between Leibniz and l'Hopital, where the interpretation of a derivative of order $1/2$ was questioned. Therefore, we can say that the birth of fractional calculus was simultaneous with ordinary calculus, although the first one only had a great development in the last decades. During this period, many famous mathematicians devoted some time to the study of fractional calculus, such as Lagrange, Laplace, Lacroix, Fourier, Abel, Liouville, Riemann, and Grünwald. In the end of the XX century, it was observed that the use of fractional calculus makes it possible to express natural phenomena more precisely when compared to ordinary calculus and, therefore, it can be useful when applied to real world systems. For example, applications in physics [17, 23], chemistry [3, 26], engineering [13, 14, 21], biology [18, 32], economics [36], and control theory [24, 29, 37, 38], have been found.

There are several definitions for fractional derivatives, although the most common are the Riemann–Liouville and the Caputo ones. However, due to the high number of different concepts, we find several works studying similar problems. One way to overcome this issue is to consider more general definitions with respect to fractional operators. In this work we intend to combine two types of existing generalizations, the fractional derivative with respect to another function [4,27] and fractional derivatives of variable order [16,30,31].

One of the areas where fractional calculus has been applied is in the calculus of variations. The classic problem of the calculus of variations is to find the minimum or maximum value of functionals, usually in the form

$$F(u) := \int_a^b L(t, u(t), u'(t))\, dt,$$

possibly subject to some boundary conditions $u(a) = U_a$, $u(b) = U_b$ for some fixed $U_a, U_b \in \mathbb{R}$. In the fractional calculus of variations, this first order derivative $u'(t)$ is replace by some kind of fractional derivative $D^\gamma u(t)$. With Riewe's pioneering work in 1996 [28], where he formulated the problem of calculus of variations and obtained the respective Euler-Lagrange equation, numerous works have emerged in this area since then. For example, in [8], the authors considered the isoperimetric problem dealing with the left and right Riemann-Liouville fractional derivatives. In [12], some variational problems were formulated, with dependence on a term, and taking its limit, we obtain the total derivative at the classical level. In the book [22] and in the paper [25], several fractional calculus of variations problems were studied in a general form, where the kernel of the fractional operators is an arbitrary function, for the Riemann–Liouville and Caputo fractional derivatives. Again, due to the large number of definitions for fractional derivatives, we find numerous works in the area of the fractional calculus of variations for different derivatives, but studying similar problems. The aim of this work is to unify some previous works, when considering this new generalized fractional derivative. With this, we generalize some previous works on fractional calculus of variations. In fact, if $g(t) = t$, then we obtain the usual variable-order fractional operators and such variational problems have been studied extensively e.g., [33,35]. If we fix the order, that is, $\gamma_n(\cdot,\cdot) = \gamma \in \mathbb{R}^+$, then the problem was considered in [5]. In addition, if $g(t) = t$, then this situation was studied in [1,10] for the Riemann–Liouville fractional derivative and in [2,9,11] for the Caputo fractional derivative. If $g(t) = \ln t$ or $g(t) = t^\sigma$ ($\sigma > 0$), then the respective variational problems were considered in [6,7,15,19]. Thus, with this paper, we intend to generalize these previous works, and for other choices of the fractional order $\gamma_n(\cdot,\cdot)$ or the kernel $g(\cdot)$, new results can be obtained.

We start by fixing some notation. For what follows, n is a positive integer, $\gamma_n : [a,b]^2 \to (n-1,n)$ is a function, and $u, g : [a,b] \to \mathbb{R}$ are two functions with $g \in C^n[a,b]$ and $g'(t) > 0$, for all $t \in [a,b]$.

Definition 1. The generalized variable-order left and right Riemann–Liouville fractional integrals of u, with respect to g and with order γ_n, are defined as

$$\mathbb{I}_{a+}^{\gamma_n} u(t) = \int_a^t \frac{1}{\Gamma(\gamma_n(t,s))} g'(s)(g(t) - g(s))^{\gamma_n(t,s)-1} u(s)\, ds,$$

$$\mathbb{I}_{b-}^{\gamma_n} u(t) = \int_t^b \frac{1}{\Gamma(\gamma_n(s,t))} g'(s)(g(s) - g(t))^{\gamma_n(s,t)-1} u(s)\, ds,$$

respectively.

For what concerns the derivatives, two different types are presented.

Definition 2. The generalized variable-order left and right Riemann–Liouville fractional derivatives of u, with respect to g and with order γ_n, are defined as

$$\mathbb{D}_{a+}^{\gamma_n}u(t) = \left(\frac{1}{g'(t)}\frac{d}{dt}\right)^n \mathbb{I}_{a+}^{n-\gamma_n}u(t) = \left(\frac{1}{g'(t)}\frac{d}{dt}\right)^n \int_a^t \frac{g'(s)}{\Gamma(n-\gamma_n(t,s))}(g(t)-g(s))^{n-1-\gamma_n(t,s)}u(s)\,ds,$$

$$\mathbb{D}_{b-}^{\gamma_n}u(t) = \left(\frac{-1}{g'(t)}\frac{d}{dt}\right)^n \mathbb{I}_{b-}^{n-\gamma_n}u(t) = \left(\frac{-1}{g'(t)}\frac{d}{dt}\right)^n \int_t^b \frac{g'(s)}{\Gamma(n-\gamma_n(s,t))}(g(s)-g(t))^{n-1-\gamma_n(s,t)}u(s)\,ds,$$

respectively.

Definition 3. The generalized variable-order left and right Caputo fractional derivatives of u, with respect to g and with order γ_n, are defined as

$$^{C}\mathbb{D}_{a+}^{\gamma_n}u(t) = \mathbb{I}_{a+}^{n-\gamma_n}\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^n u(t) = \int_a^t \frac{g'(s)}{\Gamma(n-\gamma_n(t,s))}(g(t)-g(s))^{n-1-\gamma_n(t,s)}\left(\frac{1}{g'(s)}\frac{d}{ds}\right)^n u(s)\,ds,$$

$$^{C}\mathbb{D}_{b-}^{\gamma_n}u(t) = \mathbb{I}_{b-}^{n-\gamma_n}\left(\frac{-1}{g'(t)}\frac{d}{dt}\right)^n u(t) = \int_t^b \frac{g'(s)}{\Gamma(n-\gamma_n(s,t))}(g(s)-g(t))^{n-1-\gamma_n(s,t)}\left(\frac{-1}{g'(s)}\frac{d}{ds}\right)^n u(s)\,ds,$$

respectively.

We remark that, when $g(t) = t$, the previous definitions reduce to the classical variable-order fractional operators.

Lemma 1. Suppose that the fractional order γ_n is of form $\gamma_n(t,s) = \overline{\gamma}_n(t)$, where $\overline{\gamma}_n : [a,b] \to (n-1,n)$ is a function. Then, for the function $u(t) = (g(t)-g(a))^\beta$, with $\beta > n-1$,

$$^{C}\mathbb{D}_{a+}^{\gamma_n}u(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\overline{\gamma}_n(t)+1)}(g(t)-g(a))^{\beta-\overline{\gamma}_n(t)}.$$

Proof. First observe that

$$\left(\frac{1}{g'(s)}\frac{d}{ds}\right)^n (g(s)-g(a))^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-n+1)}(g(s)-g(a))^{\beta-n}.$$

Thus,

$$\begin{aligned} ^{C}\mathbb{D}_{a+}^{\gamma_n}u(t) &= \int_a^t \frac{g'(s)\Gamma(\beta+1)}{\Gamma(n-\overline{\gamma}_n(t))\Gamma(\beta-n+1)}(g(t)-g(s))^{n-1-\overline{\gamma}_n(t)}(g(s)-g(a))^{\beta-n}\,ds \\ &= \int_a^t \frac{g'(s)\Gamma(\beta+1)}{\Gamma(n-\overline{\gamma}_n(t))\Gamma(\beta-n+1)}(g(t)-g(a))^{n-1-\overline{\gamma}_n(t)}\left(1-\frac{g(s)-g(a)}{g(t)-g(a)}\right)^{n-1-\overline{\gamma}_n(t)}(g(s)-g(a))^{\beta-n}\,ds. \end{aligned}$$

With the change of variable $\tau = \frac{g(s)-g(a)}{g(t)-g(a)}$ and recalling the definition of the Beta function $B(\cdot,\cdot)$, we get

$$\begin{aligned} ^{C}\mathbb{D}_{a+}^{\gamma_n}u(t) &= \frac{\Gamma(\beta+1)}{\Gamma(n-\overline{\gamma}_n(t))\Gamma(\beta-n+1)}(g(t)-g(a))^{\beta-\overline{\gamma}_n(t)}\int_0^1 (1-\tau)^{n-1-\overline{\gamma}_n(t)}\tau^{\beta-n}\,d\tau \\ &= \frac{\Gamma(\beta+1)}{\Gamma(n-\overline{\gamma}_n(t))\Gamma(\beta-n+1)}(g(t)-g(a))^{\beta-\overline{\gamma}_n(t)}\cdot B(n-\overline{\gamma}_n(t),\beta-n+1) \\ &= \frac{\Gamma(\beta+1)}{\Gamma(n-\overline{\gamma}_n(t))\Gamma(\beta-n+1)}(g(t)-g(a))^{\beta-\overline{\gamma}_n(t)}\cdot\frac{\Gamma(n-\overline{\gamma}_n(t))\Gamma(\beta-n+1)}{\Gamma(\beta-\overline{\gamma}_n(t)+1)}, \end{aligned}$$

proving the desired formula. □

In an analogous way, we have the following:

Lemma 2. Suppose that the fractional order γ_n is of form $\gamma_n(t,s) = \overline{\gamma}_n(s)$, where $\overline{\gamma}_n : [a,b] \to (n-1,n)$ is a function. Then, for the function $u(t) = (g(b)-g(t))^\beta$, with $\beta > n-1$,

$$ {}^C\mathbb{D}_{b-}^{\gamma_n} u(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\overline{\gamma}_n(t)+1)}(g(b)-g(t))^{\beta-\overline{\gamma}_n(t)}. $$

The paper is structured as follows: in Section 2 we present the integration by parts formulae, dealing with the previous presented fractional derivatives. These formulas will be crucial for the rest of the paper. The main result is given in Section 3, where we prove the fractional Euler–Lagrange equation, which is an important formula to determine if a given curve is a minimizer or a maximizer of a functional. Then, we extended this result by considering additional constraints in the formulation of the problem (Section 4) or in presence of higher order fractional derivatives (Section 5). The Herglotz problem will be considered in Section 6. We end with a conclusion section.

2. A fractional integration by parts formula

As a first result, we present two integration by parts formulae for the two Caputo fractional derivatives (left and right). These formulae are important in the follow-up of the work, and will be used in the proofs of the results to be presented.

Theorem 1. If $u, v \in C^n[a,b]$, then the following fractional integration by parts formulae hold:

$$ \int_a^b u(t)\, {}^C\mathbb{D}_{a+}^{\gamma_n} v(t)\,dt = \int_a^b \mathbb{D}_{b-}^{\gamma_n} \frac{u(t)}{g'(t)} \cdot g'(t)v(t)\,dt + \left[\sum_{k=0}^{n-1} \left(\frac{-1}{g'(t)}\frac{d}{dt}\right)^k \mathbb{I}_{b-}^{n-\gamma_n} \frac{u(t)}{g'(t)} \cdot \left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{n-k-1} v(t)\right]_a^b $$

and

$$ \int_a^b u(t)\, {}^C\mathbb{D}_{b-}^{\gamma_n} v(t)\,dt = \int_a^b \mathbb{D}_{a+}^{\gamma_n} \frac{u(t)}{g'(t)} \cdot g'(t)v(t)\,dt $$
$$ + \left[\sum_{k=0}^{n-1} (-1)^{n+k} \left(\frac{1}{g'(t)}\frac{d}{dt}\right)^k \mathbb{I}_{a+}^{n-\gamma_n} \frac{u(t)}{g'(t)} \cdot \left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{n-k-1} v(t)\right]_a^b. $$

Proof. Changing the order of integration, we obtain the following double integral:

$$ \begin{aligned} \int_a^b u(t)\, {}^C\mathbb{D}_{a+}^{\gamma_n} v(t)\,dt &= \int_a^b \int_a^t \frac{u(t)g'(s)}{\Gamma(n-\gamma_n(t,s))}(g(t)-g(s))^{n-1-\gamma_n(t,s)} \cdot \left(\frac{1}{g'(s)}\frac{d}{ds}\right)^n v(s)\,ds\,dt \\ &= \int_a^b \left[\int_t^b \frac{u(s)}{\Gamma(n-\gamma_n(s,t))}(g(s)-g(t))^{n-1-\gamma_n(s,t)}\,ds\right] \cdot \frac{d}{dt}\left[\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{n-1} v(t)\right] dt \\ &= \int_a^b \mathbb{I}_{b-}^{n-\gamma_n} \frac{u(t)}{g'(t)} \cdot \frac{d}{dt}\left[\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{n-1} v(t)\right] dt. \end{aligned} \tag{2.1} $$

If we integrate by parts, (2.1) becomes

$$-\int_a^b \frac{d}{dt}\mathbb{I}_{b-}^{n-\gamma_n}\frac{u(t)}{g'(t)}\cdot\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{n-1}v(t)\,dt+\left[\mathbb{I}_{b-}^{n-\gamma_n}\frac{u(t)}{g'(t)}\cdot\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{n-1}v(t)\right]_a^b$$
$$=\int_a^b\left(\frac{-1}{g'(t)}\frac{d}{dt}\right)\mathbb{I}_{b-}^{n-\gamma_n}\frac{u(t)}{g'(t)}\cdot\frac{d}{dt}\left[\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{n-2}v(t)\right]dt+\left[\mathbb{I}_{b-}^{n-\gamma_n}\frac{u(t)}{g'(t)}\cdot\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{n-1}v(t)\right]_a^b. \quad (2.2)$$

Integrating again by parts, (2.2) becomes

$$-\int_a^b \frac{d}{dt}\left(\frac{-1}{g'(t)}\frac{d}{dt}\right)\mathbb{I}_{b-}^{n-\gamma_n}\frac{u(t)}{g'(t)}\cdot\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{n-2}v(t)\,dt$$
$$+\left[\left(\frac{-1}{g'(t)}\frac{d}{dt}\right)\mathbb{I}_{b-}^{n-\gamma_n}\frac{u(t)}{g'(t)}\cdot\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{n-2}v(t)\right]_a^b+\left[\mathbb{I}_{b-}^{n-\gamma_n}\frac{u(t)}{g'(t)}\cdot\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{n-1}v(t)\right]_a^b$$
$$=\int_a^b\left(\frac{-1}{g'(t)}\frac{d}{dt}\right)^2\mathbb{I}_{b-}^{n-\gamma_n}\frac{u(t)}{g'(t)}\cdot\frac{d}{dt}\left[\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{n-3}v(t)\right]dt$$
$$+\left[\sum_{k=0}^{1}\left(\frac{-1}{g'(t)}\frac{d}{dt}\right)^k\mathbb{I}_{b-}^{n-\gamma_n}\frac{u(t)}{g'(t)}\cdot\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{n-k-1}v(t)\right]_a^b. \quad (2.3)$$

Repeating the procedure, (2.3) is written as

$$\int_a^b\left(\frac{-1}{g'(t)}\frac{d}{dt}\right)^{n-1}\mathbb{I}_{b-}^{n-\gamma_n}\frac{u(t)}{g'(t)}\cdot\frac{d}{dt}v(t)\,dt+\left[\sum_{k=0}^{n-2}\left(\frac{-1}{g'(t)}\frac{d}{dt}\right)^k\mathbb{I}_{b-}^{n-\gamma_n}\frac{u(t)}{g'(t)}\cdot\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{n-k-1}v(t)\right]_a^b,$$

and performing one last time integration by parts, we get

$$-\int_a^b\frac{d}{dt}\left(\frac{-1}{g'(t)}\frac{d}{dt}\right)^{n-1}\mathbb{I}_{b-}^{n-\gamma_n}\frac{u(t)}{g'(t)}\cdot v(t)\,dt+\left[\sum_{k=0}^{n-1}\left(\frac{-1}{g'(t)}\frac{d}{dt}\right)^k\mathbb{I}_{b-}^{n-\gamma_n}\frac{u(t)}{g'(t)}\cdot\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{n-k-1}v(t)\right]_a^b$$
$$=\int_a^b\left(\frac{-1}{g'(t)}\frac{d}{dt}\right)^{n}\mathbb{I}_{b-}^{n-\gamma_n}\frac{u(t)}{g'(t)}\cdot g'(t)v(t)\,dt+\left[\sum_{k=0}^{n-1}\left(\frac{-1}{g'(t)}\frac{d}{dt}\right)^k\mathbb{I}_{b-}^{n-\gamma_n}\frac{u(t)}{g'(t)}\cdot\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{n-k-1}v(t)\right]_a^b$$
$$=\int_a^b\mathbb{D}_{b-}^{\gamma_n}\frac{u(t)}{g'(t)}\cdot g'(t)v(t)\,dt+\left[\sum_{k=0}^{n-1}\left(\frac{-1}{g'(t)}\frac{d}{dt}\right)^k\mathbb{I}_{b-}^{n-\gamma_n}\frac{u(t)}{g'(t)}\cdot\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{n-k-1}v(t)\right]_a^b,$$

proving the first formula. The second one is obtained using similar techniques. □

Remark 1. When $n = 1$, that is, the fractional order takes values in the open interval $(0, 1)$, Theorem 1 reads as

$$\int_a^b u(t)\,{}^C\mathbb{D}_{a+}^{\gamma_n}v(t)\,dt=\int_a^b\mathbb{D}_{b-}^{\gamma_n}\frac{u(t)}{g'(t)}\cdot g'(t)v(t)\,dt+\left[\mathbb{I}_{b-}^{1-\gamma_1}\frac{u(t)}{g'(t)}\cdot v(t)\right]_a^b$$

and

$$\int_a^b u(t)\,{}^C\mathbb{D}_{b-}^{\gamma_n}v(t)\,dt=\int_a^b\mathbb{D}_{a+}^{\gamma_n}\frac{u(t)}{g'(t)}\cdot g'(t)v(t)\,dt-\left[\mathbb{I}_{a+}^{1-\gamma_1}\frac{u(t)}{g'(t)}\cdot v(t)\right]_a^b.$$

3. The fractional Euler–Lagrange equation

The purpose of this section is to present the basic problem of the fractional calculus of variations, involving the fractional derivatives presented in Definition 3. To find the candidates for minimizing or maximizing a given functional, we will have to solve a fractional differential equation, known as the Euler–Lagrange equation (see Eq (3.3)).

We will consider the following fractional calculus of variation problem: minimize or maximize the functional

$$F(u) := \int_a^b L(t, u(t), {}^C\mathbb{D}^{\gamma_1}_{a+}u(t), {}^C\mathbb{D}^{\gamma_1}_{b-}u(t))\, dt, \tag{3.1}$$

where

1) $L : [a, b] \times \mathbb{R}^3 \to \mathbb{R}$ is a function of class C^1,
2) $\gamma_1 : [a, b]^2 \to (0, 1)$ is the fractional order,
3) functional F is defined on the set $\Omega := C^1[a, b]$.

The boundary conditions

$$u(a) = U_a, \quad u(b) = U_b, \quad U_a, U_b \in \mathbb{R}, \tag{3.2}$$

may be imposed on the problem and, for abbreviation, we introduce the operator $[\cdot]$ defined by

$$[u](t) := (t, u(t), {}^C\mathbb{D}^{\gamma_1}_{a+}u(t), {}^C\mathbb{D}^{\gamma_1}_{b-}u(t)).$$

Remark 2. When γ_1 is a constant function, that is, $\gamma_1(t, s) = \gamma \in (0, 1)$, for all $(t, s) \in [a, b]^2$, functional (3.1) reduces to the one studied in [5]. If $g(t) = t$, that is, we are in presence of the usual variable order fractional operators, then the variational problem was already considered in [33–35].

Remark 3. We say that $u^\star \in \Omega$ is a local minimizer of F is there exists $\epsilon > 0$ such that, whenever $u \in \Omega$ with $\|u^\star - u\| < \epsilon$, then $F(u^\star) \leq F(u)$. If $F(u^\star) \geq F(u)$, then we say that $u^\star$ is a local maximizer of F. In such cases, we say that $u^\star$ is a local extremizer of F.

Theorem 2. Let $u^\star \in \Omega$ be a local extremizer of F as in (3.1). If the maps

$$t \mapsto \mathbb{D}^{\gamma_1}_{b-}\left(\frac{\frac{\partial L}{\partial^C\mathbb{D}^{\gamma_1}_{a+}u}[u^\star](t)}{g'(t)}\right) \quad \text{and} \quad t \mapsto \mathbb{D}^{\gamma_1}_{a+}\left(\frac{\frac{\partial L}{\partial^C\mathbb{D}^{\gamma_1}_{b-}u}[u^\star](t)}{g'(t)}\right)$$

are continuous on $[a, b]$, then the following fractional Euler–Lagrange equation is satisfied:

$$\frac{\partial L}{\partial u}[u^\star](t) + g'(t)\mathbb{D}^{\gamma_1}_{b-}\left(\frac{\frac{\partial L}{\partial^C\mathbb{D}^{\gamma_1}_{a+}u}[u^\star](t)}{g'(t)}\right) + g'(t)\mathbb{D}^{\gamma_1}_{a+}\left(\frac{\frac{\partial L}{\partial^C\mathbb{D}^{\gamma_1}_{b-}u}[u^\star](t)}{g'(t)}\right) = 0, \quad \forall t \in [a, b]. \tag{3.3}$$

If $u(a)$ my take any value, then the following fractional transversality condition

$$\mathbb{I}^{1-\gamma_1}_{b-}\left(\frac{\frac{\partial L}{\partial^C\mathbb{D}^{\gamma_1}_{a+}u}[u^\star](t)}{g'(t)}\right) = \mathbb{I}^{1-\gamma_1}_{a+}\left(\frac{\frac{\partial L}{\partial^C\mathbb{D}^{\gamma_1}_{b-}u}[u^\star](t)}{g'(t)}\right), \tag{3.4}$$

holds at $t = a$. If $u(b)$ is arbitrary, then Eq. (3.4) holds at $t = b$.

Proof. Defining function $f(\epsilon) := F(u^\star(t) + \epsilon\delta(t))$ in a neighbourhood of zero, then $f'(0) = 0$, where $\delta \in \Omega$ is a perturbing curve. If the boundary conditions (3.2) are imposed on the problem, then $\delta(a)$ and $\delta(b)$ must be both zero so that the curve $u^\star(t) + \epsilon\delta(t)$ is an admissible variation for the problem. Computing $f'(0)$, we get

$$\int_a^b \left[\frac{\partial L}{\partial u}[u^\star](t)\delta(t) + \frac{\partial L}{\partial {}^C\mathbb{D}_{a+}^{\gamma_1} u}[u^\star](t)\,{}^C\mathbb{D}_{a+}^{\gamma_1}\delta(t) + \frac{\partial L}{\partial {}^C\mathbb{D}_{b-}^{\gamma_1} u}[u^\star](t)\,{}^C\mathbb{D}_{b-}^{\gamma_1}\delta(t)\right] dt = 0.$$

Integrating by parts (Theorem 1), we prove that

$$\int_a^b \left[\frac{\partial L}{\partial u}[u^\star](t) + g'(t)\mathbb{D}_{b-}^{\gamma_1}\left(\frac{\frac{\partial L}{\partial {}^C\mathbb{D}_{a+}^{\gamma_1} u}[u^\star](t)}{g'(t)}\right) + g'(t)\mathbb{D}_{a+}^{\gamma_1}\left(\frac{\frac{\partial L}{\partial {}^C\mathbb{D}_{b-}^{\gamma_1} u}[u^\star](t)}{g'(t)}\right)\right]\delta(t)\,dt$$
$$+\left[\delta(t)\left(\mathbb{I}_{b-}^{1-\gamma_1}\left(\frac{\frac{\partial L}{\partial {}^C\mathbb{D}_{a+}^{\gamma_1} u}[u^\star](t)}{g'(t)}\right) - \mathbb{I}_{a+}^{1-\gamma_1}\left(\frac{\frac{\partial L}{\partial {}^C\mathbb{D}_{b-}^{\gamma_1} u}[u^\star](t)}{g'(t)}\right)\right)\right]_a^b = 0. \quad (3.5)$$

If, in the set of admissible functions, the boundary conditions (3.2) are imposed, then $\delta(a) = 0 = \delta(b)$ and so

$$\int_a^b \left[\frac{\partial L}{\partial u}[u^\star](t) + g'(t)\mathbb{D}_{b-}^{\gamma_1}\left(\frac{\frac{\partial L}{\partial {}^C\mathbb{D}_{a+}^{\gamma_1} u}[u^\star](t)}{g'(t)}\right) + g'(t)\mathbb{D}_{a+}^{\gamma_1}\left(\frac{\frac{\partial L}{\partial {}^C\mathbb{D}_{b-}^{\gamma_1} u}[u^\star](t)}{g'(t)}\right)\right]\delta(t)\,dt = 0,$$

and since δ may take any value in (a, b), we conclude that

$$\frac{\partial L}{\partial u}[u^\star](t) + g'(t)\mathbb{D}_{b-}^{\gamma_1}\left(\frac{\frac{\partial L}{\partial {}^C\mathbb{D}_{a+}^{\gamma_1} u}[u^\star](t)}{g'(t)}\right) + g'(t)\mathbb{D}_{a+}^{\gamma_1}\left(\frac{\frac{\partial L}{\partial {}^C\mathbb{D}_{b-}^{\gamma_1} u}[u^\star](t)}{g'(t)}\right) = 0,$$

for all $t \in [a, b]$, proving (3.3). Otherwise, δ is also arbitrary at $t = a$ and $t = b$. Replacing (3.3) into (3.5), we have

$$\left[\delta(t)\left(\mathbb{I}_{b-}^{1-\gamma_1}\left(\frac{\frac{\partial L}{\partial {}^C\mathbb{D}_{a+}^{\gamma_1} u}[u^\star](t)}{g'(t)}\right) - \mathbb{I}_{a+}^{1-\gamma_1}\left(\frac{\frac{\partial L}{\partial {}^C\mathbb{D}_{b-}^{\gamma_1} u}[u^\star](t)}{g'(t)}\right)\right)\right]_a^b = 0,$$

and depending if $u(a)$ or $u(b)$ is arbitrary, we deduce the two transversality conditions (3.4). □

For example, consider $\gamma_1 : [0, 1]^1 \to (0, 1)$ given by $\gamma_1(t, s) = \frac{t^2+1}{4}$, and $g(t) = \ln(t + 1)$. Observe that, by Lemma 1,

$${}^C\mathbb{D}_{0+}^{\gamma_1} \ln^2(t + 1) = \frac{2}{\Gamma\left(\frac{11-t^2}{4}\right)} \ln^{\frac{7-t^2}{4}}(t + 1).$$

Let

$$F(u) = \int_0^1 \Big(u(t) - \ln^2(t + 1)\Big)^2 + \left({}^C\mathbb{D}_{0+}^{\gamma_1} u(t) - \frac{2}{\Gamma\left(\frac{11-t^2}{4}\right)} \ln^{\frac{7-t^2}{4}}(t + 1)\right)^2 dt.$$

It is easy to verify that the function $u^\star(t) = \ln^2(t+1)$, $t \in [0, 1]$, is a solution of the fractional differential equations given in Theorem 2.

Remark 4. If the fractional order is constant $\gamma_1(\cdot,\cdot) = \gamma_1 \in (0,1)$ and the kernel is $g(t) = t$, that is, the generalized variable-order Caputo fractional derivatives are the usual Caputo fractional derivatives, then formulae (3.3)–(3.4) reduce to the ones proved e.g., [9].

Observe that, although the functional only depends on the Caputo fractional derivative, the Euler–Lagrange equation (3.3) also involves the Riemann–Liouville fractional derivative. So, this equation deals with four types of fractional derivatives: the left and right Caputo fractional derivatives, and the left and right Riemann–Liouville fractional derivatives. Therefore, in many situations, it is not possible to determine the exact solution of this equation and numerical methods are usually used to determine an approximation of the solution. Such fractional differential equations are useful to check if a given function may or not be a solution of the variational problem. In some particular situations, using some properties of the fractional operators, we may solve the Euler–Lagrange equation and thus produce the optimal solutions. When such a situation is not possible, then using appropriate numerical methods (for example, discretize the equation and then solve a finite dimensional system), an approximation of the solution is obtained. Then, using some sufficient conditions of optimality (e.g., convexity assumptions) we can prove that the obtained solution is indeed a minimizer or maximizer of the functional.

4. Variational problems under additional constraints

Suppose now that, in the formulation of the variational problem, an integral constraint is imposed on the set of admissible functions (what is called in the literature as an isoperimetric problem). For simplicity of the computations, we will assume from now on that the boundary conditions (3.2) are imposed when formulating the problem (if not, transversality conditions similar to Eq (3.4) are derived). The fractional isoperimetric problem is formulated in the following way: minimize or maximize functional F (as in (3.1)), subject to the boundary conditions (3.2) and to the integral constraint

$$G(u) := \int_a^b M(t, u(t), {}^C\mathbb{D}^{\gamma_1}_{a+}u(t), {}^C\mathbb{D}^{\gamma_1}_{b-}u(t))\, dt = \Upsilon, \tag{4.1}$$

where $M : [a,b]\times\mathbb{R}^3 \to \mathbb{R}$ is a C^1 function and $\Upsilon \in \mathbb{R}$ a fixed number.

Theorem 3. Let $u^\star \in \Omega$ be a local extremizer of F as in (3.1), subject to (3.2) and (4.1). Assume that the maps

$$t \mapsto \mathbb{D}^{\gamma_1}_{b-}\left(\frac{\frac{\partial L}{\partial^C\mathbb{D}^{\gamma_1}_{a+}u}[u^\star](t)}{g'(t)}\right), \quad t \mapsto \mathbb{D}^{\gamma_1}_{a+}\left(\frac{\frac{\partial L}{\partial^C\mathbb{D}^{\gamma_1}_{b-}u}[u^\star](t)}{g'(t)}\right),$$

$$t \mapsto \mathbb{D}^{\gamma_1}_{b-}\left(\frac{\frac{\partial M}{\partial^C\mathbb{D}^{\gamma_1}_{a+}u}[u^\star](t)}{g'(t)}\right), \text{ and } \quad t \mapsto \mathbb{D}^{\gamma_1}_{a+}\left(\frac{\frac{\partial M}{\partial^C\mathbb{D}^{\gamma_1}_{b-}u}[u^\star](t)}{g'(t)}\right)$$

are all continuous on $[a,b]$. Then, there exists $(\lambda_0,\lambda) \in \mathbb{R}^2 \setminus \{(0,0)\}$ such that, if we define function $H : [a,b]\times\mathbb{R}^3 \to \mathbb{R}$ as $H := \lambda_0 L + \lambda M$, the following fractional differential equation

$$\frac{\partial H}{\partial u}[u^\star](t) + g'(t)\mathbb{D}^{\gamma_1}_{b-}\left(\frac{\frac{\partial H}{\partial^C\mathbb{D}^{\gamma_1}_{a+}u}[u^\star](t)}{g'(t)}\right) + g'(t)\mathbb{D}^{\gamma_1}_{a+}\left(\frac{\frac{\partial H}{\partial^C\mathbb{D}^{\gamma_1}_{b-}u}[u^\star](t)}{g'(t)}\right) = 0, \quad \forall t \in [a,b], \tag{4.2}$$

is satisfied.

Proof. First, suppose that $u^\star$ satisfies the Euler–Lagrange equation with respect to functional G, that is,

$$\frac{\partial M}{\partial u}[u^\star](t)+g'(t)\mathbb{D}^{\gamma_1}_{b-}\left(\frac{\frac{\partial M}{\partial {}^C\mathbb{D}^{\gamma_1}_{a+}u}[u^\star](t)}{g'(t)}\right)+g'(t)\mathbb{D}^{\gamma_1}_{a+}\left(\frac{\frac{\partial M}{\partial {}^C\mathbb{D}^{\gamma_1}_{b-}u}[u^\star](t)}{g'(t)}\right)=0,\quad \forall t\in[a,b].$$

Then, the theorem is proved considering $(\lambda_0,\lambda)=(0,1)$. If not, we prove (4.2) using variational arguments. First, we prove that there exists an infinite family of variations of $u^\star$ of form $t\mapsto u^\star(t)+\epsilon_1\delta_1(t)+\epsilon_2\delta_2(t)$ satisfying the integral constraint. For that, define $f(\epsilon_1,\epsilon_2):=F(u^\star(t)+\epsilon_1\delta_1(t)+\epsilon_2\delta_2(t))$ and $g(\epsilon_1,\epsilon_2):=G(u^\star(t)+\epsilon_1\delta_1(t)+\epsilon_2\delta_2(t))-\Upsilon$, where $\delta_1,\delta_2\in\Omega$ and $\delta_i(a)=0=\delta_i(b)$, $i=1,2$. Applying the same tecniques as the ones used in the proof of Theorem 2, we get that

$$\frac{\partial g}{\partial \epsilon_2}(0,0)=\int_a^b\left[\frac{\partial M}{\partial u}[u^\star](t)+g'(t)\mathbb{D}^{\gamma_1}_{b-}\left(\frac{\frac{\partial M}{\partial {}^C\mathbb{D}^{\gamma_1}_{a+}u}[u^\star](t)}{g'(t)}\right)+g'(t)\mathbb{D}^{\gamma_1}_{a+}\left(\frac{\frac{\partial M}{\partial {}^C\mathbb{D}^{\gamma_1}_{b-}u}[u^\star](t)}{g'(t)}\right)\right]\delta_2(t)\,dt,$$

and since $u^\star$ does not satisfies the Euler–Lagrange equation for functional G, we conclude that there exists a variation curve δ_2 such that $\frac{\partial g}{\partial\epsilon_2}(0,0)\neq 0$. If we apply the Implicit Function Theorem, we conclude that there is a family of variations of $u^\star$ that satisfy the integral restriction. Also, we obtain that $\nabla g(0,0)\neq(0,0)$ and $(0,0)$ is a solution of the problem: minimize of maximize f such that $g\equiv 0$. We can apply the Lagrange multiplier rule to conclude that there exists $\lambda\in\mathbb{R}$ with $\nabla(f+\lambda g)(0,0)=(0,0)$. If we solve the equation

$$\frac{\partial(f+\lambda g)}{\partial\epsilon_1}(0,0)=0,$$

we get

$$\int_a^b\left[\frac{\partial(L+\lambda M)}{\partial u}[u^\star](t)+g'(t)\mathbb{D}^{\gamma_1}_{b-}\left(\frac{\frac{\partial(L+\lambda M)}{\partial {}^C\mathbb{D}^{\gamma_1}_{a+}u}[u^\star](t)}{g'(t)}\right)+g'(t)\mathbb{D}^{\gamma_1}_{a+}\left(\frac{\frac{\partial(L+\lambda M)}{\partial {}^C\mathbb{D}^{\gamma_1}_{b-}u}[u^\star](t)}{g'(t)}\right)\right]\delta_1(t)\,dt=0,$$

and so Eq (4.2) is deduced. □

In our next problem we add a holonomic constraint, that is, an equation that involves the spatial coordinates of the system and time as well. It is described in the following way. Let $\Omega_H:=C^1[a,b]\times C^1[a,b]$. The goal is to minimize or maximize the functional

$$F_H(u_1,u_2):=\int_a^b L_H(t,u_1(t),u_2(t),{}^C\mathbb{D}^{\gamma_1}_{a+}u_1(t),{}^C\mathbb{D}^{\gamma_1}_{a+}u_2(t),{}^C\mathbb{D}^{\gamma_1}_{b-}u_1(t),{}^C\mathbb{D}^{\gamma_1}_{b-}u_2(t))\,dt,\tag{4.3}$$

where $L_H:[a,b]\times\mathbb{R}^6\to\mathbb{R}$ is a function of class C^1, subject to the boundary conditions

$$u_1(a)=U_{a1},\ u_2(a)=U_{a2},\ u_1(b)=U_{b1},\ u_2(b)=U_{b2},\quad U_{a1},U_{a2},U_{b1},U_{b2}\in\mathbb{R},\tag{4.4}$$

and to the holonomic constrain

$$\mathcal{G}(t,u_1(t),u_2(t))=0,\quad t\in[a,b],\tag{4.5}$$

where $\mathcal{G}:[a,b]\times\mathbb{R}^2\to\mathbb{R}$ is a function of class C^1 For abbreviation,

$$u=(u_1,u_2),\ [u]_{\mathcal{G}}(t):=(t,u(t))\quad\text{and}\quad [u]_H(t):=(t,u(t),{}^C\mathbb{D}^{\gamma_1}_{a+}u(t),{}^C\mathbb{D}^{\gamma_1}_{b-}u(t)).$$

Theorem 4. Let $u^\star \in \Omega_H$ be a local extremizer of functional F_H given by (4.3), subject to the conditions (4.4)–(4.5). If the maps

$$t \mapsto \mathbb{D}^{\gamma_1}_{b-}\left(\frac{\frac{\partial L_H}{\partial^C \mathbb{D}^{\gamma_1}_{a+} u_i}[u^\star]_H(t)}{g'(t)}\right) \quad \text{and} \quad t \mapsto \mathbb{D}^{\gamma_1}_{a+}\left(\frac{\frac{\partial L_H}{\partial^C \mathbb{D}^{\gamma_1}_{b-} u_2}[u^\star]_H(t)}{g'(t)}\right)$$

are continuous on $[a, b]$, for $i = 1, 2$, and if

$$\frac{\partial \mathcal{G}}{\partial u_2}[u]_{\mathcal{G}}(t) \neq 0, \quad \forall t \in [a, b],$$

then there exists a continuous function $\lambda : [a, b] \to \mathbb{R}$ such that

$$\frac{\partial L_H}{\partial u_i}[u^\star]_H(t) + g'(t)\mathbb{D}^{\gamma_1}_{b-}\left(\frac{\frac{\partial L_H}{\partial^C \mathbb{D}^{\gamma_1}_{a+} u_i}[u^\star]_H(t)}{g'(t)}\right) + g'(t)\mathbb{D}^{\gamma_1}_{a+}\left(\frac{\frac{\partial L_H}{\partial^C \mathbb{D}^{\gamma_1}_{b-} u_i}[u^\star]_H(t)}{g'(t)}\right)$$
$$+ \lambda(t)\frac{\partial \mathcal{G}}{\partial u_i}[u]_{\mathcal{G}}(t) = 0, \quad \forall t \in [a, b],\ i = 1, 2. \quad (4.6)$$

Proof. Condition (4.6) is obviously meet for $i = 2$, if we define

$$\lambda(t) := -\frac{\frac{\partial L_H}{\partial u_2}[u^\star]_H(t) + g'(t)\mathbb{D}^{\gamma_1}_{b-}\left(\frac{\frac{\partial L_H}{\partial^C \mathbb{D}^{\gamma_1}_{a+} u_2}[u^\star]_H(t)}{g'(t)}\right) + g'(t)\mathbb{D}^{\gamma_1}_{a+}\left(\frac{\frac{\partial L_H}{\partial^C \mathbb{D}^{\gamma_1}_{b-} u_2}[u^\star]_H(t)}{g'(t)}\right)}{\frac{\partial \mathcal{G}}{\partial u_2}[u]_{\mathcal{G}}(t)}.$$

The case $i = 1$ is proven in the following way. The variation curve of $u^\star$ is given by $u^\star(t) + \epsilon\delta(t)$, where $\delta \in \Omega_H$ and $\delta(a) = \delta(b) = (0, 0)$. Since any variation must be admissible for the problem, condition (4.5) must be verified for this curve and so the equation

$$\frac{\partial \mathcal{G}}{\partial u_1}[u]_{\mathcal{G}}(t)\delta_1(t) = -\frac{\partial \mathcal{G}}{\partial u_2}[u]_{\mathcal{G}}(t)\delta_2(t), \quad \forall t \in [a, b]$$

must hold. Also, if we define $f_H(\epsilon) := F_H(u^\star(t) + \epsilon\delta(t))$, then $f'_H(0) = 0$ and so

$$\int_a^b \left[\frac{\partial L_H}{\partial u_1}[u^\star]_H(t)\delta_1(t) + \frac{\partial L_H}{\partial^C \mathbb{D}^{\gamma_1}_{a+} u_1}[u^\star]_H(t)^C\mathbb{D}^{\gamma_1}_{a+}\delta_1(t) + \frac{\partial L_H}{\partial^C \mathbb{D}^{\gamma_1}_{b-} u_1}[u^\star]_H(t)^C\mathbb{D}^{\gamma_1}_{b-}\delta_1(t)\right.$$
$$\left.+\frac{\partial L_H}{\partial u_2}[u^\star]_H(t)\delta_2(t) + \frac{\partial L_H}{\partial^C \mathbb{D}^{\gamma_1}_{a+} u_2}[u^\star]_H(t)^C\mathbb{D}^{\gamma_1}_{a+}\delta_2(t) + \frac{\partial L_H}{\partial^C \mathbb{D}^{\gamma_1}_{b-} u_2}[u^\star]_H(t)^C\mathbb{D}^{\gamma_1}_{b-}\delta_2(t)\right] dt = 0.$$

Applying Theorem 1, and since $\delta(a) = \delta(b) = (0, 0)$, we obtain

$$\int_a^b \left[\frac{\partial L_H}{\partial u_1}[u^\star]_H(t) + g'(t)\mathbb{D}^{\gamma_1}_{b-}\left(\frac{\frac{\partial L_H}{\partial^C \mathbb{D}^{\gamma_1}_{a+} u_1}[u^\star]_H(t)}{g'(t)}\right) + g'(t)\mathbb{D}^{\gamma_1}_{a+}\left(\frac{\frac{\partial L_H}{\partial^C \mathbb{D}^{\gamma_1}_{b-} u_1}[u^\star]_H(t)}{g'(t)}\right)\right]\delta_1(t)$$
$$+\left[\frac{\partial L_H}{\partial u_2}[u^\star]_H(t) + g'(t)\mathbb{D}^{\gamma_1}_{b-}\left(\frac{\frac{\partial L_H}{\partial^C \mathbb{D}^{\gamma_1}_{a+} u_2}[u^\star]_H(t)}{g'(t)}\right) + g'(t)\mathbb{D}^{\gamma_1}_{a+}\left(\frac{\frac{\partial L_H}{\partial^C \mathbb{D}^{\gamma_1}_{b-} u_2}[u^\star]_H(t)}{g'(t)}\right)\right]\delta_2(t)\, dt = 0.$$

Observing that

$$\left[\frac{\partial L_H}{\partial u_2}[u^\star]_H(t) + g'(t)\mathbb{D}^{\gamma_1}_{b-}\left(\frac{\frac{\partial L_H}{\partial^C\mathbb{D}^{\gamma_1}_{a+}u_2}[u^\star]_H(t)}{g'(t)}\right) + g'(t)\mathbb{D}^{\gamma_1}_{a+}\left(\frac{\frac{\partial L_H}{\partial^C\mathbb{D}^{\gamma_1}_{b-}u_2}[u^\star]_H(t)}{g'(t)}\right)\right]\delta_2(t)$$
$$= -\lambda(t)\frac{\partial \mathcal{G}}{\partial u_2}[u]_{\mathcal{G}}(t)\delta_2(t) = \lambda(t)\frac{\partial \mathcal{G}}{\partial u_1}[u]_{\mathcal{G}}(t)\delta_1(t),$$

we conclude that

$$\int_a^b \left[\frac{\partial L_H}{\partial u_1}[u^\star]_H(t) + g'(t)\mathbb{D}^{\gamma_1}_{b-}\left(\frac{\frac{\partial L_H}{\partial^C\mathbb{D}^{\gamma_1}_{a+}u_1}[u^\star]_H(t)}{g'(t)}\right) + g'(t)\mathbb{D}^{\gamma_1}_{a+}\left(\frac{\frac{\partial L_H}{\partial^C\mathbb{D}^{\gamma_1}_{b-}u_1}[u^\star]_H(t)}{g'(t)}\right)\right.$$
$$\left. + \lambda(t)\frac{\partial \mathcal{G}}{\partial u_1}[u]_{\mathcal{G}}(t)\right]\delta_1(t)\, dt = 0,$$

proving the case $i = 1$ in Eq (4.6). □

5. The higher-order problem

In this section we address the higher-order variational problem, by considering a sequence of functions $\gamma_i : [a,b]^2 \to (i-1, i)$, with $i = 1, \dots, n$ ($n \in \mathbb{N}$), and the functional, defined on the space $\Omega_n := C^n[a,b]$, given by

$$F_n(u) := \int_a^b L_n(t, u(t), {}^C\mathbb{D}^{\gamma_1}_{a+}u(t), \dots, {}^C\mathbb{D}^{\gamma_n}_{a+}u(t), {}^C\mathbb{D}^{\gamma_1}_{b-}u(t), \dots, {}^C\mathbb{D}^{\gamma_n}_{b-}u(t))\, dt, \tag{5.1}$$

where $L_n : [a,b] \times \mathbb{R}^{2n+1} \to \mathbb{R}$ is a function of class C^1. Define

$$[u]_n(t) := (t, u(t), {}^C\mathbb{D}^{\gamma_1}_{a+}u(t), \dots, {}^C\mathbb{D}^{\gamma_n}_{a+}u(t), {}^C\mathbb{D}^{\gamma_1}_{b-}u(t), \dots, {}^C\mathbb{D}^{\gamma_n}_{b-}u(t)).$$

The necessary condition that every extremizer of this problem must satisfy is given in the next result.

Theorem 5. If $u^\star \in \Omega_n$ is a local minimizer or maximizer of F_n (5.1), subject to the boundary conditions

$$u^{(i)}(a) = U_{a_i}, \quad u^{(i)}(b) = U_{b_i}, \quad U_{a_i}, U_{b_i} \in \mathbb{R}, \quad i = 0, \dots, n-1,$$

and if, for $i = 1, \dots, n$, the maps

$$t \mapsto \mathbb{D}^{\gamma_i}_{b-}\left(\frac{\frac{\partial L_n}{\partial^C\mathbb{D}^{\gamma_i}_{a+}u}[u^\star]_n(t)}{g'(t)}\right) \quad \text{and} \quad t \mapsto \mathbb{D}^{\gamma_i}_{a+}\left(\frac{\frac{\partial L_n}{\partial\mathbb{D}^{\gamma_i}_{b-}u}[u^\star]_n(t)}{g'(t)}\right)$$

are continuous on $[a,b]$, then

$$\frac{\partial L_n}{\partial u}[u^\star]_n(t) + \sum_{i=1}^n \left[g'(t)\mathbb{D}^{\gamma_i}_{b-}\left(\frac{\frac{\partial L_n}{\partial^C\mathbb{D}^{\gamma_i}_{a+}u}[u^\star]_n(t)}{g'(t)}\right) + g'(t)\mathbb{D}^{\gamma_i}_{a+}\left(\frac{\frac{\partial L_n}{\partial\mathbb{D}^{\gamma_i}_{b-}u}[u^\star]_n(t)}{g'(t)}\right)\right] = 0, \quad \forall t \in [a,b]. \tag{5.2}$$

Proof. A variation of the optimal curve will be given by $u^\star(t) + \epsilon\delta(t)$, where $\delta \in \Omega_n$ and $\delta^{(i)}(a) = \delta^{(i)}(b) = 0$, for each $i = 0, \ldots, n-1$, so that the variation curve satisfies the boundary conditions. Since its first variation must vanish, we obtain

$$\int_a^b \left[\frac{\partial L_n}{\partial u}[u^\star]_n(t)\delta(t) + \sum_{i=1}^n \left[\frac{\partial L_n}{\partial {}^C\mathbb{D}_{a+}^{\gamma_i} u}[u^\star]_n(t)\,{}^C\mathbb{D}_{a+}^{\gamma_i}\delta(t) + \frac{\partial L_n}{\partial {}^C\mathbb{D}_{b-}^{\gamma_i} u}[u^\star]_n(t)\,{}^C\mathbb{D}_{b-}^{\gamma_i}\delta(t) \right] \right] dt = 0.$$

Integrating by parts,

$$\int_a^b \left[\frac{\partial L_n}{\partial u}[u^\star]_n(t) + \sum_{i=1}^n \left[g'(t)\mathbb{D}_{b-}^{\gamma_i}\left(\frac{\frac{\partial L_n}{\partial {}^C\mathbb{D}_{a+}^{\gamma_i} u}[u^\star]_n(t)}{g'(t)} \right) + g'(t)\mathbb{D}_{a+}^{\gamma_i}\left(\frac{\frac{\partial L_n}{\partial \mathbb{D}_{b-}^{\gamma_i} u}[u^\star]_n(t)}{g'(t)} \right) \right] \right] \delta(t)\, dt = 0. \tag{5.3}$$

From Eq (5.3), the desired result (5.2) follows. □

Remark 5. Observe that, if $n = 1$, Theorem 5 reduces to Theorem 2. Also, additional constraints like the ones presented in Section 5 could be added and similar results as those ones are derived.

6. The Herglotz variational problem

The Herglotz variational problem is an extension of the previous problems. Instead of finding the extremals for the functional (3.1), we are interested in finding a pair $(u^\star, z^\star)$ for which function $z(\cdot)$ attains its maximum or minimum value, where functions u and z are related by the ODE

$$\begin{cases} z'(t) = L_z(t, u(t), {}^C\mathbb{D}_{a+}^{\gamma_1}u(t), {}^C\mathbb{D}_{b-}^{\gamma_1}u(t), z(t)), \quad t \in [a,b], \\ z(a) = Z_a, \quad u(a) = U_a, \quad u(b) = U_b, \quad Z_a, U_a, U_b \in \mathbb{R}, \end{cases} \tag{6.1}$$

where $L_z : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ is a function of class C^1, $u \in \Omega$ and $z \in C^1[a,b]$. This problem formulation is an extension of the one presented in Section 3. In fact, if L_z does not depend on z, then integrating both sides of Eq (6.1), we get that

$$z(b) = Z_a + \int_a^b L_z(t, u(t), {}^C\mathbb{D}_{a+}^{\gamma_1}u(t), {}^C\mathbb{D}_{b-}^{\gamma_1}u(t))\, dt.$$

Let

$$[u, z](t) := (t, u(t), {}^C\mathbb{D}_{a+}^{\gamma_1}u(t), {}^C\mathbb{D}_{b-}^{\gamma_1}u(t), z(t)).$$

Theorem 6. Let $(u^\star, z^\star) \in \Omega \times C^1[a,b]$ be a solution of problem (6.1). Define function $\lambda : [a,b] \to \mathbb{R}$ as

$$\lambda(t) = \exp\left(-\int_a^t \frac{\partial L_z}{\partial z}[u^\star, z^\star](\tau)\, d\tau \right).$$

If the maps

$$t \mapsto \mathbb{D}_{b-}^{\gamma_1}\left(\lambda(t)\frac{\frac{\partial L_z}{\partial {}^C\mathbb{D}_{a+}^{\gamma_1} u}[u^\star, z^\star](t)}{g'(t)} \right) \quad \text{and} \quad t \mapsto \mathbb{D}_{a+}^{\gamma_1}\left(\lambda(t)\frac{\frac{\partial L_z}{\partial {}^C\mathbb{D}_{b-}^{\gamma_1} u}[u^\star, z^\star](t)}{g'(t)} \right)$$

are continuous on $[a,b]$, then for all $t \in [a,b]$,

$$\lambda(t)\frac{\partial L_z}{\partial u}[u^\star, z^\star](t) + g'(t)\mathbb{D}_{b-}^{\gamma_1}\left(\lambda(t)\frac{\frac{\partial L_z}{\partial {}^C\mathbb{D}_{a+}^{\gamma_1} u}[u^\star, z^\star](t)}{g'(t)} \right) + g'(t)\mathbb{D}_{a+}^{\gamma_1}\left(\lambda(t)\frac{\frac{\partial L_z}{\partial {}^C\mathbb{D}_{b-}^{\gamma_1} u}[u^\star, z^\star](t)}{g'(t)} \right) = 0.$$

Proof. We begin by remarking that function z not only depends on time t, but also on the state function u and so we will write $z(t,u)$ instead of $z(t)$ when we need to emphasize this dependence. A variation of the curve u will be still denoted by $u^\star(t)+\epsilon\delta(t)$ ($\delta\in\Omega$ with $\delta(a)=\delta(b)=0$) and the associate variation curve of z is given by

$$Z(t)=\frac{dz^\star}{d\epsilon}(t,u^\star(t)+\epsilon\delta(t))\Big|_{\epsilon=0}.$$

The first derivative of Z is then given by

$$\begin{aligned}Z'(t)&=\frac{d}{dt}\frac{d}{d\epsilon}z^\star(t,u^\star(t)+\epsilon\delta(t))\Big|_{\epsilon=0}=\frac{d}{d\epsilon}\frac{d}{dt}z^\star(t,u^\star(t)+\epsilon\delta(t))\Big|_{\epsilon=0}\\&=\frac{d}{d\epsilon}L_z(t,u^\star(t)+\epsilon\delta(t),{}^C\mathbb{D}^{\gamma_1}_{a+}(u^\star(t)+\epsilon\delta(t)),{}^C\mathbb{D}^{\gamma_1}_{b-}(u^\star(t)+\epsilon\delta(t)),z^\star(t,u^\star(t)+\epsilon\delta(t)))\\&=\frac{\partial L_z}{\partial u}[u^\star,z^\star](t)\delta(t)+\frac{\partial L_z}{\partial {}^C\mathbb{D}^{\gamma_1}_{a+}u}[u^\star,z^\star](t){}^C\mathbb{D}^{\gamma_1}_{a+}\delta(t)+\frac{\partial L_z}{\partial {}^C\mathbb{D}^{\gamma_1}_{b-}u}[u^\star,z^\star](t){}^C\mathbb{D}^{\gamma_1}_{b-}\delta(t)+\frac{\partial L_z}{\partial z}Z(t).\end{aligned}$$

Solving this ODE, we prove that

$$\begin{aligned}&Z(b)\lambda(b)-Z(a)\lambda(a)\\&\quad=\int_a^b\lambda(t)\left[\frac{\partial L_z}{\partial u}[u^\star,z^\star](t)\delta(t)+\frac{\partial L_z}{\partial {}^C\mathbb{D}^{\gamma_1}_{a+}u}[u^\star,z^\star](t){}^C\mathbb{D}^{\gamma_1}_{a+}\delta(t)+\frac{\partial L_z}{\partial {}^C\mathbb{D}^{\gamma_1}_{b-}u}[u^\star,z^\star](t){}^C\mathbb{D}^{\gamma_1}_{b-}\delta(t)\right]dt.\end{aligned}$$

Using the fractional integration by parts formulae, and since $Z(a)=0$ ($z(a)$ is fixed) and $Z(b)=0$ ($z(b)$ attains its extremum), we get that

$$\begin{aligned}\int_a^b\Bigg[\lambda(t)\frac{\partial L_z}{\partial u}[u^\star,z^\star](t)+g'(t)\mathbb{D}^{\gamma_1}_{b-}\left(\lambda(t)\frac{\frac{\partial L_z}{\partial {}^C\mathbb{D}^{\gamma_1}_{a+}u}[u^\star,z^\star](t)}{g'(t)}\right)&\\+g'(t)\mathbb{D}^{\gamma_1}_{a+}\left(\lambda(t)\frac{\frac{\partial L_z}{\partial {}^C\mathbb{D}^{\gamma_1}_{b-}u}[u^\star,z^\star](t)}{g'(t)}\right)\Bigg]\delta(t)\,dt&=0.\end{aligned}$$

By the arbitrariness of function δ, we obtain the desired formula. □

The previous theorem can be generalized for functions of several independent variables. We denote them by $t\in[a,b]$ (time coordinate) and $s=(s_1,\dots,s_n)\in S$ (spatial coordinates), where $S=\prod_{i=1}^n[a_i,b_i]$ with $-\infty<a_i<b_i<\infty$, for all $i\in\{1,\dots,n\}$. Also, we denote

$${}^C\mathbb{D}^{\gamma_1}_{+}u(t)=({}^C\mathbb{D}^{\gamma_1}_{a+}u(t),{}^C\mathbb{D}^{\gamma_1}_{a_1+}u(t),\dots,{}^C\mathbb{D}^{\gamma_1}_{a_n+}u(t))$$

and

$${}^C\mathbb{D}^{\gamma_1}_{-}u(t)=({}^C\mathbb{D}^{\gamma_1}_{b-}u(t),{}^C\mathbb{D}^{\gamma_1}_{b_1-}u(t),\dots,{}^C\mathbb{D}^{\gamma_1}_{b_n-}u(t)),$$

where ${}^C\mathbb{D}^{\gamma_1}_{a+}u$ and ${}^C\mathbb{D}^{\gamma_1}_{b-}u$ are to be understood as the left and right partial fractional derivatives of u with respect to variable t, respectively, and for $i=1,\dots,n$, ${}^C\mathbb{D}^{\gamma_1}_{a_i+}u$ and ${}^C\mathbb{D}^{\gamma_1}_{b_i-}u$ are to be understood as the left and right partial fractional derivatives of u with respect to variable s_i, respectively.

The new problem is formulated in the following way: find a pair $(u^\star, z^\star)$ for which $z^\star(b)$ is maximum or minimum value, where u and z are related by the system

$$\begin{cases} z'(t) = \displaystyle\int_S L_{z2}(t, s, u(t,s), {}^C\mathbb{D}_+^{\gamma_1} u(t,s), {}^C\mathbb{D}_-^{\gamma_1} u(t,s), z(t))\, ds, \quad t \in [a,b], \\ z(a) = Z_a, \quad u(t,s) \text{ is fixed whenever } t \in \{a,b\} \text{ or } s \in \{a_i, b_i\}, \quad i \in \{1,\dots,n\}, Z_a \in \mathbb{R}, \end{cases} \tag{6.2}$$

where $L_{z2} : [a,b] \times \mathbb{R}^{3n+5} \to \mathbb{R}$ is a function of class C^1, $u \in \Omega_z$, $z \in C^1[a,b]$, with $\Omega_z := C^1([a,b] \times S)$. Let

$$[u,z]_2(t,s) := (t, s, u(t,s), {}^C\mathbb{D}_+^{\gamma_1} u(t,s), {}^C\mathbb{D}_-^{\gamma_1} u(t,s), z(t)).$$

Theorem 7. Let $(u^\star, z^\star) \in \Omega_z \times C^1[a,b]$ be a solution of (6.2). Let

$$\lambda(t) = \exp\left(-\int_a^t \int_S \frac{\partial L_{z2}}{\partial z}[u^\star, z^\star]_2(\tau, s)\, ds\, d\tau \right).$$

If the maps

$$(t,s) \mapsto \mathbb{D}_{b-}^{\gamma_1}\left(\lambda(t) \frac{\frac{\partial L_{z2}}{\partial {}^C\mathbb{D}_{a+}^{\gamma_1} u}[u^\star, z^\star]_2(t,s)}{g'(t)} \right), \quad (t,s) \mapsto \mathbb{D}_{a+}^{\gamma_1}\left(\lambda(t) \frac{\frac{\partial L_{z2}}{\partial {}^C\mathbb{D}_{b-}^{\gamma_1} u}[u^\star, z^\star]_2(t,s)}{g'(t)} \right),$$

$$(t,s) \mapsto \mathbb{D}_{b_i-}^{\gamma_1}\left(\lambda(t) \frac{\frac{\partial L_{z2}}{\partial {}^C\mathbb{D}_{a_i+}^{\gamma_1} u}[u^\star, z^\star]_2(t,s)}{g'(s_i)} \right), \quad \text{and} \quad (t,s) \mapsto \mathbb{D}_{a_i+}^{\gamma_1}\left(\lambda(t) \frac{\frac{\partial L_{z2}}{\partial {}^C\mathbb{D}_{b_i-}^{\gamma_1} u}[u^\star, z^\star]_2(t,s)}{g'(s_i)} \right)$$

are continuous on $[a,b] \times S$, then for all $(t,s) \in [a,b] \times S$,

$$\begin{aligned} &\lambda(t) \frac{\partial L_{z2}}{\partial u}[u^\star, z^\star]_2(t,s) + g'(t) \mathbb{D}_{b-}^{\gamma_1}\left(\lambda(t) \frac{\frac{\partial L_{z2}}{\partial {}^C\mathbb{D}_{a+}^{\gamma_1} u}[u^\star, z^\star]_2(t,s)}{g'(t)} \right) + g'(t) \mathbb{D}_{a+}^{\gamma_1}\left(\lambda(t) \frac{\frac{\partial L_{z2}}{\partial {}^C\mathbb{D}_{b-}^{\gamma_1} u}[u^\star, z^\star]_2(t,s)}{g'(t)} \right) \\ &\quad + \sum_{i=1}^n \left[g'(s_i) \mathbb{D}_{b_i-}^{\gamma_1}\left(\lambda(t) \frac{\frac{\partial L_{z2}}{\partial {}^C\mathbb{D}_{a_i+}^{\gamma_1} u}[u^\star, z^\star]_2(t,s)}{g'(s_i)} \right) + g'(s_i) \mathbb{D}_{a_i+}^{\gamma_1}\left(\lambda(t) \frac{\frac{\partial L_{z2}}{\partial {}^C\mathbb{D}_{b_i-}^{\gamma_1} u}[u^\star, z^\star]_2(t,s)}{g'(s_i)} \right) \right] = 0. \end{aligned}$$

Proof. The variation of $(u^\star, z^\star)$ is given by $(u^\star(t,s) + \epsilon\delta(t,s), Z(t))$, where $\delta \in \Omega_z$ with $\delta(t,s) = 0$ if $t \in \{a,b\}$ or $s \in \{a_i, b_i\}$, and

$$Z(t) = \frac{dz^\star}{d\epsilon}(t, u^\star(t,s) + \epsilon\delta(t,s))\Big|_{\epsilon=0}.$$

Then,

$$\begin{aligned} Z'(t) &= \frac{d}{d\epsilon}\int_S L_{z2}(t, u^\star(t,s)+\epsilon\delta(t,s), {}^C\mathbb{D}_+^{\gamma_1}(u^\star(t,s)+\epsilon\delta(t,s)), {}^C\mathbb{D}_-^{\gamma_1}(u^\star(t,s)+\epsilon\delta(t,s)),\\ &\qquad z^\star(t,u^\star(t,s)+\epsilon\delta(t,s)))\,ds\\ &= \int_S\Bigg[\frac{\partial L_{z2}}{\partial u}[u^\star,z^\star]_2(t,s)\delta(t,s)+\frac{\partial L_{z2}}{\partial z}Z(t)\\ &\quad+\frac{\partial L_{z2}}{\partial {}^C\mathbb{D}_{a+}^{\gamma_1}u}[u^\star,z^\star]_2(t,s){}^C\mathbb{D}_{a+}^{\gamma_1}\delta(t,s)+\frac{\partial L_{z2}}{\partial {}^C\mathbb{D}_{b-}^{\gamma_1}u}[u^\star,z^\star]_2(t,s){}^C\mathbb{D}_{b-}^{\gamma_1}\delta(t,s)\\ &\quad+\sum_{i=1}^n\left[\frac{\partial L_{z2}}{\partial {}^C\mathbb{D}_{a_i+}^{\gamma_1}u}[u^\star,z^\star]_2(t,s){}^C\mathbb{D}_{a_i+}^{\gamma_1}\delta(t,s)+\frac{\partial L_{z2}}{\partial {}^C\mathbb{D}_{b_i-}^{\gamma_1}u}[u^\star,z^\star]_2(t,s){}^C\mathbb{D}_{b_i-}^{\gamma_1}\delta(t,s)\right]\Bigg]ds. \end{aligned}$$

Solving this ODE, and using fractional integration by parts, we arrive at

$$\begin{aligned} &\int_a^b\int_S \lambda(t)\frac{\partial L_{z2}}{\partial u}[u^\star,z^\star]_2(t,s)\\ &\qquad+g'(t)\mathbb{D}_{b-}^{\gamma_1}\left(\lambda(t)\frac{\frac{\partial L_{z2}}{\partial {}^C\mathbb{D}_{a+}^{\gamma_1}u}[u^\star,z^\star]_2(t,s)}{g'(t)}\right)+g'(t)\mathbb{D}_{a+}^{\gamma_1}\left(\lambda(t)\frac{\frac{\partial L_{z2}}{\partial {}^C\mathbb{D}_{b-}^{\gamma_1}u}[u^\star,z^\star]_2(t,s)}{g'(t)}\right)\\ &\quad+\sum_{i=1}^n\left[g'(s_i)\mathbb{D}_{b_i-}^{\gamma_1}\left(\lambda(t)\frac{\frac{\partial L_{z2}}{\partial {}^C\mathbb{D}_{a_i+}^{\gamma_1}u}[u^\star,z^\star]_2(t,s)}{g'(s_i)}\right)+g'(s_i)\mathbb{D}_{a_i+}^{\gamma_1}\left(\lambda(t)\frac{\frac{\partial L_{z2}}{\partial {}^C\mathbb{D}_{b_i-}^{\gamma_1}u}[u^\star,z^\star]_2(t,s)}{g'(s_i)}\right)\right]\\ &\qquad\qquad\times\delta(t,s)ds\,dt=0, \end{aligned}$$

proving the desired formula by the arbitrariness of function $\delta(\cdot,\cdot)$. □

7. Conclusions

In this paper we investigated several fundamental problems of the calculus of variations, involving a fractional derivative of variable order, and with the kernel depending on an arbitrary function g. More specifically, the functional to minimize or maximize depends on time, the state function, and the left and right Caputo fractional derivatives. We have considered the fixed and free endpoint problems, as well as with additional constraints. Then the problem was generalized, first by considering fractional derivatives of any order and then the generalized Herglotz problem. Since our fractional derivative depends on an arbitrary kernel $g(\cdot)$ and the fractional order is not constant, we obtain numerous works already known in the fractional calculus of variations as particular cases of ours. Also, new ones can be produced by the arbitrariness of those functions. We believe that this is a path of research to be followed, to avoid the multiplication of works dealing with similar problems.

A question that deserves study is how to solve the fractional differential equations presented in this work. As is recognized, in most cases there is no method for analytically solving these equations and so numerical methods are used to find approximations to the optimal solution. For this type of fractional derivative, there is still no numerical method developed and this topic will be studied in a future work.

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Conflict of interest

The author declares no conflict of interest.

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